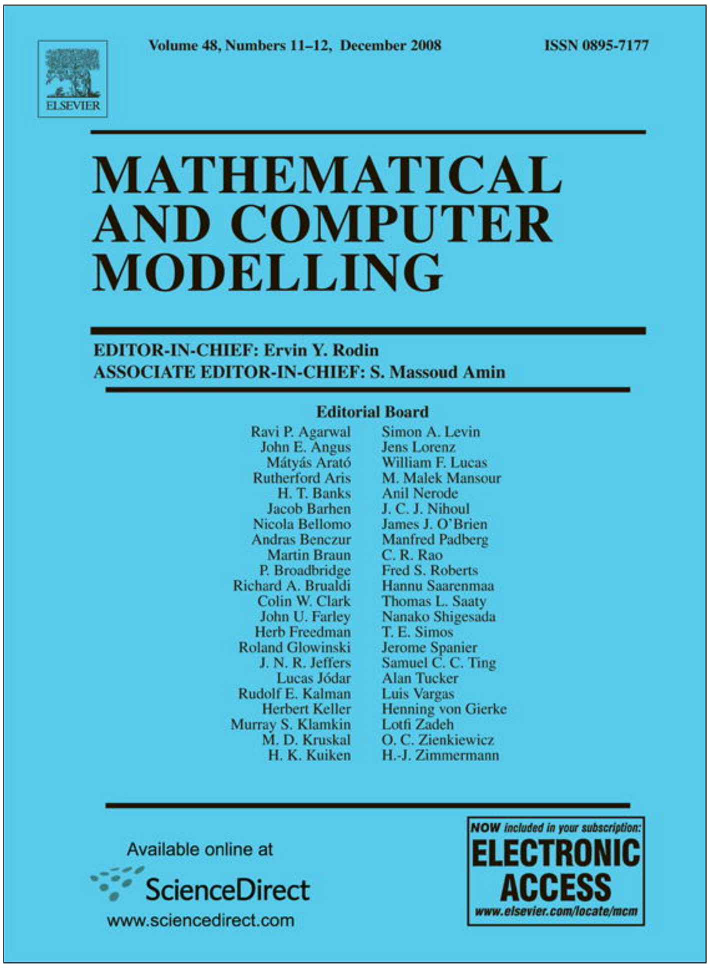
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On solvability conditions for nonlinear operator equations

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Abstract

New sufficient conditions are found for solvability and unique solvability of nonlinear operator equations in the Banach space. In particular, abstract analogues of the Conti–Opial-type theorems are established, which concern the solvability of nonlinear boundary value problems. On the basis of these results, new sufficient conditions are obtained for the solvability of a periodic problem at resonance for nonlinear higher-order functional differential equations.

Keywords: Nonlinear operator equation in a Banach space; A priori boundedness principle; Conti–Opial-type theorem; Nonlinear functional differential equation; Periodic problem at resonance

1. Formulation of the existence and uniqueness theorems

Let $\mathcal{B}$ be a Banach space with a norm $\|\cdot\|_{\mathcal{B}}$ and $h : \mathcal{B} \to \mathcal{B}$ be a completely continuous nonlinear operator. In this paper, we give theorems on the existence and uniqueness of a solution of the operator equation

$$x = h(x), \tag{1.1}$$

which generalize the results of [1–8] concerning the solvability of nonlinear boundary value problems for systems of ordinary differential and functional differential equations.

The use will be made of the following notation.

θ is the zero element of the space $\mathcal{B}$.

$\overline{D}$ is the closure of the set $D \subset \mathcal{B}$.

$\mathcal{B} \times \mathcal{B} = \{(x, y) : x \in \mathcal{B}, \ y \in \mathcal{B}\}$ is the Banach space with the norm

$$\|(x, y)\|_{\mathcal{B}\times\mathcal{B}} = \|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}}.$$

Definition 1.1. Let γ be a positive constant. $g \in \Lambda_\gamma(\mathcal{B} \times \mathcal{B})$ if $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is a completely continuous operator such that:

$$g(x, \cdot) : \mathcal{B} \to \mathcal{B} \text{ is a linear operator} \quad \text{for every } x \in \mathcal{B}, \tag{1.2}$$

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and

$$\|y\|_{\mathcal{B}} \le \gamma \|y - g(x, y)\|_{\mathcal{B}} \quad \text{for } x \text{ and } y \in B. \tag{1.3}$$

Definition 1.2. $g \in \Lambda^0(\mathcal{B} \times \mathcal{B})$ if $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is a completely continuous operator such that along with (1.2) the following conditions hold

$$\{g(x, y) : \ x \in \mathcal{B}, \ \|y\|_{\mathcal{B}} \le 1\} \quad \text{if relatively compact} \tag{1.4}$$

and

$$y \notin \overline{\{g(x, y) : \ x \in \mathcal{B}\}} \quad \text{for } y \in \mathcal{B} \text{ and } y \ne \theta. \tag{1.5}$$

Definition 1.3. Let $g \in \Lambda^0(\mathcal{B} \times \mathcal{B})$. We say that a linear bounded operator $g_0 : \mathcal{B} \to \mathcal{B}$ belongs to the set $\mathcal{L}_g$ if there exists a sequence $x_k \in \mathcal{B}$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} g(x_k, y) = g_0(y) \quad \text{for } y \in \mathcal{B}. \tag{1.6}$$

Along with $\mathcal{B}$, we consider a partially ordered Banach space $\mathcal{B}_0$ in which the partial order is generated by a cone $\mathcal{K}$, i.e., for any u and $v \in \mathcal{B}_0$, it is said that u does not exceed v, and is written as $u \le v$ if $v - u \in \mathcal{K}$.

An operator $\nu : \mathcal{B} \to \mathcal{B}_0$ is said to be *positively homogeneous* if $\nu(\lambda x) = \lambda \nu(x)$ for $\lambda \ge 0$, $x \in \mathcal{B}$.

A linear operator $\eta : \mathcal{B}_0 \to \mathcal{B}_0$ is said to be *positive* if it transforms the cone $\mathcal{K}$ into itself.

By $r(\eta)$ we denote the spectral radius of the operator η.

Set

$$\Lambda(\mathcal{B} \times \mathcal{B}) = \bigcup_{0<\gamma<\infty} \Lambda_\gamma(\mathcal{B} \times \mathcal{B}). \tag{1.7}$$

Theorem 1.1 (*A Priori Boundedness Principle*). *Let there exist an operator*

$$g \in \Lambda(\mathcal{B} \times \mathcal{B}) \tag{1.8}$$

and a positive constant ρ *such that for any* $\lambda \in]0, 1[$ *an arbitrary solution of the equation*

$$x = (1 - \lambda) g(x, x) + \lambda h(x) \tag{1.9}$$

admits the estimate

$$\|x\|_{\mathcal{B}} \le \rho. \tag{1.10}$$

Then Eq. (1.1) *has at least one solution.*

Corollary 1.1. *Let there exist positive numbers* γ *and* ρ *and an operator*

$$g \in \Lambda_\gamma(\mathcal{B} \times \mathcal{B}) \tag{1.11}$$

such that

$$\|h(x) - g(x, x)\|_{\mathcal{B}} \le \|x\|_{\mathcal{B}}/\gamma \quad \text{for } \|x\|_{\mathcal{B}} > \rho. \tag{1.12}$$

Then Eq. (1.1) *has at least one solution.*

Corollary 1.2. *Let there exist a linear completely continuous operator* $g_0 : \mathcal{B} \to \mathcal{B}$ *and a positive constant* ρ *such that the equation*

$$y = g_0(y) \tag{1.13}$$

has only a trivial solution, and for any $\lambda \in]0, 1[$ *an arbitrary solution of the equation*

$$x = (1 - \lambda) g_0(x) + \lambda h(x) \tag{1.14}$$

admits the estimate (1.10). *Then Eq.* (1.1) *has at least one solution.*

Theorem 1.2. *Let there exist an operator*

$$g \in \Lambda^0(\mathcal{B} \times \mathcal{B}), \tag{1.15}$$

a partially ordered Banach space $\mathcal{B}_0$ *with a cone* $\mathcal{K}$ *and positively homogeneous continuous operators* μ *and* $\nu : \mathcal{B} \to \mathcal{K}$ *such that*

$$\mu(y) - \nu(y - z) \notin \mathcal{K} \quad \text{for } y \neq \theta, z \in \overline{\{g(x, y) : x \in \mathcal{B}\}} \tag{1.16}$$

and

$$\nu\,(h(x) - g(x, x) - h_0(x)) \leq \mu(x) + \mu_0(x) \quad \text{for } x \in \mathcal{B}, \tag{1.17}$$

where $h_0 : \mathcal{B} \to \mathcal{B}$ *and* $\mu_0 : \mathcal{B} \to \mathcal{K}$ *satisfy the conditions*

$$\lim_{\|x\|_{\mathcal{B}} \to \infty} \frac{\|h_0(x)\|_{\mathcal{B}}}{\|x\|_{\mathcal{B}}} = 0, \qquad \lim_{\|x\|_{\mathcal{B}} \to \infty} \frac{\|\mu_0(x)\|_{\mathcal{B}_0}}{\|x\|_{\mathcal{B}}} = 0. \tag{1.18}$$

Let, moreover,

$$\left\{ \frac{1}{1 + \|x\|_{\mathcal{B}}} h(x) : x \in \mathcal{B} \right\} \quad \text{is relatively compact.} \tag{1.19}$$

Then Eq. (1.1) has at least one solution.

Corollary 1.3. *Let there exist a partially ordered Banach space* $\mathcal{B}_0$ *with a cone* $\mathcal{K}$*, a positively homogeneous operator* $\nu : \mathcal{B} \to \mathcal{K}$ *and a linear bounded positive operator* $\eta : \mathcal{B}_0 \to \mathcal{K}$ *such that*

$$r(\eta) < 1, \tag{1.20}$$

$\|\nu(x)\|_{\mathcal{B}_0} > 0$ *for* $x \neq \theta$ *and*

$$\nu\,(h(x) - h_0(x)) \leq \eta(\nu(x)) + \mu_0(x) \quad \text{for } x \in \mathcal{B}, \tag{1.21}$$

where $h_0 : \mathcal{B} \to \mathcal{B}$ *and* $\mu_0 : \mathcal{B} \to \mathcal{K}$ *are operators satisfying (1.18). Then Eq. (1.1) has at least one solution.*

Corollary 1.4. *Let there exist an operator* g *such that along with (1.15) the condition*

$$\lim_{\|x\|_{\mathcal{B}} \to \infty} \frac{\|h(x) - g(x, x)\|_{\mathcal{B}}}{\|x\|_{\mathcal{B}}} = 0 \tag{1.22}$$

holds. Then Eq. (1.1) has at least one solution.

Theorem 1.3. *Let the space* $\mathcal{B}$ *be separable. Let, moreover, there exist a completely continuous operator* $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$*, satisfying the conditions (1.2) and (1.4), a partially ordered Banach space* $\mathcal{B}_0$ *with a cone* $\mathcal{K}$*, and positively homogeneous continuous operators* μ *and* $\nu : \mathcal{B} \to \mathcal{K}$ *such that for every* $g_0 \in \mathcal{L}_g$ *the inequality*

$$\nu(y - g_0(y)) \leq \mu(y) \tag{1.23}$$

has only a trivial solution and the conditions (1.17) and (1.19) are fulfilled, where $h_0 : \mathcal{B} \to \mathcal{B}$ *and* $\mu_0 : \mathcal{B} \to \mathcal{K}$ *are operators satisfying (1.18). Then Eq. (1.1) has at least one solution.*

Corollary 1.5. *Let the space* $\mathcal{B}$ *be separable and there exist a completely continuous operator* $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ *such that along with the conditions (1.2) and (1.4) the condition (1.22) holds. Let, moreover, for every* $g_0 \in \mathcal{L}_g$ *the Eq. (1.13) have only a trivial solution. Then Eq. (1.1) has at least one solution.*

Theorem 1.1 implies a priori boundedness principles proved in [5] and [8], while Theorems 1.2 and 1.3 imply the Conti–Opial-type theorems proved in [1–4,6,7].

Theorem 1.4. *Let there exist a linear completely continuous operator* $g_0 : \mathcal{B} \to \mathcal{B}$*, a partially ordered Banach space* $\mathcal{B}_0$ *with a cone* $\mathcal{K}$ *and positive homogeneous continuous operators* μ *and* $\nu : \mathcal{B} \to \mathcal{K}$ *such that*

$$\nu\,(h(x)-h(y)-g_0(x-y)) \le \mu(x-y) \quad \text{for } x \text{ and } y \in \mathcal{B} \tag{1.24}$$

and the inequality (1.23) has only a trivial solution. Then Eq. (1.1) has one and only one solution.

Corollary 1.6. *Let there exist a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$, a positively homogeneous continuous operator $\nu : \mathcal{B} \to \mathcal{K}$ and a linear bounded positive operator $\eta : \mathcal{B}_0 \to \mathcal{B}_0$ such that $\|\nu(x)\|_{\mathcal{B}} > 0$ for $x \neq \theta$ and along with (1.20) the condition*

$$\nu\,(h(x)-h(y)) \le \eta(\nu(x-y)) \tag{1.25}$$

hold. Then Eq. (1.1) has one and only one solution.

2. Auxiliary propositions

Lemma 2.1. $\Lambda^0(\mathcal{B}\times\mathcal{B}) \subset \Lambda(\mathcal{B}\times\mathcal{B})$.

Proof. Let $g \in \Lambda^0(\mathcal{B}\times\mathcal{B})$. Then by (1.4) there exists a positive constant β such that

$$\|g(x,y)\|_{\mathcal{B}} \le \beta\|y\|_{\mathcal{B}} \quad \text{for } x \text{ and } y \in \mathcal{B}. \tag{2.1}$$

We have to prove that $g \in \Lambda(\mathcal{B}\times\mathcal{B})$. Assume the contrary. Then in view of Definition 1.1 and equality (1.7), for any natural k there exist $x_k \in \mathcal{B}$ and $y_k \in \mathcal{B}$ such that

$$\|y_k\|_{\mathcal{B}} > k\|y_k - g(x_k,y_k)\|_{\mathcal{B}}.$$

Suppose

$$\widetilde{y}_k = \frac{1}{\|y_k\|_{\mathcal{B}}}\, y_k.$$

Then

$$\|\widetilde{y}_k\|_{\mathcal{B}} = 1, \qquad \|\widetilde{y}_k - g(x_k,\widetilde{y}_k)\|_{\mathcal{B}} \le \frac{1}{k}. \tag{2.2}$$

By (1.4) the sequence $(g(x_k,\widetilde{y}_k))_{k=1}^{\infty}$ may, without restriction of generality, be assumed to converge. Suppose

$$y = \lim_{k\to\infty} g(x_k,\widetilde{y}_k).$$

Then according to (2.1) and (2.2) we have

$$\lim_{k\to\infty} \widetilde{y}_k = y, \qquad \|y\|_{\mathcal{B}} = 1$$

and

$$\|g(x_k,\widetilde{y}_k) - g(x_k,y)\|_{\mathcal{B}} = \|g(x_k,\widetilde{y}_k - y)\|_{\mathcal{B}} \le \beta\|\widetilde{y}_k - y\| \to 0 \quad \text{when } k\to\infty.$$

Therefore,

$$\|y\|_{\mathcal{B}} = 1, \qquad y \in \overline{\{g(x,y):\ x\in\mathcal{B}\}}.$$

But this contradicts the condition (1.5). The obtained contradiction proves the lemma. □

Lemma 2.2. *Let the space $\mathcal{B}$ be separable, $\mathcal{B}_0$ be a partially ordered Banach space with a cone $\mathcal{K}$ and let μ and $\nu : \mathcal{B} \to \mathcal{K}$ be positively homogeneous continuous operators. Moreover, g is an operator satisfying (1.15). Then for the condition (1.16) to be fulfilled it is necessary and sufficient that the inequality (1.23) for any $g_0 \in \mathcal{L}_g$ have only a trivial solution.*

Proof. *Necessity*. Let $g_0 \in \mathcal{L}_g$. Then by Definition 1.3,

$$g_0(y) \in \overline{\{g(x,y):\ x\in\mathcal{B}\}} \quad \text{for } y \in \mathcal{B}.$$

Therefore, according to (1.16), we have

$$\mu(y) - \nu(y - g_0(y)) \notin \mathcal{K} \quad \text{for } y \neq \theta,$$

that is the inequality (1.23) has no nontrivial solution.

Sufficiency. Assume the contrary that for any $g_0 \in \mathcal{L}_g$ the inequality (1.23) has only a trivial solution and (1.16) is violated. Then there exist y_0 and $z_0 \in \mathcal{B}$ and a sequence $x_k \in \mathcal{B}\,(k = 1, 2, \ldots)$ such that

$$y_0 \neq \theta, \quad z_0 = \lim_{k\to\infty} g(x_k, y_0) \quad \text{and} \quad \nu(y_0 - z_0) \leq \mu(y_0). \tag{2.3}$$

Since $\mathcal{B}$ is separable, there exists a countable set $\{y_1, y_2, y_3, \ldots\}$ which is dense everywhere in this space.

By (1.4), the sequence $(x_k)_{k=1}^{\infty}$ contains a subsequence $(x_{1k})_{k=1}^{\infty}$ such that $(g(x_{1k}, y_1))_{k=1}^{\infty}$ is convergent. Similarly, $(x_{1k})_{k=1}^{\infty}$ contains a subsequence $(x_{2k})_{k=1}^{\infty}$ such that $(g(x_{2k}, y_i))_{k=1}^{\infty}$ $(i = 1, 2)$ are convergent. Proceeding this process, we obtain a system of sequences $(x_{ik})_{k=1}^{\infty}$ $(i = 1, 2, \ldots)$ such that for every natural j the sequences $(g(x_{jk}, y_i))_{k=1}^{\infty}$ $(i = 1, \ldots, j)$ are convergent.

Let us consider the sequence of operators $(g(x_{kk}, \cdot))_{k=1}^{\infty}$. It is evident from the construction of $(x_{kk})_{k=1}^{\infty}$ that the sequences $(g(x_{kk}, y_i))_{k=1}^{\infty}$ $(i = 1, 2, \ldots)$ are convergent. On the other hand, because of (2.1),

$$\|g(x_{kk}, y)\|_{\mathcal{B}} \leq \beta \|y\|_{\mathcal{B}} \quad \text{for } y \in \mathcal{B}\,(k = 1, 2, \ldots).$$

From the above arguments, owing to the Banach–Steinhaus theorem ([9], Ch. VII, Section 1, Theorem 3), there follows the existence of a linear bounded operator $g_0 : \mathcal{B} \to \mathcal{B}$ such that

$$\lim_{k\to\infty} g(x_{kk}, y) = g_0(y) \quad \text{for } y \in \mathcal{B}.$$

From the Definition 1.3 and the condition (2.3) it is clear that $g_0 \in \mathcal{L}_g$, $z_0 = g_0(y_0)$, and y_0 is a nontrivial solution of the inequality (1.23). But this contradicts our assumption that for any $g_0 \in \mathcal{L}_g$ the inequality (1.23) has only a trivial solution. □

Lemma 2.3. *Let $\mathcal{B}_0$ be a Banach space with a cone $\mathcal{K}$ and let $\eta : \mathcal{B}_0 \to \mathcal{B}_0$ be a linear bounded, positive operator satisfying (1.20). Then the inequality*

$$u \leq \eta(u) \tag{2.4}$$

in the cone $\mathcal{K}$ has only a trivial solution.

Proof. Let $I : \mathcal{B}_0 \to \mathcal{B}_0$ be the unit operator. Then according to (1.20) and the fact that η is positive,

$$(I - \eta)^{-1} = \sum_{k=0}^{\infty} \eta^k$$

(see [9], Theorem 3.7.8) and, consequently, $(I - \eta)^{-1}$ is also positive.

Let u be an arbitrary element of the cone $\mathcal{K}$ satisfying (2.4). Then

$$(I - \eta)(u) \leq \theta_0,$$

where θ_0 is a zero element of $\mathcal{B}_0$. This inequality, due to the positiveness of $(I - \eta)^{-1}$, implies $u \leq \theta_0$. However, the cone $\mathcal{K}$ contains no nonpositive element except the zero one. Hence, $u = \theta_0$. □

3. Proofs of the existence and uniqueness theorems

Proof of Theorem 1.1. First of all, we note that by condition (1.8) and equality (1.7) there exists the positive constant γ, such that inequality (1.3) is fulfilled.

Suppose

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \rho \\ 2 - s/\rho & \text{for } \rho < s < 2\rho \\ 0 & \text{for } s \geq 2\rho \end{cases} \tag{3.1}$$

and for arbitrarily fixed $x \in \mathcal{B}$ let us consider the operator equation

$$y = g(x, y) + \chi(\|x\|_{\mathcal{B}})(h(x) - g(x, x)). \tag{3.2}$$

Due to Definition 1.1 and condition (1.8), $g(x, \cdot) : \mathcal{B} \to \mathcal{B}$ is a linear compact operator, and the homogeneous equation

$$y = g(x, y) \tag{3.2$_0$}$$

has only a trivial solution. This, owing to the Fredholm theorem ([9], Theorem 7.3.7), implies that for every $x \in \mathcal{B}$ the operator Eq. (3.2) has a unique solution $y = y(x) \in \mathcal{B}$. Thus we have determined the operator $y : \mathcal{B} \to \mathcal{B}$.

By condition (1.3) and equality (3.1),

$$\|y(x)\|_{\mathcal{B}} \le \gamma \chi(\|x\|_{\mathcal{B}})\|h(x) - g(x, x)\|_{\mathcal{B}} \le \rho_0 \quad \text{for } x \in \mathcal{B}, \tag{3.3}$$

where

$$\rho_0 = \gamma \sup\{\|h(x) - g(x, x)\|_{\mathcal{B}} : x \in \mathcal{B}, \|x\|_{\mathcal{B}} \le 2\rho\} < \infty.$$

Bearing in mind the conditions (1.3) and (3.3), we can conclude that $y : \mathcal{B} \to \mathcal{B}$ is a compact operator transforming the ball

$$B_{\rho_0} = \{x \in \mathcal{B} : \ \|x\|_{\mathcal{B}} \le \rho_0\}$$

into itself.

Let us now prove the continuity of y. Let $x_0 \in \mathcal{B}$ be an arbitrarily fixed point. For any $x \in \mathcal{B}$ assume

$$u(x) = y(x) - y(x_0).$$

Then from (3.2) we find

$$u(x) = g(x, u(x)) + v(x),$$

where

$$v(x) = g(x, y(x_0)) - g(x_0, y(x_0)) + \chi(\|x\|)\,(h(x) - g(x, x)) - \chi(\|x_0\|)\,(h(x_0) - g(x_0, x_0))$$

and

$$\lim_{x \to x_0} \|v(x)\|_{\mathcal{B}} = 0.$$

According to the condition (1.3) and the last equality, we have

$$\|u(x)\|_{\mathcal{B}} \le \gamma \|v(x)\|_{\mathcal{B}} \to 0 \quad \text{for } x \to x_0,$$

which proves the continuity of y.

Due to the Shauder's principle, there exists $x \in B_{\rho_0}$, such that

$$x = y(x).$$

Then from (3.1) and (3.2) it follows that x is a solution of Eq. (1.9), where

$$\lambda = \chi(\|x\|_{\mathcal{B}}) \tag{3.4}$$

and $\lambda \in [0, 1]$. The equality $\lambda = 0$ cannot take place, because in this case x is a solution of the homogeneous equation (3.2$_0$) and $x = \theta$, but on the other hand, from (3.1) and (3.4) we have $\|x\|_{\mathcal{B}} \ge 2\rho$. The inequality $0 < \lambda < 1$ likewise cannot take place because in this case, according to one of the conditions of the theorem, x admits the estimate (1.10), and on the other hand, from (3.1) and (3.4) we have $\rho < \|x\|_{\mathcal{B}} < 2\rho$. Thus we have proved that $\lambda = 1$, and hence x is a solution of Eq. (1.1). □

Proof of Corollary 1.1. Suppose to the contrary that the Corollary is invalid. Then by Theorem 1.1, for some $\lambda \in]0, 1[$, Eq. (1.9) has a solution x, such that

$$\|x\|_{\mathcal{B}} > \rho.$$

Then owing to the conditions (1.11) and (1.12), from (1.9) we find that

$$\|x\|_{\mathcal{B}} \le \gamma \|x - g(x,x)\| = \lambda\gamma \|h(x) - g(x,x)\| < \|x\|_{\mathcal{B}}.$$

The obtained contradiction proves the Corollary. □

Corollary 1.2 immediately follows from Theorem 1.1 in the case where $g(x, y) \equiv g_0(y)$.

Proof of Theorem 1.2. Assume that the theorem is invalid. Then by Theorem 1.1 and Lemma 2.1, for every natural k there exist $\lambda_k \in]0, 1[$ and $x_k \in \mathcal{B}$, such that

$$\|x_k\|_{\mathcal{B}} > k, \quad x_k = g(x_k, x_k) + \lambda_k (h(x_k) - g(x_k, x_k)).$$

Suppose

$$y_k = \frac{1}{\|x_k\|_{\mathcal{B}}} x_k.$$

Then

$$\|y_k\|_{\mathcal{B}} = 1, \quad y_k = g(x_k, y_k) + \frac{\lambda_k}{\|x_k\|_{\mathcal{B}}} (h(x_k) - g(x_k, x_k)). \tag{3.5}$$

By (1.4), (1.19) and (3.5), the sequences $(y_k)_{k=1}^{\infty}$ and $(g(x_k, y_k))_{k=1}^{\infty}$ may, without restriction of generality, be assumed to be convergent. Suppose

$$y = \lim_{k\to\infty} y_k, \qquad z = \lim_{k\to\infty} g(x_k, y_k). \tag{3.6}$$

Then by (1.3) we have

$$\lim_{k\to\infty} \|g(x_k, y_k) - g(x_k, y)\|_{\mathcal{B}} = \lim_{k\to\infty} \|g(x_k, y_k - y)\|_{\mathcal{B}} = 0.$$

Consequently,

$$\|y\|_{\mathcal{B}} = 1, \quad z \in \overline{\{g(x, y) : x \in \mathcal{B}\}}. \tag{3.7}$$

Let

$$u_k = \frac{1}{\|x_k\|_{\mathcal{B}}} h_0(x_k), \qquad v_k = \frac{1}{\|x_k\|_{\mathcal{B}}} \mu_0(x_k).$$

Then according to (1.18),

$$\lim_{k\to\infty} \|u_k\|_{\mathcal{B}_0} = 0, \qquad \lim_{k\to\infty} \|v_k\|_{\mathcal{B}_0} = 0. \tag{3.8}$$

On the other hand, by virtue of (1.17) and because of the fact that the operators μ and ν are positively homogeneous, we find from (3.5) that

$$\begin{aligned} \nu\,(y_k - g(x_k, y_k) - u_k) &= \frac{\lambda_k}{\|x_k\|_{\mathcal{B}}} \nu\,(h(x_k) - g(x_k, x_k) - h_0(x_k)) \\ &\le \frac{1}{\|x_k\|_{\mathcal{B}}} (\mu(x_k) + \mu_0(x_k)) = \mu(y_k) + v_k \end{aligned}$$

whence, according to (3.6) and (3.8), it follows that $\nu(y - z) \le \mu(y)$, i.e., $(\mu(y) - \nu(y - z)) \in \mathcal{K}$. But this, because of (1.16) and (3.7), is impossible. The obtained contradiction proves the theorem. □

Proof of Corollary 1.3. Assume $g(x, y) \equiv \theta$ and $\mu(x) = \eta(\nu(x))$. Then because of (1.21), condition (1.17) is fulfilled. By Theorem 1.2, to prove the corollary it suffices to establish that the condition

$$\eta(\nu(y)) - \nu(y) \notin \mathcal{K} \quad \text{for } y \ne \theta$$

is fulfilled.

Assume the contrary. Then there exists $y \ne \theta$ such that $\nu(y) \le \eta(\nu(y))$. Hence $u = \nu(y)$ is a nontrivial solution of the inequality (2.4). But this is impossible for, by Lemma 2.3, the above-mentioned inequality in the cone $\mathcal{K}$ has no nontrivial solution. The obtained contradiction proves the corollary. □

Proof of Corollary 1.4. Let $\mathcal{B}_0$ be the space of real numbers with the cone $\mathcal{K} = [0, +\infty[$, and let $\mu, \mu_0, \nu : \mathcal{B} \to \mathcal{K}$ and $h_0 : \mathcal{B} \to \mathcal{B}$ be the operators given by the equalities

$$\mu(x) = 0, \quad \mu_0(x) = \|h(x) - g(x,x)\|_{\mathcal{B}}, \quad \nu(x) = \|x\|_{\mathcal{B}}, \quad h_0(x) = \theta \quad \text{for } x \in \mathcal{B}.$$

Then from (1.5) and (1.22) we obtain the conditions (1.16) and (1.18), while the condition (1.17) is fulfilled automatically. Thus, all the conditions of Theorem 1.2 are fulfilled. □

To convince ourselves that Theorem 1.3 (Corollary 1.5) is true, it suffices to note that due to Lemma 2.2 the fulfilment of the conditions of the mentioned theorem (corollary) guarantees the fulfilment of the conditions of Theorem 1.2 (Corollary 1.4).

Proof of Theorem 1.4. By virtue of the identity $g(x, y) \equiv g_0(y)$, from (1.24) follows the inequality (1.17), where $h_0(x) = h(0)$, $\mu_0(x) = \theta_0$. Consequently, all the conditions of Theorem 1.3 are fulfilled. Therefore Eq. (1.1) has at least one solution.

It remains to prove that Eq. (1.1) has at most one solution. Let x and y be arbitrary solutions of this equation. Assume $z = x - y$. Then $z - g_0(z) = h(x) - h(y)$ which, owing to (1.24), implies that z is a solution of the inequality (1.23). Therefore, $z = \theta$, i.e. $x = y$. □

By Lemma 2.3, Corollary 1.6 follows from Theorem 1.4.

4. Periodic problem at resonance

Here we present one nontrivial example of application of Theorem 1.1.

Let $\omega > 0$, n be a natural number, C_ω^{n-1} be the Banach space of $(n-1)$-times continuously differentiable functions $x : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|x\|_{C_\omega^{n-1}} = \max\left\{\sum_{i=1}^{n} |u^{(i-1)}(t)| : \ 0 \le t \le \omega\right\},$$

and L_ω be the Banach space of ω-periodic, Lebesgue integrable on $[0, \omega]$ functions $y : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|y\|_{L_\omega} = \int_a^b |y(t)|\mathrm{d}t.$$

Consider the functional differential equation

$$x^{(n)}(t) = f(x)(t) + f_0(t), \tag{4.1}$$

where $f : C_\omega^{n-1} \to L_\omega$ is a continuous operator, $f_0 \in L_\omega$.

A function $u \in C_\omega^{n-1}$ is said to be an ω-periodic solution of Eq. (4.1), if $u^{(n-1)}$ is absolutely continuous and almost everywhere on $\mathbb{R}$, the equality (4.1) is satisfied.

We are interested in the case where there exists a nonnegative function $f^* \in L_\omega$ such that for an arbitrary $x \in C_\omega^{n-1}$ almost everywhere on $\mathbb{R}$ the inequality

$$|f(x)(t)| \le f^*(t) \tag{4.2}$$

is satisfied.

In this case, the problem on the existence of a periodic solution of Eq. (4.1) is at resonance because the corresponding linear homogeneous equation

$$x^{(n)} = 0$$

has nontrivial ω-periodic solutions.

Theorem 4.1. *Let the condition (4.2) be fulfilled and*

$$\int_0^\omega f_0(t)\mathrm{d}t = 0. \tag{4.3}$$

Let, moreover, there exist $\sigma \in \{-1; 1\}$ and a positive number ρ_0 such that for every $u \in C_\omega^{n-1}$, satisfying the condition

$$|u(t)| > \rho_0 \quad \textit{for } t \in \mathbb{R} \tag{4.4}$$

almost everywhere on $\mathbb{R}$, the inequality

$$\sigma f(u)(t)u(0) \geq 0 \tag{4.5}$$

holds. Then Eq. (4.1) has at least one solution.

Proof. Let

$$p = \frac{\sigma}{2}\,\omega^{1-n}. \tag{4.6}$$

Then the homogeneous problem

$$u^{(n)}(t) = pu(t); \qquad u^{(i-1)}(\omega) = u^{(i-1)}(0) \quad (i = 1, \dots, n)$$

has only a trivial solution (see, e.g., [10]). By G we denote its Green's function. It is clear that G can be assumed to be defined on $\mathbb{R} \times [0, \omega]$ and satisfying the condition

$$G(t + \omega, s) = G(t, s) \quad \text{for } t \in \mathbb{R}, s \in [0, \omega].$$

It follows from the above that the problem of the existence of a periodic solution of Eq. (4.1) is equivalent to the problem of solvability of the operator equation

$$x(t) = \int_0^\omega G(t, s)\,[f(x)(s) - pu(s) + f_0(s)]\,\mathrm{d}s$$

in the space C_ω^{n-1}, i.e., of Eq. (1.1), where

$$h(x) = \int_0^\omega G(\cdot\,, s)\,[f(x)(s) - pu(s) + f_0(s)]\,\mathrm{d}s.$$

Taking into account the fact that the operator f is continuous and also the condition (4.2), the operator $h : C_\omega^{n-1} \to C_\omega^{n-1}$ is completely continuous.

Suppose

$$\rho_1 = \|f^*\|_{L_\omega} + \|f_0\|_{L_\omega}, \qquad \rho = \rho_0 + (\omega^{1-n}\rho_0 + 2\rho_1)\sum_{i=1}^{n} \omega^{n-i}. \tag{4.7}$$

By Corollary 1.2, to prove the theorem it suffices to state that for an arbitrary $\lambda \in]0, 1[$, every solution $x \in C_\omega^{n-1}$ of the operator equation

$$x(t) = \lambda \int_0^\omega G(t, s)\,[f(x)(s) - pu(s) + f_0(s)]\,\mathrm{d}s \tag{4.8}$$

admits the estimate

$$\|u\|_{C_\omega^{n-1}} \leq \rho. \tag{4.9}$$

However, every solution of the operator Eq. (4.8) is an ω-periodic solution of the functional differential equation

$$u^{(n)} = (1 - \lambda)pu(t) + \lambda f(u)(t) + \lambda f_0(t). \tag{4.10}$$

Consequently, it remains to show that if $\lambda \in]0, 1[$ and x is an ω-periodic solution of Eq. (4.10), then the estimate (4.9) is valid.

First we show that

$$\min\{|u(t)| : t \in \mathbb{R}\} \leq \rho_0. \tag{4.11}$$

Assume the contrary that (4.11) is violated, i.e., (4.4) is fulfilled. Then according to one of the conditions of the theorem, almost everywhere on $\mathbb{R}$ the inequality (4.5) is fulfilled. If we multiply (4.10) by $\sigma u(0)$ and integrate from

0 to ω, then in view of (4.3), (4.5) and (4.6) we find

$$0 = \int_0^\omega [(1-\lambda)|p|\,|u(t)| + \sigma f(u)(t)u(0)]\,dt > (1-\lambda)|p|\rho_0 > 0.$$

The obtained contradiction proves that the estimate (4.11) is valid.

Since u is periodic, there exist points $t_i \in [0,\omega]$ $(i = 1,\dots,n-1)$ such that

$$u^{(i)}(t_i) = 0 \quad (i = 1,\dots,n-1). \tag{4.12}$$

Assume

$$\ell = \max\{|u^{(n-1)}(t)| : 0 \le t \le \omega\}.$$

Then taking into account (4.11) and (4.12), we have

$$|u(t)| \le \rho_0 + \ell\omega^{n-1}, \qquad |u^{(i)}(t)| \le \ell\omega^{n-1-i} \quad \text{for } t \in \mathbb{R}\ (i = 1,\dots,n-1). \tag{4.13}$$

If along with (4.12) and (4.13) we take into account the conditions (4.2) and (4.6), then from (4.10) we obtain

$$\ell \le \frac{\ell}{2} + \frac{\rho_0}{2}\omega^{1-n} + \rho_1,$$

and hence

$$\ell \le \omega^{1-n}\rho_0 + 2\rho_1.$$

By virtue of the above estimate and the notation (4.7), the inequalities (4.13) result in the estimate (4.9). □

A particular case of (4.1) is the differential equation with the deviating argument

$$x^{(n)}(t) = f_1(t, x(\tau(t))) + f_0(t), \tag{4.14}$$

where $f_1(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $t \in \mathbb{R}$,

$$f_1(\cdot, x) \in L_\omega \quad \text{for any } x \in \mathbb{R},\ f_0 \in L_\omega, \tag{4.15}$$

and $\tau : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\frac{\tau(t+\omega) - \tau(t)}{\omega} \quad \text{is an integer for almost all } t \in \mathbb{R}. \tag{4.16}$$

Corollary 4.1. *Let the conditions (4.3), (4.15) and (4.16) be fulfilled. Let, moreover, there exist $\sigma \in \{-1; 1\}$, a positive number ρ_0, and a nonnegative function $f^* \in L_\omega$ such that*

$$|f_1(t,x)| \le f^*(t) \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R}$$

and

$$\sigma f_1(t,x)x \ge 0 \quad \text{for } t \in \mathbb{R},\ |x| \ge \rho.$$

Then Eq. (4.14) has at least one ω-periodic solution.

To make sure that this proposition is true, it suffices to notice that if the conditions of Corollary 4.1 are fulfilled, then the operator

$$f(x)(t) = f_1(t, x(\tau_1(t)))$$

satisfies the conditions of Theorem 4.1.

Remark 4.1. If $f_1(t,x) \equiv 0$, then condition (4.3) is necessary and sufficient for the existence of an ω-periodic solution of Eq. (4.14). Consequently, condition (4.3) is essential both in Theorem 1.1 and in Corollary 4.1, and it cannot be neglected.

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