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Short Communications

Ivan Kiguradze, Alexander Lomtatidze, and Nino Partsvania

SOME MULTI–POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SINGULAR DIFFERENTIAL EQUATIONS

Abstract. For second order nonlinear differential equations with non-integrable singularities with respect to the time variable, unimprovable sufficient conditions for solvability and unique solvability of multi-point boundary value problems are established.

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Let $-\infty < a < b < +\infty$, $f :]a,b[\,\times R \to R$ be the function satisfying the local Carathéodory conditions, and let $p :]a,b[\, \to [0,+\infty[$ be the measurable function such that

$$p(t) > 0 \quad \text{almost everywhere on } \;]a,b[\,, \quad \int\limits_a^b \frac{dt}{p(t)} < +\infty.$$

In the interval $[a,b]$, we consider the differential equation

$$\big(p(t)u'\big)' = f(t,u) \tag{1}$$

with the multi-point boundary conditions

$$\sum_{i=1}^{m} \alpha_i u(a_i) = c_1, \quad \sum_{i=1}^{n} \beta_i u(b_i) = c_2. \tag{2}$$

Here m and n are natural numbers, $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n,c_1,c_2$ are real constants,

$$a \le a_i \le a_0 < b_0 \le b_j \le b \;\; (i = 1,\dots,m; \; j = 1,\dots,n).$$

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Moreover, if $m = 1$ ($n = 1$), it is assumed that $a = a_0 = a_1$ ($b = b_0 = b_1$), and if $m \geq 2$ ($n \geq 2$), then

$$a = a_1 < \cdots < a_m = a_0 \quad \big(b_0 = b_1 < \cdots < b_n = b\big).$$

We are interested, in general, in the cases where the function f with respect to the time variable has non-integrable singularities at the points a and b. In that sense the problem (1), (2) is singular.

For $m = n = 1$, the singular problem (1), (2) is investigated in detail (see [1]–[4], [9], [14]–[16] and the references therein).

The optimal conditions for the unique solvability of problems of the type (1), (2) in the case, when the equation (1) is linear, are contained in [7], [8], [11], [12].

Various particular cases of the nonlinear singular problem (1), (2) are studied in [6], [10], [13]. Nevertheless, in the general case that problem remains so far studied insufficiently. In the present paper, new and unimprovable in a certain sense sufficient conditions for solvability and unique solvability of the above-mentioned problem are given.

We will seek a solution of the problem (1), (2) in the space of continuous functions $u : [a, b] \to R$ which are absolutely continuous together with $t \to p(t)u'(t)$ on an arbitrary closed interval, contained in $]a, b[$.

We introduce the following functions:

$$f^*(t, y) = \max\big\{|f(t, x)| : \ |x| \leq y\big\} \quad \text{for } \ a < t < b, \ \ y \geq 0;$$

$$f_0(t, y) = \sup\Big\{\frac{1}{2}\big(|f(t, x)| - f(t, x)\operatorname{sgn} x\big) : \ |x| \leq y\Big\} \quad \text{for } \ a < t < b, \ \ y \geq 0;$$

$$\delta(t) = \int\limits_a^t \frac{ds}{p(s)} \quad \text{for } \ a \leq t \leq b.$$

In the statements of the main results of the present paper, besides the functions f^*, f_0, and δ, there are appearing also the functions ψ_1, ψ_2, and ψ_0, which are defined in the following manner:
if $m = 1$ ($n = 1$), then

$$\psi_1(t) = 0 \ \text{ for } \ a \leq t \leq b \quad \Big(\psi_2(t) = \beta_1(\delta(b) - \delta(t)) \ \text{ for } \ a \leq t \leq b\Big);$$

if $m > 2$, then

$$\psi_1(t) = 0 \ \text{ for } \ a \geq a_0, \quad \psi_1(t) = \psi_1(a_{k+1}) + \Big(\sum_{i=k+1}^{m} \alpha_i\Big)\big(\delta(a_{k+1}) - \delta(t)\big)$$
$$\text{for } \ a_k \leq t \leq a_{k+1} \ \ (k = 1, \dots, m-1);$$

and if $n > 2$, then

$$\psi_2(b) = 0, \quad \psi_2(t) = \psi_2(b_{k+1}) + \Big(\sum_{i=k+1}^{n} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(t)\big)$$
$$\text{for } \ b_k \leq t < b_{k+1} \ \ (k = 1, \dots, n-1),$$

$$\psi_2(t)=\psi_2(b_0)+\Big(\sum_{i=1}^n\beta_i\Big)\big(\delta(b_0)-\delta(t)\big)\ \text{ for }\ a\le t<b_0,$$

and

$$\psi_0(b)=0,\quad \psi_0(t)=\psi_0(b_{k+1})+$$
$$+\Big(\sum_{i=1}^k\beta_i\Big)\big(\delta(b_{k+1})-\delta(t)\big)\ \text{ for }\ b_k\le t<b_{k+1}\ \ (k=1,\dots,n-1),\tag{3}$$
$$\psi_0(t)=\psi_0(b_0)\quad\text{for}\ \ a\le t<b_0.$$

It is clear that

$$\sum_{i=1}^n\beta_i=0\Longrightarrow\psi_0(t)\equiv-\psi_2(t).$$

Let

$$\chi(t,s)=\begin{cases}1 & \text{for}\ \ s\le t,\\ 0 & \text{for}\ \ s>t.\end{cases}$$

The following simple lemma is valid.

Lemma 1. *The boundary value problem*

$$\big(p(t)u'\big)'=0;\quad \sum_{i=1}^m\alpha_i u(a_i)=0,\quad \sum_{i=1}^n\beta_i u(b_i)=0\tag{4}$$

has only the trivial solution if and only if

$$\Delta=\Big(\sum_{i=1}^n\beta_i\Big)\psi_1(a)-\Big(\sum_{i=1}^m\alpha_i\Big)\psi_2(a)\ne0.\tag{5}$$

Moreover, if the condition (5) *is satisfied, then the Green function of the problem* (4) *admits the representation*

$$g(t,s)=\frac{1}{\Delta}\Big[\psi_1(s)\psi_2(a)-\psi_2(s)\psi_1(a)+\Big(\psi_2(s)\sum_{i=1}^m\alpha_i-\psi_1(s)\sum_{i=1}^n\beta_i\Big)\delta(t)\Big]+$$
$$+\chi(t,s)(\delta(t)-\delta(s))$$

and

$$r=\sup\Big\{\frac{|g(t,s)|}{\delta(s)(\delta(b)-\delta(s))}:\ a\le t\le b,\ \ a<s<b\Big\}<+\infty.\tag{6}$$

We study the problem (1), (2) in the case, where

$$\int_a^b\delta(t)(\delta(b)-\delta(t))f^*(t,y)\,dt<+\infty\quad\text{for}\ \ y\ge0.\tag{7}$$

Moreover, if $a_0>a$, then it is assumed that

$$\limsup_{\tau\to t,\,y\to+\infty}\int_t^\tau\delta(s)\frac{f^*(s,y)}{y}\,ds<1\quad\text{for}\ \ a\le t<a_0,\tag{8}$$

and if $b_0 < b$, then

$$\limsup_{\tau\to t,\, y\to+\infty} \int_{\tau}^{t} (\delta(b)-\delta(s))\frac{f^*(s,y)}{y}\, ds < 1 \quad \text{for } \ b_0 < t \le b. \tag{9}$$

Along with $(1), (2)$ we consider the problem

$$\big(p(t)u'\big)' = \lambda f(t,u); \tag{10}$$

$$\sum_{i=1}^{m} \alpha_i u(a_i) = \lambda c_1, \quad \sum_{i=1}^{n} \beta_i u(b_i) = \lambda c_2, \tag{11}$$

dependent on a parameter $\lambda \in]0,1[$.

On the basis of Corollary 1.2 from [5] and Lemma 1, the following statements are proved.

Theorem 1 (The principle of a priori boundedness)**.** *Let the conditions* (5)*,* (7) *be fulfilled and let there exist a positive constant* y_0 *such that for any* $\lambda \in]0,1[$ *every solution of the problem* $(10), (11)$ *admits the estimate*

$$|u(t)| \le y_0 \quad \textit{for } \ a \le t \le b.$$

Then the problem $(1), (2)$ *has at least one solution.*

Theorem 2. *Let the inequality* (5) *hold and let there exist a positive constant* y_0 *such that*

$$r\int_{a}^{b} \delta(s)(\delta(b)-\delta(s)) f^*(s,y_0)\, ds \le y_0, \tag{12}$$

where r is a number given by the equality (6)*. Then the problem* $(1), (2)$ *has at least one solution.*

Theorem 3. *Let the inequality* (5) *hold and let in the domain* $]a,b[\times R$ *the condition*

$$\big|f(t,x_1) - f(t,x_2)\big| \le h(t)|x_1 - x_2|$$

be fulfilled, where $h :]a,b[\to [0,+\infty[$ *is a measurable function such that*

$$r\int_{a}^{b} \delta(s)(\delta(b)-\delta(s)) h(s)\, ds < 1. \tag{13}$$

If, moreover,

$$\int_{a}^{b} \delta(s)(\delta(b)-\delta(s)) |f(s,0)|\, ds < +\infty,$$

then the problem $(1), (2)$ *has one and only one solution.*

Consider now the case, where

$$\alpha_i > 0 \ \ (i = 1, \dots, m), \quad \beta_i > 0 \ \ (i = 1, \dots, n). \tag{14}$$

Then the condition (5) is satisfied since

$$\Delta < -\Big(\sum_{i=1}^{m} \alpha_i\Big) \sum_{k=1}^{n-1} \Big(\sum_{i=k+1}^{n} \beta_i\Big) \big(\delta(b_{k+1}) - \delta(b_k)\big) < 0.$$

Let g_0 be the Green function of the boundary value problem

$$\big(p(t)u'\big)' = 0; \quad u(a) = u(b) = 0,$$

i.e.,

$$g_0(t,s) = \Big(\frac{\delta(s)}{\delta(b)} - 1\Big)\delta(t) + \chi(t,s)(\delta(t) - \delta(s)).$$

The following theorem is valid.

Theorem 4. *Let the conditions* (7)–(9)*, *and* (14) *be fulfilled. Let, moreover, there exist a positive constant* y_0 *such that*

$$\int_a^b |g_0(t,s)| f_0(s,y)\, ds < y \quad \text{for } a \le t \le b, \ \ y > y_0. \tag{15}$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let the inequalities* (14) *hold. Let, moreover, in the domain* $]a,b[\,\times R$ *the inequality*

$$f(t,x)\operatorname{sgn} x \ge -h(t)|x| - h_0(t) \tag{16}$$

be fulfilled, and in the domain $\big(]a,a_0[\,\cup\,]b_0,b[\big) \times R$ *the inequality*

$$|f(t,x)| \le h_0(t)(1 + |x|) \tag{17}$$

hold, where $h :]a,b[\to [0,+\infty[$ *and* $h_0 :]a,b[\to [0,+\infty[$ *are measurable functions such that*

$$\int_a^b \delta(s)(\delta(b) - \delta(s))h(s)\, ds \le \delta(b), \tag{18}$$

$$\int_a^b \delta(s)(\delta(b) - \delta(s))h_0(s)\, ds < +\infty. \tag{19}$$

Then the problem (1), (2) *has at least one solution.*

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* For $m = 1$ ($n = 1$), the condition (7) (the condition (8)) is dropped out.

Theorem 5. *Let in the domain* $]a_0, b_0[\times R$ *the condition*

$$\big[f(t,x_1) - f(t,x_2)\big]\operatorname{sgn}(x_1 - x_2) \geq -h(t)|x_1 - x_2| \tag{20}$$

be fulfilled, and in the domain $\big(]a, a_0[\cup]b_0, b[\big) \times R$ *the condition*

$$\big|f(t,x_1) - f(t,x_2)\big| \leq \bar h(t)|x_1 - x_2| \tag{21}$$

hold, where $h :]a,b[\to [0,+\infty[$ *and* $\bar h :]a, a_0[\cup]b_0, b[\to [0,+\infty[$ *are measurable functions. If, moreover, the inequalities* (14), (18), *and* (19) *are satisfied, where*

$$h_0(t) = \begin{cases} |f(t,0)| & \text{for } t \in]a_0, b_0[\,, \\ |f(t,0)| + \bar h(t) & \text{for } t \in]a,b[\setminus]a_0, b_0[\,, \end{cases} \tag{22}$$

then the problem (1), (2) *has one and only one solution.*

Remark 1. If we take into account Example 1.1 from [4], then it becomes evident that the conditions (12), (13), (15), and (18) in Theorems 2–5 are unimprovable in the sense that they cannot be replaced, respectively, by the conditions

$$r\int_a^b \delta(s)(\delta(b) - \delta(s)) f^*(s, y_0)\, ds \leq (1+\varepsilon) y_0,$$

$$r\int_a^b \delta(s)(\delta(b) - \delta(s)) h(s)\, ds \leq 1+\varepsilon,$$

$$\int_a^b |g_0(t,s)| f_0(s,y)\, ds \leq (1+\varepsilon) y \quad \text{for } a \leq t \leq b, \;\; y \geq y_0,$$

$$\int_a^b \delta(s)(\delta(b) - \delta(s)) h(s)\, ds \leq (1+\varepsilon)\delta(b),$$

no matter how small $\varepsilon > 0$ would be.

Consider now the case, where

$$\alpha_i > 0 \;\; (i = 1, \dots, m), \quad n > 2, \quad \beta_i > 0 \;\; (i = 1, \dots, n-1), \quad \beta_n = \\ = -\sum_{i=1}^{n-1}\beta_i, \quad \sum_{k=1}^{n-1}\Big(\sum_{i=1}^{k}\beta_i\Big)\big(\delta(b_{k+1}) - \delta(b_k)\big) = 1. \tag{23}$$

In that case the inequality (5) is also satisfied since

$$\Delta = -\Big(\sum_{i=1}^{m}\alpha_i\Big)\psi_2(a) = \Big(\sum_{i=1}^{m}\alpha_i\Big)\psi_0(a) = \sum_{i=1}^{m}\alpha_i > 0.$$

Let g_1 be the Green function of the boundary value problem

$$\big(p(t)u'\big)' = 0; \quad u(a) = 0, \quad \sum_{i=1}^{n} \beta_i u(b_i) = 0.$$

Then in view of (3) and (23) we have

$$g_1(t,s) = -\psi_0(s)\delta(t) + \chi(t,s)(\delta(t) - \delta(s)).$$

Lemma 2. *If along with* (23) *the condition*

$$\sum_{k=j}^{n-1} \Big(\sum_{i=1}^{k} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(b_k)\big) \geq \frac{\delta(b) - \delta(b_j)}{\delta(b)} \quad (j = 1, \dots, n) \tag{24}$$

holds, then

$$g_1(t,s) \leq g_0(t,s) < 0 \quad \textit{for} \ \ a < t < b$$

and

$$|g_1(t,s)| \leq \delta^{\mu}(t)\delta^{1-\mu}(s)\psi_0(s) \quad \textit{for} \ \ a \leq t, \ s \leq b, \ \ 0 \leq \mu \leq 1.$$

For any $x \in R$, we suppose

$$[x]_+ = \frac{1}{2}(|x| + x).$$

On the basis of Theorem 1 and Lemma 2, the following theorems are proved.

Theorem 6. *Let the conditions* (23) *and* (24) *hold. Let, moreover, in the domains* $]a,b[\times R$ *and* $\big(]a,a_0[\cup]b_0,b[\big)\times R$ *the inequalities* (16) *and* (17) *be satisfied, respectively, where* $h :]a,b[\to [0,+\infty[$ *and* $h_0 :]a,b[\to [0,+\infty[$ *are measurable functions satisfying the conditions*

$$\int_a^b \delta^{\mu}(s)(\delta(b)-\delta(s))h(s)\,ds < +\infty, \quad \int_a^b \delta^{\mu}(s)(\delta(b)-\delta(s))h_0(s)\,ds < +\infty, \tag{25}$$

$$\int_a^b \delta(s)\psi_0(s)\Big[h(s) - \frac{\mu(1-\mu)\ell}{p(s)\psi_0(s)\delta^2(s)}\Big]_+ ds \leq 1 \tag{26}$$

for some $\mu \in]0,1]$ *and* $\ell \in]0,1]$. *Then the problem* (1), (2) *has at least one solution.*

Theorem 7. *Let the conditions* (23) *and* (24) *hold, and let in the domains* $]a,b[\times R$ *and* $\big(]a,a_0[\cup]b_0,b[\big)\times R$ *the inequalities* (20) *and* (21) *be satisfied, respectively, where* $h :]a,b[\to [0,+\infty[$ *and* $\bar{h} :]a,a_0[\cup]b_0,b[\to [0,+\infty[$ *are measurable functions. If, moreover, for some* $\mu \in]0,1]$ *and* $\ell \in]0,1]$ *the conditions* (25) *and* (26) *are satisfied, where* h_0 *is a function given by the equality* (22), *then the problem* (1), (2) *has one and only one solution.*

Remark 2. The condition (26) in Theorems 6 and 7 is unimprovable and it cannot be replaced by the condition

$$\int\limits_a^b \delta(s)\psi_0(s)\Big[h(s)-\frac{\mu(1-\mu)\ell}{p(s)\psi_0(s)\delta^2(s)}\Big]_+\,ds\le 1+\varepsilon-\ell,$$

no matter how small $\varepsilon>0$ would be.

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References

1. I. T. Kiguradze, Some singular boundary value problems for second order nonlinear ordinary differential equations. (Russian) *Differencial'nye Uravnenija* **4** (1968), No. 10, 1753–1773; English transl.: *Differ. Equations* **4** (1968), 901–910.
2. I. T. Kiguradze, The singular two-point boundary value problem. (Russian) *Differencial'nye Uravnenija* **5** (1969), No. 11, 2002–2016; English transl.: *Differ. Equations* **5** (1968), 1493–1504.
3. I. Kiguradze, On a singular boundary value problem. *J. Math. Anal. Appl.* **30** (1970), No. 3, 475–489.
4. I. Kiguradze, Some optimal conditions for the solvability of two-point singular boundary value problems. *Functional differential equations and applications* (*Beer-Sheva,* 2002). *Funct. Differ. Equ.* **10** (2003), No. 1-2, 259–281.
5. I. Kiguradze, On solvability conditions for nonlinear operator equations. *Math. Comput. Modelling* **48** (2008), No. 11-12, 1914–1924.
6. I. Kiguradze and T. Kiguradze, Optimal conditions of solvability of nonlocal problems for second-order ordinary differential equations. *Nonlinear Anal.* **74** (2011), No. 3, 757–767.
7. I. T. Kiguradze and T. I. Kiguradze, Conditions for the well-posedness of nonlocal problems for second-order linear differential equations. (Russian) *Differ. Uravn.* **47** (2011), No. 10, 1400–1411; English transl.: *Differ. Equ.* **47** (2011), No. 10, 1414–1425.
8. I. Kiguradze and A. Lomtatidze, On certain boundary value problems for second-order linear ordinary differential equations with singularities. *J. Math. Anal. Appl.* **101** (1984), No. 2, 325–347.
9. I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for second-order ordinary differential equations. (Russian) *Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 105–201, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987; English transl.: *J. Soviet Math.* **43** (1988), No. 2, 2340–2417.
10. A. G. Lomtatidze, A boundary value problem for second-order nonlinear ordinary differential equations with singularities. (Russian) *Differentsial'nye Uravneniya* **22** (1986), No. 3, 416–426; English transl.: *Differ. Equations* **22** (1986), 301–310.
11. A. G. Lomtatidze, A nonlocal boundary value problem for second-order linear ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **31** (1995), No. 3, 446–455; English transl.: *Differ. Equations* **31** (1995), No. 3, 411–420.
12. A. Lomtatidze, On a nonlocal boundary value problem for second order linear ordinary differential equations. *J. Math. Anal. Appl.* **193** (1995), No. 3, 889–908.
13. A. Lomtatidze and L. Malaguti, On a nonlocal boundary value problem for second order nonlinear singular differential equations. *Georgian Math. J.* **7** (2000), No. 1, 133–154.

14. N. Partsvania, On the solvability of boundary value problems for nonlinear second-order differential equations. (Russian) *Differencial'nye Uravnenija* **43** (2007), No. 2, 275–277; English transl.: *Differ. Equations* **43** (2007), No. 2, 286–289.
15. N. Partsvania, On extremal solutions of two-point boundary value problems for second order nonlinear singular differential equations. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **5** (2011), No. 2, 31–36.
16. N. Partsvania, On solvability and well-posedness of two-point weighted singular boundary value problems. *Mem. Differential Equations Math. Phys.* **54** (2011), 139–146.

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Authors' addresses:

I. Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.
E-mail: kig@rmi.ge

A. Lomtatidze

Mathematical Institute of the Academy of Sciences of the Czech Republic, Branch in Brno, 22 Žižkova, Brno 616 62, Czech Republic.
E-mail: bacho@math.muni.cz

N. Partsvania

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia;
2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.
E-mail: ninopa@rmi.ge