ISSN 0012-2661, Differential Equations, 2006, Vol. 42, No. 2, pp. 165–171. © Pleiades Publishing, Inc., 2006.
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ORDINARY DIFFERENTIAL EQUATIONS

On the Unique Solvability of a Periodic Boundary Value Problem for Third-Order Linear Differential Equations

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Received June 1, 2005

DOI: 10.1134/S0012266106020029

We study the periodic boundary value problem

$$u''' = p_1(t)u + p_2(t)u' + p_3(t)u'' + q(t), \tag{1}$$

$$u^{(i-1)}(b) = u^{(i-1)}(a) + c_i \qquad (i = 1, 2, 3), \tag{2}$$

where $-\infty < a < b < +\infty$, the c_i $(i = 1, 2, 3)$ are real constants, and the $p_i : [a, b] \to \mathbb{R}$ $(i = 1, 2, 3)$ and $q : [a, b] \to \mathbb{R}$ are Lebesgue integrable functions.

By $\tilde{C}$ we denote the space of absolutely continuous functions $x : [a, b] \to \mathbb{R}$, and by $\tilde{C}^1$ we denote the space of functions $x : [a, b] \to \mathbb{R}$ absolutely continuous together with their first derivatives. We write $x(t) \not\equiv y(t)$ if functions x and y differ on a set of a positive measure.

In what follows, we consider the cases in which there exists a number $\sigma \in \{-1, 1\}$ such that

$$\sigma p_1(t) \ge 0 \quad \text{for} \quad a \le t \le b, \qquad p_1(t) \not\equiv 0, \tag{3}$$

and one of the following four conditions is satisfied:

$$p_2 \in \tilde{C}, \quad p_3 \in \tilde{C}^1, \quad \sigma\,(p_2(b) - p_2(a)) \ge 0, \quad p_3(b) = p_3(a), \quad \sigma\,(p_3'(b) - p_3'(a)) \le 0, \tag{4_1}$$

$$p_1 \in \tilde{C}, \quad p_3 \in \tilde{C}^1, \quad p_1(b) \ge p_1(a), \quad p_3(b) = p_3(a), \quad \sigma\,(p_3'(b) - p_3'(a)) \le 0, \tag{4_2}$$

$$p_1 \in \tilde{C}^1, \quad p_2 \in \tilde{C}, \quad p_1(b) = p_1(a), \quad \sigma\,(p_1'(b) - p_1'(a)) \le 0, \quad \sigma\,(p_2(b) - p_2(a)) \ge 0, \tag{4_3}$$

$$p_1 \in \tilde{C}^1, \quad p_2 \in \tilde{C}, \quad p_1(b) = p_1(a), \quad \sigma\,(p_1'(b) - p_1'(a)) \ge 0, \quad \sigma\,(p_2(b) - p_2(a)) \le 0. \tag{4_4}$$

In these cases, we find earlier unknown (see [1–14] and the bibliography therein) and, in a sense, sharp criteria for the unique solvability of problem (1), (2).

Along with problem (1), (2), consider the corresponding homogeneous problem

$$u''' = p_1(t)u + p_2(t)u' + p_3(t)u'', \tag{1_0}$$

$$u^{(i-1)}(b) = u^{(i-1)}(a) \qquad (i = 1, 2, 3). \tag{2_0}$$

Suppose that this problem has a nontrivial solution u. If we consecutively multiply both sides of Eq. (1_0) by $\sigma u(t)$, $\sigma u''(t)$, and $-u'(t)$ and integrate from a to b, then, in view of (3), we obtain

$$\int_a^b |p_1(t)|\, u^2(t)dt + \sigma \int_a^b p_2(t)u'(t)u(t)dt + \sigma \int_a^b p_3(t)u''(t)u(t)dt = 0, \tag{5}$$

$$\sigma \int_a^b p_3(t)u''^2(t)dt + \sigma \int_a^b p_2(t)u'(t)u''(t)dt + \int_a^b |p_1(t)|\, u(t)u''(t)dt = 0, \tag{6}$$

$$\int_a^b u''^2(t)dt + \int_a^b p_1(t)u(t)u'(t)dt + \int_a^b p_2(t)u'^2(t)dt + \int_a^b p_3(t)u''(t)u'(t)dt = 0. \tag{7}$$

On the other hand, if $i \in \{1,2,3\}$, $j \in \{1,2\}$, and $p_i \in \tilde{C}$, then

$$\int_a^b p_i(t)u^{(j-1)}(t)u^{(j)}(t)dt = \frac{1}{2}\left(p_i(b) - p_i(a)\right)\left[u^{(j-1)}(a)\right]^2 - \frac{1}{2}\int_a^b p_i'(t)\left[u^{(j-1)}(t)\right]^2 dt.$$

If $i \in \{1,3\}$, $p_i \in \tilde{C}^1$, and $p_i(b) = p_i(a)$, then

$$\int_a^b p_i(t)u(t)u''(t)dt = \frac{1}{2}\left(p_i'(a) - p_i'(b)\right)u'^2(a) + \frac{1}{2}\int_a^b p_i''(t)u^2(t)dt - \int_a^b p_i(t)u'^2(t)dt.$$

Therefore, if condition (4_1) is satisfied, then, from (5) and (6), we obtain

$$\int_a^b \left(|p_1(t)| - \frac{\sigma}{2}p_2'(t) + \frac{\sigma}{2}p_3''(t)\right)u^2(t)dt \le \sigma\int_a^b p_3(t)u'^2(t)dt, \tag{8_1}$$

$$\sigma\int_a^b p_3(t)u''^2(t)dt \le \frac{\sigma}{2}\int_a^b p_2'(t)u'^2(t)dt - \int_a^b |p_1(t)|\,u(t)u''(t)dt, \tag{9_1}$$

and if condition (4_2) is satisfied, then relations (5) and (7) imply that

$$\int_a^b \left(|p_1(t)| + \frac{\sigma}{2}p_3''(t)\right)u^2(t)dt \le -\sigma\int_a^b p_2(t)u'(t)u(t)dt + \sigma\int_a^b p_3(t)u'^2(t)dt, \tag{8_2}$$

$$\int_a^b u''^2(t)dt \le \frac{1}{2}\int_a^b p_1'(t)u^2(t)dt - \int_a^b \left[p_2(t) - \frac{1}{2}p_3'(t)\right]u'^2(t)dt. \tag{9_2}$$

Likewise, if condition (4_3) is satisfied, then it follows from (5) and (6) that

$$\int_a^b \left(|p_1(t)| - \frac{\sigma}{2}p_2'(t)\right)u^2(t)dt \le -\sigma\int_a^b p_3(t)u''(t)u(t)dt, \tag{8_3}$$

$$\sigma\int_a^b p_3(t)u''^2(t)dt \le \int_a^b \left(|p_1(t)| + \frac{\sigma}{2}p_2'(t)\right)u'^2(t)dt - \frac{\sigma}{2}\int_a^b p_1''(t)u^2(t)dt. \tag{9_3}$$

If condition (4_4) holds, then from (6), we obtain

$$\int_a^b \left(|p_1(t)| + \frac{\sigma}{2}p_2'(t)\right)u'^2(t)dt - \frac{\sigma}{2}\int_a^b p_1''(t)u^2(t)dt - \sigma\int_a^b p_3(t)u''^2(t)dt \le 0. \tag{9_4}$$

Now let us show that

$$\int_a^b u''^2(t)dt > 0. \tag{10}$$

Indeed, otherwise we would have $u(t) \equiv c_0 = \text{const} \ne 0$ by condition (2_0) and hence $p_1(t)c_0 \equiv 0$. But this contradicts condition (3).

We have thereby proved the following assertion.

Lemma 1. *Let* p_1 *satisfy condition* (3), *and let problem* (1_0), (2_0) *have a nontrivial solution* u. *If, in addition, condition* (4_k) *is satisfied for some* $k \in \{1,2,3\}$, *then* u *satisfies inequalities* (8_k), (9_k), *and* (10). *If condition* (4_4) *holds, then* u *satisfies inequalities* (9_4) *and* (10).

We introduce the notation

$$d = \frac{b-a}{2\pi}$$

to be used throughout the following.

Theorem 1. *Let conditions* (3) *and* (4_1) *be satisfied. In addition, suppose that either*

$$\begin{aligned} &\sigma\left(p_2'(t) - p_3''(t)\right) \le 2\left|p_1(t)\right|, \\ &\sigma p_3(t) \le 0 \quad \text{for} \quad a < t < b, \\ &p_2'(t) - p_3''(t) \not\equiv 2p_1(t) \end{aligned} \tag{11}$$

or there exist constants $\delta \in \,]0,1]$, $\ell_1 > 0$, $\ell_2 \ge 0$, $\ell_3 > 0$, *and* $\ell \in \,]0,\ell_3]$ *such that*

$$\sigma\left(p_2'(t) - p_3''(t)\right) \le 2(1-\delta)\left|p_1(t)\right|, \qquad \left|p_1(t)\right| < \ell_1 \quad \text{for} \quad a < t < b, \tag{12}$$

$$\sigma p_2'(t) \le 2\ell_2, \qquad \ell \le \sigma p_3(t) \le \ell_3 \quad \text{for} \quad a < t < b, \tag{13}$$

$$d\left(\ell_1\ell_3/\delta\right)^{1/2} + d^2\ell_2 \le \ell. \tag{14}$$

Then problem (1), (2) *has exactly one solution.*

Proof. Suppose the contrary. Then the homogeneous problem (1_0), (2_0) has a nontrivial solution u, which satisfies inequalities (8_1), (9_1), and (10) by Lemma 1.

If, along with (3) and (4_1), condition (11) is satisfied, then inequality (8_1) leads to a contradiction:

$$0 < \int_a^b \left(\left|p_1(t)\right| - \frac{\sigma}{2}p_2'(t) + \frac{\sigma}{2}p_3''(t)\right) u^2(t)dt \le 0.$$

Let us proceed to the case in which, along with (3) and (4_1), conditions (12)–(14) are satisfied. Then from (8_1), we obtain the inequality

$$\delta \int_a^b \left|p_1(t)\right| u^2(t)dt \le \ell_3 \int_a^b u'^2(t)dt.$$

This, together with the Wirtinger theorem [15, Th. 258], implies that

$$\int_a^b \left|p_1(t)\right| u^2(t)dt \le \frac{\ell_3}{\delta} d^2 \int_a^b u''^2(t)dt. \tag{15}$$

If, along with (10) and (12)–(15), we use the Schwartz and Wirtinger inequalities, then from (9_1), we obtain

$$\begin{aligned} \ell \int_a^b u''^2(t)dt &\le \ell_2 \int_a^b u'^2(t)dt + \left(\int_a^b p_1^2(t)u^2(t)dt\right)^{1/2} \left(\int_a^b u''^2(t)dt\right)^{1/2} \\ &< \ell_2 \int_a^b u'^2(t)dt + \ell_1^{1/2} \left(\int_a^b \left|p_1(t)\right| u^2(t)dt\right)^{1/2} \left(\int_a^b u''^2(t)dt\right)^{1/2} \\ &\le \left[d^2\ell_2 + d\left(\frac{\ell_1\ell_3}{\delta}\right)^{1/2}\right] \int_a^b u''^2(t)dt \le \ell \int_a^b u''^2(t)dt. \end{aligned}$$

The resulting contradiction completes the proof of the theorem.

If $p_i(t) \equiv p_i = \text{const}$ $(i = 2, 3)$, i.e., Eq. (1) has the form

$$u''' = p_1(t)u + p_2 u' + p_3 u'' + q(t), \tag{1_1}$$

then Theorem 1 implies the following assertion.

Corollary 1. *Let condition* (3) *be satisfied. In addition, suppose that either* $\sigma p_3 \le 0$ *or*

$$\sigma p_3 > 0, \qquad |p_1(t)| < d^{-2}|p_3| \quad \textit{for} \quad a < t < b. \tag{16}$$

Then problem (1_1), (2) *has exactly one solution.*

Remark 1. If

$$p_1(t) \equiv d^{-2}p_3, \qquad p_2(t) \equiv -d^{-2}, \qquad p_3(t) \equiv p_3 \ne 0, \tag{17}$$

then conditions (3), (4_1), (13), and (14) are satisfied, where $\sigma = \operatorname{sgn} p_3$, $\delta = 1$, $\ell_1 = d^{-2}|p_3|$, $\ell_2 = 0$, and $\ell = \ell_3 = |p_3|$, and, instead of (12) and (16), we have

$$\sigma\left(p_2'(t) - p_3''(t)\right) \le 2(1-\delta)|p_1(t)|, \qquad |p_1(t)| \le \ell_1 \quad \text{for} \quad a < t < b, \tag{12$'$}$$

$$\sigma p_3 > 0, \qquad |p_1(t)| \le d^{-2}p_3 \quad \text{for} \quad a < t < b, \tag{16$'$}$$

respectively. Nevertheless, the homogeneous problem (1_0), (2_0) has the nontrivial solution

$$u(t) = \sin\frac{2\pi(t-a)}{b-a}.$$

Consequently, condition (12) [respectively, (16)] in Theorem 1 (respectively, Corollary 1) is sharp in the sense that it cannot be replaced by condition ($12'$) [respectively, ($16'$)].

Theorem 2. *Let conditions* (3) *and* (4_2) *be satisfied. In addition, suppose that there exist constants* $\delta \in]0, 1]$ *and* $\ell_i \ge 0$ $(i = 1, 2, 3)$, $\ell \ge 0$, *such that*

$$p_1'(t) \le 2\ell_1 |p_1(t)|, \qquad 2p_2(t) - p_3'(t) > -2\ell \quad \textit{for} \quad a < t < b, \tag{18}$$

$$\ell_1 p_2^2(t) \le \ell_2 |p_1(t)|, \qquad \sigma\ell_1 p_3(t) \le \ell_3, \qquad \ell p_3''(t) \ge -2(1-\delta)|p_1(t)| \quad \textit{for} \quad a < t < b, \tag{19}$$

$$\ell + \left(\delta^{-1}\ell_2^{1/2} + \delta^{-1/2}\ell_3^{1/2}\right)^2 \le d^{-2}. \tag{20}$$

Then problem (1), (2) *has exactly one solution.*

Proof. Suppose the contrary. Then problem (1_0), (2_0) has a nontrivial solution u, which satisfies inequalities (8_2), (9_2), and (10) by Lemma 1.

By virtue of condition (19) and the Schwartz inequality, it follows from (8_2) that

$$\begin{aligned}\ell_1 \int_a^b |p_1(t)|\,u^2(t)dt &\le \delta^{-1}\ell_2^{1/2}\ell_1^{1/2}\int_a^b |p_1(t)|^{1/2}|u(t)||u'(t)|\,dt + \delta^{-1}\ell_3\int_a^b u'^2(t)dt \\ &\le \delta^{-1}\ell_2^{1/2}\left(\ell_1\int_a^b |p_1(t)|\,u^2(t)dt\right)^{1/2}\left(\int_a^b u'^2(t)dt\right)^{1/2} + \delta^{-1}\ell_3\int_a^b u'^2(t)dt,\end{aligned}$$

and consequently,

$$\ell_1\int_a^b |p_1(t)|\,u^2(t)dt \le \left(\delta^{-1}\ell_2^{1/2} + \delta^{-1/2}\ell_3^{1/2}\right)^2\int_a^b u'^2(t)dt.$$

If, along with the last inequality, we use conditions (10), (18), and (20) and apply the Wirtinger theorem, then from (9_2), we obtain

$$\int_a^b u''^2(t)dt < \ell_1 \int_a^b |p_1(t)|\, u^2(t)dt + \ell \int_a^b u'^2(t)dt$$
$$\le \left[\ell + \left(\delta^{-1}\ell_2^{1/2} + \delta^{-1/2}\ell_3^{1/2}\right)^2\right] \int_a^b u'^2(t)dt \le \int_a^b u''^2(t)dt.$$

The resulting contradiction proves the theorem.

This implies the following assertion for the differential equation

$$u''' = p_1 u + p_2(t)u' + p_3 u + q(t), \qquad (1_2)$$

where p_1 and p_3 are constants.

Corollary 2. *If* $p_1 \neq 0$ *and*

$$p_2(t) > -d^{-2} \quad \textit{for} \quad a < t < b, \qquad (21)$$

then problem (1_2), (2) *has exactly one solution.*

Remark 2. If condition (17) is satisfied, then conditions (3), (4_2), (19), and (20) are valid, where $\sigma = \operatorname{sgn} p_3$, $\delta = 1$, $\ell_1 = \ell_2 = \ell_3 = 0$, and $\ell = d^{-2}$, and, instead of (18) and (21), we have

$$p_1'(t) \le 2\ell_1 |p_1(t)|, \qquad 2p_2(t) - p_3'(t) \ge -\ell \quad \text{for} \quad a < t < b, \qquad (18')$$
$$p_2(t) \ge -\ell \quad \text{for} \quad a < t < b, \qquad (21')$$

respectively. On the other hand, in this case, the homogeneous problem (1_0), (2_0) has the nontrivial solution $u(t) = \sin(2\pi(t-a)/(b-a))$. Consequently, condition (18) [respectively, (21)] in Theorem 2 [respectively, Corollary 2)] is sharp in the sense that it cannot be replaced by condition (18′) [respectively, (21′)].

Theorem 3. *Let conditions* (3) *and* (4_3) *be satisfied. In addition, suppose that there exist constants* $\delta \in\,]0, 1]$ *and* $\ell_i \ge 0$ $(i = 1, 2, 3)$, $\ell \ge 0$, *such that*

$$\sigma p_1''(t) \le \ell_1 |p_1(t)|, \qquad |p_1(t)| + \frac{\sigma}{2} p_2'(t) \le \ell_2 \quad \textit{for} \quad a < t < b, \qquad (22)$$
$$\sigma p_2'(t) \ge 2(1-\delta)\,|p_1(t)|, \qquad \ell_1 p_3^2(t) \le \ell_3 |p_1(t)| \quad \textit{for} \quad a < t < b, \qquad (23)$$
$$\sigma p_3(t) > \ell \quad \textit{for} \quad a < t < b, \qquad (24)$$
$$d^2\ell_2 + \delta^{-2}\ell_3 \le \ell. \qquad (25)$$

Then problem (1), (2) *has exactly one solution.*

Proof. Suppose the contrary. Then the homogeneous problem (1_0), (2_0) has a nontrivial solution u, which satisfies inequalities (8_3), (9_3), and (10) by Lemma 1.

By condition (23) and the Schwartz inequality, from (8_3), we obtain the inequality

$$\ell_1^{1/2} \int_a^b |p_1(t)|\, u^2(t)dt \le \delta^{-1}\ell_3^{1/2} \left(\int_a^b |p_1(t)|\, u^2(t)dt\right)^{1/2} \left(\int_a^b u''^2(t)dt\right)^{1/2}.$$

Therefore,

$$\ell_1 \int_a^b |p_1(t)|\, u^2(t)dt \le \delta^{-2}\ell_3 \int_a^b u''^2(t)dt.$$

If, along with this inequality, we use conditions (10), (22), (24), and (25) and apply the Wirtinger theorem, then from (9_3), we obtain

$$\ell \int_a^b u''^2(t)dt < \ell_2 \int_a^b u'^2(t)dt + \ell_1 \int_a^b |p_1(t)|\, u^2(t)dt$$

$$\leq \left[d^2\ell_2 + \delta^{-2}\ell_3\right] \int_a^b u''^2(t)dt \leq \ell \int_a^b u''^2(t)dt.$$

The resulting contradiction proves the theorem.

The following assertion can be proved by analogy with the preceding theorem.

Theorem 4. *Let conditions* (3) *and* (4_4) *be satisfied, and let*

$$\sigma p_1''(t) \leq 0, \qquad |p_1(t)| + \frac{\sigma}{2} p_2'(t) > 0, \qquad \sigma p_3(t) \leq 0 \quad \text{for} \quad a < t < b.$$

Then problem (1), (2) *has exactly one solution.*

Theorems 3 and 4 imply the following assertion for the differential equation

$$u''' = p_1 u + p_2 u' + p_3(t)u, \tag{1_3}$$

where p_1 and p_2 are constants.

Corollary 3. *Let* $p_1 \neq 0$, *and let either*

$$p_1 p_3(t) \leq 0 \quad \text{for} \quad a < t < b,$$

or

$$p_3(t) \operatorname{sgn} p_1 > d^2 |p_1| \quad \text{for} \quad a < t < b. \tag{26}$$

Then problem (1_3), (2) *has exactly one solution.*

Remark 3. As was mentioned above, if condition (17) is satisfied, then the homogeneous problem (1_0), (2_0) has a nontrivial solution. This implies that condition (24) [respectively, condition (26)] in Theorem 3 (respectively, in Corollary 3) cannot be replaced by the condition

$$\sigma p_3(t) \geq \ell \quad \text{for} \quad a < t < b \qquad \big(p_3(t) \operatorname{sgn} p_1 \geq d^2 |p_1| \quad \text{for} \quad a < t < b\big).$$

ACKNOWLEDGMENTS

The work was supported by INTAS project no. 03-51-5007.

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