On the Set of Solutions of the Cauchy Problem for Higher Order Non-Lipshitzian Ordinary Differential Equations

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In the present report, the initial value problem

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{1}$$

$$u^{(i-1)}(a) = 0 \;\; (i = 1, \dots, n) \tag{2}$$

is considered, where n is an arbitrary natural number, $-\infty < a < b < +\infty$, while $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function. We are interested in the case where the function f with respect to the phase variables does not satisfy the Lipshitz condition in the neighborhood of the point $(0, \dots, 0) \in \mathbb{R}^n$. In this case, as far as we know, the questions on the unique and multivalued solvability of problem (1), (2) remain actually open. The structure of a set of solutions of that problem is insufficiently studied as well (see, e.g., [1–5] and the references therein). The results given below fill to some extent this gap. Those cover the case where the function f admits one of the following four representations:

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)|x_i|^{\lambda_i}, \tag{3}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)\omega(|x_i|), \tag{4}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)|x_i|^{\lambda_i} + g(t), \tag{5}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)\omega(|x_i|) + g(t). \tag{6}$$

Here $\lambda_i \in]0, 1[\;\; (i = 1, \dots, n)$,

$$\omega(x) = \begin{cases} \dfrac{1}{\ln(1 + 1/x)} & \text{for } \; x > 0, \\[2mm] 0 & \text{for } \; x = 0, \end{cases}$$

while $f_0 : [a, b] \times \mathbb{R}^n \to \mathbb{R}_+$, $g_i : [a, b] \to \mathbb{R}_+$ $(i = 1, \dots, n)$, $g : [a, b] \to \mathbb{R}_+$ are continuous functions. It is also assumed that the function f_0 on the set $[a, b] \times \mathbb{R}^n$ satisfies one of the following two conditions:

$$f_0(t, 0, \dots, 0) = 0, \;\; f_0(t, x_1, \dots, x_n) \leq r\Big(1 + \sum_{i=1}^{n} |x_i|\Big), \tag{7}$$

$$f_0(t, 0, \dots, 0) = 0, \;\; \big|f_0(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big| \leq r \sum_{i=1}^{n} |x_i - y_i|, \tag{8}$$

where r is a positive constant.

We use the following notation.

$\mathbb{R}_+ = [0, +\infty[$;

$$\mathcal{D}^n(]a,b[\,;g) = \bigg\{(t, x_1, \dots, x_n) \in]a,b[\times \mathbb{R}^n : \ x_i \geq \frac{1}{(n-i)!} \int\limits_a^t (t-s)^{n-i} g(s)\,ds \ \ (i = 1, \dots, n)\bigg\};$$

$S_f([a,b];t_0)$, where $t_0 \in [a,b[$, is the set of solutions of problem (1), (2) defined on the interval $[a,b]$ and satisfying the conditions

$$u^{(i-1)}(t) = 0 \ \text{ for } \ a \leq t \leq t_0, \quad u^{(i-1)}(t) > 0 \ \text{ for } \ t_0 < t \leq b \ \ (i = 1, \dots, n);$$

$S_f([a,b])$ is the set of all nontrivial solutions of problem (1), (2) on the interval $[a,b]$.

Theorem 1. *Let*

$$f(t, 0, \dots, 0) = 0 \ \textit{ for } \ a \leq t \leq b,$$

and let on the set $[a,b] \times \mathbb{R}^n$ *one of the following two conditions*

$$\sum_{i=1}^n g_i(t)|x_i|^{\lambda_i} \leq f(t, x_1, \dots, x_n) \leq r\Big(1 + \sum_{i=1}^n |x_i|\Big),$$

$$\sum_{i=1}^n g_i(t)\omega(|x_i|) \leq f(t, x_1, \dots, x_n) \leq r\Big(1 + \sum_{i=1}^n |x_i|\Big)$$

be satisfied, where $\lambda_i \in]0,1[$ $(i = 1, \dots, n)$ *and* $r > 0$ *are constants, and* $g_i : [a,b] \to \mathbb{R}_+$ $(i = 1, \dots, n)$ *are continuous functions such that*

$$\sum_{i=1}^n g_i(t) > 0 \ \textit{ for } \ a < t < b. \tag{9}$$

Then

$$S_f([a,b];t_0) \neq \varnothing \ \textit{ for } \ a \leq t_0 < b, \quad S_f([a,b]) = \bigcup_{a \leq t_0 < b} S_f([a,b];t_0). \tag{10}$$

Corollary 1. *If the function* f *admits representation* (3) *or* (4), *then for condition* (10) *to be satisfied it is sufficient that inequalities* (7) *and* (9) *hold.*

Theorem 2. *Let there exist continuous functions* $g : [a,b] \to \mathbb{R}_+$ *and* $h_i :]a,b[\to \mathbb{R}_+$ $(i = 1, \dots, n)$ *such that the function* f *on the set* $[a,b] \times \mathbb{R}^n$ *admits the estimate*

$$f(t, x_1, \dots, x_n) \geq g(t),$$

while on the set $\mathcal{D}^n(]a,b[\,;g)$ *satisfies the Lipschitz condition*

$$\big|f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big| \leq \sum_{i=1}^n h_i(t)|x_i - y_i|.$$

If, moreover,

$$\int\limits_a^b (t-a)^{n-i} h_i(t)\,dt < +\infty \ \ (i = 1, \dots, n),$$

then problem (1), (2) *has a unique solution.*

Corollary 2. *Let the function f admit representation* (5) *and let there exist a nonnegative constant α such that along with* (8) *the conditions*

$$\liminf_{t\to a} \frac{g(t)}{(t-a)^\alpha} > 0, \tag{11}$$

$$\int_a^b (t-a)^{(n-i+1)\lambda_i-(1-\lambda_i)\alpha-1} g_i(t)\,dt < +\infty \quad (i=1,\dots,n) \tag{12}$$

are satisfied. Then problem (1), (2) *is uniquely solvable and its solution satisfies the inequalities*

$$u^{(i-1)}(t) > 0 \ \ for \ \ a < t \le b \ \ (i=1,\dots,n). \tag{13}$$

Remark 1. In view of the continuity of the functions $g_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$, for condition (12) to be satisfied it is sufficient that the constant α satisfy the inequality

$$\alpha < \min\Big\{ \frac{(n-i+1)\lambda_i}{1-\lambda_i} \ : \ i=1,\dots,n \Big\}. \tag{14}$$

Corollary 3. *Let the function f admit representation* (6) *and let there exist a nonnegative constant α such that along with* (8) *and* (11)*, the conditions*

$$\int_a^b (t-a)^{-\alpha} g_i(t)dt < +\infty \quad (i=1,\dots,n) \tag{15}$$

are satisfied. Then problem (1), (2) *is uniquely solvable and its solution satisfies inequalities* (13).

As an example, consider the differential equations

$$u^{(n)} = \sum_{i=1}^n g_i(t)|u^{(i-1)}|^{\lambda_i}, \tag{16}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)|u^{(i-1)}|^{\lambda_i} + g(t), \tag{17}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)\omega(|u^{(i-1)}|), \tag{18}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)\omega(|u^{(i-1)}|) + g(t), \tag{19}$$

where $\lambda_i \in]0,1[$ $(i=1,\dots,n)$, while $g_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$, $g : [a,b] \to \mathbb{R}_+$ are continuous functions.

From Corollaries 1 and 2 it follows

Corollary 4. *Let conditions* (9) *and* (11) *hold, where α is a nonnegative constant satisfying inequality* (14)*. Then problem* (16), (2) *has a continuum of solutions, while problem* (17), (2) *has a unique solution.*

From Corollaries 1 and 3 follows

Corollary 5. *Let conditions* (9), (11) *and* (15) *hold, where α is a nonnegative constant. Then problem* (18), (2) *has a continuum of solutions, while problem* (19), (2) *is uniquelly solvable.*

Therefore, a multivalued solvable initial value problem can be made uniquely solvable by using an arbitrarily small perturbation of the equation under consideration.

References

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