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SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH INFINITE SET OF PERIODIC SOLUTIONS

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For the differential equation $u'' = f(t, u, u')$, where the function $f\colon R \times R^2 \to R$ is periodic in the first variable and $f(t, x, 0) \equiv 0$, sufficient conditions for the existence of a continuum of nonconstant periodic solutions are found.

The problem of the existence, uniqueness, and nonuniqueness of periodic solutions of nonlinear differential equations and systems attracts attention of many mathematicians and is the subject of numerous investigations (see, e.g., [1–16] and references therein). Nevertheless, the description of classes of equations having a continuum of periodic solutions is far from being complete. The goal of the present paper is to fill this gap to a certain extent.

Below we consider the differential equation

$$u'' = f(t, u, u'), \tag{1}$$

where the function $f\colon R \times R^2 \to R$ satisfies the local Carathéodory conditions, i.e., $f(t, \cdot, \cdot)\colon R^2 \to R$ is continuous for almost all $t \in R$, $f(\cdot, x, y)\colon R \to R$ is measurable for all $(x, y) \in R^2$, and, for an arbitrary $\rho > 0$, the function f_ρ given by

$$f_\rho(t) = \max\{|f(t, x, y)|\colon |x| + |y| \le \rho\} \quad \text{for} \quad t \in R,$$

is Lebesgue integrable on every finite interval.

We are interested in the case where the following equalities are satisfied on $R \times R^2$:

$$\begin{gathered} f(t+\omega, x, y) = f(t, x, y), \quad f(-t, x, -y) = f(t, x, y), \\ f(t, -x, -y) = -f(t, x, y), \end{gathered} \tag{2}$$

$$f(t, x, 0) = 0, \tag{3}$$

where ω is a positive constant.

In view of (3), Eq. (1) has a continuum of constant solutions. There naturally arises the question of whether Eq. (1) with conditions (2) and (3) may have nonconstant periodic solutions. As stated in Theorem 1 proved below, the answer is positive.

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Let $R_+ = [0, +\infty)$, let L_ω be the space of ω-periodic real functions Lebesgue integrable on $[0, \omega]$, and let M_ω be the set of functions $\varphi\colon R \times R_+ \to R_+$ such that $\varphi(\cdot, x) \in L_\omega$ for arbitrary $x \in R_+$, $\varphi(t, \cdot)\colon R_+ \to R$ is a continuous nondecreasing function for almost all $t \in R$, $\varphi(t, 0) \equiv 0$, and

$$\int_0^\omega \varphi(t, x)\, dt > 0 \quad \text{for} \quad x > 0. \tag{4}$$

Theorem 1. *Suppose that conditions (2) and (3) are satisfied and*

$$f(t, x, y) \le -\varphi(t, x)\psi(y) \quad \text{for} \quad t \in R_+,\ x \in R_+,\ 0 \le y \le r, \tag{5}$$

where $r > 0$, $\varphi \in M_\omega$, *and* $\psi\colon [0, r] \to R_+$ *is a continuous function such that*

$$\psi(0) = 0, \quad \psi(y) > 0 \quad \text{for} \quad 0 < y \le r, \quad \int_0^r \frac{dy}{\psi(y)} < +\infty. \tag{6}$$

Then Eq. (1) has a continuum of nonconstant periodic solutions.

To prove Theorem 1 we need the following lemma:

Lemma 1. *Let inequality (5) be satisfied, where* $\varphi \in M_\omega$ *and* $\psi\colon [0, r] \to R_+$ *is a continuous function satisfying condition (6). Then, for an arbitrary* $c \in (0, r)$, *there exists* $t_c \in (0, +\infty)$ *such that, on the interval* $[0, t_c]$, *Eq. (1) has a solution* u_c *satisfying the conditions*

$$u_c(0) = 0, \quad u_c'(0) = c, \tag{7}$$

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \text{for} \quad 0 < t < t_c, \quad u_c'(t_c) = 0. \tag{8}$$

Proof. Let u_c be a solution of problem (1), (7) maximally extended to the right. Then either u_c is defined on R_+ and

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \text{for} \quad t \in R_+ \tag{9}$$

or there exists $t_c \in (0, +\infty)$ such that

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \text{for} \quad 0 < t < t_c \tag{10}$$

and

$$u_c'(t_c) \in \{0, r\}. \tag{11}$$

First, we assume that condition (9) is satisfied. Then, in view of (5), for an arbitrary fixed $a > 0$ the following inequality holds almost everywhere on $[a, +\infty)$:

$$\varphi(t, x) \le -\frac{u_c''(t)}{\psi(u_c'(t))},$$

where $x = u_c(a) > 0$. Integrating this inequality from a to $a + k\omega$, where k is an arbitrary natural number, and taking into account the ω-periodicity of $\varphi(\cdot, x)$ and condition (6), we find

$$k \int_a^{a+\omega} \varphi(t,x)\,dt \le \int_{u_c'(a+k\omega)}^{u_c'(a)} \frac{dy}{\psi(y)} < \rho,$$

where

$$\rho = \int_0^r \frac{dy}{\psi(y)} < +\infty.$$

Consequently,

$$\int_0^{\omega} \varphi(t,x)\,dt = \int_a^{a+\omega} \varphi(t,x)\,dt \le \frac{\rho}{k} \to 0 \quad \text{as} \quad k \to +\infty,$$

which contradicts condition (4). The obtained contradiction proves that the function u_c does not satisfy inequalities (9). Hence, for some $t_c \in (0, +\infty)$, conditions (10) and (11) are satisfied.

According to (5) and (10), the following inequality holds almost everywhere on $(0, t_c)$:

$$u_c''(t) \le 0.$$

Therefore, $u_c'(t_c) \le c < r$, whence, by virtue of (11), it follows that $u_c(t_c) = 0$. Thus, condition (8) is satisfied.

The lemma is proved.

Lemma 2. *Suppose that, on $R \times R^2$, equalities (2) are satisfied and the function u is a solution of Eq. (1) on some interval $[0, t_0] \subset R_+$. Then, for an arbitrary natural k, the function v given by the equality*

$$v(t) = u(k\omega - t) \quad \text{for} \quad k\omega - t_0 \le t \le k\omega$$

is a solution of Eq. (1) on $[k\omega - t_0, k\omega]$.

Proof. Indeed,

$$v''(t) = u''(k\omega - t) = f\left(k\omega - t, u(k\omega - t), u'(k\omega - t)\right)$$

$$= f\left(k\omega - t, v(t), -v'(t)\right) \quad \text{almost everywhere on} \quad [k\omega - t, k\omega].$$

Thus, according to (2), we find

$$v''(t) = f\left(-t, v(t), -v'(t)\right)$$

$$= f\left(t, v(t), v'(t)\right) \quad \text{almost everywhere on} \quad [k\omega - t, k\omega].$$

The lemma is proved.

Proof of Theorem 1. Owing to Lemma 1, for an arbitrary $c \in (0, r)$ there exists $t_c \in (0, +\infty)$ such that, on $[0, t_c]$, Eq. (1) has a solution u satisfying conditions (7) and (8). We choose a natural number k so that

$$k\omega \geq 2t_c$$

and extend u_c onto R in the following manner:

$$u_c(t) = \begin{cases} u_c(t_c) & \text{for} \quad t_c \leq t \leq k\omega - t_c, \\ u_c(k\omega - t) & \text{for} \quad k\omega - t_c \leq t \leq k\omega, \end{cases}$$

$$u_c(t + k\omega) = -u_c(t) \quad \text{for} \quad t \in R.$$

By virtue of conditions (2) and (3) and Lemma 2, the function u_c is a $2k\omega$-periodic solution of Eq. (1). On the other hand, it is clear that

$$u_{c_1}(t) \not\equiv u_{c_2}(t) \not\equiv \text{const} \quad \text{for} \quad 0 < c_1 < c_2 < r.$$

Consequently, if c runs through the interval $(0, r)$, then we obtain a continuum of periodic nonconstant solutions of Eq. (1).

The theorem is proved.

As an example, we consider the generalized Emden–Fowler equation

$$u'' = \sum_{k=1}^{m} p_k(t)|u'|^{\mu_k}|u|^{\lambda_k} \operatorname{sgn} u, \tag{12}$$

where

$$\begin{gathered} \lambda_k > 0, \quad \mu_k > 0 \quad p_k \in L_\omega, \\ p_k(-t) = p_k(t) \leq 0 \quad \text{for} \quad t \in R, \end{gathered} \tag{13}$$

$$\int_0^\omega p_k(t)\,dt < 0, \quad k = 1, \ldots, m. \tag{14}$$

The following proposition is true:

Corollary 1. *Let conditions (13) and (14) be satisfied. Then, for the existence of a continuum of periodic solutions of Eq. (12), it is necessary and sufficient that*

$$\min\{\mu_1, \ldots, \mu_n\} < 1. \tag{15}$$

Proof. Assume first that, along with (13) and (14), condition (15) is satisfied. Then, without loss of generality, we can assume that $\mu_1 < 1$. Due to condition (13), the function f given by the equality

$$f(t, x, y) = \sum_{k=1}^{m} p_k(t)|y|^{\mu_k}|x|^{\lambda_k}\operatorname{sgn} x$$

satisfies conditions (2) and (3). On the other hand, for an arbitrary $r > 0$, inequality (5) is satisfied, where

$$\varphi(t, x) = |p_1(t)|x^{\lambda_1}, \quad \psi(y) = y^{\mu_1}.$$

Moreover, $\varphi \in M_\omega$ and ψ satisfies condition (6) because

$$\int_0^\omega |p_1(t)|\, dt > 0 \quad \text{and} \quad 0 < \mu_1 < 1.$$

Thus, all conditions of Theorem 1 are satisfied, which guarantees the existence of a continuum of nonconstant ω-periodic solutions of Eq. (12).

It remains to state that if

$$\mu_k \geq 1, \quad k = 1, \ldots, m, \tag{16}$$

then an arbitrary periodic solution u of Eq. (12) is constant. Indeed, almost everywhere on R, the following equality is satisfied:

$$u''(t) = p(t)u'(t), \tag{17}$$

where

$$p(t) = \sum_{k=1}^{m} p_k(t)|u'(t)|^{\mu_k - 1}|u(t)|^{\lambda_k}\operatorname{sgn}(u(t)u'(t));$$

in addition, in view of (16), we have

$$p \in L_\omega. \tag{18}$$

On the other hand, owing to the ω-periodicity of u, there exists $t_0 \in R$ such that

$$u'(t_0) = 0.$$

Thus, it follows from (17) and (18) that $u'(t) \equiv 0$, i.e., $u(t) \equiv \text{const}$.

The corollary is proved.

Remark 1. If $p_k(t) \equiv 0$, $k = 1, \ldots, m$, then Eq. (12) does not have a nonconstant periodic solution. Consequently, condition (14) in Corollary 1 is essential and cannot be weakened.

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