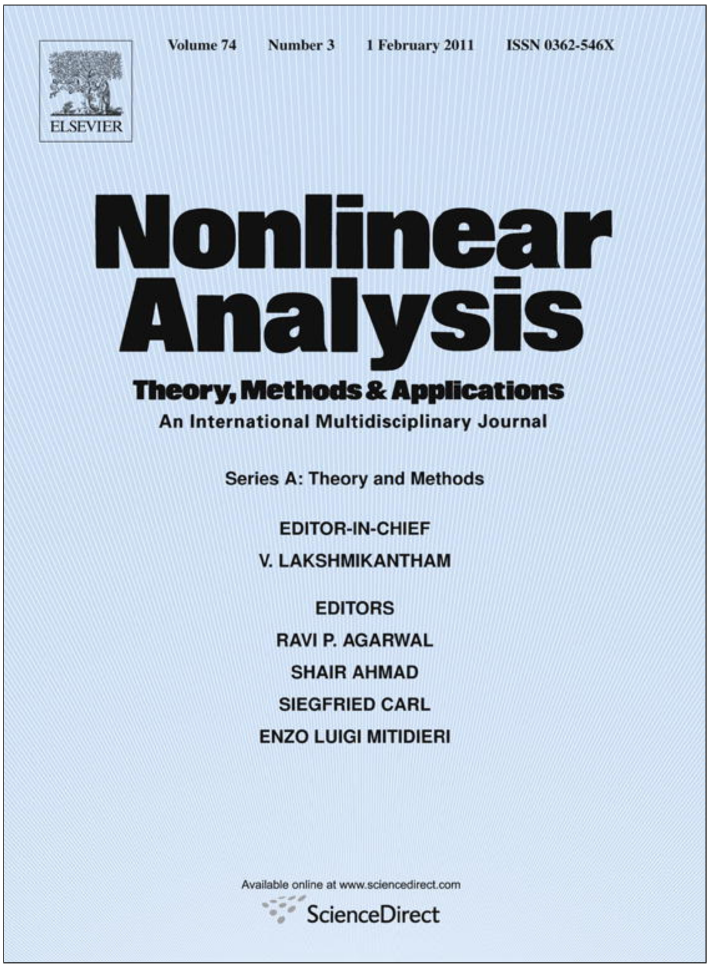
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Optimal conditions of solvability of nonlocal problems for second-order ordinary differential equations

Ivan Kiguradze [a], Tariel Kiguradze [b,*]

[a] A. Razmadze Mathematical Institute, 1 M. Aleksidze St, 0193 Tbilisi, GA, United States
[b] Florida Institute of Technology, Department of Mathematical Sciences, Melbourne, FL 32901, United States

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ABSTRACT

For the differential equation

$$u'' = f(t,u)$$

in regular as well as in singular cases there are established optimal sufficient conditions of existence for solutions satisfying nonlocal boundary conditions of the type

$$\int_a^b u^{(i-1)}(s)\,\mathrm{d}\varphi_i(s) = c_i \quad (i = 1, 2).$$

1. Formulation of the main results

1.1. Statement of the problem and basic notation

Consider the boundary value problem

$$u'' = f(t,u); \tag{1.1}$$

$$\int_a^b u^{(i-1)}(s)\,\mathrm{d}\varphi_i(s) = c_i \quad (i = 1, 2), \tag{1.2}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying local Caratheodory conditions, $c_i \in \mathbb{R}$ $(i = 1, 2)$, and $\varphi_i : [a,b] \to \mathbb{R}$ $(i = 1, 2)$ are functions of bounded variation such that

$$\varphi_i(a) = 0, \qquad \varphi_i(b) = 1 \quad (i = 1, 2). \tag{1.3}$$

Primarily we are interested in the case where, along with (1.3), one of the following four conditions holds:

$$\varphi_i(s) > 1 \quad \text{for } a < s < b \ (i = 1, 2); \tag{1.4}$$

$$\varphi_i(s) < 0 \quad \text{for } a < s < b \ (i = 1, 2); \tag{1.5}$$

$$\varphi_1(s) > 1, \qquad \varphi_2(s) < 0 \quad \text{for } a < s < b \ (i = 1, 2); \tag{1.6}$$

$$\varphi_1(s) < 0, \qquad \varphi_2(s) > 1 \quad \text{for } a < s < b \ (i = 1, 2). \tag{1.7}$$

* Corresponding address: Florida Institute of Technology, Department of Mathematical Sciences, 150 W University Blvd., Melbourne, FL 32901, United States.
E-mail addresses: kig@rmi.acnet.ge (I. Kiguradze), tkigurad@fit.edu (T. Kiguradze).

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In these cases, as far as we know, problems of the form (1.1), (1.2) are still little studied (see [1–19] and the references therein). The present paper is an attempt to fill the existing gap. In particular, there are established unimprovable, in a certain sense, sufficient conditions which guarantee: solvability of problem (1.1), (1.2); unique solvability of problem (1.1), (1.2); and the existence of at least three distinct solutions to problem (1.1), (1.2). In addition to that there is considered separately a singular case, where the function f is defined only on the set $[a,b]\times(0,+\infty)$ and, generally speaking, has no finite limit $\lim_{x\to 0} f(t,x)$. In that case there are established optimal sufficient conditions of existence of a positive solution to problem (1.1), (1.2).

Throughout the paper the following notation will be used.

$L([a,b])$ is the space of Lebesgue integrable functions.

$\mathcal{K}_{\rm loc}([a,b]\times I)$, where $I=\mathbb{R}$ or $I=(0,+\infty)$, is a set of functions $q:[a,b]\times I\to\mathbb{R}$ satisfying local Caratheodory conditions, i.e., $q\in\mathcal{K}_{\rm loc}([a,b]\times I)$ if $q(\cdot,x):[a,b]\to\mathbb{R}$ is measurable for any $x\in I$, $q(t,\cdot):I\to\mathbb{R}$ is continuous for almost every $t\in[a,b]$, and for an arbitrary compact interval $J\subset I$

$$q_J^*\in L([a,b]),\quad \text{where } q_J^*(t)=\max\{|q(t,x)|:|x|\in J\}.$$

$Z_{a,b}$ is the set of functions $q:[a,b]\times[1,+\infty)\to[0,+\infty)$ nondecreasing with respect to the second argument and satisfying the conditions

$$q(\cdot,x)\in L([a,b])\quad\text{for } 1\le x<+\infty,\qquad \lim_{x\to+\infty}\int_a^b\frac{q(t,x)}{x}\,{\rm d}t=0.$$

For arbitrary $p_i\in L([a,b])$ $(i=1,2)$, $p_1(t)\not\equiv p_2(t)$ would mean that p_1 and p_2 differ from each other on a set of positive measure.

By χ, g_0, h_0 and σ we will understand functions, an operator and a number given by the equalities

$$\chi(s,t)=\begin{cases}1 & \text{for } s\ge t,\\ 0 & \text{for } s<t;\end{cases}$$

$$g_0(t,s)=\left(\int_a^b\varphi_1(\tau)\,{\rm d}\tau+t-b\right)\varphi_2(s)-\int_a^s\varphi_1(\tau)\,{\rm d}\tau+\chi(s,t)(s-t)\quad\text{for } a\le s,\ t\le b;\tag{1.8}$$

$$h_0(q)(t)=\int_a^b g_0(t,s)q(s)\,{\rm d}s\quad\text{for } a\le t\le b,\ q\in L([a,b]);\tag{1.9}$$

$$\sigma={\rm sgn}\left(\varphi_1\Big(\frac{a+b}{2}\Big)\varphi_2\Big(\frac{a+b}{2}\Big)\right).\tag{1.10}$$

If one of the conditions (1.4)–(1.7) holds, then according to Lemma 2.2 (see below), we have

$$\sigma g_0(t,s)>0\quad\text{for } a\le t\le b,\ a<s<b.$$

Therefore

$$\sigma h_0(q)(t)>0\quad\text{for } a\le t\le b,$$

if $q\in L([a,b])$ is a nonnegative function different from zero on a set of positive measure.

In the aforementioned cases we will use the following definitions.

Definition 1.1. We say that an integrable function $p:[a,b]\to[0,+\infty)$ belongs to the set $\mathcal{G}_{\varphi_1,\varphi_2}$ if there exists an integrable function $p_0:[a,b]\to[0,+\infty)$ such that $p_0(t)\not\equiv 0$,

$$p(t)\le\frac{p_0(t)}{|h_0(p_0)(t)|}\quad\text{for } a\le t\le b,\qquad p(t)\not\equiv\frac{p_0(t)}{|h_0(p_0)(t)|}.$$

Definition 1.2. We say that $p\in\mathcal{G}^*_{\varphi_1,\varphi_2}$ if there exists an integrable function $p_0:[a,b]\to[0,+\infty)$ such that

$$p(t)=\frac{p_0(t)}{|h_0(p_0)(t)|}\quad\text{for } a\le t\le b.$$

It is clear that if $p\in\mathcal{G}^*_{\varphi_1,\varphi_2}$, then $\alpha p\in\mathcal{G}_{\varphi_1,\varphi_2}$ for any $\alpha\in[0,1)$. Consequently,

$$\mathcal{G}^*_{\varphi_1,\varphi_2}\subset\overline{\mathcal{G}}_{\varphi_1,\varphi_2}\setminus\mathcal{G}_{\varphi_1,\varphi_2},$$

where $\overline{\mathcal{G}}_{\varphi_1,\varphi_2}$ is the closure of $\mathcal{G}_{\varphi_1,\varphi_2}$ in the space $L([a,b])$.

1.2. The regular problem

In this subsection we consider problem (1.1), (1.2) in the regular case, where

$$f\in\mathcal{K}_{\rm loc}([a,b]\times\mathbb{R}).\tag{1.11}$$

A solution u of problem (1.1), (1.2) will be called positive (negative) if $u(t)>0$ for $a\le t\le b$ ($u(t)<0$ for $a\le t\le b$).

Theorem 1.1. *Let*

$$|f(t,x)| \le p(t)|x| + q(t,|x|) \quad \text{for } a \le t \le b,\ |x| \ge 1, \tag{1.12}$$

where $p \in L([a,b])$ and $q \in Z_{a,b}$. Moreover, let either

$$\int_a^b |g_0(t,s)|p(s)\,\mathrm{d}s < 1 \quad \text{for } a \le t \le b, \tag{1.13}$$

or one of the conditions (1.4)–(1.7) hold and

$$p \in \mathcal{G}_{\varphi_1,\varphi_2}. \tag{1.14}$$

Then problem (1.1), (1.2) has at least one solution.

Theorem 1.2. *Let*

$$|f(t,x) - f(t,y)| \le p(t)|x-y| \quad \text{for } a \le t \le b,\ x,y \in \mathbb{R}, \tag{1.15}$$

where $p \in L([a,b])$. Moreover, let either p satisfy (1.13), or one of the conditions (1.4)–(1.7) hold and p satisfy (1.14). Then problem (1.1), (1.2) has one and only one solution.

In Theorems 1.1 and 1.2 conditions (1.13) and (1.14) are unimprovable in a certain sense. More precisely, the following theorem is true:

Theorem 1.3. *Let*

$$c_1 \ge 0, \qquad c_2 \int_a^b \varphi_1(s)\,\mathrm{d}s \ge 0, \tag{1.16}$$

$$\sigma f(t,x) \ge p(t)|x| + q(t) \quad \text{for } a \le t \le b,\ x \in \mathbb{R}, \tag{1.17}$$

where p and $q \in L([a,b])$ are nonnegative functions and $c_1 + |c_2| + q(t) \not\equiv 0$. If, moreover, one of the conditions (1.4)–(1.7) holds and p satisfies either the inequality

$$\int_a^b |g_0(t,s)|p(s)\,\mathrm{d}s \ge 1 \quad \text{for } a \le t \le b, \tag{1.18}$$

or the inclusion

$$p \in \mathcal{G}^*_{\varphi_1,\varphi_2}, \tag{1.19}$$

then problem (1.1), (1.2) has no solution.

According to Definitions 1.1 and 1.2 and notation (1.8), (1.9), condition (1.4) (condition (1.5)) holds, if

$$p(t) < \frac{1}{|h_0(1)(t)|} \left(p(t) = \frac{1}{|h_0(1)(t)|} \right) \quad \text{for } a \le t \le b,$$

where

$$h_0(1)(t) = \left(\int_a^b \varphi_1(s)\,\mathrm{d}s + t - b \right) \int_a^b \varphi_2(s)\,\mathrm{d}s - \int_a^b (b-s)\varphi_1(s)\,\mathrm{d}s + \frac{(b-t)^2}{2}.$$

Theorem 1.4. *Let one of the conditions (1.4)–(1.7) hold and*

$$q_0(t)\omega_0(|x|) \le \sigma f(t,x)\,\mathrm{sgn}\,x \le p(t)|x| + q(t)\omega(|x|) \quad \text{for } a \le t \le b,\ x \in \mathbb{R}, \tag{1.20}$$

where p, q and $q_0 \in L([a,b])$ are nonnegative functions, $q_0(t) \not\equiv 0$, and ω_0 and $\omega : [0,+\infty) \to [0,+\infty)$ are nondecreasing functions such that

$$\lim_{x \to 0} \frac{\omega_0(x)}{x} = +\infty; \tag{1.21}$$

$$\lim_{x \to +\infty} \frac{\omega(x)}{x} = 0. \tag{1.22}$$

If, moreover, $c_1 = c_2 = 0$ and p satisfies either of the conditions (1.13) and (1.14), then problem (1.1), (1.2) along with the trivial solution has positive and negative solutions as well.

1.3. The singular problem

The results of this section concern the singular case, when

$$f \in \mathcal{K}_{\mathrm{loc}}([a,b] \times (0,+\infty)). \tag{1.23}$$

Theorem 1.5. *Let along with (1.16) the inequalities*

$$\sigma f(t,x) \geq q_0(t)\omega_0(x) \quad \text{for } a \leq t \leq b,\ x > 0, \tag{1.24}$$

$$\sigma f(t,x) \leq p(t)x + q(t,x) \quad \text{for } a \leq t \leq b,\ x \geq 1, \tag{1.25}$$

hold, where p and $q_0 \in L([a,b])$ are nonnegative functions, $c_1 + |c_2| + q_0(t) \not\equiv 0$, $q \in Z_{a,b}$, and $\omega_0 : (0,+\infty) \to (0,+\infty)$ is a nondecreasing function satisfying (1.21). If, moreover, one of the conditions (1.4)–(1.7) holds and p satisfies either (1.13) or (1.14), then problem (1.1), (1.2) has at least one positive solution.

Theorem 1.6. *Let along with (1.16) and the inequality*

$$\sigma f(t,x) \geq p(t)x + q(t)\omega(x) \quad \text{for } a < t < b,\ x > 0 \tag{1.26}$$

one of the conditions (1.4)–(1.7) hold, where p and $q \in L([a,b])$ are nonnegative functions, and $\omega : (0,+\infty) \times (0,+\infty)$ is a nondecreasing function. If, moreover, $c_1 + |c_2| + q(t) \not\equiv 0$, and p satisfies either of the conditions (1.18) and (1.19), then problem (1.1), (1.2) has no positive solutions.

1.4. Examples

Consider the differential equations

$$u'' = l(t)u + l_0(t); \tag{1.27}$$

$$u'' = l(t)|u| + l_0(t); \tag{1.28}$$

$$u'' = l(t)u + l_0(t)|u|^{\alpha} \operatorname{sgn} u; \tag{1.29}$$

$$u'' = l(t)u + l_0(t)(u^{\alpha} + u^{-\beta}), \tag{1.30}$$

where l and $l_0 \in L([a,b])$, $0 < \alpha < 1$, $\beta > 0$.

Theorems 1.1 and 1.2 imply

Corollary 1.1. *Let either*

$$\int_a^b |g_0(t,s)l(s)|\,\mathrm{d}s < 1 \quad \text{for } a \leq t \leq b, \tag{1.31}$$

or one of the conditions (1.4)–(1.7) hold and

$$|l| \in \mathcal{G}_{\varphi_1,\varphi_2}. \tag{1.32}$$

Then problem (1.29), (1.2) is solvable, while the problems (1.27), (1.2) and (1.28), (1.2) are uniquely solvable.

Theorems 1.3, 1.5 and 1.6 imply

Corollary 1.2. *Let along with (1.16) and the inequalities*

$$\sigma l(t) \geq 0, \qquad \sigma l_0(t) \geq 0 \quad \text{for } a \leq t \leq b, \qquad c_1 + |c_2| + l_0(t) \not\equiv 0$$

one of the conditions (1.4)–(1.7) hold. If, moreover, l satisfies either of the conditions (1.31) and (1.32), then:

(i) *a solution of problem (1.27), (1.2) is positive and coincides with a solution of problem (1.28), (1.2);*

(ii) *problem (1.29), (1.2), as well as problem (1.30), (1.2), has at least one positive solution, and besides, if either*

$$\int_a^b |g_0(t,s)l(s)|\,\mathrm{d}s \geq 1 \quad \text{for } a \leq t \leq b, \tag{1.33}$$

or

$$\sigma l \in \mathcal{G}^*_{\varphi_1,\varphi_2}, \tag{1.34}$$

then:

(iii) *none of the problems (1.27), (1.2); (1.29), (1.2) and (1.30), (1.2) has a positive solution;*

(iv) *problem (1.28), (1.2) has no solutions.*

Theorems 1.4 and 1.6 imply

Corollary 1.3. *Let $c_1 = c_2 = 0$ and one of the conditions (1.4)–(1.7) hold, and*

$$\sigma l(t) \geq 0, \qquad \sigma l_0(t) \geq 0 \quad \text{for } a \leq t \leq b,\ l_0(t) \not\equiv 0.$$

Then:

(i) *if l satisfies either of the conditions (1.31) and (1.32), then problem (1.29), (1.2) along with the trivial solution has positive and negative solutions;*

(ii) *if l satisfies either of the conditions (1.33) and (1.34), then problem (1.29), (1.2) has neither positive nor negative solutions.*

2. Auxiliary statements

2.1. Properties of the function g_0 and the operator h_0

Consider the differential equation

$$u'' = q(t) \tag{2.1}$$

with boundary conditions (1.2), where $q \in L([a, b])$. As above, it will be assumed that $\varphi_i : [a, b] \to \mathbb{R}$ $(i = 1, 2)$ are functions of bounded variation satisfying equalities (1.3).

Lemma 2.1. *Problem (2.1), (1.2) has a unique solution u, g_0 is its Green's function and u admits the representation*

$$u(t) = c_1 + c_2 \left(\int_a^b \varphi_1(s)\,\mathrm{d}s + t - b \right) + h_0(q)(t) \quad \text{for } a \le t \le b. \tag{2.2}$$

Proof. An arbitrary solution of Eq. (2.1) admits the representation

$$u(t) = \gamma_1 + \gamma_2(t - a) + \int_a^t (t - s)q(s)\,\mathrm{d}s, \tag{2.3}$$

where $\gamma_i \in \mathbb{R}$ $(i = 1, 2)$. In view of (1.3), u is a solution of problem (2.1), (1.2) if and only if (γ_1, γ_2) is a solution of the linear algebraic equations

$$\begin{aligned} &\gamma_1 + \gamma_2 \int_a^b (\tau - a)\,\mathrm{d}\varphi_1(\tau) + \int_a^b \left(\int_a^s (s - \tau)q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\varphi_1(s) = c_1, \\ &\gamma_2 + \int_a^b \left(\int_a^s q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\varphi_2(s) = c_2. \end{aligned} \tag{2.4}$$

However, in view (1.3) we have

$$\int_a^b (\tau - a)\,\mathrm{d}\varphi_1(\tau) = b - a - \int_a^b \varphi_1(\tau)\,\mathrm{d}\tau,$$

$$\begin{aligned} \int_a^b \left(\int_a^s (s - \tau)q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\varphi_1(s) &= \int_a^b (b - s)q(s)\,\mathrm{d}s - \int_a^b \left(\int_a^s q(\tau)\,\mathrm{d}\tau \right) \varphi_1(s)\,\mathrm{d}s \\ &= \int_a^b (b - s)q(s)\,\mathrm{d}s - \int_a^b \left(\int_s^b \varphi_1(\tau)\,\mathrm{d}\tau \right) q(s)\,\mathrm{d}s \\ &= \int_a^b \left(b - s - \int_s^b \varphi_1(\tau)\,\mathrm{d}\tau \right) q(s)\,\mathrm{d}s, \end{aligned}$$

$$\int_a^b \left(\int_a^s q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\varphi_2(s) = \int_a^b (1 - \varphi_2(s))q(s)\,\mathrm{d}s.$$

Therefore system (2.4) has a unique solution (γ_1, γ_2) given by the equalities

$$\gamma_1 = c_1 + c_2 \left(\int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + a - b \right) + \int_a^b \left[\left(\int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + a - b \right) \varphi_2(s) - \int_a^s \varphi_1(\tau)\,\mathrm{d}\tau + s - a \right] q(s)\,\mathrm{d}s,$$

$$\gamma_2 = c_2 + \int_a^b (\varphi_2(s) - 1)q(s)\,\mathrm{d}s.$$

The latter equalities together with (2.3) imply

$$\begin{aligned} u(t) = {} & c_1 + c_2 \left(\int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + t - b \right) + \int_a^b \left[\left(\int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + t - b \right) \varphi_2(s) - \int_a^s \varphi_1(\tau)\,\mathrm{d}\tau + s - t \right] q(s)\,\mathrm{d}s \\ & + \int_a^t (t - s)\,q(s)\,\mathrm{d}s. \end{aligned}$$

Hence, (1.8) and (1.9) imply equality (2.2). □

Lemma 2.2. *If one of the conditions (1.4)–(1.7) holds, then*

$$\sigma g_0(t, s) > 0 \quad \text{for } a \le t \le b,\ a < s < b \tag{2.5}$$

and, consequently,

$$\sigma h_0(q)(t) > 0 \quad \text{for } a \le t \le b \tag{2.6}$$

for any $q \in L([a,b])$ satisfying the conditions

$$q(t) \ge 0 \quad \text{for } a \le t \le b, \ q(t) \not\equiv 0. \tag{2.7}$$

Proof. Assume first that (1.4) holds. Then from (1.8) we have

$$\begin{aligned} g_0(t,s) &> \int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + t - b - \int_a^s \varphi_1(\tau)\,\mathrm{d}\tau + \chi(s,t)(s-t) \\ &= \int_s^b \varphi_1(\tau)\,\mathrm{d}\tau + t - b + \chi(s,t)(s-t) > t - s + \chi(s,t)(s-t) \ge 0 \quad \text{for } a \le t \le b, \ a < s < b. \end{aligned}$$

If (1.5) holds then

$$g_0(t,s) > \chi(s,t)(s-t) \ge 0 \quad \text{for } a \le t \le b, \ a < s < b.$$

Consequently in these two cases inequality (2.5) holds true, since $\sigma = 1$ according to (1.10).

Assume now that condition (1.6) holds. Then $\sigma = -1$, and, by (1.8), we get

$$\begin{aligned} \sigma g_0(t,s) &= \left(\int_a^b \varphi_1(\tau)\,\mathrm{d}\tau + t - b\right)|\varphi_2(s)| + \int_a^s \varphi_1(\tau)\,\mathrm{d}\tau + \chi(s,t)(t-s) \\ &> s - a + \chi(s,t)(t-s) \ge 0 \quad \text{for } a \le t \le b, \ a < s < b. \end{aligned}$$

If (1.7) holds, then again $\sigma = -1$ and

$$\begin{aligned} \sigma g_0(t,s) &= \left(\int_a^b |\varphi_1(\tau)|\,\mathrm{d}\tau + b - t\right)\varphi_2(s) - \int_a^s |\varphi_1(\tau)|\,\mathrm{d}\tau + \chi(s,t)(t-s) \\ &> \int_s^b |\varphi_1(\tau)|\,\mathrm{d}\tau + b - t + \chi(s,t)(t-s) \ge 0 \quad \text{for } a \le t \le b, \ a < s < b. \end{aligned}$$

Thus (2.5) is proved. As for the inequality (2.6), it immediately follows from (1.9), (2.5) and (2.7). □

2.2. *Lemmas on differential inequalities*

Consider the differential inequalities

$$\sigma u''(t) \ge q_0(t), \tag{2.8}$$

$$\sigma u''(t) \ge q_0(t)\omega_0(|u(t)|) \tag{2.9}$$

and

$$\sigma u''(t) \ge p(t)|u(t)| + q(t), \tag{2.10}$$

where σ is the number given by (1.10), p, q_0 and $q : [a,b] \to [0,+\infty)$ are integrable functions, and $\omega_0 : [0,+\infty) \to [0,+\infty)$ is a nondecreasing function. A function $u : [a,b] \to \mathbb{R}$ will be called a solution of the differential inequality (2.k), $k \in \{8, 9, 10\}$ if it is absolutely continuous together with its derivative and satisfies the differential inequality almost everywhere on $[a,b]$.

Lemma 2.3. *Let one of the conditions (1.4)–(1.7) hold, c_1 and c_2 satisfy inequalities (1.16) and*

$$c_1 + |c_2| + q_0(t) \not\equiv 0. \tag{2.11}$$

Then an arbitrary solution u of problem (2.8), (1.2) admits the estimate

$$u(t) \ge \delta_0(c_1 + |c_2|) + \sigma h_0(q_0)(t) > 0 \quad \text{for } a \le t \le b, \tag{2.12}$$

where

$$\delta_0 = \min\left\{1, \int_a^b |\varphi_1(s)|\,\mathrm{d}s, \left|\int_a^b \varphi_1(s)\,\mathrm{d}s + a - b\right|\right\} > 0. \tag{2.13}$$

Proof. By Lemma 2.1 the representation

$$u(t) = c_1 + c_2\left(\int_a^b \varphi_1(s)\,\mathrm{d}s + t - b\right) + h_0(u'')(t)$$

is valid. Hence in view of Lemma 2.2 and inequality (2.8) we have

$$u(t) \geq c_1 + c_2 \left(\int_a^b \varphi_1(s)\,\mathrm{d}s + t - b \right) + \sigma h_0(q_0)(t) \quad \text{for } a \leq t \leq b. \tag{2.14}$$

Moreover, if $q_0(t) \not\equiv 0$, then

$$\sigma h_0(q_0)(t) > 0 \quad \text{for } a \leq t \leq b.$$

On the other hand, inequality (1.16) and any of the conditions (1.4)–(1.7) guarantee the validity of inequality (2.13) and the estimate

$$c_1 + c_2 \left(\int_a^b \varphi_1(s)\,\mathrm{d}s + t - b \right) \geq \delta_0(c_1 + |c_2|) \quad \text{for } a \leq t \leq b.$$

Therefore estimate (2.12) follows from inequalities (2.11) and (2.14). □

Lemma 2.4. *Let, along with one of the conditions (1.4)–(1.7), conditions (1.16), (1.21) hold and*

$$q_0(t) \not\equiv 0. \tag{2.15}$$

Then there exists a positive number δ such that an arbitrary positive solution u of problem (2.9), (1.2) admits the estimate

$$u(t) > \delta \quad \text{for } a \leq t \leq b. \tag{2.16}$$

Proof. By Lemma 2.2 and condition (2.15), we have

$$\gamma = \inf\{\sigma h_0(q_0)(t) : a \leq t \leq b\} > 0.$$

On the other hand, according to condition (1.21) there exists $\delta > 0$ such that

$$\frac{x}{\omega_0(x)} < \gamma \quad \text{for } 0 < x \leq \delta. \tag{2.17}$$

Let u be a positive solution of problem (2.9), (1.2). Choose $t_0 \in [a, b]$ such that

$$\mu = \min\{u(t) : a \leq t \leq b\} = u(t_0).$$

Then the inequality

$$\sigma u''(t) \geq \omega_0(\mu) q_0(t)$$

holds almost everywhere on $[a, b]$. Hence Lemma 2.3 yields

$$\mu \geq \delta_0(c_1 + |c_2|) + \sigma h_0(\omega_0(\mu) q_0)(t_0) \geq \omega_0(\mu)\sigma h_0(q_0)(t_0) \geq \gamma\omega_0(\mu)$$

and, consequently,

$$\frac{\mu}{\omega_0(\mu)} > \gamma.$$

In view of (2.17) the latter inequality implies

$$\mu > \delta,$$

i.e., estimate (2.16) holds. □

Lemma 2.5. *Let one of the conditions (1.4)–(1.7) hold, c_1 and c_2 satisfy inequalities (1.16), and*

$$c_1 + |c_2| + q(t) \not\equiv 0. \tag{2.18}$$

If, moreover, p satisfies either of the conditions (1.18) and (1.19), then problem (2.10), (1.2) has no solution.

Proof. Assume the contrary, that problem (2.10), (1.2) has a solution u. Then by Lemma 2.3 and condition (2.18), we have

$$u(t) \geq \delta_0(c_1 + |c_2|) + \sigma h_0(p|u| + q)(t) > 0 \quad \text{for } a \leq t \leq b, \tag{2.19}$$

and, consequently,

$$\sigma h_0(p|u| + q)(t) = \sigma h_0(pu)(t) + \sigma h_0(q)(t).$$

On the other hand, by Lemma 2.2 and condition (2.18), the inequality

$$\delta(c_1 + |c_2|) + \sigma h_0(q)(t) > 0 \quad \text{for } a \leq t \leq b$$

holds, according to which (2.19) implies

$$u(t) > 0, \qquad u(t) > \sigma h_0(pu)(t) \quad \text{for } a \leq t \leq b. \tag{2.20}$$

Consider first the case where p satisfies inequality (1.18), and choose $t_0 \in [a, b]$ such that

$$u(t_0) = \min\{u(t) : a \le t \le b\}.$$

Then according to Lemma 2.2 and (2.20) we have

$$u(t_0) > u(t_0)\sigma h_0(p)(t_0) = u(t_0) \int_a^b |g_0(t_0, s)|p(s)\,\mathrm{d}s \ge u(t_0).$$

The contradiction obtained shows that if (1.18) holds, then problem (2.10), (1.2) has no solution.

Now assume that p satisfies (1.19). Then, by Definition 1.2, there exists a nonnegative function $p_0 \in L([a, b])$ such that $p_0(t) \not\equiv 0$ and

$$p(t) \equiv \frac{p_0(t)}{|h_0(p_0)(t)|}.$$

Set

$$\mu = \min\left\{\frac{u(t)}{|h_0(p_0)(t)|} : a \le t \le b\right\}$$

and choose $t_0 \in [a, b]$ such that

$$u(t_0) = \mu |h_0(p_0)(t_0)|.$$

Then by Lemma 2.2 and (2.20), again we get the contradiction

$$\mu|h_0(p_0)(t_0)| > \sigma h_0(pu)(t_0) = \sigma h_0\Big(p_0 \frac{u}{|h_0(p_0)|}\Big)(t_0)$$
$$\ge \mu\sigma h_0(p_0)(t_0) = \mu|h_0(p_0)(t_0)|.$$

Consequently, problem (2.10), (1.2) has no solution if (1.19) holds. □

2.3. *Lemma on the solvability of the regular problem (1.1), (1.2)*

The following lemma deals with the case where (1.11) holds.

Lemma 2.6. *Let inequality (1.12) hold, where $p : [a, b] \to [0, +\infty)$ is an integrable function and*

$$q \in Z_{a,b}. \tag{2.21}$$

Moreover, let the differential inequality

$$|u''(t)| \le p(t)|u(t)|, \tag{2.22}$$

subject to the homogeneous boundary conditions

$$\int_a^b u^{(i-1)}(s)\,\mathrm{d}\varphi_i(s) = 0 \quad (i = 1, 2), \tag{2.23}$$

have only the trivial solution. Then problem (1.1), (1.2) has at least one solution.

Proof. Problem (1.1), (1.2) is equivalent to problem

$$u_1' = u_2, \qquad u_2' = f(t, u_1); \tag{2.24}$$

$$\int_a^b u_i(s)\,\mathrm{d}\varphi_i(s) = c_i \quad (i = 1, 2). \tag{2.25}$$

On the other hand, according to Theorem 2.3 from [8] and conditions (1.12) and (2.21), problem (2.24), (2.25) is solvable, if the system of differential inequalities

$$|u_1'(t) - u_2(t)| \le 0, \qquad |u_2'(t)| \le p(t)|u_1(t)|, \tag{2.26}$$

subject to the homogeneous boundary conditions

$$\int_a^b u_i(s)\,\mathrm{d}\varphi_i(s) = 0 \quad (i = 1, 2), \tag{2.27}$$

has only the trivial solution. Consequently, to prove the lemma it is sufficient to show that problem (2.26), (2.27) has only the trivial solution.

Let (u_1, u_2) be an arbitrary solution of problem (2.26), (2.27). Set

$$u(t) = u_1(t) \quad \text{for } a \le t \le b.$$

Then

$$u_2(t) = u'(t) \quad \text{for } a \le t \le b,$$

and u is a solution of problem (2.22), (2.23). However, according to one of the conditions of the lemma, problem (2.22), (2.23) has only the trivial solution. Consequently, $u_i(t) \equiv u^{(i-1)}(t) \equiv 0$ $(i = 1, 2)$. $\square$

2.4. Lemma on the regularization of the singular problem (1.1), (1.2)

In this subsection consider problem (1.1), (1.2) in the singular case when (1.23) holds.

For an arbitrary $\gamma > 0$ set

$$f_\gamma(t,x) = \begin{cases} f(t,x) & \text{for } x \ge \gamma \\ f(t,\gamma) & \text{for } x \le \gamma, \end{cases} \tag{2.28}$$

and consider the differential equation

$$u'' = f_\gamma(t,u). \tag{2.29}$$

In view of (1.23) it follows from (2.28) that

$$f_\gamma \in \mathcal{K}_{\mathrm{loc}}([a,b] \times \mathbb{R}).$$

Consequently, problem (2.29), (1.2) is regular for any $\gamma > 0$.

If problem (1.1), (1.2) has a positive solution, then it is clear that it would also be a solution of problem (2.29), (1.2) for sufficiently small $\gamma > 0$. It turns out that under some additional restrictions on the function f the converse is true, i.e., solvability of regular problem (2.29), (1.2) guarantees the existence of a positive solution to singular problem (1.1), (1.2). More precisely, the following lemma holds.

Lemma 2.7. *Let inequalities (1.16) and (1.24) hold, where $q_0 : [a,b] \to [0,+\infty)$ is an integrable function and $\omega_0 : (0,+\infty) \to (0,+\infty)$ is a nondecreasing function satisfying conditions (1.21) and (2.11). If, moreover, one of the conditions (1.4)–(1.7) holds, then solvability of problem (2.29), (1.2) for an arbitrarily small $\gamma > 0$ guarantees the existence of at least one positive solution to problem (1.1), (1.2).*

Proof. If $c_1 + |c_2| > 0$, then set

$$\delta = \frac{\delta_0(c_1 + |c_2|)}{2}, \tag{2.30}$$

where δ_0 is the number given by (2.13). If $c_1 + |c_2| = 0$, then, in view of (2.11), inequality (2.15) holds. In this case by δ we will understand the number appearing in Lemma 2.4.

Let us show that if problem (2.29), (1.2) is solvable for some $\gamma \in (0, \delta]$, then its arbitrary solution u is a positive solution of problem (1.1), (1.2). For this, according to equality (2.28), it is sufficient to show that u admits estimate (2.16).

(1.24) and (2.28) imply

$$\sigma f_\gamma(t,x) \ge \omega_0(\gamma) q_0(t) \quad \text{for } a \le t \le b,\ x \in \mathbb{R} \tag{2.31}$$

and

$$\sigma f_\gamma(t,x) \ge q_0(t)\omega_0(x) \quad \text{for } a \le t \le b,\ x \ge 0. \tag{2.32}$$

By (2.31), an arbitrary solution u of problem (2.29), (1.2) is also a solution of the differential inequality

$$\sigma u''(t) \ge \omega_0(\gamma) q_0(t).$$

Hence, by Lemma 2.3 and condition (2.11), it follows that

$$u(t) \ge \delta_0(c_1 + |c_2|) + \sigma h_0(\omega_0(\gamma) q_0)(t) > 0 \quad \text{for } a \le t \le b. \tag{2.33}$$

If $c_1 + |c_2| > 0$, then (2.30) and (2.33) imply estimate (2.16). If $c_1 + |c_2| = 0$, then, in view of (2.11), inequality (2.15) holds. On the other hand from (2.32) and (2.33) it is clear that u is a positive solution of the differential inequality (2.9). Applying Lemma 2.4 we immediately get estimate (2.16). $\square$

3. Proofs of the main results

Proof of Theorem 1.1. By Lemma 2.6, it is sufficient to show that problem (2.22), (2.23) has only the trivial solution. Assume the contrary, that this problem has a nontrivial solution u. Then, by Lemma 2.1, u admits the representation

$$u(t) = \int_a^b g_0(t,s) u''(s)\, ds.$$

Therefore

$$|u(t)| \le \int_a^b |g_0(t,s)|p(s)|u(s)|\,ds \quad \text{for } a \le t \le b. \tag{3.1}$$

Consider first the case when inequality (1.13) holds. Choose $t_0 \in [a,b]$ such that

$$|u(t_0)| = \max\{|u(t)| : a \le t \le b\}.$$

Then (3.1) yields the contradiction

$$|u(t_0)| \le \int_a^b |g_0(t,s)|p(s)|u(s)|\,ds \le |u(t_0)| \int_a^b |g_0(t,s)|p(s)\,ds < |u(t_0)|.$$

It remains to consider the case where one of the conditions (1.4)–(1.7) holds and p satisfies (1.14).

By Definition 1.1, there exists a nonnegative function $p_0 \in L([a,b])$ such that $p(t) \not\equiv 0$,

$$p(t) \le \frac{p_0(t)}{|h_0(p_0)(t)|} \quad \text{for } a \le t \le b, \qquad p(t) \not\equiv \frac{p_0(t)}{|h_0(p_0)(t)|}. \tag{3.2}$$

Choose $t_0 \in [a,b]$ such that

$$\frac{|u(t_0)|}{|h_0(p_0)(t_0)|} = \max\left\{\frac{|u(t)|}{|h_0(p_0)(t)|} : a \le t \le b\right\}.$$

Then by Lemma 2.2 and inequalities (3.2), from (3.1) again we get the contradiction

$$\begin{aligned}|u(t_0)| &\le \frac{|u(t_0)|}{|h_0(p_0)(t_0)|}\int_a^b |g_0(t_0,s)|p(s)|h_0(p_0)(s)|\,ds \\ &< \frac{|u(t_0)|}{|h_0(p_0)(t_0)|}\int_a^b |g_0(t_0,s)|p_0(s)\,ds = |u(t_0)|.\end{aligned}$$

Thus it is proved that problem (2.22), (2.23) has only the trivial solution. □

Proof of Theorem 1.2. (1.15) implies inequality (1.12), where

$$q(t,x) \equiv |f(t,0)| \quad \text{and} \quad q \in Z_{a,b}.$$

Consequently, all of the conditions of Theorem 1.1 are satisfied, which guarantees solvability of problem (1.1), (1.2). It remains to prove the uniqueness of its solution.

Let u_1 and u_2 be arbitrary solutions of problem (1.1), (1.2) and

$$u(t) = u_1(t) - u_2(t).$$

Then in view of (1.15), u is a solution of problem (2.22), (2.23). However, as was shown above, in the case under consideration problem (2.22), (2.23) has only the trivial solution. Consequently, $u_1(t) \equiv u_2(t)$. □

Proof of Theorem 1.3. Assume the contrary, that problem (1.1), (1.2) has a solution u. Then in view of condition (1.17) u is a solution of problem (2.10), (1.2). However, by Lemma 2.5, in the case considered problem (2.10), (1.2) has no solution. The contradiction obtained proves the theorem. □

Proof of Theorem 1.5. Let γ be an arbitrary positive constant, and f_γ be a function given by equality (2.28). Then in view of inequalities (1.24) and (1.5) we have

$$|f(t,x)| \le p(t)|x| + q_\gamma(t,|x|) \quad \text{for } a \le t \le b,\ |x| \ge 1,$$

where

$$q_\gamma(t,x) = |f(t,\gamma)| + q(t,x) \quad \text{for } a \le t \le b,\ x \ge 1,$$

and $q_\gamma \in Z_{a,b}$. By Theorem 1.1, problem (2.29), (1.2) is solvable. However, by Lemma 2.7, solvability of the aforementioned problem for an arbitrary $\gamma > 0$ guarantees the existence of at least one positive solution of problem (1.1), (1.2). □

Proof of Theorem 1.4. From (1.20) it is clear that $f(t,0) \equiv 0$. Consequently, problem (1.1), (1.2) has the trivial solution since $c_i = 0$ $(i = 1, 2)$.

Set

$$\begin{aligned}\widetilde{f}(t,x) &= -f(t,-x) \quad \text{for } a \le t \le b,\ x \ge 0; \\ q^*(t,x) &= q(t)\omega(x) \quad \text{for } a \le t \le b,\ x \ge 1,\end{aligned} \tag{3.3}$$

and along with (1.1) consider the differential equation

$$u'' = \widetilde{f}(t, u). \tag{3.4}$$

According to (1.20) the function f along with (1.24) satisfies the inequality

$$\sigma f(t, x) \le q(t)x + q^*(t, x) \quad \text{for } a \le t \le b,\ x \ge 1,$$

and the function $\widetilde{f}$ satisfies the inequalities

$$\sigma \widetilde{f}(t, x) \ge q_0(t)\omega_0(x) \quad \text{for } a \le t \le b,\ x > 0;$$
$$\sigma \widetilde{f}(t, x) \le p(t)x + q^*(t, x) \quad \text{for } a \le t \le b,\ x \ge 1.$$

Besides, in view of (1.22), it is clear that $q^* \in Z_{a,b}$.

By Theorem 1.5, problem (1.1), (1.2), as well as problem (3.4), (1.2), has at least one positive solution.

Let u_0 be an arbitrary positive solution of problem (3.4), (1.2) and

$$u(t) = -u_0(t) \quad \text{for } a \le t \le b.$$

Then in view of condition (3.3) and the equalities $c_i = 0$ $(i = 1, 2)$, u is a solution of problem (1.1), (1.2). Consequently, problem (1.1), (1.2) along with the trivial and positive solutions also has a negative solution. □

Proof of Theorem 1.6. Assume the contrary, that problem (1.1), (1.2) has a positive solution u. Set $\delta = \min\{u(t) : a \le t \le b\}$. Then in view of (1.26) the function u is a solution of the differential inequality

$$\sigma u''(t) \ge p(t)|u(t)| + q_1(t), \tag{3.5}$$

where $q_1(t) = \omega(\delta)q(t) \ge 0$ for $a \le t \le b$ and $c_1 + |c_2| + q_1(t) \not\equiv 0$. However, by Lemma 2.5, problem (3.5), (1.2) has no solution. The contradiction obtained proves the theorem. □

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