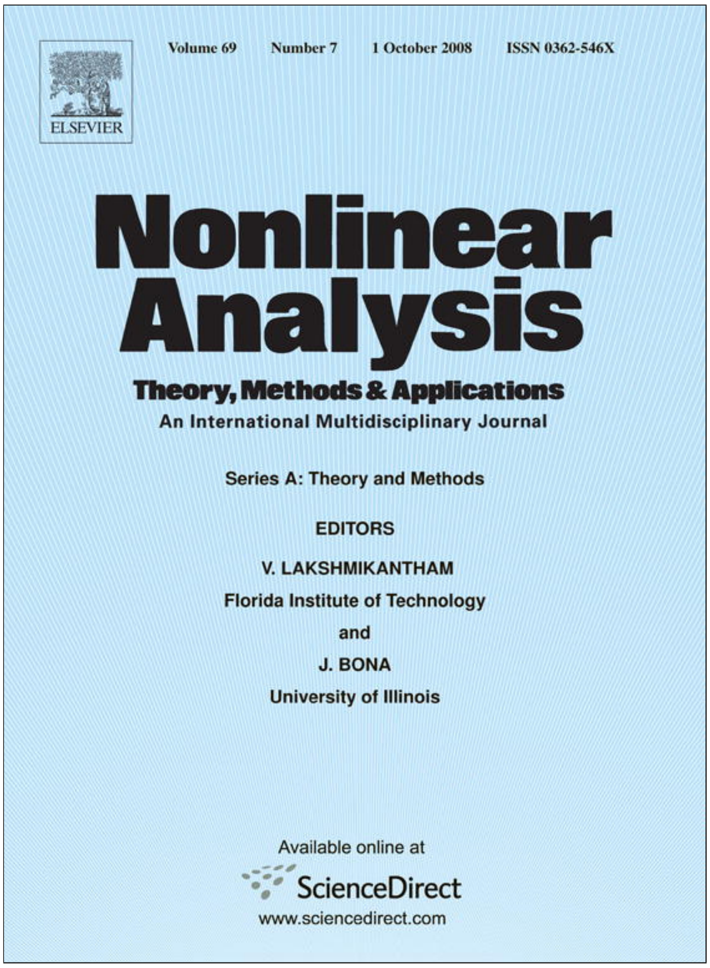
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On solvability of boundary value problems for higher order nonlinear hyperbolic equations☆

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Abstract

In the rectangle $\Omega = [0, a] \times [0, b]$ for the nonlinear hyperbolic equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + f(x, y, u, \dots, u^{(m-1,n-1)})$$

the boundary value problems of the type

$$l_{1i}(u(\cdot, y)) = 0 \quad (i = 1, \dots, m), \qquad l_{2k}(u(x, \cdot)) = 0 \quad (k = 1, \dots, n)$$

are considered, where $l_{1i} : C^{m-1}([0, a]) \to \mathbb{R}$ $(i = 1, \dots, m)$ and $l_{2k} : C^{n-1}([0, b]) \to \mathbb{R}$ $(k = 1, \dots, n)$ are linear bounded functionals.

Sufficient conditions of solvability and unique solvability of the general problem and its particular cases (Nicoletti type, Dirichlet, Lidstone and Periodic problems) are established.

MSC: 35L35; 35B10

Keywords: Nonlinear; Higher order; Hyperbolic equation; Dirichlet; Lidstone; Periodic; Boundary value problem

0. Introduction

In the rectangle $\Omega = [0, a] \times [0, b]$ consider the hyperbolic equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + f(x, y, u, \dots, u^{(m-1,n-1)}) \tag{0.1}$$

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with the functional boundary conditions

$$l_{1i}(u(\cdot, y)) = 0 \quad (i = 1, \ldots, m), \qquad l_{2k}(u(x, \cdot)) = 0 \quad (k = 1, \ldots, n). \tag{0.2}$$

Here $h_{1i} : [0, a] \to \mathbb{R}$ $(i = 1, \ldots, m)$, $h_{2k} : [0, b] \to \mathbb{R}$ $(k = 1, \ldots, n)$, $f : \Omega \times \mathbb{R}^{mn} \to \mathbb{R}$ are continuous functions, $l_{1i} : C^{m-1}([0, a]) \to \mathbb{R}$ $(i = 1, \ldots, m)$, $l_{2k} : C^{n-1}([0, b]) \to \mathbb{R}$ $(k = 1, \ldots, n)$ are linear bounded functionals, and

$$u^{(i,k)}(x, y) = \frac{\partial^{i+k} u(x, y)}{\partial x^i \partial y^k}.$$

Throughout the paper the following notations will be used.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}^k$ is the k-dimensional Euclidean space.

$C^k(I)$, where I is a compact interval, is the Banach space of k-times continuously differentiable functions $u : I \to \mathbb{R}$ with the norm

$$\|u\|_{C^k(I)} = \max\left\{\sum_{i=0}^{k} |u^i(s)| : x \in I\right\}.$$

$C^{m,n}(\Omega)$ is the Banach space of continuous functions $u : \Omega \to \mathbb{R}$ having continuous partial derivatives $u^{(i,k)}$ $(i = 0, \ldots, m;\ k = 0, \ldots, n)$, with the norm

$$\|u\|_{C^{m,n}(\Omega)} = \max\left\{\sum_{i=0}^{m}\sum_{k=0}^{n} |u^{(i,k)}(x, y)| : (x, y) \in \Omega\right\}.$$

By a solution of problem (0.1), (0.2) we understand a function $u \in C^{m,n}(\Omega)$ satisfying Eq. (0.1) and conditions (0.2) everywhere on Ω.

Previously problem (0.1), (0.2) was studied basically in the following cases:

(i) conditions (0.2) are initial-boundary, i.e.,

$$u^{(i-1,0)}(0, y) = 0 \quad (i = 1, \ldots, m), \qquad l_{2k}(u(x, \cdot)) = 0 \quad (k = 1, \ldots, n)$$

(see [1–18]);

(ii) $m = n = 1$ and conditions (0.2) have the form

$$u(0, y) = u(a, y), \qquad u(x, 0) = u(x, b)$$

(see [6,19–21]);

(iii) $m = n = 2$, Eq. (0.1) is linear and (0.2) are either periodic conditions, i.e.,

$$u^{(i-1,0)}(0, y) = u^{(i-1,0)}(a, y) \quad (i = 1, 2), \qquad u^{(0,k-1)}(x, 0) = u^{(0,k-1)}(x, b) \quad (k = 1, 2),$$

or the Dirichlet conditions

$$u(0, y) = u(a, y) = 0, \qquad u(x, 0) = u(x, b) = 0$$

(see [22–24]).

For some classes of linear hyperbolic equations the Dirichlet problem was studied in [25].

In the general case problem (0.1), (0.2) has been actually unstudied. The present paper is an attempt to fill this gap.

The paper is organized as follows: in Section 1 a class of linear boundary value problem with the Fredholm property is described; in Section 2 a theorem on solvability of a general nonlinear boundary value problem is proved (a priori boundedness principle), on the basis of which effective and unimprovable in a sense sufficient conditions of solvability of Nicoletti type nonlocal problems, Dirichlet and Lidstone type problems, and periodic problems are established in Sections 3–5.

1. A general linear problem

In this section we consider the problem

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x) u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y) u^{(m,k)} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f_{ik}(x, y) u^{(i,k)} + f_0(x, y), \tag{1.1}$$

$$l_{1i}(u(\cdot, y)) = 0 \quad (i = 1, \dots, m), \qquad l_{2k}(u(x, \cdot)) = 0 \quad (k = 1, \dots, n), \tag{1.2}$$

where

$$h_{1i} \in C([0, a]), \qquad h_{2k} \in C([0, b]) \quad (i = 0, \dots, m-1;\ k = 0, \dots, n-1), \tag{1.3}$$
$$f_{ik} \in C(\Omega) \quad (i = 0, \dots, m-1;\ k = 0, \dots, n-1), \qquad f_0 \in C(\Omega),$$

and $l_{1i} : C^{m-1}([0, a]) \to \mathbb{R}$, $l_{2k} : C^{n-1}([0, b]) \to \mathbb{R}$ $(i = 1, \dots, m, k = 1, \dots, n)$ are linear bounded functionals.

Along with (1.1) consider the corresponding homogeneous equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f_{ik}(x, y)u^{(i,k)}. \tag{1.1$_0$}$$

Problem (1.1), (1.2) is closely related to the linear homogeneous boundary value problems for ordinary differential equations

$$v^{(m)} = \sum_{i=0}^{m-1} h_{1i}(x)v^{(i)}, \qquad l_{1j}(v) = 0 \quad (j = 1, \dots, m) \tag{1.4}$$

and

$$w^{(n)} = \sum_{i=0}^{n-1} h_{2k}(y)w^{(k)}, \qquad l_{2k}(w) = 0 \quad (j = 1, \dots, n). \tag{1.5}$$

Lemma 1.1. *Let both problem (1.4) and problem (1.5) have only trivial solutions. Then for an arbitrary $h \in C(\Omega)$ the differential equation*

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} h_{1i}(x)h_{2k}(y)u^{(i,k)} + h(x, y) \tag{1.6}$$

has a unique solution satisfying conditions (1.2) and this solution admits the representation

$$u(x, y) = \int_0^b \int_0^a g_1(x, s)g_2(y, t)h(s, t)\mathrm{d}s\mathrm{d}t, \tag{1.7}$$

where g_1 is the Green's function of problem (1.4), and g_2 is the Green's function of problem (1.5).

Proof. First show that if problem (1.6), (1.2) has a solution u, then it admits representation (1.7).

Let $y \in [0, b]$ be arbitrarily fixed and

$$v(x) = u^{(0,n)}(x, y) - \sum_{k=0}^{n-1} h_{2k}(y)u^{(0,k)}(x, y) \quad \text{for } x \in [0, a].$$

Then v is a solution of the problem

$$v^{(m)} = \sum_{i=0}^{m-1} h_{1i}(x)v^{(i)} + h(x, y), \qquad l_{1j}(v) = 0 \quad (j = 1, \dots, m).$$

Since the corresponding homogeneous problem (1.4) has only a trivial solution, the latter problem has a unique solution

$$v(x) = \int_0^a g_1(x, s)h(s, y)\mathrm{d}s \quad \text{for } x \in [0, a]$$

(see [26], Theorem 1.1). Consequently,

$$u^{(0,n)}(x, y) = \sum_{k=0}^{n-1} h_{2k}(y)u^{(0,k)}(x, y) + \int_0^a g_1(x, s)h(s, y)\mathrm{d}s \quad \text{for } (x, y) \in \Omega.$$

Therefore for any fixed $x \in [0, a]$ the function

$$w(y) = u(x, y)$$

is a solution of the problem

$$w^{(n)} = \sum_{k=0}^{n-1} h_{2k}(y) w^{(k)} + \int_0^a g_1(x, s) h(s, y) \mathrm{d}s, \qquad l_{2j}(w) = 0 \quad (j = 1, \dots, n).$$

Hence, by the above mentioned theorem from [26] we get

$$w(y) = \int_0^b \int_0^a g_1(x, s) g_2(y, t) h(s, t) \mathrm{d}s \mathrm{d}t \quad \text{for } y \in [0, b].$$

Thus the validity of (1.7) is proved.

Finally notice that the function u given by (1.7) is a solution of problem (1.6), (1.2). □

Theorem 1.1. *Let problems (1.4) and (1.5), and problem* (1.1_0), *(1.2) have only trivial solutions. Then problem (1.1), (1.2) is uniquely solvable and its solution admits the representation*

$$u(x, y) = \mathcal{G}(f_0)(x, y) \quad \text{for } (x, y) \in \Omega, \tag{1.8}$$

where $\mathcal{G} : C(\Omega) \to C^{m,n}(\Omega)$ *is a linear bounded operator.*

Proof. For arbitrary $z \in C(\Omega)$ and $u \in C^{m-1,n-1}(\Omega)$ set

$$\mathcal{G}_0(z)(x, y) = \int_0^b \int_0^a g_1(x, s) g_2(y, t) z(s, t) \mathrm{d}s \mathrm{d}t$$

and

$$\mathcal{P}(u)(x, y) = \mathcal{G}_0 \left(\sum_{i=0}^{m-1} \sum_{k=0}^{n-1} (h_{1i} h_{2k} + f_{ik}) u^{(i,k)} \right),$$

where g_1 and g_2 are the Green's functions of problems (1.4) and (1.5), respectively. Then $\mathcal{G}_0 : C(\Omega) \to C^{m,n}(\Omega)$ and $\mathcal{P} : C^{m-1,n-1}(\Omega) \to C^{m,n}(\Omega)$ are linear bounded operators, and hence compact operators from $C(\Omega)$ to $C^{m-1,n-1}(\Omega)$ and from $C^{m-1,n-1}(\Omega)$ to $C^{m-1,n-1}(\Omega)$, respectively.

By Lemma 1.1, problem (1.1), (1.2) is equivalent to the operator equation

$$u = \mathcal{P}(u) + q \tag{1.9}$$

in the space $C^{m-1,n-1}(\Omega)$, where

$$q(x, y) = \mathcal{G}_0(f_0)(x, y). \tag{1.10}$$

On the other hand, the homogeneous equation

$$u = \mathcal{P}(u)$$

has only a trivial solution, since it is equivalent to the homogeneous problem (1.1_0), (1.2) which has only a trivial solution according to one of the conditions of Theorem 1.1.

By Fredholm's theorem for operator equations, Eq. (1.9) and, consequently, problem (1.1), (1.2) have a unique solution

$$u = \mathcal{P}_0(q),$$

where $\mathcal{P}_0 : C^{m-1,n-1}(\Omega) \to C^{m-1,n-1}(\Omega)$ is a linear bounded operator. Since $\mathcal{P} : C^{m-1,n-1}(\Omega) \to C^{m,n}(\Omega)$ is a bounded linear operator, then taking into account (1.9) we find out that actually $\mathcal{P}_0$ is a linear bounded operator from $C^{m,n}(\Omega)$ to $C^{m,n}(\Omega)$. The latter formula and notation (1.10) yield representation (1.8), where

$$\mathcal{G}(f_0)(x, y) = \mathcal{P}_0(\mathcal{G}_0(f_0))(x, y). \quad \square$$

2. General nonlinear problem

In this section we consider the problem

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + f(x, y, u, \dots, u^{(m-1,n-1)}), \tag{2.1}$$

$$l_{1i}(u(\cdot, y)) = 0 \quad (i = 1, \dots, m), \qquad l_{2k}(u(x, \cdot)) = 0 \quad (k = 1, \dots, n), \tag{2.2}$$

where the functions h_{1i} $(i = 0, \dots, m-1)$ and $h_{2k}(k = 0, \dots, n-1)$ satisfy conditions (1.3), $f : \Omega \times \mathbb{R}^{mn} \to \mathbb{R}$ is a continuous function and $l_{1i} : C^{m-1}([0, a]) \to \mathbb{R}$ $(i = 1, \dots, m)$ and $l_{2k} : C^{n-1}([0, b]) \to \mathbb{R}$ $(k = 1, \dots, n)$ are linear bounded functionals.

Theorem 2.1. *Let problems (1.4) and (1.5) have only trivial solutions. Moreover, let there exist a positive number ϱ and functions $f_{ik} \in C(\Omega)$ $(i = 0, \dots, m-1;\ k = 0, \dots, n-1)$ such that:*(i) *problem (1.1_0), (1.2) has only a trivial solution;* (ii) *for any $\lambda \in (0, 1)$ every solution of the differential equation*

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + (1-\lambda) \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f_{ik}(x, y)u^{(i,k)} + \lambda f(x, y, u, \dots, u^{(m-1,n-1)}) \tag{2.3}$$

satisfying the boundary conditions (2.2) admits the estimate

$$\|u\|_{C^{m-1,n-1}} \le \varrho. \tag{2.4}$$

Then problem (2.1), (2.2) has at least one solution.

Proof. Let

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \le s \le \varrho \\ 2 - \dfrac{s}{\varrho} & \text{for } \varrho < s < 2\varrho \\ 0 & \text{for } s > 2\varrho. \end{cases} \tag{2.5}$$

For an arbitrary $u \in C^{m-1,n-1}(\Omega)$ set

$$f_\varrho(u)(x, y) = \chi(\|u\|_{C^{m-1,n-1}}) \left(f(x, y, u(x, y), \dots, u^{(m-1,n-1)}(x, y)) - \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f_{ik}(x, y)u^{(i,k)} \right), \tag{2.6}$$

and consider the functional differential equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f_{ik}(x, y)u^{(i,k)} + f_\varrho(u)(x, y) \tag{2.7}$$

with the boundary conditions (1.2).

By Theorem 1.1, there exists a linear bounded operator $\mathcal{P} : C(\Omega) \to C^{m,n}(\Omega)$ such that problem (2.7), (2.2) is equivalent to the operator equation

$$u = \mathcal{F}(u) \tag{2.8}$$

in the space $C^{m-1,n-1}(\Omega)$, where

$$\mathcal{F}(u)(x, y) = \mathcal{P}(f_\varrho(u)(x, y)), \tag{2.9}$$

i.e., every solution of problem (2.7), (2.2) is a solution of Eq. (2.8) and vice versa, every solution of Eq. (2.8) is a solution of problem (2.7), (2.2).

According to (2.5) and (2.6) the operator $f_\varrho : C^{m-1,n-1}(\Omega) \to C(\Omega)$ is continuous and for an arbitrary $u \in C^{m-1,n-1}(\Omega)$ satisfies the inequality

$$|f_\varrho(u)(x, y)| \le \varrho_0$$

on Ω, where

$$\varrho_0 = \max\left\{|f(x,y,z_{00},\ldots,z_{m-1n-1})| : (x,y)\in\Omega, \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}|z_{ik}| \le 2\varrho\right\} + 2\varrho\max\left\{\sum_{i=1}^{m-1}\sum_{k=0}^{n-1}|f_{ik}(x,y)| : (x,y)\in\Omega\right\}.$$

Therefore it follows from (2.9) that $\mathcal{F} : C^{m-1,n-1(\Omega)} \to c^{m-1,n-1}(\Omega)$ is a compact operator mapping the ball

$$\mathbf{B}(\varrho_1) = \{u \in C^{m-1,n-1}(\Omega) : \|u\|_{C^{m-1,n-1}} \le \varrho_1\},$$

where $\varrho_1 = \|\mathcal{P}\|\varrho_0$ and $\|\mathcal{P}\|$ is the norm of the operator $\mathcal{P}$, into itself. By Schauder's theorem, Eq. (2.8) and, consequently, problem (2.7), (2.2) has at least one solution $u \in \mathbf{B}(\varrho_1)$.

To complete the proof of the theorem we need to show that an arbitrary solution of problem (2.7), (2.2) is at the same time a solution of (2.1), (2.2). Assume the contrary that problem (2.7), (2.2) has a solution u which is not a solution of problem (2.1), (2.2). Then in view of (2.5) and (2.6) either

$$\|u\|_{C^{m-1,n-1}} \ge 2\varrho, \tag{2.10}$$

or

$$\varrho < \|u\|_{C^{m-1,n-1}} < 2\varrho. \tag{2.11}$$

Inequality (2.10) may not be the case because then $f_\varrho(u)(x,y) \equiv 0$ and, consequently, u is a solution of the homogeneous problem (1.1_0), (1.2) which has only a trivial solution. If (2.11) holds, then in view of (2.5) and (2.6), u is a solution of problem (2.3), (2.2), where

$$\lambda = \chi(\|u\|_{C^{m-1,n-1}}) \in (0,1).$$

But this is impossible again since, by one of the conditions of the theorem, every solution of problem (2.3), (2.2) admits estimate (2.4). The obtained contradiction proves the theorem. ☐

3. Nicoletti type nonlocal problem

Consider the problem

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + f(x,y,u,\ldots,u^{(m-1,n-1)}), \tag{3.1}$$

$$\begin{aligned}&\int_0^a u^{(i,0)}(s,y)\mathrm{d}\varphi_{1i}(s) = 0 \quad (i=0,\ldots,m-1),\\ &\int_0^b u^{(0,k)}(x,t)\mathrm{d}\varphi_{2k}(t) = 0 \quad (k=0,\ldots,n-1),\end{aligned} \tag{3.2}$$

where $\varphi_{1i} : [0,a] \to \mathbb{R}$ and $\varphi_{2k} : [0,b] \to \mathbb{R}$ are nondecreasing functions such that

$$\varphi_{1i}(a) > \varphi_{1i}(0) \quad (i=0,\ldots,m-1), \qquad \varphi_{2k}(b) > \varphi_{2k}(0) \quad (i=0,\ldots,n-1). \tag{3.3}$$

As above, the functions $h_{1i} : [0,a] \to \mathbb{R}$, $h_{2k} : [0,b] \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^{mn} \to \mathbb{R}$ are considered to be continuous.

The boundary conditions

$$u^{(i,0)}(x_i,y) = 0 \quad (i=0,\ldots,m-1), \qquad u^{(0,k)}(x,y_k) = 0 \quad (k=0,\ldots,n-1),$$

where $0 \le x_i \le a$, $0 \le y_k \le b$, are a particular case of (3.2). Similar conditions for ordinary differential equations are called Nicoletti conditions (see [26] and the literature quoted therein). Therefore it is natural to call (3.1), (3.2) a Nicoletti type nonlocal problem.

Theorem 3.1. *Let there exist nonnegative constants* $\alpha_{1i}, \alpha_{2k}, \beta_{ik}$ $(i = 0, \dots, m-1; k = 0, \dots, n-1)$ *and* γ *such that*

$$\sum_{i=0}^{m-1} \left(\frac{2a}{\pi}\right)^{m-i} \alpha_{1i} + \sum_{k=0}^{n-1} \left(\frac{2b}{\pi}\right)^{n-k} \alpha_{2k} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \left(\frac{2a}{\pi}\right)^{m-i} \left(\frac{2b}{\pi}\right)^{n-k} \beta_{ik} < 1, \tag{3.4}$$

and the inequalities

$$|h_{1i}(x)| \le \alpha_{1i} \quad (i = 0, \dots, m-1), \qquad |h_{2k}(y)| \le \alpha_{2k} \quad (k = 0, \dots, n-1), \tag{3.5}$$

$$|f(x, y, z_{00}, \dots, z_{m-1m-1})| \le \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \beta_{ik}|z_{ik}| + \gamma \tag{3.6}$$

hold on $\Omega \times \mathbb{R}^{mn}$. *Then problem* (3.1), (3.2) *has at least one solution.*

To prove this theorem we will need the following three lemmas. This first of them is about a priori estimates of solutions of the differential inequality

$$|u^{(m,n)}(x, y)| \le \sum_{i=0}^{m-1} \alpha_{1i}|u^{(i,n)}(x, y)| + \sum_{k=0}^{n-1} \alpha_{2k}|u^{(m,k)}(x, y)| + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \beta_{ik}|u^{(i,k)}(x, y)| + \gamma \tag{3.7}$$

subject to the boundary conditions (3.2).

Everywhere below by $\|z\|_{L^2}$ we denote the L^2-norm of the function $z \in L^2(\Omega)$.

Lemma 3.1. *Let* $\alpha_{1i}, \alpha_{2k}, \beta_{ik}$ $(i = 0, \dots, m-1; k = 0, \dots, n-1)$ *be constants satisfying inequality* (3.4). *Then there exists a positive number* r *such that for an arbitrary* $\gamma \ge 0$ *every solution of problem* (3.7), (3.2) *admits the estimate*

$$\|u\|_{C^{m-1,n-1}} \le r\gamma. \tag{3.8}$$

Proof. According to condition (3.4) the number

$$\delta = \sum_{i=0}^{m-1} \left(\frac{2a}{\pi}\right)^{m-i} \alpha_{1i} + \sum_{k=0}^{n-1} \left(\frac{2b}{\pi}\right)^{n-k} \alpha_{2k} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \left(\frac{2a}{\pi}\right)^{m-i} \left(\frac{2b}{\pi}\right)^{n-k} \beta_{ik}$$

is less than 1. Set

$$r_0 = (1-\delta)^{-1}(ab)^{\frac{1}{2}}, \qquad r = r_0 \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} a^{m-i} b^{n-k}.$$

Let u be an arbitrary solution of problem (3.7), (3.2). Then by the Minkowski inequality, from (3.7) we have

$$\|u^{(m,n)}\|_{L^2} \le \sum_{i=0}^{m-1} \alpha_{1i}\|u^{(i,n)}\|_{L^2} + \sum_{k=0}^{n-1} \alpha_{2k}\|u^{(m,k)}\|_{L^2} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \beta_{ik}\|u^{(i,k)}\|_{L^2} + (ab)^{\frac{1}{2}}\gamma. \tag{3.9}$$

On the other hand in view of (3.3) and monotonicity of the functions φ_{1i} and φ_{2k} it follows from (3.2) that

$$\min\{|u^{(i,k)}(x, y)| : 0 \le x \le a\} = 0 \quad \text{for } 0 \le y \le b \ (i = 0, \dots, m-1; k = 0, \dots, n) \tag{3.10}$$

and

$$\min\{|u^{(i,k)}(x, y)| : 0 \le y \le b\} = 0 \quad \text{for } 0 \le x \le a \ (i = 0, \dots, m; k = 0, \dots, n-1). \tag{3.11}$$

Therefore

$$\begin{aligned} |u^{(i,k)}(x, y)| &\le a^{m-i} b^{n-k} \int_0^a \int_0^b |u^{(m,n)}(s, t)| \mathrm{d}s\mathrm{d}t \\ &\le (ab)^{\frac{1}{2}} a^{m-i} b^{n-k} \|u^{(m,n)}\|_{L^2} \quad \text{for } (x, y) \in \Omega \ (i = 0, \dots, n-1;\ k = 0, \dots, n-1). \end{aligned} \tag{3.12}$$

By Wirtinger's inequality (see [27]) and conditions (3.10) and (3.11), for arbitrary $i \in \{0, \dots, m\}$, $k \in \{0, \dots, n\}$ and $(x, y) \in \Omega$ the inequalities

$$\int_0^a |u^{(i,k)}(s, y)|^2 \mathrm{d}s \le \left(\frac{2a}{\pi}\right)^{2m-2i} \int_0^a |u^{(m,k)}(s, y)|^2 \mathrm{d}s,$$
$$\int_0^b |u^{(i,k)}(x, t)|^2 \mathrm{d}t \le \left(\frac{2b}{\pi}\right)^{2n-2k} \int_0^b |u^{(i,n)}(x, t)|^2 \mathrm{d}t$$

hold and, consequently,

$$\|u^{(i,k)}\|_{L^2} \le \left(\frac{2a}{\pi}\right)^{m-i} \left(\frac{2b}{\pi}\right)^{n-k} \|u^{(m,n)}\|_{L^2} \quad (i = 0, \dots, m;\ k = 0, \dots, n).$$

Therefore from (3.9) we find

$$\|u^{(m,n)}\|_{L^2} \le \delta \|u^{(m,n)}\|_{L^2} + (ab)^{\frac{1}{2}} \gamma$$

and

$$\|u^{(m,n)}\|_{L^2} \le r_0 \gamma.$$

If along with this we take into account inequalities (3.12), then validity of the estimate (3.8) becomes evident. □

Along with (3.1), (3.2) consider the auxiliary differential equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x) u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y) u^{(m,k)} \tag{3.13}$$

and the auxiliary boundary value problems

$$v^{(m)} = \sum_{i=0}^{m-1} h_{1i}(x) v^{(i)}, \qquad \int_0^a v^{(j)}(s) \mathrm{d}\varphi_{1j}(s) = 0 \quad (j = 0, \dots, m-1), \tag{3.14}$$

$$w^{(n)} = \sum_{k=0}^{n-1} h_{2k}(y) w^{(k)}, \qquad \int_0^b w^{(j)}(t) \mathrm{d}\varphi_{2j}(t) = 0 \quad (j = 0, \dots, n-1). \tag{3.15}$$

As above it will be assumed that $\varphi_{1i} : [0, a] \to \mathbb{R}$ and $\varphi_{2k} : [0, b] \to \mathbb{R}$ are nondecreasing functions satisfying conditions (3.3).

Lemma 3.2. *Let conditions (3.5) hold on Ω, where α_{1i} $(i = 0, \dots, m-1)$ and α_{2k} $(k = 0, \dots, n-1)$ are constants satisfying the inequality*

$$\sum_{i=0}^{m-1} \left(\frac{2a}{\pi}\right)^{m-i} \alpha_{1i} + \sum_{k=0}^{n-1} \left(\frac{2b}{\pi}\right)^{n-k} \alpha_{2k} < 1. \tag{3.16}$$

Then problem (3.13), (3.2), as well as problems (3.14) and (3.15) have only trivial solutions.

Proof. Let u be an arbitrary solution on problem (3.13), (3.2). Then in view of (3.5) it is also a solution of problem (3.7), (3.2), where $\beta_{ik} = 0$ $(i = 0, \dots, m-1; k = 0, \dots, n-1)$ and $\gamma = 0$. Hence by Lemma 3.1 and inequality (3.16), it follows that $u(x, y) \equiv 0$.

On the other hand, with the same reasoning that we used in the proof of Lemma 3.1 one can prove that both problems (3.14) and (3.15) have only trivial solutions provided that inequalities (3.5) and (3.16) hold. □

Proof of Theorem 3.1. By Theorem 2.1 and Lemma 3.2, to prove Theorem 3.1 it is sufficient to find a positive number ϱ such that for any $\lambda \in (0, 1)$ every solution of the differential equation

$$u^{(m,n)} = \sum_{i=0}^{m-1} h_{1i}(x)u^{(i,n)} + \sum_{k=0}^{n-1} h_{2k}(y)u^{(m,k)} + \lambda f(x, y, u, \dots, u^{(m-1,n-1)}) \tag{3.17}$$

subject to the boundary conditions (3.2) admits the estimate (2.4).

Let r be the number appearing in Lemma 3.1 and $\varrho = r\gamma$. By (3.5) and (3.6), every solution of problem (3.17), (3.2) is a solution of problem (3.7), (3.2) as well. Hence by Lemma 3.1 and inequality (3.4), we immediately get estimate (2.4). □

Theorem 3.2. *Let (3.5) hold on Ω, and the condition*

$$|f(x, y, z_{00}, \dots, z_{m-1n-1}) - f(x, y, \bar{z}_{00}, \dots, \bar{z}_{m-1n-1})| \le \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \beta_{ik}|z_{ik} - \bar{z}_{ik}| \tag{3.18}$$

hold on $\Omega \times \mathbb{R}^{mn}$, where $\alpha_{1i}, \alpha_{2k}, \beta_{ik}$ $(i = 0, \dots, m-1; k = 0, \dots, n-1)$ are nonnegative constants satisfying inequality (3.4). Then problem (3.1), (3.2) has one and only one solution.

Proof. Inequality (3.6) follows from (3.18), where $\gamma = \max\{|f(x, y, 0, \dots, 0)| : (x, y) \in \Omega\}$. Consequently all of the conditions of Theorem 3.1 are fulfilled that guarantees solvability of problem (3.1), (3.2).

All we need is to show is that the problem under consideration has at most one solution. Let u_1 and u_2 be its arbitrary solutions. Then in view of (3.18) the function

$$u(x, y) = u_1(x, y) - u_2(x, y)$$

is a solution of the problem (3.7), (3.2) with $\gamma = 0$. Hence by Lemma 3.1 and inequality (3.4) it follows that $u(x, y) \equiv 0$, i.e., $u_1(x, y) \equiv u_2(x, y)$. □

In Theorems 3.1 and 3.2 condition (3.4) is unimprovable in the sense that it cannot be replaced by the inequality

$$\sum_{i=0}^{m-1}\left(\frac{2a}{\pi}\right)^{m-i}\alpha_{1i} + \sum_{k=0}^{n-1}\left(\frac{2b}{\pi}\right)^{n-k}\alpha_{2k} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\left(\frac{2a}{\pi}\right)^{m-i}\left(\frac{2b}{\pi}\right)^{n-k}\beta_{ik} \le 1. \tag{3.19}$$

Indeed, if m and n are even numbers, then the problem

$$\begin{aligned} &u^{(m,n)} = (-1)^{\frac{m+n}{2}}\left(\frac{\pi}{2a}\right)^m\left(\frac{\pi}{2b}\right)^n u + \sin\frac{\pi t}{2a}\sin\frac{\pi t}{2b},\\ &u^{(2i,0)}(0, y) = u^{(2i+1,0)}(a, y) = 0 \quad \left(i = 0, \dots, \frac{m}{2} - 1\right),\\ &u^{(0,2k)}(x, 0) = u^{(0,2k+1)}(x, b) = 0 \quad \left(i = 0, \dots, \frac{n}{2} - 1\right) \end{aligned}$$

has no solution, although it satisfies all of the conditions of Theorem 3.2 except (3.4), which is replaced by (3.19).

4. Dirichlet and Lidstone type problems

In this section we consider the differential equation of even order

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} + f(x, y, u, \dots, u^{(m-1,n-1)}) \tag{4.1}$$

with the boundary conditions of one of the following three types

$$\begin{aligned} &u^{(i,0)}(0, y) = u^{(i,0)}(a, y) = 0 \quad (i = 0, \dots, m-1),\\ &u^{(0,k)}(x, 0) = u^{(0,k)}(x, b) = 0 \quad (k = 0, \dots, n-1); \end{aligned} \tag{4.2}$$

$$\begin{aligned} &u^{(i,0)}(0, y) = u^{(i,0)}(a, y) = 0 \quad (i = 0, \dots, m-1),\\ &u^{(0,2k)}(x, 0) = u^{(0,2k)}(x, b) = 0 \quad (k = 0, \dots, n-1); \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&u^{(2i,0)}(0,y)=u^{(2i,0)}(a,y)=0 \quad (i=0,\dots,m-1),\\
&u^{(0,2k)}(x,0)=u^{(0,2k)}(x,b)=0 \quad (k=0,\dots,n-1).
\end{aligned} \tag{4.4}
$$

It is reasonable to call problem (4.1), (4.2) the Dirichlet problem, and problems (4.1), (4.3) and (4.1), (4.4) the Dirichlet–Lidstone and the Lidstone problems, respectively, since similar problems for ordinary differential equations are called namely in that way (see e.g. [28]).

Everywhere below the functions $h_{1i} : [0,a] \to \mathbb{R}$ $(i = 0,\dots,m)$, $h_{2k} : [0,b] \to \mathbb{R}$ $(k = 0,\dots,n)$ and $f : \Omega \times \mathbb{R}^{mn} \to \mathbb{R}$ are assumed to be continuous.

Theorem 4.1. *Let the conditions*

$$
\begin{aligned}
&(-1)^m h_{10}(x) \le \alpha_{10}, \qquad |h_{1i}(x)| \le \alpha_{1i} \quad (i=1,\dots,m),\\
&(-1)^n h_{20}(y) \le \alpha_{20}, \qquad |h_{2k}(y)| \le \alpha_{2k} \quad (k=1,\dots,n)
\end{aligned} \tag{4.5}
$$

and

$$
(-1)^{m+n} f(x,y,z_{00},\dots,z_{m-1n-1})\operatorname{sgn} z_{00} \le \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\beta_{ik}|z_{ik}| + \gamma \tag{4.6}
$$

hold on Ω and $\Omega \times \mathbb{R}^{mn}$, respectively, where α_{1i}, α_{2k}, β_{ik} and γ are nonnegative constants such that

$$
\sum_{i=0}^{m}\left(\frac{a}{\pi}\right)^{2m-i}\alpha_{1i} + \sum_{k=0}^{n}\left(\frac{b}{\pi}\right)^{2n-k}\alpha_{2k} + \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\left(\frac{a}{\pi}\right)^{2m-i}\left(\frac{b}{\pi}\right)^{2n-k}\beta_{ik} < 1. \tag{4.7}
$$

Then for any $j \in \{2,3,4\}$ problem (4.1), $(4.j)$ *has at least one solution.*

To prove this theorem we will need a lemma on a priori estimates of solutions of the differential inequality

$$
\begin{aligned}
&(-1)^{m+n}\left(u^{(2m,2n)}(x,y) - \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)}(x,y) - \sum_{k=0}^{n} h_{2k}u^{(2m,k)}(x,y)\right)\operatorname{sgn} u(x,y)\\
&\quad \le \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}\beta_{ik}|u^{(i,k)}(x,y)| + \gamma
\end{aligned} \tag{4.8}
$$

subject to appropriate boundary conditions, and also lemmas on unique solvability of auxiliary homogeneous boundary value problems. In particular, we consider the auxiliary differential equation

$$
u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} \tag{4.9}
$$

and the auxiliary boundary value problems

$$
v^{(2m)} = \sum_{i=0}^{m} h_{1i}(x)v^{(i)}, \qquad v^{(k)}(0) = v^{(k)}(a) = 0 \quad (k=0,\dots,m-1); \tag{4.10}
$$

$$
v^{(2m)} = \sum_{i=0}^{m} h_{1i}(x)v^{(i)}, \qquad v^{(2k)}(0) = v^{(2k)}(a) = 0 \quad (k=0,\dots,m-1); \tag{4.11}
$$

$$
w^{(2n)} = \sum_{i=0}^{n} h_{2i}(y)w^{(i)}, \qquad w^{(k)}(0) = w^{(k)}(b) = 0 \quad (k=0,\dots,n-1); \tag{4.12}
$$

$$
w^{(2n)} = \sum_{i=0}^{n} h_{2i}(x)w^{(i)}, \qquad w^{(2k)}(0) = w^{(2k)}(b) = 0 \quad (k=0,\dots,n-1). \tag{4.13}
$$

We also make use of the following Wirtinger's lemma [27].

Lemma 4.1. *Let k be a positive integer, k_0 be the integer part of $\frac{k-1}{2}$, $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, +\infty)$. Then an arbitrary function $z \in C^k([t_0, t_1])$ satisfying the boundary conditions*

$$z^{(2j)}(t_0) = z^{(2j)}(t_1) = 0 \quad (j = 0, \dots, k_0)$$

satisfies the inequalities

$$\int_{t_0}^{t_1} |z^{(i)}(t)|^2 \mathrm{d}t \le \left(\frac{t_1 - t_0}{\pi}\right)^{2(k-i)} \int_{t_0}^{t_1} |z^{(k)}(t)|^2 \mathrm{d}t \quad (i = 0, \dots, k-1).$$

This lemma immediately implies

Lemma 4.2. *Let m_0 and n_0 be the integer parts of $\frac{m-1}{2}$ and $\frac{n-1}{2}$ respectively. Then an arbitrary function $u \in C^{m,n}(\Omega)$ satisfying the boundary conditions*

$$u^{(2i,0)}(0, y) = u^{(2i,0)}(a, y) = 0 \quad (i = 0, \dots, m_0),$$
$$u^{(0,2k)}(x, 0) = u^{(0,2k)}(x, b) = 0 \quad (k = 0, \dots, n_0),$$

satisfies the inequalities

$$\|u^{(i,k)}\|_{L^2} \le \left(\frac{a}{\pi}\right)^{m-i} \left(\frac{b}{\pi}\right)^{n-k} \|u^{(m,n)}\|_{L^2} \quad (i = 0, \dots, m;\ k = 0, \dots, n). \tag{4.14}$$

Lemma 4.3. *Let α_{1i} $(i = 0, \dots, m)$, α_{2k} $(k = 0, \dots, n)$ and β_{ik} $(i = 0, \dots, m-1; k = 0, \dots, n-1)$ be nonnegative numbers satisfying condition (4.7). Moreover, let inequalities (4.5) hold on Ω. Then there exists a positive number r such that for any $j \in \{2, 3, 4\}$ and $\gamma \ge 0$ every solution of problem (4.8), $(4.j)$ admits estimate (3.8).*

Proof. Let δ be the number from inequality (4.7) and

$$r = (1 - \delta)^{-1} ab \sum_{i=1}^{m} \sum_{k=1}^{n} \left(\frac{a}{\pi}\right)^{2m-i} \left(\frac{b}{\pi}\right)^{2n-k}. \tag{4.15}$$

For an arbitrary function $u \in C^{2m,2n}(\Omega)$ satisfying condition $(4.j)$ have

$$(-1)^{m+n} \int_0^a \int_0^b u^{(2m,2n)}(x, y) u(x, y) \mathrm{d}x\mathrm{d}y = \int_0^a \int_0^b |u^{(m,n)}(x, y)|^2 \mathrm{d}x\mathrm{d}y, \tag{4.16}$$

$$\begin{aligned} &(-1)^{m+n} \int_0^a \int_0^b h_{1i}(x) u^{(i,2n)}(x, y) u(x, y) \mathrm{d}x\mathrm{d}y = (-1)^m \int_0^a \int_0^b h_{1i}(x) u^{(i,n)}(x, y) u^{(0,n)}(x, y) \mathrm{d}x\mathrm{d}y, \\ &(-1)^{m+n} \int_0^a \int_0^b h_{2k}(y) u^{(2m,k)}(x, y) u(x, y) \mathrm{d}x\mathrm{d}y = (-1)^n \int_0^a \int_0^b h_{2k}(y) u^{(m,k)}(x, y) u^{(m,0)}(x, y) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.17}$$

On the other hand, by Lemma 4.2, the function u satisfies inequalities (4.14).

Multiplying both sides of inequality (4.8) by $|u(x, y)|$, integrating over Ω and utilizing conditions (4.5), (4.16), (4.17) and Schwartz's inequality we obtain

$$\begin{aligned} \|u^{(m,n)}\|_{L^2}^2 \le{} & \sum_{i=0}^{m} \alpha_{1i} \|u^{(i,n)}\|_{L^2} \|u^{(0,n)}\|_{L^2} + \sum_{k=0}^{n} \alpha_{2k} \|u^{(m,k)}\|_{L^2} \|u^{(m,0)}\|_{L^2} \\ & + \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \beta_{ik} \|u^{(i,k)}\|_{L^2} \|u\|_{L^2} + (ab)^{\frac{1}{2}} \gamma \|u\|_{L^2}. \end{aligned}$$

Hence by inequalities (4.7) and (4.14), it follows that

$$\|u^{(m,n)}\|_{L^2} \le \delta \|u^{(m,n)}\|_{L^2} + \left(\frac{a}{\pi}\right)^m \left(\frac{b}{\pi}\right)^n (ab)^{\frac{1}{2}} \gamma$$

and

$$\|u^{(m,n)}\|_{L^2} \le (1-\delta)^{-1}\left(\frac{a}{\pi}\right)^m\left(\frac{b}{\pi}\right)^n (ab)^{\frac{1}{2}}\gamma. \tag{4.18}$$

In view of $(4.j)$ we have

$$\min\{|u^{(i,k)}(x,y)| : 0 \le x \le a\} = 0 \quad \text{for } 0 \le y \le b \ (i = 0,\dots,m-1;\ k = 0,\dots,n),$$
$$\min\{|u^{(i,k)}(x,y)| : 0 \le y \le b\} = 0 \quad \text{for } 0 \le x \le a \ (i = 0,\dots,m;\ k = 0,\dots,n-1).$$

Therefore

$$|u^{(i,k)}(x,y)| \le \int_0^a \int_0^b |u^{(i+1,k+1)}(s,t)|\mathrm{d}s\mathrm{d}t \le (ab)^{\frac{1}{2}}\|u^{(i+1,k+1)}\|_{L^2}$$
$$\text{for } (x,y) \in \Omega \ (i = 0,\dots,m-1; k = 0,\dots,n-1).$$

If along with this we take into account inequalities (4.14), (4.18) and equality (4.15), then validity of estimate (3.8) becomes obvious. □

Lemma 4.4. *Let inequalities (4.5) hold on Ω, where α_{1i} $(i = 0,\dots,m)$ and α_{2k} $(k = 0,\dots,n)$ are nonnegative numbers such that*

$$\sum_{i=0}^{m}\left(\frac{a}{\pi}\right)^{2m-i}\alpha_{1i} + \sum_{k=0}^{n}\left(\frac{b}{\pi}\right)^{2n-k}\alpha_{2k} < 1. \tag{4.19}$$

Then for any $j \in \{2,3,4\}$ problem (4.9), $(4.j)$ has only a trivial solution. Moreover, each of the four problems (4.10)–(4.13) has only a trivial solution.

Proof. Let u be a solution of problem (4.9), $(4.j)$. Then it is a solution of problem (4.8), $(4.j)$ as well, where $\beta_{ik} = \gamma = 0$ $(i = 0,\dots,m-1; k = 0,\dots,n-1)$. Hence Lemma 4.3 and conditions (4.5) and (4.19) imply that $u(x,y) \equiv 0$.

We prove the second part of the lemma for problem (4.10) only, since for problems (4.11)–(4.13) it can be proved similarly. Let v be an arbitrary solution of problem (4.10). Multiplying both sides of the equation under consideration by $(-1)^m v(x)$ and integrating over $[0,a]$, by inequalities (4.5) and Lemma 4.1, we get

$$\int_0^a |v^{(m)}(x)|^2\mathrm{d}x \le \sum_{i=0}^{m}\alpha_{1i}\left(\int_0^a |v^{(i)}(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^a |v(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le \left(\sum_{i=0}^{m}\left(\frac{a}{\pi}\right)^{2m-i}\alpha_{1i}\right)\int_0^a |v^{(m)}(x)|^2\mathrm{d}x.$$

Hence (4.19) and equalities $v^{(i)}(a) = 0$ $(i = 0,\dots,m-1)$ imply that $v(x) \equiv 0$. □

Proof of Theorem 4.1. We prove solvability of problem (4.1), (4.2) only, since solvability of problems (4.1), (4.3) and (4.1), (4.4) can be proved similarly.

By Theorem 1.1 and Lemma 4.3, there exists a linear bounded operator $\mathcal{G} : C(\Omega) \to C^{2m,2n}(\Omega)$ such that for any $f_0 \in C(\Omega)$ a solution of the differential equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} + f_0(x,y)$$

subject to the boundary conditions (4.2) admits representation (1.8).

Let $\|\mathcal{G}\|$ be the norm of the operator $\mathcal{G}$, r be the number from Lemma 4.3 and

$$\varrho_0 = \max\left\{|f(x,t,z_{00},\dots,z_{m-1n-1})| : (x,y) \in \Omega, \sum_{i=0}^{m-1}\sum_{k=0}^{n-1}|z_{ik}| \le r\gamma\right\} \tag{4.20}$$

and

$$\varrho = \varrho_0 \|\mathcal{G}\|. \tag{4.21}$$

By Theorem 2.1 and Lemma 4.4, to prove solvability of problem (4.1), (4.2) it is sufficient to show that for any $\lambda \in (0, 1)$ every solution of the differential equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x) u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y) u^{(2m,k)} + \lambda f(x, y, u, \ldots, u^{(m-1,n-1)}) \tag{4.22}$$

subject to boundary conditions (4.2) admits the estimate

$$\|u\|_{C^{2m-1,2n-1}} \le \varrho. \tag{4.23}$$

According to (4.6) every solution of problem (4.22), (4.2) is also a solution of problem (4.8), (4.2). Hence, by Lemma 4.3 and conditions (4.5) and (4.7), we get estimate (3.8). On the other hand, by Theorem 1.1, every such solution admits the representation

$$u(x, y) = \lambda \mathcal{G}(z)(x, y),$$

where $z(x, y) = f(x, y, u(x, y), \ldots, u^{(m-1,n-1)}(x, y))$. Taking into account (3.8), (4.20) and (4.21), the validity of estimate (4.23) becomes evident. □

Theorem 4.1 and Lemma 4.3 imply

Theorem 4.2. *Let conditions (4.5) hold on Ω and the conditions*

$$(-1)^{m+n}(f(x, y, z_{00}, \ldots, z_{m-1n-1}) - f(x, y, \bar{z}_{00}, \ldots, \bar{z}_{m-1n-1}))\mathrm{sgn}(z_{00} - \bar{z}_{00}) \le \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \beta_{ik}|z_{ik} - \bar{z}_{ik}|$$

hold on $\Omega \times \mathbb{R}^{mn}$, where α_{1i}, α_{2k}, β_{ik} are nonnegative constants satisfying inequality (4.7). Then for any $j \in \{2, 3, 4\}$ problem (4.1), (4.j) has one and only one solution.

Note that Theorems 4.1 and 4.2 cover equations having an arbitrary growth order with respect to phase arguments. Indeed, consider the following examples of differential equations

$$u^{(2m,2n)} = (-1)^{m+n} h(x, y, u, \ldots, u^{(m-1,n-1)})|u|^{\mu(x,y)} \mathrm{sgn}\, u + q(x, y), \tag{4.24}$$

$$u^{(2m,2n)} = (-1)^{m+n} h_0(x, y)|u|^{\mu(x,y)} \mathrm{sgn}\, u + q(x, y), \tag{4.25}$$

where $h : \Omega \times \mathbb{R}^{mn} \to (-\infty, 0]$, $h_0 : \Omega \to (-\infty, 0]$, $\mu : \Omega \to (0, +\infty)$ and $q : \Omega \to \mathbb{R}$ are continuous functions. By Theorems 4.1 and 4.2, for any $j \in \{2, 3, 4\}$ problem (4.24), (4.j) has at least one solution, and problem (4.25), (4.j) has one and only one solution.

In conclusion of this section consider one more example

$$u^{(2m,2n)} = (-1)^{m+n} h \left(\frac{\pi}{a}\right)^{2m} \left(\frac{\pi}{b}\right)^{2n} u + \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{4.26}$$

where h is a constant. If $h < 1$, then by Theorem 4.2, for any $j \in \{2, 3, 4\}$ problem (4.26), (4.j) has one and only one solution. Let us show that if $h = 1$, then problem (4.26), (4.4) has no solutions. Assume the contrary that problem has a solution u. Then by the formula of integration by parts we get

$$\int_0^a\int_0^b u^{(2m,2n)}(x, y) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \mathrm{d}x\mathrm{d}y = (-1)^{m+n}\left(\frac{\pi}{a}\right)^{2m}\left(\frac{\pi}{b}\right)^{2n} \int_0^a\int_0^b u(x, y) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \mathrm{d}x\mathrm{d}y.$$

Therefore multiplying (4.26) by $\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$ and integrating over Ω we get the contradiction

$$\int_0^a\int_0^b \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{b} \mathrm{d}x\mathrm{d}y = 0.$$

Consequently, problem (4.26), (4.4) has no solution.

This example demonstrates that in Theorems 4.1 and 4.2 the strong inequality (4.7) cannot be replaced by an unstrict one.

5. Periodic problem

Consider the differential equation of even order

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} + f(x, y, u, u^{(1,1)}, \ldots, u^{(m-1,n-1)}) \tag{5.1}$$

with the periodic boundary conditions

$$\begin{aligned} &u^{(i,0)}(0, y) = u^{(i,0)}(a, y) \quad (i = 0, \ldots, 2m-1), \\ &u^{(0,k)}(x, 0) = u^{(0,k)}(x, b) \quad (k = 0, \ldots, 2n-1), \end{aligned} \tag{5.2}$$

where $h_{1i} : [0, a] \to \mathbb{R}$ $(i = 0, \ldots, m)$, $h_{2k} : [0, b] \to \mathbb{R}$ $(k = 0, \ldots, n)$ and $f : \Omega \times \mathbb{R}^{1+(m-1)(n-1)}$ are continuous functions.

Note that if either $m = 1$ or $n = 1$, then by (5.1) we understand the equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} + f(x, y, u).$$

Theorem 5.1. *Let the inequalities*

$$\begin{aligned} &(-1)^m h_{10}(x) \le -\alpha_1, \quad |h_{1i}(x)| \le \alpha_{1i} \ (i = 1, \ldots, m), \\ &(-1)^n h_{20}(y) \le -\alpha_2, \quad |h_{2k}(y)| \le \alpha_{2k} \ (k = 1, \ldots, n), \end{aligned} \tag{5.3}$$

$$(-1)^{m+n} f(x, y, z, z_{11}, \ldots, z_{m-1n-1})\,\mathrm{sgn}\, z \le -\beta|z| + \sum_{i=1}^{m-1}\sum_{k=1}^{n-1} \beta_{ik}|z_{ik}| + \gamma, \tag{5.4}$$

hold on Ω and $\Omega \times \mathbb{R}^{1+(m-1)(n-1)}$, respectively, where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, $\alpha_{1i} \ge 0$, $\alpha_{2k} \ge 0$, $\beta_{ik} \ge 0$ and $\gamma \ge 0$ are constants such that

$$\eta = \frac{1}{4\alpha_1}\left(\sum_{i=1}^{m}\left(\frac{a}{2\pi}\right)^{m-i}\alpha_{1i}\right)^2 + \frac{1}{4\alpha_2}\left(\sum_{k=1}^{n}\left(\frac{b}{2\pi}\right)^{n-k}\alpha_{2k}\right)^2 < 1, \tag{5.5}$$

$$\sum_{i=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{a}{2\pi}\right)^{m-i}\left(\frac{b}{2\pi}\right)^{n-k}\beta_{ik} < 2\,(\beta(1-\eta))^{\frac{1}{2}}. \tag{5.6}$$

Then problem (5.1), (5.2) has at least one solution.

To prove Theorem 5.1 we need to study the differential inequality

$$\begin{aligned} &(-1)^{m+n}\left(u^{(2m,2n)}(x, y) - \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)}(x, y) - \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)}(x, y)\right)\mathrm{sgn}\, u(x, y) \\ &\quad \le -\beta|u(x, y)| + \sum_{i=1}^{m-1}\sum_{k=1}^{n-1} \beta_{ik}|u^{(i,k)}(x, y)| + \gamma \end{aligned} \tag{5.7}$$

and the differential equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} - (-1)^{m+n}\beta u \tag{5.8}$$

subject to conditions (5.2); and also the auxiliary problems

$$v^{(2m)} = \sum_{i=0}^{m} h_{1i}(x)v^{(i)}, \qquad v^{(k)}(0) = v^{(k)}(a) \quad (k = 0, \ldots, 2m-1); \tag{5.9}$$

$$w^{(2n)} = \sum_{i=0}^{n} h_{2i}(y) w^{(i)}, \qquad w^{(k)}(0) = w^{(k)}(b) = 0 \quad (k = 0, \dots, 2n-1). \tag{5.10}$$

Note that in Theorem 5.1 and everywhere below it is assumed that if either $m = 1$ or $n = 1$, then

$$\sum_{i=1}^{m-1} \sum_{k=1}^{n-1} \beta_{ik} z_{ik} \equiv 0.$$

Lemma 5.1. *Let $k \geq 2$ be a natural number, $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, +\infty)$. Then an arbitrary function $z \in C^k([t_0, t_1])$ satisfying conditions*

$$z^{(i)}(t_0) = z^{(i)}(t_1) \quad (i = 0, \dots, k-1),$$

satisfies the inequalities

$$\int_{t_0}^{t_1} |z^{(i)}(t)|^2 \mathrm{d}t \leq \left(\frac{t_1 - t_0}{2\pi}\right)^{2(k-i)} \int_{t_0}^{t_1} |z^{(k)}(t)|^2 \mathrm{d}t \quad (i = 1, \dots, k-1).$$

This lemma follows directly from Wirtinger's theorem on periodic functions (see [29]).

Lemma 5.1 itself implies

Lemma 5.2. *Let $u \in C^{m,n}(\Omega)$ and*

$$u^{(i,0)}(0, y) = u^{(i,0)}(a, y), \qquad u^{(0,k)}(x, 0) = u^{(0,k)}(x, b) \quad \textit{for } (x, y) \in \Omega$$
$$(i = 0, \dots, m-1;\ k = 0, \dots, n-1).$$

Then

$$\|u^{(i,k)}\|_{L^2} \leq \left(\frac{a}{2\pi}\right)^{m-i} \left(\frac{b}{2\pi}\right)^{n-k} \|u^{(m,n)}\|_{L^2} \quad (i = 1, \dots, m;\ k = 1, \dots, n). \tag{5.11}$$

Lemma 5.3. *If $u \in C^{1,1}(\Omega)$, then*

$$\|u\|_C \leq (ab)^{-\frac{1}{2}} \|u\|_{L^2} + \left(\frac{b}{a}\right)^{\frac{1}{2}} \|u^{(0,1)}\|_{L^2} + \left(\frac{a}{b}\right)^{\frac{1}{2}} \|u^{(1,0)}\|_{L^2} + 2(ab)^{\frac{1}{2}} \|u^{(1,1)}\|_{L^2}.$$

Proof. Set

$$v(x) = \int_0^b |u^{(0,1)}(x, t)| \mathrm{d}t, \qquad w(y) = \int_0^a |u^{(1,0)}(s, y)| \mathrm{d}s$$

and choose points $(x_0, y_0) \in \Omega$, $x_1 \in [0, a]$ and $y_1 \in [0, b]$ in such a way that

$$|u(x_0, y_0)| = \min\{|u(x, y)| : (x, y) \in \Omega\}, \qquad v(x_1) = \min\{v(x) : 0 \leq x \leq a\},$$
$$w(y_1) = \min\{w(y) : 0 \leq y \leq b\}.$$

Then

$$|u(x_0, y_0)| \leq (ab)^{-\frac{1}{2}} \|u\|_{L^2},$$
$$v(x_1) \leq a^{-1} \int_0^a \int_0^b |u^{(0,1)}(s, t)| \mathrm{d}s \mathrm{d}t \leq \left(\frac{b}{a}\right)^{\frac{1}{2}} \|u^{(0,1)}\|_{L^2},$$
$$w(y_1) \leq b^{-1} \int_0^a \int_0^b |u^{(1,0)}(s, t)| \mathrm{d}s \mathrm{d}t \leq \left(\frac{a}{b}\right)^{\frac{1}{2}} \|u^{(1,0)}\|_{L^2}.$$

On the other hand

$$v(x) \le v(x_1) + \int_0^b |u^{(0,1)}(x,t) - u^{(0,1)}(x_1,t)|\mathrm{d}t$$
$$\le v(x_1) + \int_0^a \int_0^b |u^{(1,1)}(s,t)|\mathrm{d}s\mathrm{d}t \le \left(\frac{b}{a}\right)^{\frac{1}{2}} \|u^{(0,1)}\|_{L^2} + (ab)^{\frac{1}{2}} \|u^{(1,1)}\|_{L^2},$$
$$|w(y)| \le \left(\frac{a}{b}\right)^{\frac{1}{2}} \|u^{(1,0)}\|_{L^2} + (ab)^{\frac{1}{2}} \|u^{(1,1)}\|_{L^2},$$
$$|u(x,y)| \le |u(x_0,y_0)| + |u(x,y) - u(x,y_0)| + |u(x,y_0) - u(x_0,y_0)|$$
$$\le |u(x_0,y_0)| + v(x) + w(y_0) \le (ab)^{-\frac{1}{2}} \|u\|_{L^2} + \left(\frac{b}{a}\right)^{\frac{1}{2}} \|u^{(0,1)}\|_{L^2}$$
$$+ \left(\frac{a}{b}\right)^{\frac{1}{2}} \|u^{(1,0)}\|_{L^2} + 2(ab)^{\frac{1}{2}} \|u^{(1,1)}\|_{L^2}. \qquad \square$$

Lemma 5.4. *Let $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, $\alpha_{1i} \ge 0$, $\alpha_{2k} \ge 0$ and $\beta_{ik} \ge 0$ be constants satisfying inequalities (5.5) and (5.6). Moreover, let inequality (5.3) hold on Ω. Then there exists a positive number r such that for any $\gamma \ge 0$ every solution of problem (5.7), (5.2) admits estimate (3.8).*

Proof. According to (5.6) there exists a number $\delta \in (0,1)$ such that

$$\sum_{i=1}^{m-1} \sum_{k=1}^{n-1} \left(\frac{a}{2\pi}\right)^{m-i} \left(\frac{b}{2\pi}\right)^{n-k} \beta_{ik} < 2(1-\delta)\,(\beta(1-\eta))^{\frac{1}{2}}\,. \tag{5.12}$$

Set

$$r_0 = \left(\frac{ab}{2\delta^2\beta(1-\eta)}\right)^{\frac{1}{2}} \sum_{i=1}^{m-1} \sum_{k=1}^{n-1} \left(\frac{a}{2\pi}\right)^{m-i} \left(\frac{b}{2\pi}\right)^{n-k} + \frac{(ab)^{\frac{1}{2}}}{\delta\beta}, \tag{5.13}$$

$$r = (ab)^{-\frac{1}{2}}(1 + a + b + ab)r_0. \tag{5.14}$$

In view of (5.2) equalities (4.16) and (4.17) hold. On the other hand, Lemma 5.2 implies inequalities (5.11). Multiplying both sides of inequality (5.7) by $|u(x,y)|$, integrating over Ω and taking into account (4.16), (4.17), (5.11), (5.12) and Schwartz's inequality, we obtain

$$\|u^{(m,n)}\|_{L^2}^2 \le -\alpha_1 \|u^{(0,n)}\|_{L^2}^2 + \sum_{i=1}^{m} \alpha_{1i} \|u^{(i,n)}\|_{L^2} \|u^{(0,n)}\|_{L^2} - \alpha_2 \|u^{(m,0)}\|_{L^2}^2$$
$$+ \sum_{k=1}^{n} \alpha_{2k} \|u^{(m,k)}\|_{L^2} \|u^{(m,0)}\|_{L^2} + \sum_{i=1}^{m} \sum_{k=1}^{n} \beta_{ik} \|u^{(i,k)}\|_{L^2} \|u\|_{L^2} + (ab)^{\frac{1}{2}} \gamma \|u\|_{L^2}$$
$$\le -\alpha_1 \|u^{(0,n)}\|_{L^2}^2 + \left(\sum_{i=1}^{m} \left(\frac{a}{2\pi}\right)^{m-i} \alpha_{1i}\right) \|u^{(m,n)}\|_{L^2} \|u^{(0,n)}\|_{L^2}$$
$$- \alpha_2 \|u^{(m,0)}\|_{L^2}^2 + \left(\sum_{k=1}^{n} \left(\frac{b}{2\pi}\right)^{n-k} \alpha_{2k}\right) \|u^{(m,n)}\|_{L^2} \|u^{(m,0)}\|_{L^2}$$
$$+ 2(1-\delta)\,(\beta(1-\eta))^{\frac{1}{2}} \|u^{(m,n)}\|_{L^2} \|u\|_{L^2} + (ab)^{\frac{1}{2}} \gamma \|u\|_{L^2}. \tag{5.15}$$

However,

$$\left(\sum_{i=1}^{m} \left(\frac{a}{2\pi}\right)^{m-i} \alpha_{1i}\right) \|u^{(m,n)}\|_{L^2} \|u^{(0,n)}\|_{L^2} \le \alpha_1 \|u^{(0,n)}\|_{L^2}^2 + \frac{1}{4\alpha_1} \left(\sum_{i=1}^{m} \left(\frac{a}{2\pi}\right)^{m-i} \alpha_{1i}\right)^2 \|u^{(m,n)}\|_{L^2}^2,$$
$$\left(\sum_{k=1}^{n} \left(\frac{b}{2\pi}\right)^{n-k} \alpha_{2k}\right) \|u^{(m,n)}\|_{L^2} \|u^{(m,0)}\|_{L^2} \le \alpha_2 \|u^{(m,0)}\|_{L^2}^2 + \frac{1}{4\alpha_2} \left(\sum_{k=1}^{n} \left(\frac{b}{2\pi}\right)^{n-k} \alpha_{2k}\right)^2 \|u^{(m,n)}\|_{L^2}^2,$$

$$2(1-\delta)\,(\beta(1-\eta))^{\frac{1}{2}}\,\|u^{(m,n)}\|_{L^2}\|u\|_{L^2} \le (1-\delta)(1-\eta)\|u^{(m,n)}\|_{L^2}^2 + (1-\delta)\beta\|u\|_{L^2}^2,$$

$$(ab)^{\frac{1}{2}}\gamma\|u\|_{L^2} \le \frac{\delta\beta}{2}\|u\|_{L^2}^2 + \frac{ab}{2\delta\beta}\gamma^2.$$

If along with this we take into account condition (5.5), then from (5.15) we get

$$\delta(1-\eta)\|u^{(m,n)}\|_{L^2}^2 + \frac{\delta\beta}{2}\|u\|_{L^2}^2 \le \frac{ab}{2\delta\beta}\gamma^2.$$

Hence (5.11) and (5.13) imply the inequality

$$\sum_{i=0}^{m}\sum_{k=0}^{n}\|u^{(i,k)}\|_{L^2} \le r_0\gamma.$$

By Lemma 5.3, estimate (3.8) directly follows from the latter inequality and (5.14). □

Lemma 5.5. *Let conditions (5.3) hold on Ω, where $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_{1i} \ge 0$, $\alpha_{2k} \ge 0$ are constants satisfying inequality (5.5). Then problems (5.9) and (5.10) have only trivial solutions. Moreover, for any $\beta > 0$ problem (5.8), (5.2) has only a trivial solution.*

Proof. Let u be a solution of problem (5.8), (5.2). Then it is a solution of problem (5.7), (5.2) as well, where $\beta_{ik} = 0$ $(i = 1, \dots, m-1;\, k = 1, \dots, n-1)$ and $\gamma = 0$. Hence Lemma 5.4 and conditions (5.3) and (5.5) imply that $u(x, y) \equiv 0$.

Now consider problem (5.9). According to (5.5) there exists $\delta \in (0, \alpha_1)$ such that

$$\left(\sum_{i=1}^{m}\left(\frac{a}{2\pi}\right)\alpha_{1i}\right)^2 < 4(\alpha_1-\delta). \tag{5.16}$$

Let v be an arbitrary solution of problem (5.9). Multiplying both sides of the corresponding differential equation by $(-1)^m v(x)$, integrating over $[0, a]$, and taking into account conditions (5.3) we get

$$\int_0^a |v^{(m)}(x)|^2\mathrm{d}x \le -\alpha_1\int_0^a |v(x)|^2\mathrm{d}x + \sum_{i=1}^{m}\alpha_{1i}\left(\int_0^a |v^{(i)}(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^a |v(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.17}$$

On the other hand, by Lemma 5.1 and inequality (5.16), we have

$$\sum_{i=1}^{m}\alpha_{1i}\left(\int_0^a |v^{(i)}(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^a |v(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \le \left(\sum_{i=0}^{m}\left(\frac{a}{2\pi}\right)^{m-i}\alpha_{1i}\right)\left(\int_0^a |v^{(m)}(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^a |v(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le \frac{1}{4(\alpha_1-\delta)}\left(\sum_{i=0}^{m}\left(\frac{a}{2\pi}\right)^{m-i}\alpha_{1i}\right)^2\int_0^a |v^{(m)}(x)|^2\mathrm{d}x + (\alpha_1-\delta)\int_0^a |v(x)|^2\mathrm{d}x$$
$$\le \int_0^a |v^{(m)}(x)|^2\mathrm{d}x + (\alpha_1-\delta)\int_0^a |v(x)|^2\mathrm{d}x.$$

Therefore (5.17) yields

$$\delta\int_0^a |v(x)|^2\mathrm{d}x \le 0,$$

and, consequently, $v(x) \equiv 0$.

Similarly one can prove that problem (5.10) has only a trivial solution. □

Proof of Theorem 5.1. By Theorem 1.1 and Lemma 5.5, there exists a linear bounded operator $\mathcal{G} : C(\Omega) \to C^{2m,2n}(\Omega)$ such that for any $f_0 \in C(\Omega)$ a solution of the differential equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} - (-1)^{m+n}\beta u + f_0(x, y)$$

subject to the boundary conditions (5.2) admits the representation (1.8).

Let $\|\mathcal{G}\|$ be the norm of the operator $\mathcal{G}$, r be the number from Lemma 5.4 and

$$\varrho_0 = \max\left\{|f(x,t,z,z_{11},\ldots,z_{m-1n-1})| : (x,y)\in\Omega,\ \sum_{i=1}^{m}\sum_{k=1}^{n}|z_{ik}|\le r\gamma\right\} + \beta r\gamma, \qquad \varrho = \varrho_0\|\mathcal{G}\|. \tag{5.18}$$

By Theorem 2.1 and Lemma 5.5, to prove the theorem it is sufficient to show that for any $\lambda\in(0,1)$ every solution of the differential equation

$$u^{(2m,2n)} = \sum_{i=0}^{m} h_{1i}(x)u^{(i,2n)} + \sum_{k=0}^{n} h_{2k}(y)u^{(2m,k)} - (-1)^{m+n}(1-\lambda)\beta u + \lambda f(x,y,u,u^{(1,1)},\ldots,u^{(m-1,n-1)}) \tag{5.19}$$

subject to boundary conditions (5.2) admits estimate (4.23).

According to (5.4) an arbitrary solution u of problem (5.19), (5.2) is also a solution of problem (5.7), (5.2). Hence, by Lemma 5.4 and conditions (5.3), (5.5) and (5.6) we get estimate (3.8). On the other hand, by Theorem 1.1, u admits the representation

$$u(x,y) = \lambda\mathcal{G}(z)(x,y),$$

where

$$z(x,y) = f(x,y,u(x,y),u^{(1,1)}(x,y),\ldots,u^{(m-1,n-1)}(x,y)) - (-1)^{m+n}\beta u(x,y).$$

Hence (3.8) and (5.18) imply (4.23). □

Theorem 5.2. *Let conditions (5.3) hold on Ω, and the inequalities*

$$(-1)^{m+n}\left(f(x,y,z,z_{11},\ldots,z_{m-1n-1}) - f(x,y,\overline{z},\overline{z}_{11},\ldots,\overline{z}_{m-1n-1})\right)\operatorname{sgn}(z-\overline{z}) \le -\beta|z-\overline{z}| + \sum_{i=1}^{m-1}\sum_{k=1}^{n-1}\beta_{ik}|z_{ik}-\overline{z}_{ik}|,$$

hold on $\Omega\times\mathbb{R}^{1+(m-1)(n-1)}$, where $\alpha_1>0$, $\alpha_2>0$, $\beta>0$, $\alpha_{1i}\ge 0$, $\alpha_{2k}\ge 0$, $\beta_{ik}\ge 0$ are constants satisfying inequalities (5.5) and (5.6). Then problem (5.1), (5.2) has one and only one solution.

Consider the following examples of differential equations

$$u^{(2m,2n)} = -(-1)^m\left(\frac{2\pi}{a}\right)^{2m}\delta_0 u^{(0,2n)} - (-1)^n\left(\frac{2\pi}{b}\right)^{2n}\delta_0 u^{(2m,0)} + (-1)^{m+i}2\left(\frac{2\pi}{a}\right)^{2m-2i}\delta_0 u^{(2i,2n)} + (-1)^{n+k}2\left(\frac{2\pi}{b}\right)^{2n-2k}\delta_0 u^{(2m,2k)} - (-1)^{m+n}\left(\frac{2\pi}{a}\right)^{2m}\left(\frac{2\pi}{b}\right)^{2n}\delta u + \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}, \tag{5.20}$$

$$u^{(2m,2n)} = -(-1)^m\left(\frac{2\pi}{a}\right)^{2m}\delta u^{(0,2n)} - (-1)^n\left(\frac{2\pi}{b}\right)^{2n}\delta u^{(2m,0)} - (-1)^{m+n}\left(\frac{2\pi}{a}\right)^{2m}\left(\frac{2\pi}{b}\right)^{2n} u + (-1)^{m+n+i+k}\left(\frac{2\pi}{a}\right)^{2m-2i}\left(\frac{2\pi}{b}\right)^{2n-2k}\delta_0 u^{(2i,2k)} + \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}, \tag{5.21}$$

where $m\ge 2i\ge 1$, $n\ge 2k\ge 1$, $\delta_0>0$, $\delta\ge 0$. If $\delta_0<\frac{1}{2}$ ($\delta_0<2$), then by Theorem 5.2, problem (5.20), (5.2) (problem (5.21), (5.2)) has one and only one solution for arbitrary $\delta>0$. If $\delta_0>\frac{1}{2}$ and $\delta=2\delta_0-1$ ($\delta_0>2$ and $\delta=\delta_0-2$), then problem (5.20), (5.2) (problem (5.21), (5.2)) has no solution at all. These examples demonstrate that in Theorem 5.1 (Theorem 5.2) in the righthand side of inequality (5.5) (inequality (5.6)) the constant 1 (constant $2(\beta(1-\eta))^{\frac{1}{2}}$) cannot be replaced by $1+\varepsilon$ (by $2(\beta(1-\eta))^{\frac{1}{2}}+\varepsilon$) however small $\varepsilon>0$ may be.

The equation

$$u^{(2m,2n)} = h_1(x)u^{(0,2n)} + h_2(y)u^{(2m,0)} + f(x,y,u) \tag{5.22}$$

is a particular case of Eq. (5.1), where $h_1 : [0, a] \to \mathbb{R}$, $h_2 : [0, b] \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Besides,

$$(-1)^m h_1(x) < 0, \qquad (-1)^n h_2(y) < 0, \tag{5.23}$$

and the function f satisfies either of the conditions

$$(-1)^{m+n} f(x, y, z) \operatorname{sgn} z \le -\beta|z| + \gamma \tag{5.24}$$

and

$$(-1)^{m+n} \left(f(x, y, z) - f(x, y, \overline{z})\right) \operatorname{sgn}(z - \overline{z}) \le -\beta|z - \overline{z}|. \tag{5.25}$$

Theorems 5.1 and 5.2 imply

Corollary 5.1. *Let inequality (5.23) hold on Ω, and let condition (5.24) (condition (5.25)) hold on $\Omega \times \mathbb{R}$, where $\beta > 0$, $\gamma \ge 0$. Then problem (5.22), (5.2) has at least one (one and only one) solution.*

It is obvious that the function

$$f(x, y, z) = (-1)^{m+n+1} h_0(x, y) \exp(z^2) z + h(x, y),$$

where h_0 and $h : \Omega \to \mathbb{R}$ are continuous functions and $h_0(x, y) \ge \beta > 0$ for $(x, y) \in \Omega$, satisfies condition (5.25). This example demonstrates that Theorems 5.1 and 5.2 cover equations with righthand sides having an arbitrary growth order with respect to phase variables.

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