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THE CAUCHY WEIGHTED PROBLEM FOR SINGULAR IN TIME AND PHASE VARIABLES HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. Unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability, unique solvability and unsolvability of the Cauchy weighted problem for singular in time and phase variables higher order delay ordinary differential equations.

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1 Statement of the problem and formulation of the main results

The Cauchy problem for ordinary differential and functional differential equations has been studied in sufficient detail for both regular case and the case where these equations have nonintegrable singularity in an independent variable at the initial point (see [1–8] and the references therein). In [9], optimal sufficient conditions are established for the solvability of that problem for singular in phase variables higher order differential equations. As for singular in phase variables functional differential equations and, namely, for delay differential equations, for them the Cauchy problem still remains unstudied. The present paper is devoted to filling this existing gap.

We use the following notation.

$\mu! = 1$ for $\mu \in]-1,0]$, and $\mu! = \prod_{i=0}^{m}(i+\mu_0)$ for $\mu = m+\mu_0$, where $\mu_0 \in]0,1[$, and m is a nonnegative integer;

$\mathbb{R}_+ = [0,+\infty[$; $\mathbb{R}_{0+} =]0,+\infty[$;

$L([a,b])$ is the space of Lebesgue integrable on $[a,b]$ real functions, while $L_{loc}(]a,b])$ is the space of real functions which are integrable on $[a+\varepsilon,b]$ for any $\varepsilon \in]0,b-a[$;

$\widetilde{C}^m([a,b])$ is the space of m-times continuously differentiable on $[a,b]$ real functions whose mth order derivative is absolutely continuous.

We study the delay differential equation

$$u^{(n)}(t) = f\big(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\big) \tag{1.1}$$

with the weighted initial conditions

$$\limsup_{t\to a} \frac{u^{(i-1)}(t)}{(t-a)^{n-i+\alpha}} < +\infty \quad (i=1,\dots,n). \tag{1.2}$$

Here $f :]a,b[\times\mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a measurable in the first and continuous in the last n arguments function, $\tau_i : [a,b] \to [a,b]$ $(i=1,\dots,n)$ are continuous functions satisfying the inequalities

$$a < \tau_i(t) \le t \ \text{ for } \ a < t \le b \ \ (i=1,\dots,n), \tag{1.3}$$

and α is a positive constant.

An important particular case of Eq. (1.1) is the differential equation

$$u^{(n)}(t) = \sum_{i=1}^{n} \big(p_i(t)u^{(i-1)}(\tau_i(t)) + q_i(t)(u^{(i-1)}(\tau_i(t)))^{-\lambda_i}\big) + q_0(t), \tag{1.4}$$

with nonnegative coefficients $p_i \in L_{loc}(]a,b])$ $(i=1,\dots,n)$, $q_i \in L([a,b])$ $(i=0,\dots,n)$, and with positive exponents λ_i $(i=1,\dots,n)$.

A function $u \in \widetilde{C}^{n-1}([a,b])$ is said to be **a solution of Eq.** (1.1) if it satisfies the inequalities

$$u^{(i-1)}(t) > 0 \ \text{ for } \ a < t \le b \ \ (i=1,\dots,n), \tag{1.5}$$

and satisfies Eq. (1.1) almost everywhere on $]a,b[$. A solution of Eq. (1.1), satisfying the initial conditions (1.2), is said to be **a solution of problem** (1.1), (1.2).

For an arbitrary function $q \in L([a,b])$ we put

$$w_i(q)(t) = \frac{1}{(n-i)!}\int_a^t (t-s)^{n-i} q(s)\,ds \ \ (i=1,\dots,n), \tag{1.6}$$

$$D_q = \big\{(t,x_1,\dots,x_n) : \ a<t<b,\ w_1(q)(\tau_1(t)) \le x_1 < +\infty, \dots, w_n(q)(\tau_n(t)) \le x_n < +\infty\big\}. \tag{1.7}$$

Theorems proved below on the solvability and unique solvability of problem (1.1), (1.2) concern the cases when there exists a nonnegative function $q \in L([a,b])$ such that the function f in the domain $]a,b[\times\mathbb{R}^n_{0+}$ admits the lower estimate

$$f(t,x_1,\dots,x_n) \ge q(t), \tag{1.8}$$

and on the domain D_q either admits the upper estimate

$$f(t, x_1, \dots, x_n) \le \sum_{i=1}^{n} p_i(t) x_i + q_0(t), \tag{1.9}$$

or satisfies the Lipschitz condition

$$\big|f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big| \le \sum_{i=1}^{n} p_i(t)|x_i - y_i|. \tag{1.10}$$

The solvability and unsolvability of the above-mentioned problems are investigated separately in the cases where the function f in the domain $]a, b[\times\mathbb{R}^n_{0+}$ admits one of the following two estimates

$$q(t) \le f(t, x_1, \dots, x_n) \le \sum_{i=1}^{n} \big(p_i(t) x_i + q_i(t) x_i^{-\lambda_i}\big) + q_0(t), \tag{1.11}$$

$$f(t, x_1, \dots, x_n) \ge \sum_{i=1}^{n} \big(p_i(t) x_i + q_i(t) x_i^{-\lambda_i}\big) + q_0(t). \tag{1.12}$$

Here and everywhere below it is assumed that λ_i $(i = 1, \dots, n)$ are positive constants, $p_i \in L_{loc}(]a, b])$ $(i = 1, \dots, n)$, $q_i \in L([a, b])$ $(i = 0, \dots, n)$, $q \in L([a, b])$ are nonnegative functions, and

$$\int_a^t q(s)\, ds > 0 \ \text{ for } \ a < t \le b.$$

Theorem 1.1. *Let the function f in the domain $]a, b[\times\mathbb{R}^n_{0+}$ admit estimate* (1.8)*, and on the domain D_q admit estimate* (1.9)*. If, moreover,*

$$\limsup_{t\to a} \bigg(\sum_{i=1}^{n} \frac{\alpha!(t-a)^{-\alpha}}{(n-i+\alpha)!} \int_a^t (\tau_i(s) - a)^{n-i+\alpha} p_i(s)\, ds\bigg) < 1, \tag{1.13}$$

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t q_0(s)\, ds\bigg) < +\infty, \tag{1.14}$$

then problem (1.1), (1.2) *has at least one solution.*

Corollary 1.1. *Let the function f in the domain $]a, b[\times\mathbb{R}^n_{0+}$ admit estimate* (1.11)*, and let inequality* (1.13) *hold. Let, moreover, there exist a number $\beta \ge \alpha$ such that*

$$\liminf_{t\to a} \bigg((t-a)^{-\beta} \int_a^t q(s)\, ds\bigg) > 0, \tag{1.15}$$

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t \Big(\sum_{i=1}^{n} (\tau_i(s) - a)^{-(n-i+\beta)\lambda_i} q_i(s) + q_0(s)\Big)\, ds\bigg) < +\infty. \tag{1.16}$$

Then problem (1.1), (1.2) *has at least one solution.*

The restrictions imposed on the function f in Theorem 1.1 and its corollary are optimal in a certain sense. The following theorem is valid.

Theorem 1.2. *Let the function f in the domain $]a, b[\times\mathbb{R}^n_{0+}$ admit estimate* (1.12)*, and let, moreover, either the condition*

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t \Big(\sum_{i=1}^{n} (\tau_i(s) - a)^{-(n-i+\alpha)\lambda_i} q_i(s) + q_0(s)\Big)\, ds\bigg) = +\infty \tag{1.17}$$

hold, or there exist numbers $b_0 \in]a,b]$ *and* $\delta > 0$ *such that in the interval* $]a,b_0]$ *the inequalities*

$$\sum_{i=1}^{n} \frac{\alpha!(t-a)^{-\alpha}}{(n-i+\alpha)!} \int\limits_a^t (\tau_i(s)-a)^{n-i+\alpha} p_i(s)\,ds \geq 1, \quad (t-a)^{-\alpha} \int\limits_a^t q_0(s)\,ds \geq \delta \tag{1.18}$$

are fulfilled. Then problem (1.1), (1.2) *has no solution.*

The following two corollaries of Theorems 1.1 and 1.2 contain conditions guaranteeing, respectively, the solvability and unsolvability of problem (1.4), (1.2).

Corollary 1.2. *If for some* $\beta \geq \alpha$ *along with the conditon*

$$\liminf_{t\to a} \bigg((t-a)^{-\beta} \int\limits_a^t q_0(s)\,ds \bigg) > 0 \tag{1.19}$$

inequalities (1.13) *and* (1.16) *are satisfied, then problem* (1.4), (1.2) *has at least one solution. And if condition* (1.17) *holds, or for some* $b_0 \in]a,b]$ *and* $\delta > 0$ *in the interval* $]a,b_0]$ *inequalities* (1.18) *are satisfied, then problem* (1.4), (1.2) *has no solution.*

Corollary 1.3. *Let*

$$\liminf_{t\to a} \bigg((t-a)^{-\alpha} \int\limits_a^t q_0(s)\,ds \bigg) > 0, \tag{1.20}$$

and let there exist numbers $b_0 \in]a,b]$ *and* $\ell \geq 0$ *such that the equality*

$$\sum_{i=1}^{n} \frac{(\alpha-1)!}{(n-i+\alpha)!}\, (\tau_i(t)-a)^{n-i+\alpha} p_i(t) = \ell(t-a)^{\alpha-1} \tag{1.21}$$

is satisfied almost everywhere on $]a,b_0[$. *Then for problem* (1.4), (1.2) *to be solvable, it is necessary and sufficient that the inequalities*

$$\ell < 1, \quad \limsup_{t\to a} \bigg((t-a)^{-\alpha} \int\limits_a^t \Big(\sum_{i=1}^{n} (\tau_i(s)-a)^{-(n-i+\alpha)\lambda_i} q_i(s) + q_0(s) \Big)\,ds \bigg) < +\infty \tag{1.22}$$

hold.

The last theorem of this section and its corollaries concern the unique solvability of problems (1.1), (1.2) and (1.4), (1.2).

Theorem 1.3. *Let the function* f *in the domain* $]a,b[\times \mathbb{R}^n_{0+}$ *admit estimate* (1.8), *and on the domain* D_q *satisfy the Lipschitz condition* (1.10). *If, moreover,*

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha} \int\limits_a^t f\big(s, w_1(q)(\tau_1(s)), \dots, w_n(q)(\tau_n(s))\big)\,ds \bigg) < +\infty, \tag{1.23}$$

and inequality (1.13) *holds, then problem* (1.1), (1.2) *has a unique solution.*

Corollary 1.4. *Let for some* $\beta \geq \alpha$ *along with* (1.19) *the condition*

$$\lim_{t\to a} \bigg(\sum_{i=1}^{n} (t-a)^{-\alpha} \int\limits_a^t (\tau_i(s)-a)^{-(n-i)\lambda_i-(1+\lambda_i)\beta+\alpha} q_i(s)\,ds \bigg) = 0 \tag{1.24}$$

be satisfied. If, moreover, the functions p_i $(i=1,\dots,n)$ *satisfy inequality* (1.13)*, then for problem* (1.4), (1.2) *to be uniquely solvable, it is necessary and sufficient that the condition*

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q_0(s)\,ds\bigg)<+\infty \tag{1.25}$$

hold.

Corollary 1.5. *Let along with* (1.20) *the condition*

$$\lim_{t\to a}\bigg(\sum_{i=1}^n (t-a)^{-\alpha}\int\limits_a^t (\tau_i(s)-a)^{-(n-i+\alpha)\lambda_i}q_i(s)\,ds\bigg)=0$$

be satisfied, and let there exist numbers $b_0\in\,]a,b]$ *and* $\ell\geq 0$ *such that equality* (1.21) *holds almost everywhere on* $]a,b_0[\,$*. Then for problem* (1.4), (1.2) *to be uniquely solvable, it is necessary and sufficient that along with* (1.25) *the inequality*

$$\ell<1$$

hold.

As an example, we consider the differential equation

$$u^{(n)}(t)=\sum_{i=1}^n \frac{\ell_{0i}(t-a)^{\alpha-1}}{(\tau_i(t)-a)^{n-i+\alpha}}\,u^{(i-1)}(\tau_i(t))+\sum_{i=1}^m \frac{\ell_i(t-a)^{\alpha-1}(\tau_{n_i}(t)-a)^{\gamma_i}}{(u^{(n_i-1)}(\tau_{n_i}(t)))^{\mu_i}}+\ell_0(t-a)^{\beta-1}, \tag{1.26}$$

where $\ell_{0i}>0$ $(i=1,\dots,n)$, $m\in\{1,\dots,n\}$, $n_i\in\{1,\dots,n\}$ $(i=1,\dots,m)$, $n_i<n_j$ for $i<j$, $\ell_i>0$, $\gamma_i>0$, $\mu_i>0$ $(i=1,\dots,m)$, $\ell_0>0$, $\beta\geq\alpha$.

For this equation, the following statements are valid.

Corollary 1.6. *If* $\beta>\alpha$ $(\beta=\alpha)$*, then for problem* (1.26), (1.2) *to be solvable, it is sufficient (necessary and sufficient) that the inequalities*

$$\sum_{i=1}^n \frac{(\alpha-1)!}{(n-i+\alpha)!}\,\ell_{0i}<1, \tag{1.27}$$

$$\gamma_i\geq(n_i-1+\beta)\mu_i \ \ (i=1,\dots,m)$$

be satisfied.

Corollary 1.7. *If along with* (1.27) *the inequalities*

$$\gamma_i>(n_i-1+\beta)\mu_i \ \ (i=1,\dots,m)$$

hold, then problem (1.26), (1.2) *has a unique solution.*

Remark 1.1. Conditions for the solvability (unique solvability) of the weighted initial value problem in Theorem 1.1 (in Theorem 1.3) and its corollaries cover the case where the differential equation under consideration has a singularity of inifinite order with respect to the time variable at the initial point of the interval $]a,b[\,$, i.e. the case, where

$$\int\limits_a^t f\big(t,(t-a)^{k_1}x_1,\dots,(t-a)^{k_n}x_n\big)\,dt=+\infty \ \text{ for } \ k_i>0, \ \ x_i>0 \ \ (i=1,\dots,n). \tag{1.28}$$

Indeed, let

$$\tau_i(t)=a+\exp\Big(-\frac{r_i}{t-a}\Big)(t-a) \ \text{ for } \ a<t\leq b \ \ (i=1,\dots,n),$$

$$f_i(t,x_1,\dots,x_n)=\sum_{i=1}^n \ell_{0i}(t-a)^{i-n-1}\exp\Big(\frac{(n-i+\alpha)r_i}{t-a}\Big)x_i$$
$$+\sum_{i=1}^m \ell_i(t-a)^{\gamma_i+\alpha-1}\exp\Big(-\frac{\gamma_i r_{n_i}}{t-a}\Big)x_{n_i}^{-\mu_i}+\ell_0(t-a)^{\beta-1} \ \text{ for } \ a<t\leq b \ \ (i=1,\dots,n),$$

where r_i $(i = 1, \dots, n)$ are positive constants, and m, n_i, γ_i, μ_i $(i = 1, \dots, m)$, ℓ_0, β are numbers, satisfying the conditions of Corollary 1.6 (Corollary 1.7). Then condition (1.28) holds but nevertheless problem (1.1), (1.2) has a unique solution.

2 Auxiliary propositions

In this section, we study the differential inequality

$$|u^{(n)}(t)| \leq \sum_{i=1}^{n} p_i(t) |u^{(i-1)}(\tau_i(t))| + q_0(t), \tag{2.1}$$

and the auxiliary differential equation

$$u^{(n)}(t) = f_0(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))) \tag{2.2}$$

with the weighted initial conditions

$$\limsup_{t \to a} \frac{|u^{(i-1)}(t)|}{(t-a)^{n-i+\alpha}} < +\infty \ \ (i = 1, \dots, n). \tag{2.3}$$

Here, as in the first section, it is assumed that $p_i \in L_{loc}(]a, b])$ $(i = 1, \dots, n)$, $q_0 \in L([a, b])$ are nonnegative functions, $\tau_i : [a, b] \to [a, b]$ $(i = 1, \dots, n)$ are continuous functions satisfying inequalities (1.3), and α is a positive constant. As for the function $f_0 :]a, b[\times \mathbb{R}^n \to \mathbb{R}$, it is measurable in the first argument, continuous in the last n arguments and in the domain $]a, b[\times \mathbb{R}^n$ admits the estimate

$$|f_0(t, x_1, \dots, x_n)| \leq \sum_{i=1}^{n} p_i(t) |x_i| + q_0(t). \tag{2.4}$$

By a solution of problem (2.1), (2.3) (problem (2.2), (2.3)) we mean a function $u \in \widetilde{C}^{n-1}([a, b])$ which satisfies the differential inequality (2.1) (the differential equation (2.2)) almost everywhere on $]a, b[$ and along with this satisfies the initial conditions (2.3).

2.1 Lemma on a priori estimate of solutions of problem (2.1), (2.3)

Lemma 2.1. *If inequality* (1.13) *holds, then there exists a positive constant* r *such that for any nonnegative function* $q_0 \in L([a, b])$*, satisfying condition* (1.14)*, an arbitrary solution of problem* (2.1), (2.3) *admits the estimates*

$$|u^{(i-1)}(t)| \leq r \nu(q_0) (t-a)^{n-i+\alpha} \ \ (i = 1, \dots, n) \ \text{ for } \ a \leq t \leq b, \tag{2.5}$$

where

$$\nu(q_0) = \sup \left\{ (t-a)^{-\alpha} \int_a^t q_0(s)\, ds : \ a < t \leq b \right\}. \tag{2.6}$$

Proof. In view of (1.13), there exist numbers $b_0 \in]a, b[$ and $\delta \in]0, 1[$ such that

$$\sum_{i=1}^{n} \frac{\alpha!}{(n-i+\alpha)!} \int_a^t (\tau_i(s) - a)^{n-i+\alpha} p_i(s)\, ds \leq \delta (t-a)^{\alpha} \ \text{ for } \ a \leq t \leq b_0. \tag{2.7}$$

Put

$$r = \frac{1}{1-\delta} \Big(\frac{b-a}{b_0 - a} \Big)^{\alpha} \exp\Big(\sum_{i=1}^{n} \int_{b_0}^{b} \frac{(\tau_i(s) - a)^{n-i}}{(n-i)!}\, p_i(s)\, ds \Big). \tag{2.8}$$

Let u be a solution of problem (2.1), (2.3). Then

$$\rho = \sup\big\{(t-a)^{-\alpha}|u^{(n-1)}(t)| :\ a < t \le b_0\big\} < +\infty,$$
$$|u^{(i-1)}(t)| \le \frac{\alpha!\,\rho}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha} \ \ (i=1,\dots,n) \ \text{ for } \ a \le t \le b_0.$$

If along with these estimates we take into account notation (2.6) and condition (2.7), then from the differential inequality (2.1) we find

$$|u^{(n-1)}(t)| \le \sum_{i=1}^{n} \int_a^t p_i(s)|u^{(i-1)}(\tau_i(s))|\,ds + \int_a^t q_0(s)\,ds$$
$$\le \rho\bigg(\sum_{i=1}^{n} \frac{\alpha!}{(n-i+\alpha)!} \int_a^t (\tau_i(s)-a)^{n-i+\alpha} p_i(s)\,ds\bigg) + (t-a)^{\alpha}\nu(q_0) \le (t-a)^{\alpha}(\delta\rho+\nu(q_0)) \ \text{ for } a \le t \le b_0.$$

Hence we get

$$\rho \le \frac{\nu(q_0)}{1-\delta}\,,$$

and, therefore,

$$|u^{(i-1)}(t)| \le \frac{\nu(q_0)\,\alpha!}{(1-\delta)(n-i+\alpha)!}\,(t-a)^{n-i+\alpha} \ \ (i=1,\dots,n) \ \text{ for } \ a \le t \le b_0. \tag{2.9}$$

We introduce the function

$$v(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}|u^{(n)}(s)|\,ds \ \text{ for } \ a \le t \le b.$$

It is clear that

$$|u^{(i-1)}(t)| \le v^{(i-1)}(t) \le \frac{(t-a)^{n-i}}{(n-i)!}\,v^{(n-1)}(t) \ \ (i=1,\dots,n) \ \text{ for } \ a \le t \le b, \tag{2.10}$$

$$v^{(n-1)}(t) \le \sum_{i=1}^{n} \int_a^t p_i(s)|u^{(i-1)}(\tau_i(s))|\,ds + (b-a)^{\alpha}\nu(q_0) \ \text{ for } \ a \le t \le b. \tag{2.11}$$

By virtue of conditions (2.7), (2.9) and (2.10), from (2.11) we obtain

$$v^{(n-1)}(t) \le \frac{\nu(q_0)}{1-\delta}\bigg(\sum_{i=1}^{n} \frac{\alpha!}{(n-i+\alpha)!} \int_a^{b_0} (\tau_i(s)-a)^{n-i+\alpha} p_i(s)\,ds\bigg)$$
$$+ \int_{b_0}^t \bigg(\sum_{i=1}^{n} \frac{(\tau_i(s)-a)^{n-i}}{(n-i)!}\,p_i(s)v^{(n-1)}(\tau_i(s))\bigg)\,ds + (b-a)^{\alpha}\nu(q_0)$$
$$\le \frac{(b-a)^{\alpha}}{1-\delta}\,\nu(q_0) + \int_{b_0}^t \bigg(\sum_{i=1}^{n} \frac{(\tau_i(s)-a)^{n-i}}{(n-i)!}\,p_i(s)\bigg)v^{(n-1)}(s)\,ds \ \text{ for } \ b_0 \le t \le b.$$

According to the Gronwall–Bellman lemma, the last inequality yields

$$v^{(n-1)}(t) \le \frac{(b-a)^{\alpha}}{1-\delta}\,\nu(q_0)\exp\bigg(\sum_{i=1}^{n} \int_{b_0}^b \frac{(\tau_i(s)-a)^{n-i}}{(n-i)!}\,p_i(s)\,ds\bigg) \ \text{ for } \ b_0 \le t \le b.$$

Thus

$$v^{(n-1)}(t) \le r\,\nu(q_0)(t-a)^{\alpha} \ \text{ for } \ b_0 \le t \le b,$$

where r is a number given by equality (2.8). The estimate obtained together with inequalities (2.9) and (2.10) guarantees the validity of estimates (2.5), where r is a positive constant independent of the functions q_0 and u. $\square$

Example 2.1. Let $p_i :]a,b[\to \mathbb{R}_+$ $(i=1,\dots,n)$ be measurable functions satisfying the equality

$$\sum_{i=1}^{n} \frac{(\alpha-1)!}{(n-i+\alpha)!}\,(\tau_i(t)-a)^{n-i+\alpha} p_i(t) = \ell(t-a)^{\alpha-1}$$

almost everywhere on $]a,b[$, where ℓ is a positive constant. Then for condition (1.13) to be satisfied, it is necessary and sufficient that the inequality

$$\ell < 1$$

hold. On the other hand, if $\ell \ge 1$ and $q_0 \in L([a,b])$ is a nonnegative function, satisfying condition (1.14), then for any $c>0$ the function

$$u(t) \equiv c(t-a)^{n-1+\alpha}$$

is a solution of problem (2.1), (2.3). Consequently, there is no positive constant r such that an arbitrary solution of problem (2.1), (2.3) admits estimates (2.5).

The above-given example shows that condition (1.13) in Lemma 2.1 is unimprovable and it cannot be replaced by the condition

$$\limsup_{t\to a} \bigg(\sum_{i=1}^{n} \frac{\alpha!(t-a)^{-\alpha}}{(n-i+\alpha)!} \int\limits_a^t (\tau_i(s)-a)^{n-i+\alpha} p_i(s)\,ds \bigg) \le 1.$$

2.2 Lemma on the solvability of problem (2.2), (2.3)

Lemma 2.2. *If along with* (2.4) *conditions* (1.13) *and* (1.14) *hold, then problem* (2.2), (2.3) *has at least one solution.*

Proof. Let $r>0$ be a number appearing in Lemma 2.1. We introduce the functions

$$r_i(t) = r\nu(q_0)(\tau_i(t)-a)^{n-i+\alpha} \ \ (i=1,\dots,n),$$

$$\varphi_i(t,x) = \begin{cases} x & \text{for } |x| \le r_i(t) \\ r_i(t)\,\mathrm{sgn}(x) & \text{for } |x| > r_i(t) \end{cases} \quad (i=1,\dots,n),$$

$$f_1(t,x_1,\dots,x_n) = f_0\big(t,\varphi_1(t,x_1),\dots,\varphi_n(t,x_n)\big),$$

$$q_1(t) = q_0(t) + \sum_{i=1}^{n} r_i(t)p_i(t),$$

and consider the initial value problem

$$u^{(n)}(t) = f_1\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big), \tag{2.12}$$

$$u^{(i-1)}(a) = 0 \ \ (i=1,\dots,n). \tag{2.13}$$

According to estimate (2.4), the function f_1 in the domain $]a,b[\,\times\mathbb{R}^n$ admits the estimates

$$|f_1(t,x_1,\dots,x_n)| \le q_1(t), \tag{2.14}$$

$$|f_1(t,x_1,\dots,x_n)| \le \sum_{i=1}^{n} p_i(t)|x_i| + q_0(t). \tag{2.15}$$

On the other hand, by conditions (1.13) and (1.14) we have

$$q_1 \in L([a,b]), \tag{2.16}$$

$$\limsup_{t\to a}\left((t-a)^{-\alpha}\int_a^t q_1(s)\,ds\right) < +\infty. \tag{2.17}$$

By virtue of the Schauder principle, conditions (2.14) and (2.16) guarantee the solvability of problem (2.12), (2.13). Let u be a solution of that problem. Then in view of (2.15), it is a solution of the differential inequality (2.1) as well. On the other hand, due to (2.14) and (2.17) we have

$$\limsup_{t\to a}\frac{|u^{(i-1)}(t)|}{(t-a)^{n-i+\alpha}} \leq \limsup_{t\to a}\left(\frac{(t-a)^{-\alpha}}{(n-i)!}\int_a^t\left(\frac{t-s}{t-a}\right)^{n-i}q_1(s)\,ds\right)$$
$$\leq \limsup_{t\to a}\left((t-a)^{-\alpha}\int_a^t q_1(s)\,ds\right) < +\infty \ \ (i=1,\dots,n).$$

Consequently, u is a solution of problem (2.1), (2.3).

By virtue of Lemma 2.1 and the choice of a number r, the function u admits estimates (2.5), i.e.,

$$|u^{(i-1)}(\tau_i(t))| \leq r_i(t) \ \ (i=1,\dots,n) \ \text{ for } \ a \leq t \leq b.$$

Thus the equality

$$f_1\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big) = f_0\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big)$$

holds almost everywhere on $]a,b[$. Therefore, u is a solution of problem (2.2), (2.3). □

3 Proof of the main results

Proof of Theorem 1.1. Suppose

$$\chi_i(t,x) = \begin{cases} x & \text{for } x \geq w_i(q)(\tau_i(t)) \\ w_i(q)(\tau_i(t)) & \text{for } x < w_i(q)(\tau_i(t)) \end{cases} \quad (i=1,\dots,n),$$
$$f_0(t,x_1,\dots,x_n) = f\big(t,\chi_1(t,x_1),\dots,\chi_n(t,x_n)\big).$$

In view of conditions (1.7)–(1.9), the function f_0 in the domain $]a,b[\times\mathbb{R}^n$ along with (2.4) admits the estimate

$$f_0(t,x_1,\dots,x_n) \geq q(t). \tag{3.1}$$

By Lemma 2.2, problem (2.2), (2.3) has a solution u. In view of conditions (1.6) and (3.1), we have

$$u^{(i-1)}(t) \geq w_i(q)(t) > 0 \ \text{ for } \ a < t \leq b \ \ (i=1,\dots,n).$$

Thus $\chi_i(t,u^{(i-1)}(\tau_i(t))) \equiv u^{(i-1)}(\tau_i(t))$, and the equality

$$f_0\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big) = f\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big)$$

holds almost everywhere on $]a,b[$. Therefore, u is a solution of problem (1.1), (1.2). □

Proof of Corollary 1.1. Due to conditions (1.6), (1.15), there exists a positive number δ such that

$$w_i(q)(t) \geq \delta(t-a)^{n-i+\beta} \ \text{ for } \ a \leq t \leq b \ \ (i=1,\dots,n). \tag{3.2}$$

If along with these estimates we take into account conditions (1.7) and (1.11), then it becomes clear that the function f in the domain $]a,b[\times\mathbb{R}^n_{0+}$ admits estimate (1.8), and on the domain D_q admits the estimate

$$f(t,x_1,\dots,x_n)\le\sum_{i=1}^n p_i(t)x_i+\overline{q}_0(t),$$

where

$$\overline{q}_0(t)=\sum_{i=1}^n\delta^{-\lambda_i}(\tau_i(t)-a)^{-(n-i+\beta)\lambda_i}q_i(t)+q_0(t).$$

On the other hand, according to (1.16) we have

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t\overline{q}_0(s)\,ds\bigg)<+\infty.$$

Consequently, all the conditions of Theorem 1.1 are satisfied which guarantees the solvability of problem (1.1), (1.2). $\square$

Proof of Theorem 1.2. Assume the contrary that problem (1.1), (1.2) has a solution u.

According to inequalities (1.2) and (1.5), there exists a number $r>1$ such that

$$0<u^{(i-1)}(t)\le r(t-a)^{n-i+\alpha}\ \text{ for }\ a\le t\le b\ \ (i=1,\dots,n).$$

By virtue of these estimates and condition (1.12), the inequality

$$\sum_{i=1}^n(\tau_i(t)-a)^{-(n-i+\alpha)\lambda_i}q_i(t)+q_0(t)\le r_0u^{(n)}(t)$$

holds almost everywhere on $]a,b[$, where

$$r_0=\max\{r^{\lambda_1},\dots,r^{\lambda_n}\}.$$

Therefore,

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t\Big(\sum_{i=1}^n(\tau_i(s)-a)^{-(n-i+\alpha)\lambda_i}q_i(s)+q_0(s)\Big)\,ds\bigg)\le r_0\limsup_{t\to a}\frac{|u^{(n-1)}(t)|}{(t-a)^{\alpha}}\le r_0r.$$

Consequently, inequality (1.17) is violated, and it remains to consider the case where for some $b_0\in\,]a,b]$ and $\delta>0$ inequalities (1.18) are satisfied in $]a,b_0]$.

Put

$$\rho=\inf\Big\{\frac{u^{(n-1)}(t)}{(t-a)^{\alpha}}\ :\ a<t\le b_0\Big\}.$$

Then

$$u^{(i-1)}(t)\ge\frac{\rho\alpha!}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha}\ \text{ for }\ a\le t\le b_0\ \ (i=1,\dots,n).$$

By these estimates and inequalities (1.12), (1.18), we have

$$u^{(n-1)}(t)\ge\int\limits_a^t\Big(\sum_{i=1}^n p_i(s)u^{(i-1)}(\tau_i(s))+q_0(s)\Big)\,ds\ge(\rho+\delta)(t-a)^{\alpha}\ \text{ for }\ a\le t\le b_0.$$

Hence we get

$$\rho\ge\rho+\delta.$$

The contradiction obtained proves that problem (1.1), (1.2) has no solution. $\square$

Corollaries 1.2 and 1.3 immediately follow from Corollary 1.1 and Theorem 1.2 in the case, where

$$f(t,x_1,\dots,x_n) \equiv \sum_{i=1}^{n} \big(p_i(t)x_i + q_i(t)x_i^{-\lambda_i}\big) + q_0(t). \tag{3.3}$$

Proof of Theorem 1.3. In view of conditions (1.8), (1.10), the inequality

$$q(t) \le f\big(t, w_1(q)(t),\dots,w_n(q)(t)\big)$$

holds almost everywhere on $]a,b[$, and the function f admits estimate (1.9) on the domain D_q, where

$$q_0(t) = f\big(t, w_1(q)(t),\dots,w_n(q)(t)\big) + \sum_{i=1}^{n} p_i(t)w_i(q)(t). \tag{3.4}$$

On the other hand, by virtue of inequality (1.23), there exists a positive number r such that

$$\int_a^t q(s)\,ds \le \int_a^t f\big(s, w_1(q)(s),\dots,w_n(q)(s)\big)\,ds \le r(t-a)^{\alpha} \ \text{ for } \ a \le t \le b.$$

Thus

$$w_i(q)(t) \le \frac{r\alpha!}{(n-i+\alpha)!}(t-a)^{n-i+\alpha} \ \text{ for } \ a \le t \le b \ \ (i=1,\dots,n).$$

If along with these estimates we take into account inequality (1.13), then from equality (3.4) we find

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha}\int_a^t q_0(s)\,ds\bigg) \le r + r\limsup_{t\to a}\bigg(\sum_{i=1}^{n}\frac{\alpha!(t-a)^{-\alpha}}{(n-i+\alpha)!}\int_a^t (\tau_i(s)-a)^{n-i+\alpha}p_i(s)\,ds\bigg) < 2r.$$

Consequently, all the conditions of Theorem 1.1 are satisfied which guarantees the solvability of problem (1.1), (1.2).

It remains to prove that the problem we are considering has at most one solution.

Let u_1 and u_2 be solutions of problem (1.1), (1.2), and let

$$u(t) = u_2(t) - u_1(t).$$

In view of (1.8), the inequalities

$$u_k^{(n)}(t) \ge q(t) \ \ (k=1,2)$$

hold almost everywhere on $]a,b[$. Thus

$$u_k^{(i-1)}(t) \ge w_i(q)(t) \ \text{ for } \ a \le t \le b \ \ (i=1,\dots,n;\ k=1,2),$$

and, consequently,

$$\big(t, u_k(\tau_1(t)),\dots,u_k^{(n-1)}(\tau_n(t))\big) \in D_q \ \text{ for } \ a < t < b \ \ (k=1,2).$$

If now we take into account the fact that the function f on the domain D_q satisfies the Lipschitz condition (1.10), then it becomes evident that the function u is a solution of the differential inequality

$$|u^{(n)}(t)| \le \sum_{i=1}^{n} p_i(t)|u^{(i-1)}(\tau_i(t))| \tag{3.5}$$

under the weighted initial conditions (1.2). However, by Lemma 2.1, problem (3.5), (1.2) has only a trivial solution. Therefore, $u_1(t) \equiv u_2(t)$. □

Proof of Corollary 1.4. By virtue of Corollary 1.2, for problem (1.4), (1.2) to be solvable, it is necessary that condition (1.25) hold. Thus it remains to consider the case when that condition is satisfied.

Let f be a function given by equality (3.3). Then Eq. (1.4) coincides with Eq. (1.1), and in the domain $]a,b[\,\times\mathbb{R}^n_{0+}$ inequality (1.8) holds, where

$$q(t)=q_0(t).$$

On the other hand, in view of (1.6), (1.19) and (1.25), there exists a number $\delta\in\,]0,1[$ such that the functions $w_i(q)$ along with (3.2) admit the estimates

$$w_i(q)(t)\leq\delta^{-1}(t-a)^{n-i+\alpha}\ \text{ for }\ a\leq t\leq b\ \ (i=1,\dots,n). \tag{3.6}$$

Let $(t,x_1,\dots,x_n)$ and $(t,y_1,\dots,y_n)$ be any two points from the domain D_q. Then due to (1.7) and (3.2) we have

$$|x_i^{-\lambda_i}-y_i^{-\lambda_i}|\leq\lambda_i\delta^{-1-\lambda_i}(\tau_i(t)-a)^{-(n-i+\beta)(1+\lambda_i)}|x_i-y_i|\ \ (i=1,\dots,n).$$

Thus from (3.3) it follows that

$$\big|f(t,x_1,\dots,x_n)-f(t,y_1,\dots,y_n)\big|\leq\sum_{i=1}^n\overline{p}_i(t)|x_i-y_i|, \tag{3.7}$$

where

$$\overline{p}_i(t)=p_i(t)+\lambda_i\delta^{-1-\lambda_i}(\tau_i(t)-a)^{-(n-i+\beta)(1+\lambda_i)}q_i(t)\ \ (i=1,\dots,n).$$

On the other hand, in view of (3.2), (3.3) and (3.6), the inequality

$$\begin{aligned}f\big(t,w_1(q)(\tau_1(t)),\dots,w_n(q)(\tau_n(t))\big)&\\ \leq\delta^{-1}\sum_{i=1}^n(\tau_i(t)-a)^{n-i+\alpha}p_i(t)&+\sum_{i=1}^n\delta^{-\lambda_i}(\tau_i(t)-a)^{-(n-i+\beta)\lambda_i}q_i(t)+q_0(t)\end{aligned}$$

holds almost everywhere on $]a,b[\,$. If now we take into account conditions (1.13), (1.24) and (1.25), then it becomes obvious that along with (1.23) the inequality

$$\limsup_{t\to a}\bigg(\sum_{i=1}^n\frac{\alpha!(t-a)^{-\alpha}}{(n-i+\alpha)!}\int\limits_a^t(\tau_i(s)-a)^{n-i+\alpha}\overline{p}_i(s)\,ds\bigg)<1 \tag{3.8}$$

is satisfied. However, by Theorem 1.3, conditions (1.8), (1.23), (3.7) and (3.8) guarantee the unique solvability of problem (1.4), (1.2). $\square$

Corollaries 1.5–1.7 immediately follow from Corollaries 1.1–1.4.

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