FUNCTIONAL DIFFERENTIAL EQUATIONS

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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS [*]

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Dedicated to Professor A.D. Myshkis
on the occasion of his jubilee

Abstract. For the functional differential equation

$$u''(t) = f(u)(t)$$

with the continuous operator $f : C^1_{loc}(]a, b[) \to L_{loc}(]a, b[)$ the unimprovable, in a certain sense, sufficient conditions for the solvability and unique solvability of the two-point boundary value problems

$$u(a+) = 0, \quad u(b-) = 0$$

and

$$u(a+) = 0, \quad u'(b-) = 0$$

are established. These conditions cover the case when for an arbitrary $u \in C^1_{loc}(]a, b[)$ the function $f(u)(\cdot) :]a, b[\to R$ is not integrable on $[a, b]$, having singularities at the points a and b.

Key Words. second order singular functional differential equation, two-point boundary value problems, solvability, unique solvability, stability

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§ 1. Statement of the Basic Results.

1.1. Formulation of the problem, main notation and definitions. On a finite interval $]a,b[$ we consider the functional differential equation

$$u''(t) = f(u)(t), \tag{1.1}$$

where f is the operator acting from the space $C^1_{loc}(]a,b[)$ to the space $L_{loc}(]a,b[)$. We are mainly interested in the case when f is a singular operator, i.e. when $f(u)(\cdot) \notin L([a,b])$ for an arbitrary $u \in C^1_{loc}(]a,b[)$. A simple, but important particular case of (1.1) is the differential equation with deviating arguments

$$u''(t) = f_0(t, u(\tau_1(t)), u'(\tau_2(t))), \tag{1.1$'$}$$

where $f_0 :]a,b[\times R^2 \to R$ is the function satisfying the local Carathéodory conditions, and $\tau_i :]a,b[\to]a,b[$ $(i = 1,2)$ are continuous functions. In the present paper, for the singular equations (1.1) and (1.1$'$) we investigate the two-point boundary value problems

$$u(a+) = 0, \quad u(b-) = 0 \tag{1.2$_1$}$$

and

$$u(a+) = 0, \quad u'(b-) = 0 \tag{1.2$_2$}$$

with the additional condition

$$\int_a^b u'^2(s)\,ds < +\infty. \tag{1.3}$$

If f is the Nemytski operator, i.e. if $f(u)(t) \equiv f_0(t, u(t), u'(t))$, then the singular problems $(1.1), (1.2_i)$ $(i = 1,2)$ are studied with sufficient thoroughness (see, e.g., [1]–[3], [5]–[9], [15], [16], [18], [20]). If f is the operator of general type, or $\tau_i(t) \not\equiv t$ $(i = 1,2)$, then the problems $(1.1), (1.2_i)$ $(i = 1,2)$ and $(1.1'), (1.2_i)$ $(i = 1,2)$ are studied only in the so-called weakly singular cases, when

$$\int_a^b (t-a)(b-t)|f(u)(t)|\,dt < +\infty \ \text{ for } \ u \in C^1([a,b]),$$

or

$$\int_a^b (t-a)(b-t)|f_0(t,x,y)|\,dt < +\infty \ \text{ for } \ x \ \text{ and } \ y \in R$$

(see [4], [11]-[14], [19], [21]–[24]) and the references therein). In strongly singular cases these problems remained in fact unstudied. The present paper is meant to fill up the existing gap to some extent.

Throughout the paper we will use the following notation.

$R =]-\infty, +\infty[\,$; $R_+ = [0, +\infty[\,$; $I =]a, b[\,$, or $I =]a, b]$.

$u(t_0+)$ and $u_0(t-)$ are, respectively, the right and the left limits of the function u at the point t_0.

$C^1_{loc}(I)$ is the topological space of continuously differentiable functions $u : I \to R$ in which the sequence $(u_k)_{k=1}^{\infty}$ is assumed to be converging to u if

$$\lim_{k\to+\infty} u_k(t) = u(t), \quad \lim_{k\to+\infty} u_k'(t) = u'(t)$$

uniformly on every compact interval contained in I.

$$D_1(]a, b[) = \Big\{u \in C^1_{loc}(]a, b[) : u(a+) = u(b-) = 0, \ \int_a^b u'^2(s)\, ds < +\infty\Big\}.$$

$$D_2(]a, b]) = \Big\{u \in C^1_{loc}(]a, b]) : u(a+) = u'(b) = 0, \ \int_a^b u'^2(s)\, ds < +\infty\Big\}.$$

$\widetilde{C}^1_{loc}(I)$ is the space of functions $u : I \to R$, absolutely continuous together with their first derivative on every compact interval contained in I.

$L_{loc}(I)$ is the topological space of functions $v : I \to R$, Lebesgue integrable on every compact interval contained in I, in which the sequence $(v_k)_{k=1}^{+\infty}$ is assumed to be converging to v if

$$\lim_{k\to+\infty} \int_{t_1}^{t_2} |v_k(t) - v(t)|\, dt = 0 \ \text{ for } \ t_i \in I \ \ (i = 1, 2).$$

$L^2([a, b])$ is the space of Lebesgue square integrable functions $v : [a, b] \to R$ with the norm

$$\|v\|_{L^2} = \Big(\int_a^b v^2(t)\, dt\Big)^{1/2}.$$

$L^2_{loc}(I)$ is the space of functions $v : I \to R$, Lebesgue square integrable on every compact interval contained in I.

$L^2_{\alpha,\beta}(]a, b[)$ is the space of square integrable with the weight $(t-a)^\alpha(b-t)^\beta$ functions $v :]a, b[\to R$ with the norm

$$\|v\|_{L^2_{\alpha,\beta}} = \Big(\int_a^b (t-a)^\alpha (b-t)^\beta v^2(t)\, dt\Big)^{1/2}.$$

By **a solution of Eq. (1.1)** is understood the function $u \in \widetilde{C}^1_{loc}(]a, b[)$ which almost everywhere on $]a, b[$ satisfies that equation. The solution of Eq. (1.1) satisfying conditions (1.2_1) and (1.3) (conditions (1.2_2) and (1.3)) is called **the solution of problem (1.1), (1.2_1), (1.3)** (**the solution of problem (1.1), (1.2_2), (1.3)**).

Along with (1.1), we consider the perturbed equation

$$u''(t) = f(u)(t) + h(t) \tag{1.4}$$

and introduce the following definition.

DEFINITION 1.1. *Problem* $(1.1), (1.2_1), (1.3)$ (*problem* $(1.1), (1.2_2), (1.3)$) *is called stable with respect to a small perturbation of the right-hand member of Eq.* (1.1) *if there exists a positive number* r *such that for any* $h \in L^2_{2,2}(]a,b[)$ (*for any* $h \in L^2_{2,0}(]a,b]))$ *problem* $(1.4), (1.2_1), (1.3)$ (*problem* $(1.4), (1.2_2), (1.3)$) *is uniquely solvable and*

$$\|u_h' - u_0'\|_{L^2} \le r\|h\|_{L^2_{2,2}} \quad \Big(\|u_h' - u_0'\|_{L^2} \le r\|h\|_{L^2_{2,0}} \Big),$$

where u_h *and* u_0 *are the solutions of problems* $(1.4), (1.2_1), (1.3)$ *and* $(1.1), (1.2_1)$, (1.3) (*of problems* $(1.4), (1.2_2), (1.3)$ *and* $(1.1), (1.2_2), (1.3)$).

DEFINITION 1.2. *We say that the operator* $f : C^1_{loc}(]a,b[) \to L_{loc}(]a,b[)$ (*the operator* $f : C^1_{loc}(]a,b]) \to L_{loc}(]a,b]))$ *belongs to the set* $\mathcal{K}_1(]a,b[)$ (*to the set* $\mathcal{K}_2(]a,b])$) *if it is continuous and there exists a continuous function* $\omega :]a,b[\times]a,b[\times R_+ \to R_+$ $(\omega :]a,b]\times]a,b]\times R_+ \to R_+)$ *such that*

$$\omega(t,t,\rho) = 0 \ \ for \ \ a < t < b, \ \ \rho \in R_+ \tag{1.5}$$

and for an arbitrary $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ *the inequality*

$$\int\limits_s^t |f(u)(\xi)|\, d\xi \le \omega(s,t,\|u'\|_{L^2}) \ \ for \ \ a < s \le t < b \tag{1.6}$$

is fulfilled.

In the case, where

$$f \in \mathcal{K}_1(]a,b[) \quad (\, f \in \mathcal{K}_2(]a,b])\,), \tag{1.7}$$

we have proved a general theorem on the solvability of problem $(1.1), (1.2_1)$, (1.3) (of problem $(1.1), (1.2_2), (1.3)$) and called it the principle of a priori boundedness. Using this principle, we have found effective and optimal in a certain sense conditions which guarantee, respectively, the solvability and unique solvability of problems $(1.1), (1.2_i), (1.3)$ and $(1.1'), (1.2_i), (1.3)$ $(i = 1, 2)$ and their stability with respect to small perturbations of the right-hand member of the equation under consideration.

1.2. The principle of a priori boundedness. Let $(a_k)_{k=1}^{+\infty}$ and $(b_k)_{k=1}^{+\infty}$ be the number sequences such that

$$a < a_k < b_k < b \;\; (k = 1, 2, \dots), \quad \lim_{k\to+\infty} a_k = a, \quad \lim_{k\to+\infty} b_k = b. \tag{1.8}$$

For an arbitrary $u \in C^1_{loc}(]a, b[)$ and natural k we put

$$f_k(u)(t) = \begin{cases} 0 & \text{for } \; t \in [a, a_k] \cup [b_k, b] \\ f(u)(t) & \text{for } \; a_k < t < b_k \end{cases}, \tag{1.9}$$

and consider the auxiliary functional differential equation

$$u''(t) = \lambda f_k(u)(t) \tag{1.10}$$

with a parameter $\lambda \in [0, 1[$.

Theorem 1.1. *Let condition* (1.7) *be fulfilled, and let there exist a positive constant r_0 and sequences $(a_k)_{k=1}^{+\infty}$, $(b_k)_{k=1}^{+\infty}$ satisfying conditions* (1.8), *such that for arbitrary $\lambda \in [0, 1[$ and natural k every solution of problem* $(1.10), (1.2_1)$ (*of problem* $(1.10), (1.2_2)$) *admits the estimate*

$$\int_a^b u'^2(s)\, ds \le r_0^2. \tag{1.11}$$

Then problem $(1.1), (1.2_1), (1.3)$ (*problem* $(1.1), (1.2_2), (1.3)$) *has at least one solution.*

Note that the analogous result for regular boundary value problems is contained in [10].

1.3. The existence and uniqueness theorems for Eq. (1.1).

Theorem 1.2. *Let condition* (1.7) *be fulfilled, and let there exist constants $\ell \in [0, 1[$, $\ell_0 \ge 0$, $a_0 \in]a, b[$ and $b_0 \in]a_0, b[$ such that for an arbitrary $u \in D_1(]a, b[)$ (for an arbitrary $u \in D_2(]a, b])$) the inequality*

$$\int_{t_0}^t f(u)(s)u(s)\, ds \ge -\ell \int_a^b u'^2(s)\, ds - \ell_0 \;\; \text{for } \; a < t_0 \le a_0, \;\; b_0 \le t < b \tag{1.12}$$

is fulfilled. Then problem $(1.1), (1.2_1), (1.3)$ (*problem* $(1.1), (1.2_2), (1.3)$) *has at least one solution.*

Corollaries of that theorem given below deal with the functional differential equation

$$u''(t) = f_1(u)(t)u(t) + f_2(u)(t)u'(t) + f_0(u)(t), \tag{1.13}$$

where $f_i : C^1_{loc}(]a,b[) \to L_{loc}(]a,b[)$ $(i=0,1,2)$ are continuous operators. Assume

$$f^*_{i1}(t;\rho) = \sup\Big\{|f_i(u)(t)| : \ u \in D_1(]a,b[), \ \|u'\|_{L^2} \le \rho\Big\} \ \ (i=0,1,2),$$
$$f^*_{i2}(t;\rho) = \sup\Big\{|f_i(u)(t)| : \ u \in D_2(]a,b]), \ \|u'\|_{L^2} \le \rho\Big\} \ \ (i=0,1,2).$$

We are interested in the cases where the conditions

$$(1.14_1) \qquad f^*_{01}(\cdot,\rho) \in L^2_{2,2}(]a,b[) \ \text{ for } \ \rho \in R_+, \quad \lim_{\rho\to+\infty} \rho^{-1}\|f^*_{01}(\cdot,\rho)\|_{L^2_{2,2}} = 0,$$

$$(1.15_1) \qquad f^*_{11}(\cdot,\rho) \in L_{loc}(]a,b[), \ \ f^*_{21}(\cdot,\rho) \in L^2_{loc}(]a,b[) \ \text{ for } \ \rho \in R_+,$$

or the conditions

$$(1.14_2) \qquad f^*_{02}(\cdot,\rho) \in L^2_{2,0}(]a,b[) \ \text{ for } \ \rho \in R_+, \quad \lim_{\rho\to+\infty} \rho^{-1}\|f^*_{02}(\cdot,\rho)\|_{L^2_{2,0}} = 0,$$

$$(1.15_2) \qquad f^*_{12}(\cdot,\rho) \in L_{loc}(]a,b]), \quad f^*_{22}(\cdot,\rho) \in L^2_{loc}(]a,b]) \ \text{ for } \ \rho \in R_+$$

are fulfilled.

Corollary 1.1. *Let conditions* (1.14_1), (1.15_1) (*conditions* (1.14_2), (1.15_2)) *be fulfilled, and let there exist a constant* $\lambda \in]0,4[$ *such that for an arbitrary* $u \in D_1(]a,b[)$ (*for an arbitrary* $u \in D_2(]a,b])$) *almost everywhere on* $]a,b[$ *the inequality*

$$(1.16) \qquad f_2^2(u)(t) \le \lambda f_1(u)(t)$$

is fulfilled. Then problem $(1.13), (1.2_1), (1.3)$ (*problem* $(1.13), (1.2_2), (1.3)$) *has at least one solution.*

Example 1.1. *Consider the differential equation*

$$(1.17) \qquad u''(t) = \frac{\lambda_1(t)|u(\tau(t))|^{2\mu}}{(t-a)^{2\alpha_1}(b-t)^{2\beta_1}}\, u(t) + \frac{\lambda_2(t)|u(\tau(t))|^{\mu}}{(t-a)^{\alpha_1}(b-t)^{\beta_1}}\, u'(t) + \\ + \frac{\lambda_0(t)}{(t-a)^{\alpha_0}(b-t)^{\beta_0}}\,,$$

where α_i, β_i $(i=0,1,2)$, μ *are nonnegative constants, while* $\lambda_i : [a,b] \to R$ $(i=0,1,2)$ *and* $\tau :]a,b[\to]a,b[$ *are continuous functions. According to Corollary* 1.1, *if*

$$\alpha_0 < \frac{3}{2}\,, \ \ \beta_0 < \frac{3}{2} \ \ \Big(\alpha_0 < \frac{3}{2}\,, \ \ \beta_0 = \beta_1 = \beta_2 = 0\Big) \ \ \text{and} \ \ \lambda_2^2(t) < 4\lambda_1(t),$$

then problem $(1.17), (1.2_1), (1.3)$ (*problem* $(1.1), (1.2_2), (1.3)$) *has at least one solution. Consequently, under the conditions of Corollary* 1.1 *Eq.* (1.13) *may have singularities of arbitrary order at the points* a *and* b (*at the point* a).

Example 1.2. *If* $f_1(u)(t) \equiv \frac{1}{4(t-a)^2}$, $f_2(u)(t) \equiv -\frac{1}{t-a}$, *and* $f_0(u)(t) \equiv 15$, *then for Eq.* (1.13) *all the conditions of Corollary* 1.1 *hold, except* (1.16)*, instead of which we have*

$$f_2^2(u)(t) \le 4 f_1(u)(t).$$

On the other hand, in this case Eq. (1.13) *has the form*

$$u''(t) = \frac{u(t)}{4(t-a)^2} - \frac{u'(t)}{t-a} + 15$$

and its arbitrary solution admits the representation

$$u(t) = c_1(t-a)^{1/2} + c_2(t-a)^{-1/2} + 4(t-a)^2,$$

where $c_i \in R$ $(i = 1, 2)$*. Hence it is evident that both problems* $(1.13), (1.2_1), (1.3)$ *and* $(1.13), (1.2_2), (1.3)$ *have no solution.*

The above constructed example shows that the condition $\ell \in]0,1[$ $(\lambda \in]0,4[)$ in Theorem 1.2 (in Corollary 1.1) is unimprovable and it cannot be replaced by the condition $\ell = 1$ $(\lambda = 4)$.

Now we proceed to the consideration of the case when condition (1.16) is violated, but f_1 and f_2 satisfy either the conditions

$$f_1(u)(t) \ge -\frac{\ell_1}{(t-a)^2(b-t)^2}\,, \quad |f_2(u)(t)| \le \frac{\ell_2}{(t-a)(b-t)}\,, \tag{1.18_1}$$

where

$$\frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} < 1, \tag{1.19_1}$$

or the conditions

$$f_1(u)(t) \ge -\frac{\ell_1}{(t-a)^2}\,, \quad |f_2(u)(t)| \le \frac{\ell_2}{(t-a)}\,, \tag{1.18_2}$$

where

$$4\ell_1 + 2\ell_2 < 1. \tag{1.19_2}$$

Corollary 1.2. *Let conditions* (1.14_1) *(conditions* (1.14_2)*) be fulfilled and* $f^*_{11}(\cdot,\rho) \in L_{loc}(]a,b[)$ $(f^*_{12}(\cdot,\rho) \in L_{loc}(]a,b]))$ *for* $\rho \in R_+$*. Let, moreover, there exist nonnegative constants* ℓ_1, ℓ_2 *satisfying inequality* (1.19_1) *(inequality* (1.19_2)*), such that for an arbitrary* $u \in D_1(]a,b[)$ *(for an arbitrary* $u \in D_2(]a,b])$*) inequalities* (1.18_1) *(inequalities* (1.18_2)*) are fulfilled almost everywhere on* $]a,b[$*. Then problem* $(1.13)\,(1.2_1), (1.3)$ *(problem* $(1.13), (1.2_2), (1.3)$*) has at least one solution.*

Example 1.3. *Consider the differential equation*

$$u''(t) = -\frac{\lambda_1}{(t-a)^2}\,u(t) - \frac{\lambda_2}{t-a}\,u'(t) + 2 + \lambda_1 + \lambda_2, \tag{1.20}$$

where λ_1 *and* λ_2 *are positive constants. This equation is obtained from* (1.13) *in the case where*

$$f_1(u)(t) \equiv -\frac{\lambda_1}{(t-a)^2}\,, \quad f_2(u)(t) \equiv -\frac{\lambda_2}{t-a}\,, \quad f_0(u)(t) \equiv 2+\lambda_1+\lambda_2.$$

It is clear that in this case conditions (1.18_1), (*conditions* (1.18_2)), *where* $\ell_1 = \lambda_1(b-a)^2$, $\ell_2 = \lambda_2(b-a)$ $(\ell_1 = \lambda_1,\ \ell_2 = \lambda_2)$, *are fulfilled. By Corollary* 1.2, *it follows that the inequality*

$$4\lambda_1 + 2\lambda_2 < 1$$

guarantees the solvability of problems $(1.20), (1.2_1), (1.3)$ *and* $(1.20), (1.2_2), (1.3)$. *On the other hand, if*

$$4\lambda_1 + 2\lambda_2 = 1,$$

then both problems $(1.20), (1.2_1), (1.3)$ *and* $(1.20), (1.2_2), (1.3)$ *have no solution since an arbitrary solution of Eq.* (1.20) *has the form*

$$u(t) = c_1(t-a)^{1/2} + c_2(t-a)^{1/2-\lambda_2} + (t-a)^2,$$

where $c_i \in R$ $(i=1,2)$.

The above constructed example shows that condition (1.19_1) (condition (1.19_2)) in Corollary 1.2 is unimprovable and it cannot be replaced by the condition

$$\frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} \le 1 \quad (\,4\ell_1 + 2\ell_2 \le 1\,). \tag{1.21}$$

Theorem 1.3. *Let condition* (1.7) *be fulfilled, and let there exist constants* $\ell \in [0,1[$, $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$ *such that for arbitrary* $u_i \in D_1(]a,b[)$ (*for arbitrary* $u_i \in D_2(]a,b])$) $(i=1,2)$ *the inequality*

$$\begin{aligned} &\int_{t_0}^{t} (f(u_2)(s) - f(u_1)(s))(u_2(s) - u_1(s))\,ds \ge \\ &\ge -\ell \int_a^b (u_2'(s) - u_1'(s))^2\,ds \quad \text{for } a < t_0 \le a_0,\ \ b_0 \le t < b \end{aligned} \tag{1.22}$$

is fulfilled. Let, moreover,

$$f(0)(\cdot) \in L^2_{2,2}(]a,b[) \quad (\,f(0)(\cdot) \in L^2_{2,0}(]a,b])\,). \tag{1.23}$$

Then problem $(1.1), (1.2_1), (1.3)$ (*problem* $(1.1), (1.2_2), (1.3)$) *is uniquely solvable and stable with respect to small perturbations of the right-hand member of Eq.* (1.1).

REMARK 1.1. *By Examples* 1.2 *and* 1.3, *the condition* $\ell \in [0,1[$ *in Theorem* 1.3 *is unimprovable and it cannot be replaced by the condition* $\ell = 1$.

REMARK 1.2. *Under the conditions of Theorem* 1.3, *problem* $(1.1),(1.2_1)$ (*problem* $(1.1),(1.2_2)$) *may have an infinite set of solutions. Indeed, if* $\lambda_1 > 0$, $\lambda_2 \geq 0$, *and* $4\lambda_1 + 2\lambda_2 < 1$, *then for Eq.* (1.20) *all the conditions of Theorem* 1.3 *are fulfilled, and hence problems* $(1.20),(1.2_1),(1.3)$ *and* $(1.20),(1.2_2),(1.3)$ *are uniquely solvable. On the other hand, it is clear that both problems* $(1.20),(1.2_1)$ *and* $(1.20),(1.2_2)$ *have an infinite set of solutions.*

1.4. The existence and uniqueness theorems for Eq. (1.1′). Everywhere in this section we assume that the function $f_0 :]a,b[\times R^2 \to R$ is measurable in the first and continuous in the two last arguments. As for the functions $\tau_i :]a,b[\to]a,b[$ $(i=1,2)$, they are continuously differentiable and

$$\tau_i'(t) \neq 0 \ \text{ for } \ a<t<b \ \ (i=1,2).$$

Of special interest is the case when in $]a,b[\times R^2$ either the inequality

$$(1.24_1) \qquad |f_0(t,x,y)| \leq \frac{\ell_1|\tau_1'(t)|^{1/2}|x|}{(\tau_1(t)-a)(b-\tau_1(t))(t-a)(b-t)} + \\ + \frac{\ell_2|\tau_2'(t)|^{1/2}|y|}{(t-a)(b-t)} + q(t,(t-a)^{-1/2}(b-t)^{-1/2}|x|),$$

or the inequality

$$(1.24_2) \qquad |f_0(t,x,y)| \leq \frac{\ell_1|\tau_1'(t)|^{1/2}|x|}{(\tau_1(t)-a)(t-a)} + \frac{\ell_2|\tau_2'(t)|\,|y|}{t-a} + q(t,(t-a)^{-1/2}|x|)$$

is fulfilled. Here ℓ_1 and ℓ_2 are nonnegative constants, and $q :]a,b[\times R_+ \to R_+$ is a nondecreasing in the second argument function, satisfying the conditions

$$(1.25_1) \qquad q(\cdot,\rho) \in L^2_{2,2}(]a,b[) \ \text{ for } \ \rho \in R_+, \quad \lim_{\rho\to+\infty} \rho^{-1}\|q(\cdot,\rho)\|_{L^2_{2,2}} = 0,$$

or

$$(1.25_2) \qquad q(\cdot,\rho) \in L^2_{2,0}(]a,b[) \ \text{ for } \ \rho \in R_+, \quad \lim_{\rho\to+\infty} \rho^{-1}\|q(\cdot,\rho)\|_{L^2_{2,0}} = 0.$$

THEOREM 1.4. *Let there exist positive constants* ℓ_1, ℓ_2 *and a nondecreasing in the second argument function* $q :]a,b[\times R_+ \to R_+$ *satisfying conditions* (1.19_1) *and* (1.25_1) (*conditions* (1.19_2), (1.25_2)) *such that in* $]a,b[\times R^2$ *condition* (1.24_1) (*condition* (1.24_2)) *is fulfilled. Then problem* $(1.1'),(1.2_1),(1.3)$ (*problem* $(1.1'),(1.2_2),(1.3)$) *has at least one solution.*

The theorem on the unique solvability of problems $(1.1'),(1.2_1),(1.3)$ and $(1.1'),(1.2_2),(1.3)$ concerns the cases where instead of (1.24_1) the condition

$$(1.26_1) \qquad |f_0(t,x_1,y_1) - f_0(t,x_2,y_2)| \leq$$

$$\leq \frac{\ell_1|\tau_1'(t)|^{1/2}|x_1-x_2|}{(\tau_1(t)-a)(b-\tau_1(t))(t-a)(b-t)}+\frac{\ell_2|\tau_2'(t)|^{1/2}|y_1-y_2|}{(t-a)(b-t)}$$

is fulfilled, and instead of (1.24_2) the condition

$$|f_0(t,x_1,y_1)-f_0(t,x_2,y_2)|\leq \tag{1.26$_2$}$$

$$\leq \frac{\ell_1|\tau_1'(t)|^{1/2}|x_1-x_2|}{(\tau_1(t)-a)(t-a)}+\frac{\ell_2|\tau_2'(t)|^{1/2}|y_1-y_2|}{t-a}$$

is fulfilled.

THEOREM 1.5. *Let there exist positive constants ℓ_1, ℓ_2 satisfying inequality* (1.19_1) (*inequality* (1.19_2)) *such that in* $]a,b[\times R^2$ *condition* (1.26_1) (*condition* (1.26_2)) *is fulfilled. Moreover, let*

$$f_0(\cdot,0,0)\in L^2_{2,2}(]a,b[) \quad (\, f_0(\cdot,0,0)\in L^2_{2,0}(]a,b[)\,). \tag{1.27}$$

Then problem $(1.1'),(1.2_1),(1.3)$ (*problem* $(1.1'),(1.2_2),(1.3)$) *is uniquely solvable, and its solution is stable with respect to small perturbations of the right-hand member of Eq.* $(1.1')$.

REMARK 1.3. *According to Example* 1.3, *condition* (1.19_1) (*condition* (1.19_2)) *in Theorems* 1.4 *and* 1.5 *is unimprovable and it cannot be replaced by condition* (1.21).

§ 2. Auxiliary Propositions.

2.1. Lemmas on integral inequalities. Let $\tau_i:]a,b[\to]a,b[$ $(i=1,2)$ be continuously differentiable monotone functions. For an arbitrary $u\in C^1_{loc}(]a,b[)$ we put

$$\begin{aligned} w_{10}(u)(t)&=\frac{|\tau_1'(t)|^{1/2}|u(\tau_1(t))|}{(t-a)(b-t)(\tau_1(t)-a)(b-\tau_1(t))}\,,\\ w_{11}(u)(t)&=\frac{|\tau_2'(t)|^{1/2}|u'(\tau_2(t))|}{(t-a)(b-t)}\,; \end{aligned} \tag{2.1}$$

$$w_{20}(u)(t)=\frac{|\tau_1'(t)|^{1/2}|u(\tau_1(t))|}{(t-a)(\tau_1(t)-a)}\,,\quad w_{21}(u)(t)=\frac{|\tau_2'(t)|^{1/2}|u'(\tau_2(t))|}{t-a}\,. \tag{2.2}$$

LEMMA 2.1. *If $u\in D_1(]a,b[)$, then*

$$\int_a^b |u(s)|w_{1i}(u)(s)\,ds\leq\Big(\frac{2}{b-a}\Big)^{2-i}\int_a^b u'^2(s)\,ds\ \ (i=0,1). \tag{2.3}$$

If $u\in D_2(]a,b])$, then

$$\int_a^b |u(s)|w_{2i}(u)(s)\,ds\leq 2^{2-i}\int_a^b u'^2(s)\,ds\ \ (i=0,1). \tag{2.4}$$

To prove this lemma, we need the following

LEMMA 2.2 (V. I. LEVIN [17]). *If* $u \in D_1(]a,b[)$, *then*

$$\int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2} \le \frac{4}{(b-a)^2}\int_a^b u'^2(s)\,ds.$$

If $u \in D_2(]a,b])$, *then*

$$\int_a^b \frac{u^2(s)\,ds}{(s-a)^2}\,ds \le 4\int_a^b u'^2(s)\,ds.$$

Proof of Lemma 2.1. Let $u \in D_1(]a,b[)$. Then by virtue of Lemma 2.2 and the Schwartz inequality we have

$$\int_a^b |u(s)|w_{10}(u)(s)\,ds \le$$

$$\le \bigg(\int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2}\bigg)^{1/2}\bigg(\int_a^b \frac{|\tau_1'(s)|u^2(\tau_1(s))\,ds}{(\tau_1(s)-a)^2(b-\tau_1(s))^2}\bigg)^{1/2} =$$

$$= \bigg(\int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2}\bigg)^{1/2}\bigg|\int_{\tau_1(a)}^{\tau_2(b)} \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2}\bigg|^{1/2} \le$$

$$\le \int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2} \le \frac{4}{(b-a)^2}\int_a^b u'^2(s)\,ds,$$

$$\int_a^b |u(s)|w_{11}(u)(s)\,ds \le$$

$$\le \bigg(\int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2}\bigg)^{1/2}\bigg(\int_a^b |\tau_2'(s)|u'^2(\tau_2(s))\,ds\bigg)^{1/2} =$$

$$= \bigg(\int_a^b \frac{u^2(s)\,ds}{(s-a)^2(b-s)^2}\bigg)^{1/2}\bigg|\int_{\tau_2(a)}^{\tau_2(b)} u'^2(s)\,ds\bigg|^{1/2} \le \frac{2}{b-a}\int_a^b u'^2(s)\,ds.$$

Consequently, inequalities (2.3) are valid.

Analogously we can show that for $u \in D_2(]a,b])$ inequalities (2.4) are fulfilled. □

Lemma 2.1 immediately yields the following lemma.

LEMMA 2.3. *Let ℓ_1 and ℓ_2 be nonnegative constants and*

$$w_i(u)(t) = \ell_1 w_{i0}(u)(t) + \ell_2 w_{i1}(u)(t) \ \ (i = 1,2). \tag{2.5}$$

If, moreover, $u \in D_1(]a,b[)$, then

$$\int\limits_a^b |u(s)| w_1(u)(s)\, ds \le \Big(\frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a}\Big) \int\limits_a^b u'^2(s)\, ds.$$

If $u \in D_2(]a,b])$, then

$$\int\limits_a^b |u(s)| w_2(u)(s)\, ds \le (4\ell_1 + 2\ell_2) \int\limits_a^b u'^2(s)\, ds.$$

LEMMA 2.4. *Let*

$$u \in D_1(]a,b[), \ \ q \in L^2_{2,2}(]a,b[) \ \ \Big(u \in D_2(]a,b]), \ \ q \in L^2_{2,0}(]a,b[)\Big).$$

Then for an arbitrary $\varepsilon > 0$ the inequality

$$\int\limits_a^b |q(s)u(s)|\, ds \le \rho_\varepsilon(q) + \varepsilon \int\limits_a^b u'^2(s)\, ds \tag{2.6}$$

holds, where

$$\rho_\varepsilon(q) = (b-a)^{-2} \varepsilon^{-1} \|q\|^2_{L^2_{2,2}} \ \ \Big(\rho_\varepsilon(q) = \varepsilon^{-1} \|q\|^2_{L^2_{2,0}}\Big).$$

Proof. Obviously,

$$|q(s)u(s)| \le \varepsilon^{-1}(b-a)^{-2}(s-a)^2(b-s)^2 q^2(s) + \frac{\varepsilon(b-a)^2}{4}\, \frac{u^2(s)}{(s-a)^2(b-s)^2}$$

and

$$|q(s)u(s)| \le \varepsilon^{-1}(s-a)^2 q^2(s) + \frac{\varepsilon}{4}\, (s-a)^2 q^2(s).$$

If we integrate the first (the second) of these last two inequalities from a to b, and apply Lemma 2.2, then we get inequality (2.6). □

LEMMA 2.5. *Let ℓ_1, ℓ_2 be nonnegative constants satisfying inequality* (1.19_1) (*inequality* (1.19_2)), *and let $q :]a,b[\times R_+ \to R_+$ be a nondecreasing in the second argument function, satisfying conditions* (1.25_1) (*conditions* (1.25_2)). *Let,* *moreover, the operator* $f : C^1_{loc}(]a,b[) \to L_{loc}(]a,b[)$ *be such that for an arbitrary* $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ *almost everywhere on* $]a,b[$ *the inequality*

$$f(u)(t)\text{sgn}u(t) \geq -w_1(u)(t) - q(t, \|u'\|_{L^2})$$

$$\Big(f(u)(t)\text{sgn}u(t) \geq -w_2(u)(t) - q(t, \|u'\|_{L^2})\Big) \tag{2.7}$$

holds, where w_1 *and* w_2 *are the operators, given by equality* (2.5)*. Then there exist constants* $\ell \in]0,1[$ *and* $\ell_0 \geq 0$ *such that for arbitrary* $a_0 \in]a,b[$, $b_0 \in]a_0,b[$ *and* $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ *inequality* (1.12) *is satisfied.*

Proof. By conditions (1.19_1) and (1.25_1) (conditions (1.19_2) and (1.25_2)), there exist constants $\varepsilon \in]0,1[$ and $\ell_0 \geq 0$ such that

$$\ell = \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} + \varepsilon < 1 \quad \Big(\ell = 4\ell_1 + 2\ell_2 + \varepsilon < 1\Big)$$

and

$$2(b-a)^{-2}\varepsilon^{-1}\|q(\cdot,\rho)\|^2_{L^2_{2,2}} \leq \frac{\varepsilon}{2}\rho^2 + \ell_0 \ \text{ for } \ \rho \in R_+,$$

$$\Big(2\varepsilon^{-1}\|q(\cdot,\rho)\|^2_{L^2_{2,0}} \leq \frac{\varepsilon}{2}\rho^2 + \ell_0 \ \text{ for } \ \rho \in R_+\Big). \tag{2.8}$$

Due to Lemma 2.3, from (2.7) we find

$$\int_{t_0}^{t} f(u)(s)u(s)\,ds \geq -(\ell-\varepsilon)\int_a^b u'^2(s)\,ds -$$

$$-\int_a^b q(s, \|u'\|_{L^2})|u(s)|\,ds \ \text{ for } \ a < t_0 < t < b.$$

On the other hand, by Lemma 2.4 and condition (2.8), we get

$$\int_a^b q(s, \|u'\|_{L^2})|u(s)|\,ds \leq$$

$$\leq \frac{\varepsilon}{2}\int_a^b u'^2(s)\,ds + 2(b-a)^{-2}\varepsilon^{-1}\|q(\cdot, \|u'\|_{L^2})\|^2_{L^2_{2,2}} \leq \varepsilon\int_a^b u'^2(s)\,ds + \ell_0$$

$$\Big(\int_a^b q(s, \|u'\|_{L^2})|u(s)|\,ds \leq$$

$$\leq \frac{\varepsilon}{2}\int_a^b u'^2(s)\,ds + 2\varepsilon^{-1}\|q(\cdot, \|u'\|_{L^2})\|^2_{L^2_{2,0}} \leq \varepsilon\int_a^b u'^2(s)\,ds + \ell_0\Big).$$

Therefore, for arbitrary $a_0 \in]a,b[$, $b_0 \in]a_0,b[$ and $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ inequality (1.12) holds. □

2.2. Lemmas on a priori estimates.

Lemma 2.6. *Let*

$$u \in D_1(]a,b[) \cup D_2(]a,b]) \tag{2.9}$$

and

$$\liminf_{t_0\to a,\, t\to b} \int_{t_0}^{t} u''(s)u(s)\,ds \geq -\ell \int_a^b u'^2(s)\,ds - \ell_0, \tag{2.10}$$

where $\ell \in [0,1[$ *and* $\ell_0 \geq 0$. *Then*

$$\int_a^b u'^2(s)\,ds \leq \frac{\ell_0}{1-\ell}\,. \tag{2.11}$$

To prove the above lemma, we need the following

Lemma 2.7. *If condition* (2.9) *is fulfilled, then*

$$\liminf_{t\to a} |u'(t)u(t)| = 0, \tag{2.12}$$

$$\liminf_{t\to b} |u'(t)u(t)| = 0. \tag{2.13}$$

Proof. Suppose that equality (2.12) is violated. Then there exist $a_0 \in]a,b[$ and $\delta > 0$ such that

$$|u'(t)u(t)| > \delta \ \text{ for } \ a < t \leq a_0,$$

whence with regard for the equality $u(a+) = 0$, we find

$$u^2(t) > 2\delta(t-a) \ \text{ for } \ a < t \leq a_0.$$

On the other hand, by (2.9) we have

$$2\delta \leq (t-a)^{-1}u^2(t) = (t-a)^{-1}\bigg(\int_a^t u'(s)\,ds\bigg)^2 \leq \int_a^t u'^2(s)\,ds \to 0 \ \text{ as } \ t \to a.$$

The obtained contradiction shows that equality (2.12) is valid.

If $u \in D_2(]a,b])$, then equality (2.13) is obvious. If, however, $u \in D_1(]a,b[)$, then it can be proved just in the same way as (2.12). □.

Proof of Lemma 2.6. By Lemma 2.7, there exist sequences $(t_{0k})_{k=1}^{+\infty}$ and $(t_k)_{k=1}^{+\infty}$ such that

$$a < t_{0k} < t_k < b \ \ (k = 1, 2, \dots), \quad \lim_{k\to+\infty} t_{0k} = a, \quad \lim_{k\to+\infty} t_k = b$$

and

$$\lim_{k\to+\infty} u'(t_{0k})u(t_{0k}) = \lim_{k\to+\infty} u'(t_k)u(t_k) = 0.$$

On the other hand,

$$\int_{t_{0k}}^{t_k} u''(s)u(s)\,ds = u'(t_k)u(t_k) - u'(t_{0k})u(t_{0k}) - \int_{t_{0k}}^{t_k} u'^2(s)\,ds.$$

Therefore,

$$\lim_{k\to+\infty} \int_{t_{0k}}^{t_k} u''(s)u(s)\,ds = -\int_a^b u'^2(s)\,ds.$$

This equality, according to (2.10), implies

$$-\int_a^b u'^2(s)\,ds \geq -\ell \int_a^b u'^2(s)\,ds - \ell_0.$$

Consequently, estimate (2.11) is true. □

Lemma 2.8. *Let*

$$u \in D_1(]a,b[),\quad q \in L^2_{2,2}(]a,b[) \quad \Big(u \in D_2(]a,b]),\quad q \in L^2_{2,0}(]a,b[) \Big)$$

and

$$\liminf_{t_0\to a,\, t\to b} \int_{t_0}^{t} u''(s)u(s)\,ds \geq -\ell \int_a^b u'^2(s)\,ds - \int_a^b q(s)|u(s)|\,ds, \tag{2.14}$$

where $\ell \in [0,1[$. *Then*

$$\|u'\|_{L^2} \leq \frac{2}{(1-\ell)(b-a)} \|q\|_{L^2_{2,2}} \quad \Big(\|u'\|_{L^2} \leq \frac{2}{1-\ell} \|q\|_{L^2_{2,0}} \Big). \tag{2.15}$$

Proof. By Lemma 2.4, from (2.14) we find

$$\liminf_{t_0\to a,\, t\to b} \int_{t_0}^{t} u''(s)u(s)\,ds \geq -(\ell+\varepsilon) \int_a^b u'^2(s)\,ds - \rho_\varepsilon(q),$$

where $\varepsilon = \frac{1-\ell}{2}$. Hence by Lemma 2.6 we have

$$\int_a^b u'^2(s)\,ds \leq \frac{\rho_\varepsilon(q)}{1-\ell-\varepsilon} = \varepsilon^{-1}\rho_\varepsilon(q).$$

Therefore, estimate (2.15) is valid.

§ 3. Proofs of the Basic Results.

Below, under C_0^1 we will mean the Banach space of continuously differentiable functions $u : [a,b] \to R$, satisfying the conditions

$$u(a) = 0, \quad u(b) = 0, \tag{3.1}$$

with the norm

$$\|u\|_{C_0^1} = \max\{\|u'(t)\| : \ a \le t \le b\}.$$

Proof of Theorem 1.1. We will consider only the case where $f \in \mathcal{K}_1(]a,b[)$ and prove the solvability of problem $(1.1), (1.2_1), (1.3)$, since in the case where $f \in \mathcal{K}_2(]a,b])$, the solvability of problem $(1.1), (1.2_2), (1.3)$ is proved analogously.

By Definition 1.2, there exists a continuous function $\omega :]a,b[\times]a,b[\times R_+ \to R_+$ satisfying identity (1.5), such that for an arbitrary $u \in D_1(]a,b[)$ inequality (1.6) is fulfilled.

First we prove that for an arbitrary natural k the functional differential equation

$$u''(t) = f_k(u)(t) \tag{3.2}$$

has at least one solution u_k satisfying the conditions

$$u_k(a) = u_k(b) = 0, \quad \|u_k'\|_{L^2} \le r_0. \tag{3.3}$$

Suppose

$$\eta(\rho) = \begin{cases} 1 & \text{for } 0 \le \rho \le r_0 \\ 2 - \dfrac{\rho}{r_0} & \text{for } r_0 < \rho < 2r_0 \\ 0 & \text{for } \rho \ge 2r_0 \end{cases}, \quad \xi_k(t) = \begin{cases} a_k & \text{for } t \le a_k \\ t & \text{for } a_k < t < b_k \\ b_k & \text{for } t \ge b_k \end{cases},$$

$$\widetilde{f}_k(u)(t) = \eta(\|u'\|_{L^2}) f_k(u)(t), \quad \omega_k(s,t) = \omega(\xi_k(s), \xi_k(t), 2r_0), \quad r_k = \omega_k(a_k, b_k),$$

$$B_k = \Big\{u \in C_0^1 : \ \|u\|_{C_0^1} \le r_k, \ |u'(t) - u'(s)| \le \omega_k(s,t) \text{ for } a \le s \le t \le b\Big\}.$$

Then the function $\omega_k : [a,b] \times [a,b] \to R_+$ is continuous, and

$$\omega_k(t,t) = 0.$$

Consequently, B_k is the compact set of the space C_0^1. On the other hand, by conditions (1.6) and (1.8), for an arbitrary $u \in C_0^1$ the inequalities

$$\int_s^t |\widetilde{f}_k(u)(\xi)|\, d\xi \le \omega_k(s,t) \ \text{ for } \ a \le s \le t \le b, \quad \int_a^b |\widetilde{f}_k(u)(s)|\, ds \le r_k \tag{3.4}$$

are fulfilled.

To prove the solvability of problem (3.2), (3.1), we need to consider the problem on the existence of a solution of the functional differential equation

$$u''(t) = \widetilde{f}_k(u)(t), \tag{3.5}$$

satisfying the boundary conditions (3.1). This problem is equivalent to the following operator equation in the space C_0^1,

$$u(t) = g_k(u)(t), \tag{3.6}$$

where

$$g_k(u)(t) = \int_a^b g(t,s)\widetilde{f}_k(u)(s)\,ds,$$

and g is the Green function of the boundary value problem

$$u'' = 0, \quad u(a) = u(b) = 0.$$

The continuity of the operator $f : C^1_{loc}(]a,b[) \to L_{loc}(]a,b[)$ implies that of the operator $g_k : C_0^1 \to C_0^1$. On the other hand, by conditions (3.4), for an arbitrary $u \in B_k$ the function $v = g_k(v)$ satisfies the inequalities

$$\|v\|_{C_0^1} \le \int_a^b |\widetilde{f}_k(u)(s)|\,ds \le r_k,$$

$$|v'(t) - v'(s)| = \left| \int_s^t \widetilde{f}_k(u)(\xi)\,d\xi \right| \le \omega_k(s,t) \ \text{ for } \ a \le s \le t \le b.$$

Consequently, the operator $\widetilde{g}_k$ transforms the convex compact B_k into itself. By the Schauder principle, this implies the existence of a solution u_k of Eq. (3.5), belonging to the set B_k. Obviously, u_k is a solution of problem (3.5), (3.1) as well, i.e. a solution of problem (1.10), (1.2_1), where

$$\lambda = \eta_k(\|u_k'\|_{L^2}) \in [0,1]$$

and $\lambda < 1$ for $\|u_k'\|_{L^2} > r_0$. However, by the condition of the theorem, for an arbitrary $\lambda \in [0,1[$ every solution of problem (1.10), (1.2_1) admits estimate (1.11). Therefore,

$$\|u_k'\|_{L}^{2} \le r_0,$$

whence by the definition of $\widetilde{f}_k$ it follows that $\widetilde{f}_k(u)(t) \equiv f_k(u_k)(t)$. Thus we have proved that u_k is a solution of the functional differential equation (3.2), satisfying conditions (3.3).

By virtue of conditions (1.6), (3.3), for every natural k the function u_k satisfies the inequalities

$$|u_k(t)| \leq \frac{2r_0}{b-a}\,(t-a)^{1/2}(b-t)^{1/2} \ \text{ for } \ a \leq t \leq b,$$
$$|u_k(t)-u_k(s)| \leq r_0|t-s|^{1/2} \ \text{ for } \ a \leq s \leq t \leq b,$$
$$\min\Big\{|u_k'(t)| : \ \frac{3a+b}{4} \leq t \leq \frac{a+b}{2}\Big\} \leq 2r_0(b-a)^{-1/2},$$
$$|u_k'(t)-u_k'(s)| = \Big|\int\limits_s^t f_k(u)(\xi)\,d\xi\Big| \leq \int\limits_s^t |f(u)(\xi)|\,d\xi \leq \omega(s,t,r_0) \ \text{ for } \ a < s \leq t < b.$$

Hence, by the Arzela–Ascoli lemma follows the existence of a subsequence $(u_{k_j})_{j=1}^{+\infty}$ of the sequence $(u_k)_{k=1}^{+\infty}$ and a function $u \in D_1(]a,b[)$ such that

$$\lim_{j\to+\infty} u_{k_j}(t) = u(t), \quad \lim_{j\to+\infty} u_{k_j}'(t) = u'(t)$$

uniformly on every compact interval contained in $]a,b[$. Consequently, $(u_{k_j})_{j=1}^{+\infty}$ converges to u due to the topology of the space $C^1_{loc}(]a,b[)$.

To complete the proof of the theorem, it remains to show that u is a solution of Eq. (1.1). Indeed, let t_0 and t be arbitrarily fixed points from $]a,b[$. By virtue of condition (1.8), there exists a natural number $m(t,t_0)$ such that

$$t_0 \in]a_{k_j}, b_{k_j}[\,, \ \ t \in]a_{k_j}, b_{k_j}[\ \text{ for } \ j \geq m(t,t_0).$$

Owing to this fact and condition (1.9), we have

$$u_{k_j}'(t) = u_{k_j}'(t_0) + \int\limits_{t_0}^t f(u_{k_j})(s)\,ds \ \text{ for } \ j \geq m(t,t_0).$$

If in this equality we pass to the limit as $j \to +\infty$ and take into account the continuity of the operator $f : C^1_{loc}(]a,b[) \to L_{loc}(]a,b[)$, then we get

$$u'(t) = u'(t_0) + \int\limits_{t_0}^t f(u)(s)\,ds.$$

Hence, due to the arbitrariness of $t \in]a,b[$, it follows that u is a solution of Eq. (1.1). □.

Proof of Theorem 1.2. Let

$$r_0 = \Big(\frac{\ell_0}{1-\ell}\Big)^{1/2}, \quad a_k = a + \frac{a_0-a}{2k}\,, \quad b_k = b - \frac{b-b_0}{2k} \ \ (k=1,2,\dots),$$

and u be a solution of problem $(1.10),(1.2_1)$ (of problem $(1.10),(1.2_2)$) for some $\lambda\in[0,1]$ and natural k. Then by condition (1.12) we have

$$\lim_{t\to b,\,t_0\to a}\int_{t_0}^{t}u''(s)u(s)\,ds=\lambda\int_{a_k}^{b_k}f(u)(s)u(s)\,ds\geq-\ell\int_a^b u'^2(s)\,ds-\ell_0.$$

However, according to Lemma 2.6, the last inequality guarantees the validity of estimate (1.11). Thus we have proved that all the conditions of Theorem 1.1 are fulfilled and, consequently, problem $(1.1),(1.2_1),(1.3)$ (problem $(1.1),(1.2_2),(1.3)$) is solvable. □

Proof of Corollary 1.1. Let us choose $\varepsilon\in]0,1[$ such that

$$\ell\overset{def}{=}\frac{\lambda}{4}+\varepsilon<1.$$

By condition (1.14_1) (condition (1.14_2)), there exists a nonnegative constant ℓ_0 such that

$$\frac{2}{(b-a)^2\varepsilon}\,\|f_{01}^*(\cdot,\rho)\|^2_{L^2_{2,2}}\leq\frac{\varepsilon}{2}\,\rho^2+\ell_0\quad\left(\frac{1}{\varepsilon}\,\|f_{01}^*(\cdot,\rho)\|^2_{L^2_{2,0}}\leq\frac{\varepsilon}{2}\,\rho^2+\ell_0\right)\quad\text{for }\ \rho\in R_+.\tag{3.7}$$

Let

$$f(u)(t)=f_1(u)(t)u(t)+f_2(u)(t)u'(t)+f_0(u)(t).\tag{3.8}$$

Owing to the continuity of the operators $f_i:C^1_{loc}(]a,b[)\to L_{loc}(]a,b[)$ $(i=0,1,2)$ and by conditions (1.14_1), (1.15_1) (by conditions (1.14_2), (1.15_2)), the operator $f:C^1_{loc}(]a,b[)\to L_{loc}(]a,b[)$ is continuous and for an arbitrary $u\in D_1(]a,b[)$ $(u\in D_2(]a,b]))$ inequality (1.6) holds, where

$$\omega(s,t,\rho)=\int_s^t\big(f_{02}^*(\xi,\rho)+(b-a)^{1/2}\rho f_{11}^*(\xi,\rho)\big)\,d\xi+\rho\Big(\int_s^t f_{21}^{*2}(\xi,\rho)\,d\xi\Big)^{1/2}$$

$$\left(\omega(s,t,\rho)=\int_s^t\big(f_{02}^*(\xi,\rho)+(b-a)^{1/2}\rho f_{12}^*(\xi,\rho)\big)\,d\xi+\rho\Big(\int_s^t f_{22}^{*2}(\xi,\rho)\,d\xi\Big)^{1/2}\right).$$

Moreover, the function $\omega:]a,b[\times]a,b[\to R_+$ $(\omega:]a,b]\times]a,b]\to R_+)$ is continuous and satisfies identity (1.5). Hence the operator f satisfies condition (1.7).

It follows from (3.8) that

$$\int_{t_0}^{t}f(u)(s)u(s)\,ds\geq\int_{t_0}^{t}\Big(f_1(u)(s)u^2(s)-|f_2(u)(s)u'(s)u(s)|\Big)\,ds-$$
$$-\int_a^b|f_0(u)(s)u(s)|\,ds\ \text{ for }\ a<t_0\leq t<b.$$

However, by virtue of Lemma 2.4 and condition (3.7), for an arbitrary $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ the inequality

$$\int_a^b |f_0(u)(s)u(s)|\,ds \le \int_a^b f_{01}^*(s,\|u'\|_{L^2})|u(s)|\,ds +$$

$$+\frac{2}{(b-a)^2\varepsilon}\|f_{01}^*(\cdot,\|u'\|_{L^2})\|^2_{L^2_{2,2}} + \frac{\varepsilon}{2}\int_a^b u'^2(s)\,ds \le \varepsilon\int_a^b u'^2(s)\,ds + \ell_0$$

$$\left(\int_a^b |f_0(u)(s)u(s)|\,ds \le \int_a^b f_{02}^*(s,\|u'\|_{L^2})|u(s)|\,ds \le \varepsilon\int_a^b u'^2(s)\,ds + \ell_0\right)$$

is fulfilled. On the other hand, by condition (1.16) we have

$$f_1(u)(s)u^2(s) - |f_2(u)(s)u'(s)u(s)| \ge$$
$$\ge \Big(f_1(u)(s) - \frac{1}{\lambda}\,f_2^2(u)(s)\Big)u^2(s) - \frac{\lambda}{4}\,u'^2(s) \ge -\frac{\lambda}{4}\,u'^2(s).$$

Therefore it is clear that for arbitrary $a_0 \in]a,b[\,$, $b_0 \in]a_0,b[$ and $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ inequality (1.12) is fulfilled.

Thus the operator f, given by equality (3.8), satisfies all the conditions of Theorem 1.1, which guarantees the solvability of problem $(1.1),(1.2_1),(1.3)$ (of problem $(1.1),(1.2_2),(1.3)$). $\square$

Proof of Corollary 1.2. Let f be the operator given by equality (3.8). Then, as it is proved above, condition (1.7) is fulfilled. On the other hand, by conditions (1.14_1) and (1.18_1) (by conditions (1.14_2) and (1.18_2)) for an arbitrary $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ almost everywhere on $]a,b[$ inequality (2.7) is fulfilled, where w_i is the operator given by equality (2.5),

$$w_{10}(u)(t) = \frac{|u(t)|}{(t-a)^2(b-t)^2}\,, \quad w_{11}(t) = \frac{|u'(t)|}{(t-a)(b-t)}$$
$$\left(w_{20}(u)(t) = \frac{|u(t)|}{(t-a)^2}\,, \quad w_{21}(u)(t) = \frac{|u'(t)|}{t-a}\right),$$
$$q(t,\rho) = f_{01}^*(t,\rho) \quad \Big(q(t,\rho) = f_{02}^*(t,\rho)\Big),$$

and the function q satisfies conditions (1.25_1) (conditions (1.25_2)). By virtue of Lemma 2.5, this implies that there exist constants $\ell \in]0,1[$ and $\ell_0 \ge 0$ such that for arbitrary $a_0 \in]a,b[\,$, $b_0 \in]a_0,b[$ and $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ inequality (1.12) is fulfilled.

Consequently, the operator f satisfies all the conditions of Theorem 1.2, which guarantees the solvability of problem $(1.1),(1.2_1),(1.3)$ (of problem (1.1), $(1.2_2),(1.3)$). $\square$

Proof of Theorem 1.3. Let us choose $\varepsilon > 0$ such that

$$\widetilde{\ell} = \ell + \varepsilon < 1.$$

Let $h \in L^2_{2,2}(]a,b[)$ $(h \in L^2_{2,0}(]a,b[))$ and

$$h_0(t) = f(0)(t) + h(t).$$

Then by condition (1.23),

$$h_0 \in L^2_{2,2}(]a,b[) \quad \Big(h \in L^2_{2,0}(]a,b[)\Big)$$

On the other hand, if $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$, then by Lemma 2.4, we have

$$\int_a^b |h_0(s)u(s)|\,ds \le \ell_0 + \varepsilon \int_a^b u'^2(s)\,ds,$$

where

$$\ell_0 = \frac{1}{(b-a)^2\varepsilon}\,\|h_0\|^2_{L^2_{2,2}} \quad \Big(\ell_0 = \frac{1}{\varepsilon}\,\|h_0\|^2_{L^2_{2,0}}\Big).$$

This inequality and condition (1.22) imply that for an arbitrary $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ the inequality

$$\int_{t_0}^t [f(u)(s) + h(s)]u(s)\,ds = \int_{t_0}^t [f(u)(s) - f(0)(s)]u(s)\,ds +$$
$$+ \int_{t_0}^t h_0(s)u(s)\,ds \ge -\ell \int_a^b u'^2(s)\,ds - \int_a^b |h_0(s)u(s)|\,ds \ge$$
$$\ge -\widetilde{\ell} \int_a^b u'^2(s)\,ds - \ell_0 \ \text{ for } \ a < t_0 \le a_0, \ \ b_0 \le t < b$$

is fulfilled. Using now Theorem 1.2, we can see that problem $(1.4), (1.2_1), (1.3)$ (problem $(1.4), (1.2_2), (1.3)$) is solvable.

Let u_1 and u_2 be two arbitrary solutions of that problem. Assume $u(t) = u_2(t) - u_1(t)$. Then by virtue of condition (1.22), we have

$$\liminf_{t_0 \to a,\, t \to b} \int_{t_0}^t u''(s)u(s)\,ds \ge -\ell \int_a^b u'^2(s)\,ds.$$

This inequality, by Lemma 2.6 and equality $u(a+) = 0$, yields $u(t) \equiv 0$. Therefore, problem $(1.4), (1.2_1), (1.3)$ (problem $(1.4), (1.2_2), (1.3)$) is uniquely solvable for an arbitrary $h \in L^2_{2,2}(]a,b[)$ $(h \in L^2_{2,0}(]a,b[))$.

Let u_0 and u_h be, respectively, the solutions of problems $(1.1), (1.2_1), (1.3)$ and $(1.4), (1.2_1), (1.3)$ (of problems $(1.1), (1.2_2), (1.3)$ and $(1.4), (1.2_2), (1.3)$). Then by condition (1.22), we obtain

$$\liminf_{t_0\to a,\, t\to b} \int\limits_{t_0}^{t} (u_h''(s) - u_0''(s))(u_h(s) - u_0(s))\,ds \geq$$
$$\geq -\ell \int\limits_a^b (u_h'(s) - u_0'(s))^2\,ds - \int\limits_a^b |h(s)|\,|u_h(s) - u_0(s)|\,ds.$$

Hence, by Lemma 2.8, we have the estimate

$$\|u_h' - u_0'\|_{L^2} \leq r\|h\|_{L^2_{2,2}} \quad \Big(\|u_h' - u_0'\|_{L^2} \leq r\|h\|_{L^2_{2,0}}\Big),$$

where

$$r = \frac{2}{(1-\ell)(b-a)} \quad \Big(r = \frac{2}{1-\ell}\Big)$$

is a number, independent of h. Consequently, the problem under consideration is stable with respect to small perturbations of the right-hand member of Eq. (1.1). □

Proof of Theorem 1.4. Suppose

$$f(u)(t) = f_0(t, u(\tau_1(t)), u'(\tau_2(t))). \tag{3.9}$$

Conditions (1.24_1) (conditions (1.24_2)) and the continuity of the function f_0 in the last two arguments imply that the operator f satisfies condition (1.7). On the other hand, according to inequality (1.24_1) (inequality (1.24_2)), for an arbitrary $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ almost everywhere on $]a,b[$ the inequality

$$|f(u)(t)| \leq w_1(u)(t) + q\Big(t, \Big(\frac{2}{b-a}\Big)^{1/2}\|u'\|_{L^2}\Big)$$
$$\Big(|f(u)(t)| \leq w_2(u)(t) + q(t, \|u'\|_{L^2})\Big)$$

holds, where w_1 and w_2 are the operators, given by equalities (2.1), (2.2) and (2.5). Hence, by conditions (1.19_1) and (1.25_1) (conditions (1.19_2) and (1.25_2)) and Lemma 2.5 follows the existence of constants $\ell \in]0,1[$ and $\ell_0 \geq 0$ such that for arbitrary $a_0 \in]a,b[$, $b_0 \in]a_0,b[$ and $u \in D_1(]a,b[)$ $(u \in D_2(]a,b]))$ inequality (1.12) holds.

Therefore, the operator f, given by equality (3.9), satisfies all the conditions of Theorem 1.2, which guarantees the solvability of problem $(1.1'), (1.2_1), (1.3)$ (problem $(1.1'), (1.2_2), (1.3)$). □

Proof of Theorem 1.5. Suppose f is an operator, given by equality (3.9). Then, by virtue of conditions (1.26_1) and (1.27) (conditions (1.26_2) and (1.27)), the operator f satisfies condition (1.7). On the other hand, according to condition (1.26_1) (condition (1.26_2)), for arbitrary $u_i \in D_1(]a,b[)$ $(i=1,2)$ (for arbitrary $u_i \in D_2(]a,b])$ $(i=1,2)$) almost everywhere on $]a,b[$ the inequality

$$(f(u_2)(t)-f(u_1)(t))(u_2(t)-u_1(t)) \geq -w_1(u_2-u_1)(t)|u_2(t)-u_1(t)|$$
$$\Big((f(u_2)(t)-f(u_1)(t))(u_2(t)-u_1(t)) \geq -w_2(u_2-u_1)(t)|u_2(t)-u_1(t)|\Big)$$

is fulfilled, where w_1 and w_2 are the operators, given by equalities (2.1), (2.2) and (2.5). Hence, by inequality (1.19_1) (inequality (1.19_2)) and Lemma 2.3, follows that for arbitrary $a_0 \in]a,b[$, $b_0 \in]a_0,b[$ and $u_i \in D_1(]a,b[)$ $(u_i \in D_2(]a,b]))$ $(i=1,2)$ inequality (1.22) holds, where

$$\ell = \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} < 1 \quad \Big(4\ell_1 + 2\ell_2 < 1\Big).$$

If now we apply Theorem 1.3, then the validity of Theorem 1.5 becomes evident.

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