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On Nonlinear Boundary-Value Problems for Higher Order Ordinary Differential Equations

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Abstract. Sufficient conditions are established for the solvability and unique solvability of nonlinear boundary-value problems of the type $u^{(n)} = f(t, u, \dots, u^{(n-1)})$, $\sum\limits_{k=1}^{n} \big(\alpha_{ik}(u)u^{(k-1)}(a) + \beta_{ik}(u)u^{(k-1)}(b)\big) = \gamma_i(u)$ $(i = 1, \dots, n)$, where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a function from the Carathéodory class, and α_{ik}, $\beta_{ik} : \mathbb{C}^{n-1} \to \mathbb{R}$ $(i, k = 1, \dots, n)$ are nonlinear continuous functionals.

1 Statement of the Problem and Formulation of the Main Results

We investigate the nonlinear differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1.1}$$

with the nonlinear boundary conditions

$$\sum_{k=1}^{n} \big(\alpha_{ik}(u)u^{(k-1)}(a) + \beta_{ik}(u)u^{(k-1)}(b)\big) = \gamma_i(u) \quad (i = 1, \dots, n). \tag{1.2}$$

Throughout the paper, we assume that $-\infty < a < b < +\infty$, $\mathbb{C}^{n-1}$ is the space of $n-1$ times continuously differentiable functions $u : [a, b] \to \mathbb{R}$ with the norm $\|u\|_{\mathbb{C}^{n-1}} = \max\Big\{ \sum\limits_{k=1}^{n} |u^{(k-1)}(t)| : \quad a \le t \le b \Big\}$, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a function, satisfying the local Carathéodory conditions, $\alpha_{ik} : \mathbb{C}^{n-1} \to \mathbb{R}$, $\beta_{ik} : \mathbb{C}^{n-1} \to \mathbb{R}$ $(i, k = 1, \dots, n)$ are functionals, continuous and bounded on every bounded set of the space $\mathbb{C}^{n-1}$, and $\gamma_i : \mathbb{C}^{n-1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous functionals such that

$$\sup\big\{|\gamma_i(v)| : \quad v \in \mathbb{C}^{n-1}\big\} < +\infty \quad (i = 1, \dots, n). \tag{1.3}$$

By a solution of (1.1) we mean the function $u \in \mathbb{C}^{n-1}$ having absolutely continuous $(n-1)$th derivative and almost everywhere on $[a,b]$ satisfying (1.1).

A solution of (1.1) satisfying the conditions (1.2) is called *a solution of the problem* (1.1), (1.2).

Set

$$\nu_n(x_1,\dots,x_n;y_1,\dots,y_n) = \begin{cases} \displaystyle\sum_{k=1}^{m}(-1)^k(x_{n-k+1}x_k - y_{n-k+1}y_k) & \text{for } n=2m, \\ \displaystyle\sum_{k=1}^{m}(-1)^k(x_{n-k+1}x_k - y_{n-k+1}y_k) - \\ \qquad\qquad -\dfrac{(-1)^m}{2}\,(x_{m+1}^2 - y_{m+1}^2) & \text{for } n=2m+1. \end{cases} \tag{1.4}$$

Below we will consider the case when there exist numbers $j \in \{1,2\}$ and $\mu > 0$ such that for any $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$ $(i=1,\dots,n)$ and $v \in \mathbb{C}^{n-1}$ the functionals α_{ik}, β_{ik} $(i,k=1,\dots,n)$ satisfy the inequalities

$$(-1)^{m+j}\nu_n(x_1,\dots,x_n,y_1,\dots,y_n) \le \mu\sum_{k=1}^{n}\big(|x_k|+|y_k|\big)\sum_{i=1}^{n}\Big|\sum_{k=1}^{n}\big(\alpha_{ik}(v)x_k+\beta_{ik}(v)y_k\big)\Big|. \tag{1.5_j}$$

As for the function f, on the set $[a,b]\times\mathbb{R}^n$ it satisfies the condition

$$p(t)h(|x_1|) - q(t) \le (-1)^{m+j}f(t,x_1,\dots,x_n)\operatorname{sgn} x_1 \le p^*\big(t,|x_1|\big), \tag{1.6_j}$$

where p and $q : [a,b] \to [0,+\infty[$ are integrable functions, $h : [0,+\infty[\to[0,+\infty[$ is a nondecreasing function, and $p^* : [a,b]\times[0,+\infty[\to[0,+\infty[$ is an integrable in the first argument and nondecreasing in the second argument function. Moreover,

$$\int_a^b p(t)\,dt > 0, \quad \lim_{x\to+\infty} h(x) = +\infty. \tag{1.7}$$

For $n=2m$, the problems

$$\begin{gathered} \alpha_i(u)u^{(i-1)}(a) + \alpha_{m+i}(u)u^{(n-i)}(a) = \gamma_i(u), \\ \beta_i(u)u^{(i-1)}(b) + \beta_{m+i}(u)u^{(n-i)}(b) \quad (i=1,\dots,m), \end{gathered} \tag{1.8}$$

$$u^{(i-1)}(a) = \eta_i(u)u^{(i-1)}(b) + \gamma_i(u), \quad u^{(n-i)}(a) = \frac{u^{(n-i)}(b)}{\eta_i(u)} + \gamma_{m+i}(u) \quad (i=1,\dots,m) \tag{1.9}$$

are considered separately.

For $n=2m+1$, to the boundary conditions (1.8) (to the boundary conditions (1.9)) we add one of the following two conditions:

$$u^{(m)}(a) = \eta(u)u^{(m)}(b) + \gamma_n(u) \tag{1.10_1}$$

or

$$u^{(m)}(b) = \eta(u)u^{(m)}(a) + \gamma_n(u). \tag{1.10$_2$}$$

Here $\alpha_i : \mathbb{C}^{n-1} \to \mathbb{R}$, $\beta_i : \mathbb{C}^{n-1} \to \mathbb{R}$ $(i = 1, \dots, 2m)$, $\eta_i : \mathbb{C}^{n-1} \to \mathbb{R}$ $(i = 1, \dots, m)$, and $\eta : \mathbb{C}^{n-1} \to \mathbb{R}$ are continuous and bounded on every bounded set of the space $\mathbb{C}^{n-1}$ functionals such that

$$\begin{gathered}(-1)^{m+i+j}\alpha_i(v)\alpha_{m+i}(v) \ge 0, \quad (-1)^{m+i+j}\beta_i(v)\beta_{m+i}(v) \le 0, \\ \inf\big\{|\alpha_i(v)| + |\alpha_{m+i}(v)| : \ v \in \mathbb{C}^{n-1}\big\} > 0, \\ \inf\big\{|\beta_i(v)| + |\beta_{m+i}(v)| : \ v \in \mathbb{C}^{n-1}\big\} > 0 \ (i = 1, \dots, m);\end{gathered} \tag{1.11$_j$}$$

$$\inf\big\{|\eta_i(v)| : \ v \in \mathbb{C}^{n-1}\big\} > 0 \quad (i = 1, \dots, m), \tag{1.12}$$

$$|\eta(v)| \le 1 \tag{1.13}$$

for any $v \in \mathbb{C}^{n-1}$.

The class of boundary conditions under consideration involves the well-known boundary conditions

$$u^{(i-1)}(b) = u^{(i-1)}(a) + c_i \quad (i = 1, \dots, n); \tag{1.14}$$

$$\begin{aligned} u^{(n-i)}(b) &= c_{1i} \ \ (i = 1, \dots, m + j - 1), \\ u^{(n-i)}(a) &= c_{2i} \ \ (i = 1, \dots, n - m - j + 1), \end{aligned} \tag{1.15$_j$}$$

$$\begin{aligned} u^{(i-1)}(a) &= c_{1i} \ \ (i = 1, \dots, m + j - 1), \\ u^{(i-1)}(b) &= c_{2i} \ \ (i = 1, \dots, n - m - j + 1), \end{aligned} \tag{1.16$_j$}$$

$$\begin{aligned} u^{(i-1)}(b) &= c_{1i} \ \ (i = 1, \dots, m + j - 1), \\ u^{(n-i)}(a) &= c_{2i} \ \ (i = 1, \dots, n - m - j + 1), \end{aligned} \tag{1.17$_j$}$$

where c_i, c_{1i} and $c_{2i} \in \mathbb{R}$. A vast literature is devoted to the problems (1.1), (1.14); (1.1), (1.15$_j$); (1.1), (1.16$_j$) and (1.1), (1.17$_j$) (see, e.g., [1–13, 15–20] and the references therein), but the problem (1.1), (1.2) in the general case remains still studied insufficiently. The present paper is devoted to fill this gap.

Theorem 1.1. *Let $n = 2m$, $j = 1$ ($n = 2m + 1$, $j \in \{1, 2\}$) and let the conditions* (1.3), (1.5$_j$), (1.6$_j$) *and* (1.7) *be fulfilled. Then the problem* (1.1), (1.2) *has at least one solution.*

Corollary 1.2. *Let $n = 2m$, and let the conditions* (1.3), (1.6$_1$), (1.7) *and* (1.11$_1$) (*the conditions* (1.3), (1.6$_1$), (1.7) *and* (1.12)) *be fulfilled. Then the problem* (1.1), (1.8) (*the problem* (1.1), (1.9)) *has at least one solution.*

Corollary 1.3. *Let $n = 2m + 1$, $j \in \{1, 2\}$, and let the conditions* (1.3), (1.6$_j$), (1.7), (1.11$_j$) *and* (1.13) (*the conditions* (1.3), (1.6$_j$), (1.7), (1.12) *and* (1.13)) *be fulfilled. Then the problem* (1.1), (1.8), (1.10$_j$) (*the problem* (1.1), (1.9), (1.10$_j$)) *has at least one solution.*

Corollary 1.4. *Let $n = 2m$, $j = 1$ ($n = 2m+1$, $j \in \{1, 2\}$) and let the conditions* (1.6$_j$) *and* (1.7) *be fulfilled. Then every of the problems problems* (1.1), (1.14); (1.1), (1.15$_j$); (1.1), (1.16$_j$) *and* (1.1), (1.17$_j$) *has at least one solution.*

We will now proceed to considering the case when the right part of (1.1) does not contain intermediate derivatives, and the functionals α_{ik}, β_{ik}, γ_i, α_i, β_i, η_i and η are constant, i.e. when (1.1) and the above-mentioned boundary conditions have, respectively, the form

$$u^{(n)} = f(t,u), \tag{1.18}$$

$$\sum_{k=1}^{n} \big(\alpha_{ik}u^{(i-1)}(a) + \beta_{ik}u^{(i-1)}(b)\big) = \gamma_i \ \ (i=1,\dots,n); \tag{1.19}$$

$$\begin{aligned} \alpha_i u^{(i-1)}(a) + \alpha_{m+i}u^{(n-i)}(a) = \gamma_i, \ \ \beta_i u^{(i-1)}(b) + \beta_{m+i}u^{(n-i)}(b) = \\ = \gamma_{m+i} \ \ (i=1,\dots,m); \end{aligned} \tag{1.20}$$

$$u^{(i-1)}(a) = \eta_i u^{(i-1)}(b) + \gamma_i, \quad u^{(n-i)}(a) = \frac{u^{(n-i)}(b)}{\eta_i} + \gamma_{m+i} \ \ (i=1,\dots,m); \tag{1.21}$$

$$u^{(m)}(a) = \eta u^{(m)}(b) + \gamma_n; \tag{1.22$_1$}$$

$$u^{(m)}(b) = \eta u^{(m)}(a) + \gamma_n. \tag{1.22$_2$}$$

As for the inequalities (1.5$_j$) and (1.11$_j$), they take the form

$$(-1)^{m+j}\nu_n(x_1,\dots,x_n;y_1,\dots,y_n) \le \mu \sum_{k=1}^{n}\big(|x_k|+|y_k|\big) \sum_{i=1}^{n}\Big|\sum_{k=1}^{n}(\alpha_{ik}x_k + \beta_{ik}y_k)\Big|; \tag{1.23$_j$}$$

$$\begin{gathered} (-1)^{m+i+j}\alpha_i\alpha_{m+i} \ge 0, \ \ (-1)^{m+i+j}\beta_i\beta_{m+i} \le 0, \\ |\alpha_i| + |\alpha_{m+i}| > 0, \ \ |\beta_i| + |\beta_{m+i}| > 0 \ \ (i=1,\dots,m). \end{gathered} \tag{1.24$_j$}$$

Just as above, we assume that $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is the function from the Carathéodory class, satisfying on $[a,b]\times\mathbb{R}$ the inequality

$$(-1)^{m+j} f(t,x)\operatorname{sgn} x \ge p(t)h(|x|) - q(t), \tag{1.25$_j$}$$

where p and $q : [a,b] \to [0,+\infty[$ are integrable, and $h : [0,+\infty[\to [0,+\infty[$ is a nondecreasing function. Moreover,

$$(-1)^{m+j}\big(f(t,x) - f(t,y)\big) > 0 \ \text{ for } \ x > y. \tag{1.26$_j$}$$

Theorem 1.5. *Let $n = 2m$, $j = 1$ ($n = 2m+1$, $j \in \{1,2\}$) and let the conditions* (1.23$_j$), (1.25$_j$), (1.26$_j$) *and* (1.7) *be fulfilled. Then the problem* (1.18), (1.19) *has one and only one solution.*

Corollary 1.6. *Let $n = 2m$, and let the conditions* (1.25$_1$), (1.26$_1$) *and* (1.7) *be fulfilled. If, moreover, the inequalities* (1.24$_1$) (*the inequalities $\eta_i \ne 0$ ($i = 1,\dots,m$)*) *hold, then the problem* (1.18), (1.20) (*the problem* (1.18), (1.21)) *has one and only one solution.*

Corollary 1.7. *Let $n = 2m+1$, $j \in \{1,2\}$, and let the conditions* (1.25$_j$), (1.26$_j$) *and* (1.7) *be fulfilled. If, moreover, $|\eta| \le 1$ and the inequalities* (1.24$_j$) (*the inequalities $\eta_i \ne 0$ ($i = 1,\dots,m$)*) *hold, then the problem* (1.18), (1.20), (1.22$_j$) (*the problem* (1.18), (1.21), (1.22$_j$)) *has one and only one solution.*

Corollary 1.8. *Let $n = 2m$, $j = 1$ ($n = 2m+1$, $j \in \{1, 2\}$) and let the conditions* (1.25_j), (1.26_j) *and* (1.7) *be fulfilled. Then every of the problems* $(1.18), (1.14)$; $(1.18), (1.15_j)$; $(1.18), (1.16_j)$ *and* $(1.18), (1.17_j)$) *has one and only one solution.*

As an example, let us consider the differential equation

$$u^{(n)} = g_0(t) f_0(u) + g(t), \tag{1.27}$$

where g_0 and $g : [a, b] \to \mathbb{R}$ are integrable and $f_0 : \mathbb{R} \to \mathbb{R}$ is a continuous, increasing function. By Corollary 1.8, if $n = 2m$, $j = 1$ ($n = 2m + 1$, $j \in \{1, 2\}$),

$$\begin{gathered} (-1)^{m+j} g_0(t) > 0 \;\text{ for }\; a < t < b, \\ \lim_{x \to -\infty} f_0(x) = -\infty, \quad \lim_{x \to +\infty} f_0(x) = +\infty, \end{gathered} \tag{1.28}$$

then each of the problems $(1.27), (1.14)$; $(1.27), (1.15_j)$; $(1.27), (1.16_j)$ and $(1.27), (1.17_j)$ has one and only one solution. On the other hand, it is clear that if

$$|f_0(x)| \le \ell \;\text{ for }\; x \in \mathbb{R} \;\text{ and }\; g(t) > \ell |g_0(t)| \;\text{ for }\; a < t < b, \tag{1.29}$$

then just as the problem $(1.27), (1.14)$, the problem $(1.27), (1.15_j)$ has no solution.

The above example shows that the restriction (1.7) in Theorems 1.1, 1.5 and in their corollaries is in some sense optimal and cannot be weakened.

2 Auxiliary Propositions

2.1 Lemmas on a priori estimates

Consider the system of differential inequalities:

$$(-1)^{m+j} u^{(n)}(t) \operatorname{sgn} u(t) \ge p(t) h\big(|u(t)|\big) - q(t), \tag{2.1_j}$$

$$|u^{(n)}(t)| \le p^*\big(t, |u(t)|\big) \tag{2.2}$$

with the boundary condition

$$(-1)^{m+j} \nu_n\big(u(a), \dots, u^{(n-1)}(a); u(b), \dots, u^{(n-1)}(b)\big) \le \mu_0 \|u\|. \tag{2.3_j}$$

Here $n = 2m$, $j = 1$ ($n = 2m + 1$, $j \in \{1, 2\}$), $\mu_0 \ge 0$, p and $q : [a, b] \to [0, +\infty[$ are integrable functions, $p^* : [a, b] \times [0, +\infty[\to [0, +\infty[$ is a function, integrable in the first and nondecreasing in the second argument, and ν_n is a function given by the equality (1.4).

By a solution of the problem $(2.1_j), (2.2), (2.3_j)$ we mean the function $u \in \mathbb{C}^{n-1}$ having absolutely continuous $(n-1)$th derivative and satisfying both the system of differential inequalities (2.1_j), (2.2) almost everywhere on $[a, b]$ and the condition (2.3_j).

Lemma 2.1. *If the condition* (1.7) *holds, then there exists a positive constant r such that an arbitrary solution u of the problem* $(2.1_j), (2.2), (2.3_j)$ *admits the estimate*

$$\|u\| \le r. \tag{2.4}$$

Proof. By virtue of (1.7), there exist numbers $\delta \in]0,1[\,$, $a_k \in [a,b[\,$, $b_k \in]a_k,b]$ $(k = 1,\dots,n)$, and $r_1 > 0$ such that

$$a_{k+1} - b_k > \delta \quad (k = 1,\dots,n-1), \tag{2.5}$$

$$h(r_1)\int_{a_k}^{b_k} p(t)\,dt > \varepsilon \quad (k = 1,\dots,n), \tag{2.6}$$

where

$$\varepsilon = \delta^{n-1}(1+b-a)^{1-n}\big(2(n+2)!(1+\mu_1)\big)^{-1} \tag{2.7}$$

and

$$\mu_1 = \mu_0 + 2\int_a^b q(t)\,dt.$$

Suppose

$$r_2 = \frac{2(1+\mu_1)r_1}{\varepsilon}, \quad r = \frac{2r_1\Big(1+\int_a^b p^*(t,r_2)\,dt\Big)}{\varepsilon}. \tag{2.8}$$

Let u be a solution of the problem $(2.1_j), (2.2), (2.3_j)$. Then almost everywhere on $[a,b]$ the inequality

$$\eta(t) \overset{def}{=} (-1)^{m+j}u^{(n)}(t)u(t) - p(t)h\big(|u(t)|\big)|u(t)| + q(t)|u(t)| \geq 0 \tag{2.9}$$

is satisfied.

On the other hand, according to (1.4), we have

$$\int_a^b u^{(n)}(t)u(t)\,dt(-1)^m\sigma_n\int_a^b |u^{(m)}(t)|^2\,dt + \nu_n\big(u(a),\dots,u^{(n-1)}(a);u(b),\dots,u^{(n-1)}(b)\big), \tag{2.10}$$

where $\sigma_n = 1$ for $n = 2m$ and $\sigma_n = 0$ for $n = 2m+1$. Therefore,

$$\begin{aligned}
&\int_a^b |u^{(n)}(t)u(t)|\,dt \leq \int_a^b \big(\eta(t) + p(t)h(|u(t)|)|u(t)|\big)\,dt + \int_a^b q(t)|u(t)|\,dt,\\
&\int_a^b p(t)h(|u(t)|)|u(t)|\,dt \leq \int_a^b \big(\eta(t) + p(t)h(|u(t)|)|u(t)|\big)\,dt\\
&\qquad = (-1)^{m+j}\int_a^b u^{(n)}(t)u(t)\,dt + \int_a^b q(t)|u(t)|\,dt\\
&\qquad \leq (-1)^{m+j}\nu\big(u(a),\dots,u^{(n-1)}(a);u(b),\dots,u^{(n-1)}(b)\big) + \|u\|\int_a^b q(t)\,dt.
\end{aligned} \tag{2.11}$$

Taking now into account the inequality (2.3_j), we can see that

$$\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt \le \mu_1\|u\|, \tag{2.12}$$

$$\int_a^b \big|u^{(n)}(t)u(t)\big|\,dt \le \mu_1\|u\|. \tag{2.13}$$

For every $k \in \{1,\dots,n\}$, we choose $t_k \in [a_k,b_k]$ so that

$$|u(t_k)| = \min\big\{|u(t)| :\ a_k \le t \le b_k\big\}. \tag{2.14}$$

If $|u(t_k)| \ge r_1$, then by (2.6) we have

$$\int_{a_k}^{b_k} p(t)h\big(|u(t)|\big)|u(t)|\,dt \ge |u(t_k)|h(r_1)\int_{a_k}^{b_k} p(t)\,dt > \frac{|u(t_k)|}{\varepsilon}\,. \tag{2.15}$$

Consequently,

$$|u(t_k)| < r_1 + \varepsilon\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt \quad (k=1,\dots,n). \tag{2.16}$$

On the other hand, it follows from (2.5) that

$$t_{k+1} - t_k > \delta \quad (k=1,\dots,n-1). \tag{2.17}$$

Therefore,

$$\begin{aligned}\min\big\{|u^{(i-1)}|(t)| :\ a \le t \le b\big\} &\le i!\delta^{1-i}\max\big\{|u(t_k)| :\ k=1,\dots,n\big\}\\ &< i!\delta^{1-i}\Big(r_1 + \varepsilon\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt\Big) \quad (i=1,\dots,n),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\|u\| < (n+2)!(1+b-a)^{n-1}\delta^{1-n}\Big(r_1 + \varepsilon\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt\Big)\\ + n(1+b-a)^{n-1}\int_a^b |u^{(n)}(t)|\,dt.\end{aligned} \tag{2.19}$$

With regard for (2.7) and (2.19), from (2.12) we find

$$\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt < \frac{r_1}{2\varepsilon} + \frac{1}{2}\int_a^b p(t)h(|u(t)|)|u(t)|\,dt + \frac{r_1}{2\varepsilon}\int_a^b |u^{(n)}(t)|\,dt \tag{2.20}$$

and consequently,

$$\int_a^b p(t)h\big(|u(t)|\big)|u(t)|\,dt < \frac{\Big(1+\int_a^b |u^{(n)}(t)|\,dt\Big)r_1}{\varepsilon}\,. \tag{2.21}$$

By virtue of the above estimate and the equality (2.7), from (2.13) and (2.19) we obtain

$$\|u\| < \frac{\Big(1+\int_a^b |u^{(n)}(t)|\,dt\Big)r_1}{\varepsilon}\,, \tag{2.22}$$

$$\int_a^b \big|u^{(n)}(t)u(t)\big|\,dt \le \frac{\Big(1+\int_a^b |u^{(n)}(t)|\,dt\Big)\mu_1 r_1}{\varepsilon}\,. \tag{2.23}$$

Let

$$I_1 = \big\{t\in[a,b]:\ |u(t)|\le r_2\big\}, \qquad I_2 = \big\{t\in[a,b]:\ |u(t)|> r_2\big\}. \tag{2.24}$$

Then by means of (2.2) and (2.23) we get

$$\begin{aligned}\int_a^b |u^{(n)}(t)|\,dt &= \int_{I_1} |u^{(n)}(t)|\,dt + \int_{I_2} |u^{(n)}(t)|\,dt\\ &\le \int_{I_1} p^*(t,r_2)\,dt + \frac{1}{r_2}\int_{I_1} \big|u^{(n)}(t)u(t)\big|\,dt\\ &< \int_a^b p^*(t,r_2)\,dt + \frac12 + \frac12\int_a^b |u^{(n)}(t)|\,dt,\end{aligned} \tag{2.25}$$

and consequently,

$$\int_a^b |u^{(n)}(t)|\,dt < 1 + 2\int_a^b p^*(t,r_2)\,dt. \tag{2.26}$$

According to the latter inequality, from (2.22) follows the estimate (2.4), where r is the positive, independent of u constant given by the equalities (2.8). □

Let $n = 2m$, $j = 1$ ($n = 2m+1$, $j \in \{1,2\}$), and let $p : [a,b] \to [0,+\infty[$ be an integrable function, different from zero on the set of positive measure. For arbitrary $c_i \in \mathbb{R}$ $(i = 1,\dots,n)$, $v \in \mathbb{C}^{n-1}$ and integrable function $g : [a,b] \to \mathbb{R}$, consider the linear boundary-value problem

$$u^{(n)} = (-1)^{m+j}p(t)u + g(t), \tag{2.27}$$

$$\sum_{k=1}^{n}\Big(\alpha_{ik}(v)u^{(k-1)}(a) + \beta_{ik}(v)u^{(k-1)}(b)\Big) = c_i \quad (i = 1,\dots,n). \tag{2.28}$$

Analogously to Lemma 2.1 we can prove

Lemma 2.2. *Let the condition* (1.5_j) *be fulfilled, where* μ *is an independent of* x_k, y_k $(k = 1, \dots, n)$ *and* v *constant. Then there exists an independent of* c_i $(i = 1, \dots, n)$, v *and* g *positive constant* r_0 *such that an arbitrary solution* u *of the problem* (2.27), (2.28) *admits the estimate*

$$\|u\| \le r_0\bigg(\sum_{i=1}^{n} |c_i| + \int_a^b |g(t)|\, dt\bigg). \tag{2.29}$$

2.2 Lemma on the solvability of the problem (1.1), (1.2)

From Theorem 1 of [14] and Lemma 2.2 follows

Lemma 2.3. *Let* $n = 2m$, $j = 1$ $(n = 2m+1,\ j \in \{1, 2\})$, *and let* $p : [a, b] \to [0, +\infty[$ *be an integrable function, different from zero on the set of positive measure. Let, moreover, the condition* (1.5_j) *be fulfilled and there exist a positive constant* r *such that for every* $\lambda \in]0, 1[$ *an arbitrary solution* u *of the boundary-value problem*

$$u^{(n)} = (-1)^{m+j}(1 - \lambda)p(t)u + \lambda f\big(t, u, \dots, u^{(n-1)}\big), \tag{2.30}$$

$$\sum_{k=1}^{n} \big(\alpha_{ik}(u)u^{(k-1)}(a) + \beta_{ik}(u)u^{(k-1)}(b)\big) = \lambda\gamma_i(u) \quad (i = 1, \dots, n) \tag{2.31}$$

admits the estimate (2.4)*. Then the problem* (1.1), (1.2) *has at least one solution.*

3 Proof of the Main Results

Proof of Theorem 1.1. By the condition (1.6_j), without loss of generality, we can assume that on $[a, b] \times \mathbb{R}^n$ the inequalities

$$h(|x_1|) \le |x_1|, \qquad \big|f(t, x_1, \dots, x_n)\big| + p(t)|x_1| \le p^*\big(t, |x_1|\big) \tag{3.1}$$

are satisfied. On the other hand, by (1.3) we have

$$\mu_0 = 2\mu \sup\bigg\{\sum_{i=1}^{n} |\gamma_i(v)| : \ v \in \mathbb{C}^{n-1}\bigg\} < +\infty. \tag{3.2}$$

Let $\lambda \in]0, 1[$, and let u be an arbitrary solution of the problem (2.30), (2.31). Then by virtue of the conditions (1.5_j), (1.6_j), (3.1) and (3.2), the function u is likewise the solution of the problem (2.1_j), (2.2), (2.3_j). From the above reasoning, by Lemma 2.1 we obtain the estimate (2.4), where r is the positive constant, independent of u and λ. Using now Lemma 2.3, it is not difficult to see that Theorem 1.1 is valid. □

Proof of Theorem 1.5. By Theorem 1.1, the conditions (1.23_j), (1.25_j) and (1.7) guarantee the solvability of the problem (1.18), (1.19). Therefore it remains to prove that this problem does not have more than one solution. Assume the contrary that the problem (1.18), (1.19) has two different solutions u_1 and u_2. Suppose

$$u(t) = u_2(t) - u_1(t), \qquad g(t) = (-1)^{m+j}\big(f(t, u_2(t)) - f(t, u_1(t))\big)u(t). \tag{3.3}$$

Then

$$g(t) = (-1)^{m+j} u^{(n)}(t) u(t), \tag{3.4}$$

$$\sum_{k=1}^{m} \big(\alpha_{ik} u^{(k-1)}(a) + \beta_{ik} u^{(k-1)}(b)\big) = 0 \quad (i = 1, \dots, n). \tag{3.5}$$

Integrating both parts of the identity (3.4) from a to b, by virtue of the conditions (1.23_j), (1.26_j) and (3.5) we find that

$$0 < \int_a^b g(t)\, dt = (-1)^{m+j} \nu_n\big(u(a), \dots, u^{(n-1)}(a); u(b), \dots, u^{(n-1)}(b)\big) \leq 0. \tag{3.6}$$

The obtained contradiction proves the theorem. □

Proof of Corollary 1.2. We choose a number $\delta \in]0, 1[$ such that for arbitrary $v \in \mathbb{C}^{n-1}$ and $k \in \{1, \dots, m\}$ the inequalities

$$|\alpha_k(v)| + |\alpha_{m+k}(v)| \geq 2\delta, \qquad |\beta_k(v)| + |\beta_{m+k}(v)| \geq 2\delta \qquad \big(\, |\eta_k(v)| \geq \delta\, \big) \tag{3.7}$$

are satisfied.

For arbitrarily fixed $x_k \in \mathbb{R}$, $y_k \in \mathbb{R}$ $(k = 1, \dots, n)$, and $v \in \mathbb{C}^{n-1}$ we put

$$\begin{gathered} \alpha_k(v) x_k + \alpha_{m+k}(v) x_{n-k+1} = z_k, \quad \beta_k(v) y_k + \beta_{m+k}(v) y_{n-k+1} = \\ = z_{m+k} \quad (k = 1, \dots, m), \\ \left(x_k - \eta_k(v) y_k = z_k, \qquad x_{n-k+1} - \frac{y_{n-k+1}}{\eta_k(v)} = z_{m+k} \quad (k = 1, \dots, m) \right). \end{gathered} \tag{3.8}$$

Then by virtue of the conditions (1.11_1) and (3.7) we have

$$\begin{aligned} &(-1)^{m+1+k} (x_{n-k+1} x_k - y_{n-k+1} y_k) \\ &\qquad \leq \delta^{-1} \big(|x_k| + |x_{n-k+1}| + |y_k| + |y_{n-k+1}|\big) \big(|z_k| + |z_{m+k}|\big) \\ &\left(|x_{n-k+1} x_k - y_{n-k+1} y_k| \leq (1 + \delta^{-1}) \big(|x_k| + |y_{n-k+1}|\big) \big(|z_k| + |z_{m+k}|\big) \right). \end{aligned} \tag{3.9}$$

Hence with regard for the notation (1.4) we find

$$(-1)^{m+1+k} \nu(x_1, \dots, x_n; y_1, \dots, y_n) \leq \mu \sum_{k=1}^{n} \big(|x_k| + |y_k|\big) \sum_{i=1}^{n} |z_i|, \tag{3.10}$$

where $\mu = 1 + \delta^{-1}$ is the constant, independent of x_k, y_k $(k = 1, \dots, n)$ and v.

Applying now Theorem 1.1, the validity of Corollary 1.2 becomes obvious. □

Corollaries 1.3, 1.6 and 1.7 can be proved analogously.

Corollary 1.4 follows directly from Corollaries 1.2 and 1.3, while Corollary 1.8 follows from Corollaries 1.6 and 1.7.

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