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ON BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT. A general theorem (principle of a priori boundedness) on solvability of the boundary value problem

$$\frac{dx(t)}{dt} = f(x)(t), \quad h(x) = 0$$

is established, where

$$f : C([a, b]; R^n) \rightarrow L([a, b]; R^n) \quad \text{and} \quad h : C([a, b]; R^n) \rightarrow R^n$$

are continuous operators. As an application, a two-point boundary value problem for the system of ordinary differential equations is considered.

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$$\frac{dx(t)}{dt} = f(x)(t), \quad h(x) = 0$$

სასაზღვრო ამოცანის ამოხსნადობის შესახებ, სადაც $f : C([a, b]; R^n) \rightarrow L([a, b]; R^n)$ და $h : C([a, b]; R^n) \rightarrow R^n$ უწყვეტი ოპერატორებია. ამ თეორემის საფუძველზე გამოკვლეულია ორწერტილოვანი სასაზღვრო ამოცანა ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისათვის.

1. STATEMENT OF THE PROBLEM AND MAIN NOTATION

Let n be a natural number, $I = [a, b]$ be a segment of the real axis, and let $f : C(I; R^n) \rightarrow L(I; R^n)$ and $h : C(I; R^n) \rightarrow R^n$ be continuous operators satisfying for every $\rho \in]0, +\infty[$ the conditions

$$\begin{aligned} \sup \{ \|f(x)(\cdot)\| : x \in C(I; R^n), \|x\|_C \leq \rho \} &\in L(I; R), \\ \sup \{ \|h(x)\| : x \in C(I; R^n), \|x\|_C \leq \rho \} &< +\infty. \end{aligned}$$

Consider the functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

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with the boundary condition

$$h(x) = 0. \quad (2)$$

Under the solution of the equation (1) we mean an absolutely continuous vector function $x : I \rightarrow R^n$ which almost everywhere on I satisfies this equation, and under the solution of the problem (1), (2) we mean a solution of the equation (1) satisfying (2).

The theorem on the existence of a solution of the problem (1), (2) which will be proved below and be called the principle of a priori boundedness, generalizes Conti–Opial type theorems [2, 3, 7, 10–13] and supplements earlier known criteria for the solvability of boundary value problems for systems of ordinary differential and functional differential equations [1–14].

On the basis of the above-mentioned principle of a priori boundedness, we have obtained effective criteria for the solvability of the boundary value problem

$$\frac{dx(t)}{dt} = f_0(t, x(t)), \quad (3)$$

$$x(t_1(x)) = A(x) x(t_2(x)) + c_0, \quad (4)$$

where $f_0 : I \times R^n \rightarrow R^n$ is a vector function satisfying the local Carathéodory conditions, $c_0 \in R^n$ and $t_i : C(I; R^n) \rightarrow I$ ($i = 1, 2$) and $A : C(I; R^n) \rightarrow R^n$ are continuous operators.

The use is made of the following notation:

$$I = [a, b], R =] - \infty, +\infty[, R_+ = [0, +\infty[;$$

R^n is the space of n -dimensional column vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in R$ ($i = 1, \dots, n$) and the norm $\|x\| = \sum_{i=1}^n |x_i|$;

$$\text{if } x = (x_i)_{i=1}^n, \text{ then } \text{sgn}(x) = (\text{sgn } x_i)_{i=1}^n;$$

$$x \cdot y \text{ is the scalar product of the vectors } x \text{ and } y \in R^n;$$

$R^{n \times n}$ is the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with the components $x_{ik} \in R$ ($i, k = 1, \dots, n$) and the norm $\|X\| = \sum_{i,k=1}^n |x_{ik}|$;

$C(I; R^n)$ is the space of continuous vector functions $x : I \rightarrow R^n$ with the norm $\|x\|_C = \max\{\|x(t)\| : t \in I\}$;

$L(I; R^n)$ is the space of summable vector functions $x : I \rightarrow R^n$ with the norm $\|x\|_L = \int_a^b \|x(t)\| dt$.

2. THE PRINCIPLE OF A PRIORI BOUNDEDNESS

To formulate our basic result, we will need the following

Definition 1. The pair (p, l) of continuous operators $p : C(I; R^n) \times C(I; R^n) \rightarrow L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \rightarrow R^n$ is said to be *consistent* if:

(i) for any fixed $x \in C(I; R^n)$ the operators $p(x, \cdot) : C(I; R^n) \rightarrow L(I; R^n)$ and $l(x, \cdot) : C(I; R^n) \rightarrow R^n$ are linear;

(ii) for any x and $y \in C(I; R^n)$ and for almost all $t \in I$ the inequalities

$$\|p(x, y)(t)\| \leq \alpha(t, \|x\|_C) \|y\|_C, \quad \|l(x, y)\| \leq \alpha_0(\|x\|_C) \|y\|_C$$

are fulfilled, where $\alpha_0 : R_+ \rightarrow R_+$ is nondecreasing and $\alpha : I \times R_+ \rightarrow R_+$ is summable in the first argument and nondecreasing in the second one;

(iii) there exists a positive number β such that for any $x \in C(I; R^n)$, $q \in C(I; R^n)$ and $c_0 \in R^n$ an arbitrary solution y of the boundary value problem

$$\frac{dy(t)}{dt} = p(x, y)(t) + q(t), \quad l(x, y) = c_0 \quad (5)$$

admits the estimate

$$\|y\|_C \leq \beta(\|c_0\| + \|q\|_L). \quad (6)$$

Theorem 1. *Let there exist a positive number ρ and a consistent pair (p, l) of continuous operators $p : C(I; R^n) \times C(I; R^n) \rightarrow L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \rightarrow R^n$ such that for any $\lambda \in]0, 1[$ an arbitrary solution of the problem*

$$\frac{dx(t)}{dt} = p(x, x)(t) + \lambda[f(x)(t) - p(x, x)(t)], \quad (7)$$

$$l(x, x) = \lambda[l(x, x) - h(x)] \quad (8)$$

admits the estimate

$$\|x\|_C \leq \rho. \quad (9)$$

Then the problem (1), (2) is solvable.

Proof. Let α , α_0 and β be the functions and numbers appearing in Definition 1. Set

$$\gamma(t) = 2\rho\alpha(t, 2\rho) + \sup \{\|f(x)(t)\| : x \in C(I; R^n), \|x\|_C \leq 2\rho\},$$

$$\gamma_0 = 2\rho\alpha_0(2\rho) + \sup \{\|h(x)\| : x \in C(I; R^n), \|x\|_C \leq 2\rho\},$$

$$\sigma(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \rho \\ 2 - s/\rho & \text{for } \rho < s < 2\rho, \\ 0 & \text{for } s \geq 2\rho \end{cases} \quad (10)$$

$$\begin{aligned} q(x)(t) &= \sigma(\|x\|_C)[f(x)(t) - p(x, x)(t)], \\ c_0(x) &= \sigma(\|x\|_C)[l(x, x) - h(x)]. \end{aligned} \quad (11)$$

Then $\gamma \in L(I; R)$, $\gamma_0 < +\infty$, and for every $x \in C(I; R^n)$ and almost all $t \in I$, the inequalities

$$\|q(x)(t)\| \leq \gamma(t), \quad \|c_0(x)\| \leq \gamma_0. \quad (12)$$

hold.

For an arbitrarily fixed $x \in C(I; R^n)$, let us consider the linear boundary value problem

$$\frac{dy(t)}{dt} = p(x, y)(t) + q(x)(t), \quad l(x, y) = c_0(x). \quad (13)$$

By virtue of the condition (iii) from Definition 1, the homogeneous problem

$$\frac{dy(t)}{dt} = p(x, y)(t), \quad l(x, y) = 0 \quad (13_0)$$

has only the trivial solution. However, by Theorem 1.1 in [9], the conditions (i) and (ii) from Definition 1 and the absence of nontrivial solutions of the problem (13₀) guarantee the unique solvability of the problem (13). On the other hand, by virtue of the conditions (ii) and (iii) from Definition 1 and the inequalities (12), the solution y of the problem (11) admits the estimate

$$\|y\|_C \leq \rho_0, \quad \|y'(t)\| \leq \gamma^*(t) \quad \text{for almost all } t \in I, \quad (14)$$

where $\rho_0 = \beta(\gamma_0 + \|\gamma\|_L)$, $\gamma^*(t) = \alpha(t, \rho_0)\rho_0 + \gamma(t)$.

Let $u : C(I; R^n) \rightarrow C(I; R^n)$ be an operator which to every $x \in C(I; R^n)$ assigns the solution y of the problem (13). Due to Corollary 1.6 from [9], the operator u is continuous. On the other hand, by (14) we have

$$\|u(x)\|_C \leq \rho_0, \quad \|u(x)(t) - u(x)(s)\| \leq \left| \int_s^t \gamma^*(\xi) d\xi \right| \quad \text{for } s \text{ and } t \in I.$$

Consequently, the operator u continuously maps the ball $C_{\rho_0} = \{x \in C(I; R^n) : \|x\|_C \leq \rho_0\}$ into its own compact subset. Therefore, owing to Schauder's principle, there exists $x \in C_{\rho_0}$ such that $u(x)(t) = x(t)$ for $t \in I$. By the equalities (11), x is obviously a solution of the problem (7), (8), where

$$\lambda = \sigma(\|x\|_C). \quad (15)$$

Let us show that x admits the estimate (9). Suppose the contrary. Then either

$$\rho < \|x\|_C \leq 2\rho, \quad (16)$$

or

$$\|x\|_C > 2\rho. \quad (17)$$

If we assume that the inequality (16) is fulfilled, then because of (10) and (15) we have $\lambda \in]0, 1[$. However, by the conditions of the theorem, in this case we have the estimate (9) which contradicts (16). Suppose now that (17) is fulfilled. Then by (10) and (15), we have $\lambda = 0$. Hence x is a solution of the problem (13₀). But this is impossible because (13₀) has only

the trivial solution. The above-obtained contradiction proves the validity of the estimate (9).

By virtue of (9), (10) and (15), it is clear that $\lambda = 1$ and hence x is a solution of the problem (1), (2). ■

Following [10], we introduce

Definition 2. Let $p : C(I; R^n) \times C(I; R^n) \rightarrow L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \rightarrow R^n$ be arbitrary, while $p_0 : C(I; R^n) \rightarrow L(I; R^n)$ and $l_0 : C(I; R^n) \rightarrow R^n$ be linear operators. We say that the pair (p_0, l_0) belongs to the set $\mathcal{E}_{p,l}^n$ if there exists a sequence $x_k \in C(I; R^n)$ ($k = 1, 2, \dots$) such that for every $y \in C(I; R^n)$ the following conditions are fulfilled:

$$\lim_{k \rightarrow \infty} \int_0^t p(x_k, y)(s) ds = \int_0^t p_0(y)(s) ds \quad \text{uniformly on } I,$$

$$\lim_{k \rightarrow \infty} l(x_k, y) = l_0(y).$$

Definition 3. We say that the pair (p, l) of continuous operators $p : C(I; R^n) \times C(I; R^n) \rightarrow L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \rightarrow R^n$ belongs to the *Opial class* O_0^n if:

(i) for any fixed $x \in C(I; R^n)$ the operators $p(x, \cdot) : C(I; R^n) \rightarrow L(I; R^n)$ and $l(x, \cdot) : C(I; R^n) \rightarrow R^n$ are linear;

(ii') for any x and $y \in C(I; R^n)$ and for almost all $t \in I$, the inequalities

$$\|p(x, y)(t)\| \leq \alpha(t)\|y\|_C, \quad \|l(x, y)\| \leq \alpha_0\|y\|_C$$

are fulfilled, where $\alpha : I \rightarrow R_+$ is summable and $\alpha_0 \in R_+$;

(iii') for every $(p_0, l_0) \in \mathcal{E}_{p,l}^n$ the problem

$$\frac{dy(t)}{dt} = p_0(y)(t), \quad l_0(y) = 0 \tag{18}$$

has only the trivial solution.

By Lemma 2.2 from [10], if $(p, l) \in O_0^n$, then the pair (p, l) is consistent. Therefore from Theorem 1 we have

Corollary 1. *Let there exist a positive number ρ and a pair of operators $(p, l) \in O_0^n$ such that for every $\lambda \in]0, 1[$ an arbitrary solution of the problem (7), (8) admits the estimate (9). Then the problem (1), (2) is solvable.*

Definition 4. The linear operator $p_0 : C(I; R^n) \rightarrow L(I; R^n)$ is said to be *strongly bounded* if there exists a summable function $\alpha : I \rightarrow R_+$ such that for every $y \in C(I; R^n)$, the inequality $\|p_0(y)(t)\| \leq \alpha(t)\|y\|_C$ is fulfilled almost everywhere on I .

Let $p(x, y)(t) \equiv p_0(y)(t)$ and $l(x, y) \equiv l_0(y)$, where $p_0 : C(I; R^n) \rightarrow L(I; R^n)$ is a strongly bounded linear operator and $l_0 : C(I; R^n) \rightarrow R^n$ is a bounded linear operator. Then by Definition 3, for the condition $(p, l) \in O_0^n$ to be fulfilled, it is necessary and sufficient that the problem (18) have only the trivial solution. Therefore from Corollary 1 follows

Corollary 2. *Let there exist a positive number ρ , a linear strongly bounded operator $p_0 : C(I; R^n) \rightarrow L(I; R^n)$ and a linear bounded operator $l_0 : C(I; R^n) \rightarrow R^n$ such that the problem (18) has only the trivial solution and for every $\lambda \in]0, 1[$ an arbitrary solution of the problem*

$$\frac{dx(t)}{dt} = p_0(x)(t) + \lambda[f(x)(t) - p_0(x)(t)], \quad l_0(x) = \lambda[l_0(x) - h(x)]$$

admits the estimate (9). Then the problem (1), (2) is solvable.

3. THEOREM ON THE SOLVABILITY OF THE PROBLEM (3), (4)

As is mentioned in Section 1, we investigate the problem (3), (4) under the assumptions that the vector function $f_0 : I \times R^n \rightarrow R^n$ satisfies the local Carathéodory conditions, and the operators $t_i : C(I; R^n) \rightarrow I$ ($i = 1, 2$) and $A : C(I; R^n) \rightarrow R^{n \times n}$ are continuous.

Assume

$$I_0 = \{t_1(x) : x \in C(I; R^n)\},$$

$$\|A(x)\|_0 = \max \{\|A(x)y\| : y \in R^n, \|y\| = 1\}.$$

The following theorem holds.

Theorem 2. *Let there exist summable functions $g_1 : I \rightarrow R$, $g_2 : I \rightarrow R_+$ and a number $\delta \in]0, 1[$ such that*

$$\begin{aligned} f_0(t, x) \cdot \operatorname{sgn}[(t - t_0)x] &\leq \\ &\leq g_1(t)\|x\| + g_2(t) \quad \text{for } t \in I, t_0 \in I_0, x \in R^n \end{aligned} \quad (19)$$

and

$$\begin{aligned} \exp \left[\int_{t_1(x)}^{t_2(x)} g_1(t) dt \cdot \operatorname{sgn}(t_2(x) - t_1(x)) \right] \|A(x)\|_0 &\leq \\ &\leq \delta \quad \text{for } x \in C(I; R^n). \end{aligned} \quad (20)$$

Then the problem (3), (4) is solvable.

Proof. For every x and $y \in C(I; R^n)$, we suppose $f(x)(t) = f_0(t, x(t))$, $h(x) = x(t_1(x)) - A(x)x(t_2(x)) - c_0$,

$$p(x, y)(t) = [g_1(t) \operatorname{sgn}(t - t_1(x))]y(t), \quad l(x, y) = y(t_1(x)).$$

Obviously, the operators $p : C(I; R^n) \times C(I; R^n) \rightarrow L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \rightarrow R^n$ are continuous and the pair (p, l) is consistent.

By Theorem 1, to prove Theorem 2 it suffices to establish the uniform with respect to $\lambda \in]0, 1[$ a priori boundedness of solutions of the problem

$$\begin{aligned} \frac{dx(t)}{dt} &= (1 - \lambda)[g_1(t) \operatorname{sgn}(t - t_1(x))]x(t) + \lambda f_0(t, x(t)), \\ x(t_1(x)) &= \lambda[A(x)x(t_2(x)) + c_0]. \end{aligned}$$

Let x be an arbitrary solution of this problem for some $\lambda \in]0, 1[$. Suppose $u(t) = \|x(t)\|$. Then by (19),

$$u'(t) \operatorname{sgn}(t - t_1(x)) \leq g_1(t)u(t) + g_2(t) \quad \text{for } t \in I. \quad (21)$$

On the other hand,

$$u(t_1(x)) \leq \|A(x)\|_0 u(t_2(x)) + \|c_0\|. \quad (22)$$

The inequality (21) implies

$$u(t) \leq \exp\left(\int_{t_1(x)}^t g_1(s) \operatorname{sgn}(s - t_1(x)) ds\right) u(t_1(x)) + \rho_1 \quad \text{for } t \in I, \quad (23)$$

where $\rho_1 = \exp(\|g_1\|_L) \|g_2\|_L$. This, with regard for (20) and (22), yields

$$u(t_2(x)) \leq \delta u(t_2(x)) + \|c_0\| \exp(\|g_1\|_L) + \rho_1$$

and, consequently,

$$u(t_2(x)) \leq \rho_2, \quad (24)$$

where $\rho_2 = (1 - \delta)^{-1}[\|c_0\| \exp(\|g_1\|_L) + \rho_1]$. However, as it is clear from (20),

$$\|A(x)\|_0 \leq \delta \exp(\|g_1\|_L).$$

According to this inequality, from (22)–(24) there follows the estimate (9), where $\rho_0 = \delta \exp(2\|g_1\|_L)(\delta \rho_2 + \|c_0\|) + \rho_1$ is a positive constant, which does not depend on λ and x . ■

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