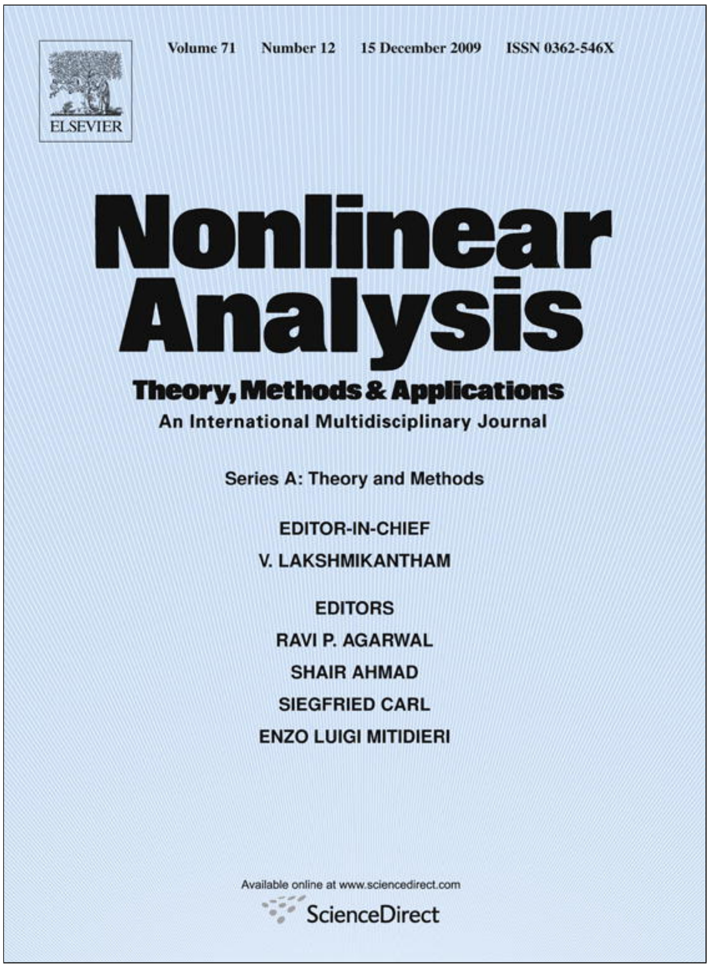

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On boundary value problems with conditions at infinity for nonlinear differential systems

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ABSTRACT

Optimal sufficient conditions for the solvability and well-posedness of the boundary value problem

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n),$$
$$x_i(0) = c_i \quad (i = 1, \ldots, m), \qquad \limsup_{t\to+\infty} |x_i(t)| < +\infty \quad (i = m+1, \ldots, n)$$

are established.

1. Statement of the main results

In the present paper, the boundary value problem

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n), \tag{1.1}$$

$$x_i(0) = c_i \quad (i = 1, \ldots, m), \qquad \limsup_{t\to+\infty} |x_i(t)| < +\infty \quad (i = m+1, \ldots, n) \tag{1.2}$$

is investigated on the interval $\mathbb{R}_+ = [0, +\infty[$. Here $n \geq 2$, $m \in \{1, \ldots, n-1\}$, $c_i \in \mathbb{R} (i = 1, \ldots, m)$, and $f_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} (i = 1, \ldots, n)$ are functions satisfying the local Carathéodory conditions.

The previous well-known results on the solvability of such problems do not cover the wide class of nonlinear differential systems with right-hand sides rapidly growing with respect to the phase variables. As for the well-posedness of the problem (1.1), (1.2), and the behavior of its solutions at $+\infty$, they have remained practically unstudied (see, e.g, [1–6] and the references therein). Theorems 1.1–1.5 below fill this gap to some extent. Theorems 1.1–1.3 and 1.5 contain unimprovable in a sense conditions guaranteeing solvability and well-posedness of the problem (1.1), (1.2). In Theorem 1.4 we give optimal sufficient conditions under which every solution of that problem vanishes at infinity.

We use the following notation.

$\mathbb{R}^n$ is the n-dimensional real Euclidean space;

$x = (x_i)_{i=1}^n \in \mathbb{R}^n$ is the vector with components x_i $(i = 1, \ldots, n)$;

δ_{ik} is Kronecker's symbol;

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$-matrix with components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \ldots, n)$ and with the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

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$r(X)$ is the spectral radius of X; E is the unit matrix;

A_s is the set of asymptotically stable, quasi-nonnegative $n \times n$-matrices, i.e. $H = (h_{ik})_{i,k=1}^n \in A_s$ if and only if $h_{ik} \geq 0$ for $i \neq k$ and real parts of eigenvalues of H are negative;

$\widetilde{C}_{\text{loc}}(\mathbb{R}+)$ is the space of functions $x : \mathbb{R}_+ \to \mathbb{R}$, absolutely continuous on every compact interval containing in $\mathbb{R}_+$;

$L_{\text{loc}}(\mathbb{R}_+)$ is the space of functions $x : \mathbb{R}_+ \to \mathbb{R}$, Lebesgue integrable on every compact interval containing in $\mathbb{R}_+$;

$L^\infty(\mathbb{R}_+)$ is the space of essentially bounded measurable functions $x : \mathbb{R}_+ \to \mathbb{R}$ with the norm

$$\|x\|_{L^\infty} = \text{ess sup}\,\{|x(t)| : t \in \mathbb{R}_+\};$$

$\mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)$ is the set of functions $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, satisfying the local Carathéodory[1] conditions, i.e., $f \in \mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)$ iff $f(t, \cdot, \ldots, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous for almost all $t \in \mathbb{R}_+$, $f(\cdot, x_1, \ldots, x_n) \in L_{\text{loc}}(\mathbb{R}_+)$ for any $(x_i)_{i=1}^n \in \mathbb{R}^n$ and the function f_ρ^*, given by the equality

$$f_\rho^*(t) = \max\left\{|f(t, x_1, \ldots, x_n)| : \sum_{i=1}^n |x_i| \leq \rho\right\},$$

belongs to the space $L_{\text{loc}}(\mathbb{R}_+)$ for any $\rho \in \mathbb{R}_+$.

Throughout the paper, it is supposed that

$$f_i \in \mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n) \quad (i = 1, \ldots, n). \tag{1.3}$$

By a solution of the system (1.1), defined on the interval $\mathbb{R}_+$, we understand a vector function $(x_i)_{i=1}^n : \mathbb{R}_+ \to \mathbb{R}^n$ with components $x_i \widetilde{C}_{\text{loc}}(\mathbb{R}_+)$ $(i = 1, \ldots, n)$ satisfying that system almost everywhere on $\mathbb{R}_+$.

A solution $(x_i)_{i=1}^n$ of the system (1.1), defined on $\mathbb{R}_+$ and satisfying the boundary conditions (1.2), is called a solution of the problem (1.1), (1.2).

Along with the problem (1.1), (1.2) we consider the auxiliary problem

$$\frac{dx_i}{dt} = \lambda f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n), \tag{1.4}$$

$$x_i(0) = c_i \quad (i = 1, \ldots, m), \qquad x_i(a) = c_i \quad (i = m+1, \ldots, n), \tag{1.5}$$

depending on parameters $\lambda \in]0, 1]$ and $a \in]0, +\infty[$.

The following theorems are valid.

Theorem 1.1 (*Principle of a priori Boundedness*). *Let there exists a non-decreasing function $\rho_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for arbitrary $\lambda \in]0, 1]$, $b \in]0, +\infty[$, and $(c_i)_{i=1}^n \in \mathbb{R}^n$, every solution $(x_i)_{i=1}^n$ of the problem (1.4), (1.5) admits the estimate*

$$\sum_{i=1}^n |x_i(t)| \leq \rho_0\left(\sum_{i=1}^n |c_i|\right) \quad \text{for } 0 \leq t \leq b. \tag{1.6}$$

Then for any $(c_i)_{i=1}^m \in \mathbb{R}^m$, the problem (1.1), (1.2) is solvable, and every solution of this problem is bounded on $\mathbb{R}_+$.

Theorem 1.2. *Let there exist nonnegative functions $g_i \in \mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n) (i = 1, \ldots, n)$, $h \in L^\infty(\mathbb{R}_+)$ and a matrix $H = (h_{ik})_{i,k=1}^n \in A_s$ such that on the set $\mathbb{R}_+ \times \mathbb{R}^n$ the inequalities*

$$\sigma_i f_i(t, x_1, \ldots, x_n)\,\text{sgn}(x_i) \leq g_i(t, x_1, \ldots, x_n)\left(\sum_{k=1}^n h_{ik}|x_k| + h(t)\right) \quad (i = 1, \ldots, n), \tag{1.7}$$

where $\sigma_1 = \cdots = \sigma_m = 1$ and $\sigma_{m+1} = \cdots = \sigma_n = -1$, are satisfied. Then for arbitrary $(c_i)_{i=1}^m \in \mathbb{R}^m$ the problem (1.1), (1.2) has at least one solution, and every solution of this problem is bounded on $\mathbb{R}_+$.

It is known (see [3], Theorem 1.18) that the quasi-nonnegative matrix $H = (h_{ik})_{i,k=1}^n$ belongs to the set A_s iff

$$h_{ii} < 0 \quad (i = 1, \ldots, n) \quad \text{and} \quad r(H_0) < 1, \tag{1.8}$$

where

$$H_0 = \left((1 - \delta_{ik})\frac{h_{ik}}{|h_{ii}|}\right)_{i,k=1}^n. \tag{1.9}$$

Note that the condition $H \in A_s$ in Theorem 1.2 and in other theorems below is unimprovable and it cannot be weakened. It can be replaced by the equivalent condition (1.8) but not by the condition

$$h_{ii} < 0 \quad (i = 1, \ldots, n), \qquad r(H_0) \leq 1. \tag{1.10}$$

[1] The Greek spelling is $K\alpha\rho\alpha\theta\varepsilon o\delta\omega\rho\eta$.

Indeed, consider the problem

$$\frac{dx_i}{dt} = (-1)^i(x_1 + x_2) + i - 1 \quad (i = 1, 2), \tag{1.11}$$

$$x_i(0) = c_i, \qquad \limsup_{t\to+\infty} |x_2(t)| < +\infty. \tag{1.12}$$

For this problem all the conditions of Theorem 1.2 hold except $H \in A_s$ instead of which the condition (1.10) holds, since

$$H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Nevertheless the problem (1.11), (1.12) does not have a solution since general solution of the system (1.11) has the form

$$x_1(t) = \alpha_1 - (\alpha_1 + \alpha_2)t - \frac{t^2}{2}, \qquad x_2(t) = \alpha_2 + (\alpha_1 + \alpha_2 + 1)t + \frac{t^2}{2},$$

where α_1 and α_2 are arbitrary real numbers.

For any $H \in A_s$ suppose that

$$\mu(H) = \left\|(E - H_0)^{-1}\right\| \left(1 + \sum_{i=1}^{n} |h_{ii}|^{-1}\right), \tag{1.13}$$

where H_0 is the matrix given by the equality (1.9).

Theorem 1.3. *Let the conditions of Theorem 1.2 be fulfilled and*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i = m + 1, \dots, n), \tag{1.14}$$

where

$$p_i(t) = \inf\left\{g_i(t, x_1, \dots, x_n) : (x_k)_{k=1}^n \in \mathbb{R}^n\right\}. \tag{1.15}$$

Then every solution of the problem (1.1), (1.2) admits the estimate

$$\sum_{k=1}^{n} |x_k(t)| \le \mu(H) \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty}\right) \quad \text{for } t \in \mathbb{R}_+, \tag{1.16}$$

where $\mu(H)$ is the number, given by the equality (1.13).

From the estimate (1.16) it, in particular, follows that if the conditions of Theorem 1.3 are fulfilled, then an arbitrary solution of the system (1.1), satisfying the conditions

$$x_i(0) = c_i \quad (i = 1, \dots, n) \quad \text{and} \quad \sum_{k=1}^{n} |c_k| > \mu(H) \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty}\right),$$

is either unbounded or blowing-up.

Theorem 1.4. *Let the conditions of Theorem 1.2 be fulfilled, $h(t) \to 0$ as $t \to +\infty$, and*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i = 1, \dots, n), \tag{1.17}$$

where each p_i is the function given by the equality (1.15). Then an arbitrary solution of the problem (1.1), (1.2) is vanishing at infinity, i.e.,

$$\lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1, \dots, n). \tag{1.18}$$

Now along with the problem (1.1), (1.2) we consider the perturbed problem

$$\frac{dy_i}{dt} = f_i(t, y_1, \dots, y_n) + q_i(t, y_1, \dots, y_n) \quad (i = 1, \dots, n), \tag{1.19}$$

$$y_i(0) = c_i + \delta_i \quad (i = 1, \dots, m), \qquad \limsup_{t\to+\infty} |y_i(t)| < +\infty \quad (i = m + 1, \dots, n), \tag{1.20}$$

where $(\delta_i)_{i=1}^m \in \mathbb{R}^m$, and $q_i \in \mathcal{K}_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)(i = 1, \dots, n)$ are functions satisfying the conditions

$$|q_i(t, y_1, \dots, y_n)| \le p_i(t)q_0(t) \quad (i = 1, \dots, n), \; q_0 \in L^\infty(\mathbb{R}_+). \tag{1.21}$$

The case, where

$$\lim_{t\to+\infty} q_0(t) = 0, \tag{1.22}$$

is considered separately.

Let us introduce the following definition.

Definition 1.1. Suppose $p_i \in L_{\text{loc}}(\mathbb{R}_+)(i = 1, \dots, n)$ are nonnegative functions. The problem (1.1), (1.2) is said to be well-posed with the weight $(p_i)_{i=1}^n$ if for any $(\delta_i)_{i=1}^m \in \mathbb{R}^m$ and functions $q_i \in \mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)(i = 1, \dots, n)$, satisfying the conditions (1.21), the problem (1.19), (1.20) is solvable and there exists a positive constant ρ such that arbitrary solutions $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ of the problems (1.1), (1.2), and (1.19), (1.20) admit the estimate

$$\sum_{i=1}^n |y_i(t) - x_i(t)| \le \rho \left(\sum_{i=1}^m |\delta_i| + \|q_0\|_{L^\infty} \right) \quad \text{for } t \in \mathbb{R}_+. \tag{1.23}$$

From this definition it is clear that if the problem (1.1), (1.2) is well-posed, then it has a unique solution.

Definition 1.2. The problem (1.1), (1.2) is said to be asymptotically well-posed with the weight $(p_i)_{i=1}^n$ if it is well-posed and for any $(\delta_i)_{i=1}^m \in \mathbb{R}^m$ and functions $q_i \in \mathcal{K}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)(i = 1, \dots, n)$, satisfying the conditions (1.21) and (1.22), an arbitrary solution $(y_i)_{i=1}^n$ of the problem (1.19), (1.20) satisfies the equalities

$$\lim_{t\to+\infty} (y_i(t) - x_i(t)) = 0 \quad (i = 1, \dots, n), \tag{1.24}$$

where $(x_i)_{i=1}^n$ is a solution of the problem (1.1), (1.2).

Theorem 1.5. *Let there exist nonnegative functions $p_i \in L_{\text{loc}}(\mathbb{R}_+)(i = 1, \dots, n)$, $h \in L^\infty(\mathbb{R}_+)$, and a matrix $H = (h_{ik})_{i,k=1}^n \in A_s$ such that, respectively, on $\mathbb{R}_+ \times \mathbb{R}^n$ and $\mathbb{R}_+$ the conditions*

$$\sigma_i \left(f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n) \right) \operatorname{sgn}(x_i - y_i) \le p_i(t) \sum_{k=1}^n h_{ik}|x_k - y_k| \quad (i = 1, \dots, n), \tag{1.25}$$

$$|f_i(t, 0, \dots, 0)| \le h(t) p_i(t) \quad (i = 1, \dots, n), \tag{1.26}$$

where $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, are satisfied. If, moreover, the equalities (1.14) (the equalities (1.17)) hold, then the problem (1.1), (1.2) is well-posed (asymptotically well-posed) with the weight $(p_i)_{i=1}^n$.

Note that if

$$f_i(t, x_1, \dots, x_n) = -\sigma_i p_i(t) \quad (i = 1, \dots, n),$$

where $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, and $p_i \in L_{\text{loc}}(\mathbb{R})(i = 1, \dots, n)$ are nonnegative functions, then the problem (1.1), (1.2) is well-posed (asymptotically well-posed) with the weight $(p_i)_{i=1}^n$ if and only if the equalities (1.14) (the equalities (1.17)) are satisfied.

Consequently, the condition (1.14) (the condition (1.17)) in Theorem 1.5 is unimprovable.

2. Auxiliary propositions

2.1. Lemmas on a priori estimates

Consider the system of differential inequalities

$$\sigma_i u_i'(t) \le h_i(t) \left(\sum_{k=1}^n h_{ik} u_k(t) + h(t) \right) \quad (i = 1, \dots, n), \tag{2.1}$$

where

$$\sigma_1 = \dots = \sigma_m = 1, \qquad \sigma_{m+1} = \dots = \sigma_n = -1, \qquad H = (h_{ik})_{i,k=1}^n \in A_s, \tag{2.2}$$

$$h_i \in L_{\text{loc}}(\mathbb{R}_+) \quad (i = 1, \dots, n) \quad \text{and} \quad h \in L^\infty(\mathbb{R}_+) \quad \text{are nonnegative functions.} \tag{2.3}$$

Let I be some interval from $\mathbb{R}_+$. A vector function $(u_i)_{i=1}^n$ with nonnegative components $u_i \in \widetilde{C}_{\text{loc}}(I)$ $(i = 1, \dots, n)$ is said to be a nonnegative solution of the system (2.1) if it satisfies this system almost everywhere on I.

Lemma 2.1. *Let conditions (2.2) and (2.3) be fulfilled and $(u_i)_{i=1}^n$ be a nonnegative solution of the system (2.1) on some interval $[0, a] \subset \mathbb{R}_+$. Then*

$$\sum_{i=1}^{n} u_i(t) \le \mu(H)\left(\sum_{i=1}^{m} u_i(0) + \sum_{i=m+1}^{n} u_i(a) + \|h\|_{L^\infty}\right) \quad \text{for } 0 \le t \le a, \tag{2.4}$$

where $\mu(H)$ *is the number given by the equality* (1.13).

To prove this lemma, we need the following

Lemma 2.2. *Let* γ_i, γ_{0i} *and* $h_{0ik}(i,k = 1,\dots,n)$ *be nonnegative numbers such that*

$$\gamma_i \le \sum_{k=1}^{n} h_{0ik}\gamma_k + \gamma_{0i} \quad (i = 1,\dots,n) \tag{2.5}$$

and

$$r(H_0) < 1, \quad \text{where } H_0 = (h_{0ik})_{i,k=1}^{n}. \tag{2.6}$$

Then

$$\sum_{i=1}^{n} \gamma_i \le \|(E - H_0)^{-1}\| \sum_{i=1}^{n} \gamma_{0i}. \tag{2.7}$$

Proof. If we suppose that

$$\gamma = (\gamma_i)_{i=1}^{n}, \qquad \gamma_0 = (\gamma_{0i})_{i=1}^{n},$$

then the system of inequalities (2.5) takes the form

$$\gamma \le H_0\gamma + \gamma_0.$$

Consequently,

$$(E - H_0)\gamma \le \gamma_0. \tag{2.8}$$

However, in view of (2.6), the matrix $E - H_0$ is non-degenerate and $(E - H_0)^{-1}$ is a nonnegative matrix. If we multiply the vector inequality (2.8) by $(E - H_0)^{-1}$, then we get

$$\gamma \le (E - H_0)^{-1}\gamma_0.$$

Hence we obtain the estimate (2.7). □

As we already said above, the condition $H \in A_s$ guarantees the condition (1.8), where H_0 is a matrix given by the equality (1.9). Consequently, the following lemma is valid.

Lemma 2.3. *If* $H = (h_{ik})_{i,k=1}^{n} \in A_s$ *and*

$$h_{0ik} = (1 - \delta_{ik})|h_{ii}|^{-1}h_{ik} \quad (i,k = 1,\dots,n), \tag{2.9}$$

then the condition (2.6) *is fulfilled.*

Proof of Lemma 2.1. Suppose

$$t_i = 0 \quad (i = 1,\dots,m), \qquad t_i = a \quad (i = m+1,\dots,n). \tag{2.10}$$

Then due to (2.1)–(2.3) the inequalities

$$u_i(t) \le u_i(t_i)\exp\left(-\left|h_{ii}\int_{t_i}^{t} h_i(\tau)\mathrm{d}\tau\right|\right) + \left|\int_{t_i}^{t} \exp\left(-\left|h_{ii}\int_{s}^{t} h_i(\tau)\mathrm{d}\tau\right|\right) h_i(s)\left[\sum_{k=1}^{n}(1-\delta_{ik})h_{ik}|u_k(s)| + h(s)\right]\mathrm{d}s\right| \quad (i = 1,\dots,n) \tag{2.11}$$

are satisfied on $[0, a]$. Hence we get the inequalities (2.5) with

$$\gamma_i = \max\{|u_i(t)| : 0 \le t \le a\}, \qquad \gamma_{0i} = |u_i(t_i)| + |h_{ii}|^{-1}\|h\|_{L^\infty},$$

where $h_{0ik}(i,k = 1,\dots,n)$ are numbers given by the equalities (2.9). On the other hand, by Lemmas 2.2 and 2.3, the inequalities (2.5) result in the inequality (2.7). However,

$$\sum_{i=1}^{n}\gamma_{0i} = \sum_{i=1}^{m}u_i(0) + \sum_{i=m+1}^{n}u_i(a) + \|h\|_{L^\infty}\sum_{i=1}^{n}|h_{ii}|^{-1}$$
$$\leq \left(1+\sum_{i=1}^{n}|h_{ii}|^{-1}\right)\left(\sum_{i=1}^{m}u_i(0) + \sum_{i=m+1}^{n}u_i(a) + \|h\|_{L^\infty}\right).$$

Taking into account this fact and the notation (1.13), from (2.7) we obtain the estimate (2.4). □

Lemma 2.4. *Let the conditions (2.2) and (2.3) be fulfilled and $(u_i)_{i=1}^n$ be a nonnegative solution of the system (2.1) on $\mathbb{R}_+$ such that*

$$\limsup_{t\to+\infty} u_i(t) < +\infty \quad (i = m+1, \dots, n). \tag{2.12}$$

Then

$$\sup\left\{\sum_{i=1}^{n}u_i(t) : t\in\mathbb{R}_+\right\} < +\infty. \tag{2.13}$$

Proof. By Lemma 2.1, for an arbitrary $a\in\mathbb{R}_+$ the estimate (2.4) is valid, from which due to (2.12) it follows the inequality (2.13). □

Lemma 2.5. *Let, along with (2.2) and (2.3), the condition*

$$\int_0^{+\infty} h_i(s)\mathrm{d}s = +\infty \quad (i = m+1, \dots, n) \tag{2.14}$$

hold. Let, moreover, $(u_i)_{i=1}^n$ be a nonnegative solution of the system (2.1) on $\mathbb{R}_+$, satisfying the condition (2.12). Then

$$\sum_{i=1}^{n}u_i(t) \leq \mu(H)\left(\sum_{i=1}^{m}u_i(0) + \|h\|_{L^\infty}\right) \quad \text{for } t\in\mathbb{R}_+. \tag{2.15}$$

Proof. By Lemma 2.4,

$$\gamma_i = \sup\{u_i(t) : t\in\mathbb{R}_+\} < +\infty \quad (i = 1, \dots, n). \tag{2.16}$$

On the other hand, in view of (2.1)–(2.3) for any $a\in]0,+\infty[$ the inequalities (2.11) are satisfied on the interval $[0, a]$, where $t_i (i = 1, \dots, m)$ are numbers given by the equalities (2.10). Therefore,

$$u_i(t) \leq u_i(0) + |h_{ii}|^{-1}\|h\|_{L^\infty} + \sum_{k=1}^{n}h_{0ik}\gamma_k \quad \text{for } 0\leq t\leq a \quad (i = 1, \dots, m),$$
$$u_i(t) \leq \gamma_i \exp\left(-|h_{ii}|\int_t^a h_i(s)\mathrm{d}s\right) + |h_{ii}|^{-1}\|h\|_{L^\infty} + \sum_{k=1}^{n}h_{0ik}\gamma_k \quad \text{for } 0\leq t\leq a \ (i = m+1, \dots, n),$$

where $h_{0ik} (i, k = 1, \dots, n)$ are numbers given by the equalities (2.9). If we pass to the limit in these inequalities as $a\to+\infty$, then due to (2.14) we obtain

$$u_i(t) \leq \gamma_{0i} + \sum_{i=1}^{n}h_{0ik}\gamma_k \quad \text{for } t\in\mathbb{R}_+,$$

where

$$\gamma_{0i} = u_i(0) + |h_{ii}|^{-1}\|h\|_{L^\infty} \quad (i = 1, \dots, m), \qquad \gamma_{0i} = |h_{ii}|^{-1}\|h\|_{L^\infty} \quad (i = m+1, \dots, n). \tag{2.17}$$

Consequently, the inequalities (2.5) are satisfied. Hence by Lemma 2.2 and 2.3 we obtain the inequality (2.7). If along with (2.7) we take into account (2.16) and (2.17), then the validity of the estimate (2.15) becomes evident. □

Lemma 2.6. *Let along with (2.2) and (2.3) the condition*

$$\int_0^{+\infty} h_i(s)\mathrm{d}s = +\infty \quad (i = 1, \dots, n) \tag{2.18}$$

be fulfilled and $h(t)\to 0$ as $t\to+\infty$. Let, moreover, $(u_i)_{i=1}^n$ be a nonnegative solution of the system (2.1) on $\mathbb{R}_+$, satisfying the condition (2.12). Then along with (2.15) the condition

$$\lim_{t\to+\infty} u_i(t) = 0 \quad (i = 1, \dots, n) \tag{2.19}$$

holds.

Proof. By Lemma 2.4,

$$\gamma_i = \limsup_{t\to+\infty} |u_i(t)| < +\infty \quad (i = 1, \dots, n). \tag{2.20}$$

For any $\varepsilon > 0$, choose $a = a(\varepsilon) > 0$ so that

$$h(t) < \varepsilon, \qquad u_i(t) < \gamma_i + \varepsilon \quad (i = 1, \dots, n) \text{ for } t \geq a. \tag{2.21}$$

On the other hand, in view of (2.1)–(2.3) for any $b \in]a, +\infty[$ the inequalities (2.11) hold on $[a, b]$, where

$$t_i = a \quad (i = 1, \dots, m), \qquad t_i = b \quad (i = m + 1, \dots, n).$$

On account of (2.21) from (2.11) we find that

$$u_i(t) \leq u_i(a) \exp\left(-|h_{ii}| \int_a^t h_i(s)\mathrm{d}s\right) + \sum_{k=1}^n h_{0ik}\gamma_i + \ell_i\varepsilon \quad \text{for } t \geq a \ (i = 1, \dots, n), \tag{2.22}$$

$$u_i(t) \leq u_i(b) \exp\left(-|h_{ii}| \int_t^b h_i(s)\mathrm{d}s\right) + \sum_{k=1}^n h_{0ik}\gamma_i + \ell_i\varepsilon \quad \text{for } a \leq t \leq b \ (i = m + 1, \dots, n), \tag{2.23}$$

where $h_{0ik} (i = 1, \dots, n)$ are numbers given by the equalities (2.9) and

$$\ell_i = \sum_{k=1}^n h_{0ik} + |h_{ii}|^{-1} \quad (i = 1, \dots, n).$$

By (2.18) and (2.20), from (2.22) we get

$$\gamma_i \leq \sum_{k=1}^n h_{0ik}\gamma_i + \ell_i\varepsilon \quad (i = 1, \dots, m). \tag{2.24}$$

If we pass to the limit in the inequalities (2.23) as $b \to +\infty$, then by (2.18) we find that

$$u_i(t) \leq \sum_{k=1}^n h_{0ik}\gamma_i + \ell_i\varepsilon \quad \text{for } t \in \mathbb{R}_+ \ (i = m + 1, \dots, n).$$

Therefore,

$$\gamma_i \leq \sum_{k=1}^n h_{0ik}\gamma_i + \ell_i\varepsilon \quad (i = m + 1, \dots, n). \tag{2.25}$$

By virtue of Lemmas 2.2 and 2.3, the inequalities (2.24) and (2.25) imply the estimate

$$\sum_{i=1}^n \gamma_i \leq \ell\varepsilon, \quad \text{where } \ell = \left\|(E - H_0)^{-1}\right\| \sum_{i=1}^n \ell_i.$$

Hence, in view of the arbitrariness of ε and the nonnegativeness of $\gamma_i (i = 1, \dots, n)$, we obtain

$$\gamma_i = 0 \quad (i = 1, \dots, n).$$

Consequently, the equalities (2.19) are valid. □

2.2. Lemma on the solvability of the problem (1.1), (1.5)

From Corollary 2 in [7] it follows

Lemma 2.7. *If the conditions of Theorem 1.1 are satisfied, then for any $a \in]0, +\infty[$ and $(c_i)_{i=1}^n \in \mathbb{R}^n$ the problem (1.1), (1.5) is solvable and each of its solution admits the estimate (1.6).*

3. Proof of the main results

Proof of Theorem 1.1. Suppose $(c_i)_{i=1}^m \in \mathbb{R}^m$ is fixed arbitrarily. By Lemma 2.7, for any natural k the system (1.1) has a solution $(x_{ik})_{i,k=1}^n$ in the interval $[0, k]$, satisfying the boundary conditions

$$x_{ik}(0) = c_i \quad (i = 1, \dots, m), \qquad x_{ik}(k) = 0 \quad (i = m + 1, \dots, n) \tag{3.1}$$

and admitting the estimate

$$\sum_{i=1}^{n} |x_{ik}(t)| \le \rho \quad \text{for } 0 \le t \le k, \tag{3.2}$$

where $\rho = \rho_0 \left(\sum_{i=1}^{m} c_i\right)$. In view of (1.3) and (3.2), the inequality

$$\sum_{i=1}^{n} |f_i\,(t, x_{1k}(t), \dots, x_{nk}(t))| \le f^*(t) \tag{3.3}$$

is satisfied almost everywhere on $[0, k]$, where

$$f^*(t) = \max\left\{\sum_{i=1}^{n} |f_i(t, x_1, \dots, x_n)| : \sum_{j=1}^{n} |x_j| \le \rho\right\} \quad \text{and} \quad f^* \in L_{\mathrm{loc}}(\mathbb{R}_+).$$

Thus

$$\sum_{i=1}^{n} |x_i(t) - x_i(s)| \le \int_s^t f^*(\tau)\mathrm{d}\tau \quad \text{for } 0 \le s \le t \le k. \tag{3.4}$$

Suppose

$$x_{ik}(t) = 0 \quad \text{for } t \ge k \quad (i = 1, \dots, n).$$

Then, according to the conditions (3.1), (3.2) and (3.4), the sequence of vector functions $((x_{ik})_{i=1}^{n})_{k=1}^{\infty}$ is uniformly bounded and equicontinuous on each compact interval from $\mathbb{R}_+$. By the Arzela–Ascoli lemma, from this sequence we can choose a subsequence $((x_{ikj})_{i=1}^{n})_{j=1}^{\infty}$ which is uniformly convergent on each compact interval from $\mathbb{R}_+$.

Let

$$x_i(t) = \lim_{j\to+\infty} x_{ikj}(t) \quad \text{for } t \in \mathbb{R}_+ \quad (i = 1, \dots, m). \tag{3.5}$$

If we apply the Lebesgue dominant theorem, then in view of the conditions (1.3), (3.3) and (3.5), from the equalities

$$x_{ikj}(t) = x_{ikj}(0) + \int_0^t f_i\left(s, x_{1kj}(s), \dots, x_{nkj}(s)\right) \mathrm{d}s \quad \text{for } 0 \le t \le k_j (i = 1, \dots, n)$$

we find

$$x_i(t) = x_i(0) + \int_0^t f_i\,(s, x_1(s), \dots, x_n(s))\,\mathrm{d}s \quad \text{for } t \in \mathbb{R}_+ \ (i = 1, \dots, n).$$

Consequently, $(x_i)_{i=1}^{n}$ is a solution of the system (1.1) on $\mathbb{R}_+$. On the other hand, by (3.5), from (3.1) and (3.2) it follows that $(x_i)_{i=1}^{n}$ satisfies the boundary conditions (1.2). Thus the solvability of the problem (1.1), (1.2) is proved.

It remains to show that an arbitrary solution $(x_i)_{i=1}^{n}$ of the problem (1.1), (1.2) is bounded on $\mathbb{R}_+$. According to (1.2),

$$\gamma = \sup\left\{\sum_{i=m+1}^{n} |x_i(t)| : t \in \mathbb{R}_+\right\} < +\infty.$$

On the other hand, by Lemma 2.7, for an arbitrary $a \in\,]0, +\infty[$ we have

$$\sum_{i=1}^{n} |x_i(t)| \le \rho_0\left(\sum_{i=1}^{m} |c_i| + \sum_{i=m+1}^{n} |x_i(a)|\right) \le \rho_0\left(\sum_{i=1}^{m} |c_i| + \gamma\right) \quad \text{for } 0 \le t \le a.$$

Hence due to the arbitrariness of a it follows that $(x_i)_{i=1}^{n}$ is bounded on $\mathbb{R}_+$. □

Proof of the Theorem 1.2. Let $\mu(H)$ be the number given by the equality (1.13) and

$$\rho_0(x) = \mu(H)\,(x + \|h\|_{L^\infty}) \quad \text{for } x \in \mathbb{R}_+. \tag{3.6}$$

According to Theorem 1.1, to prove Theorem 1.2 it suffices to state that for any $\lambda \in\,]0, 1]$, $a \in\,]0, +\infty[$ and $(c_i)_{i=1}^{n} \in \mathbb{R}^n$, every solution $(x_i)_{i=1}^{n}$ of the problem (1.4), (1.5) admits the estimate (1.6).

Suppose

$$u_i(t) = |x_i(t)| \quad (i = 1, \dots, n). \tag{3.7}$$

Then in view of (1.7) the conditions

$$\sigma_i u_i'(t) = \sigma_i \lambda f_i\,(t, x_1(t), \dots, x_n(t))\,\mathrm{sgn}(x_i(t)) \le h_i(t)\left(\sum_{k=1}^{n} h_{ik} u_k(t) + h(t)\right) \quad (i = 1, \dots, n)$$

are satisfied almost everywhere on $[0, a]$, where

$$h_i(t) = \lambda g_i\,(t, x_1, (t), \dots, x_n(t)) \quad (i = 1, \dots, n).$$

Therefore, $(u_i)_{i=1}^n$ is a nonnegative solution of the system of differential inequalities (2.1) on $[0, a]$. Moreover, $\sigma_i, h_i, h_{ik} (i, k = 1, \dots, n)$ and h satisfy the conditions (2.2) and (2.3), which by Lemma 2.1 guarantees the validity of the estimate (2.4). If now we take into account conditions (1.5) and the notations (3.6) and (3.7), then the validity of the estimate (1.6) becomes evident. □

Proof of Theorem 1.3 (Theorem 1.4). Let $(x_i)_{i=1}^n$ be an arbitrary solution of the problem (1.1), (1.2). Then in view of (1.7) the vector function $(u_i)_{i=1}^n$, whose components are given by the equalities (3.7), satisfies the condition (2.12) and is a nonnegative solution of the system of differential inequalities (2.1) on $\mathbb{R}_+$, where

$$h_i(t) = g_i\,(t, x_1(t), \dots, x_n(t)) \quad (i = 1, \dots, n). \tag{3.8}$$

Moreover, $\sigma_i, h_i, h_{ik} (i, k = 1, \dots, n)$ and h satisfy the conditions (2.2), (2.3) (and $h(t) \to 0$ as $t \to +\infty$). On the other hand, in view of (3.8) from (1.14) and (1.15) (from (1.15) and (1.17)) it follow the equalities (2.14) (the equalities (2.18)). By Lemma 2.5 (by Lemma 2.6), the vector function $(u_i)_{i=1}^n$ admits the estimate (2.15) (satisfies the equalities (2.19)). Consequently, $(x_i)_{i=1}^n$ admits the estimate (1.16) (satisfies the equalities (1.18)). □

Proof of Theorem 1.5. Let $q_i \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)(i = 1, \dots, n)$ be arbitrary functions satisfying the condition (1.21). Then in view of (1.25) and (1.26) the inequalities

$$\sigma_i\,(f_i(t, x_1, \dots, x_n) + q_i(t, x_1, \dots, x_n))\,\mathrm{sgn}(x_i) \le p_i(t)\left(\sum_{k=1}^{n} h_{ik}|x_k| + h(t) + q_0(t)\right) \quad (i = 1, \dots, n) \tag{3.9}$$

are satisfied on $\mathbb{R}_+ \times \mathbb{R}^n$. Moreover, $h \in L^\infty(\mathbb{R}_+)$,

$$p_i \in L_{\mathrm{loc}}(\mathbb{R}_+) \quad (i = 1, \dots, n), \qquad q_0 \in L^\infty(\mathbb{R}_+) \tag{3.10}$$

and the condition (2.2) holds. Hence by Theorem 1.2 it follows that the problem (1.19), (1.20) is solvable for any $(\delta_i)_{i=1}^m \in \mathbb{R}^m$.

Let $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ be arbitrary solutions of the problems (1.1), (1.2) and (1.19), (1.20), and

$$u_i(t) = |x_i(t) - y_i(t)| \quad (i = 1, \dots, n).$$

Then, due to (1.21) and (1.25), the vector function $(u_i)_{i=1}^n$ is a nonnegative solution of the system of differential inequalities

$$\sigma_i u_i'(t) \le p_i(t)\left(\sum_{k=1}^{n} h_{ik} u_k(t) + q_0(t)\right) \quad (i = 1, \dots, n),$$

satisfying the conditions

$$u_i(0) = |\delta_i| \quad (i = 1, \dots, m), \qquad \limsup_{t\to+\infty} u_i(t) < +\infty \quad (i = m + 1, \dots, n).$$

If, along with (2.2) and (3.10), the condition (1.14) (the conditions (1.17) and (1.22)) holds, then by Lemma 2.5 (by Lemma 2.6) we have

$$\sum_{i=1}^{n} u_i(t) \le \mu(H)\left(\sum_{i=1}^{m} |\delta_i| + \|q_0\|_{L^\infty}\right) \quad \text{for } t \in \mathbb{R}_+ \left(\lim_{t\to+\infty} u_i(t) = 0\ (i = 1, \dots, n)\right).$$

Thus we have proved that the estimate (1.23) is valid (along with the estimate (1.23) the equalities (1.24) are valid), where $\rho = \mu(H)$. Therefore the problem (1.1), (1.2) is well-posed (asymptotically well-posed) with the weight $(p_i)_{i=1}^n$. □

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