ON PERIODIC-TYPE SOLUTIONS OF SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

I. KIGURADZE

We establish nonimprovable, in a certain sense, sufficient conditions for the existence of a unique periodic-type solution for systems of linear ordinary differential equations.

1. Formulation of the problem and statement of the main results

Let n_1 and n_2 be natural numbers, $\omega > 0$, $\Lambda_i \in \mathbb{R}^{n_i \times n_i}$ $(i = 1, 2)$ nonsingular matrices, and $\mathcal{P}_{ik} : \mathbb{R} \to \mathbb{R}^{n_i \times n_k}$ $(i, k = 1, 2)$ and $q_i : \mathbb{R} \to \mathbb{R}^{n_i}$ $(i = 1, 2)$ matrix and vector functions whose components are Lebesgue integrable on each compact interval. We consider the problem on the existence and uniqueness of a solution of the linear differential system

$$\frac{dx_i}{dt} = \mathcal{P}_{i1}(t)x_1 + \mathcal{P}_{i2}(t)x_2 + q_i(t) \quad (i = 1, 2), \tag{1.1}$$

satisfying the conditions

$$x_i(t + \omega) = \Lambda_i x_i(t) \quad \text{for } t \in \mathbb{R} \ (i = 1, 2). \tag{1.2}$$

When Λ_1 and Λ_2 are unit matrices, this problem becomes the well-known problem on a periodic solution which has been the subject of numerous studies (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein).

In this paper, sufficient conditions for the unique solvability of problem (1.1), (1.2) are established, which are nonimprovable in a certain sense and in particular provide new results on the existence of a unique ω-periodic solution of system (1.1).

The following notation is used in the paper:

(1) $\mathbb{R}$ is the set of real numbers;
(2) $\mathbb{R}^n$ is the n-dimensional real Euclidean space;
(3) $x = (\xi_i)_{i=1}^n \in \mathbb{R}^n$ is the column vector with components $\xi_1, \ldots, \xi_n$,

$$|x| = (|\xi_i|)_{i=1}^n, \qquad \|x\| = \left(\sum_{i=1}^n \xi_i^2 \right)^{1/2}; \tag{1.3}$$

(4) $x \cdot y$ is the scalar product of vectors $x, y \in \mathbb{R}^n$;

(5) $\mathbb{R}^{m \times n}$ is the space of $m \times n$ matrices $X = (\xi_{ik})_{i,k=1}^{m,n}$ with components ξ_{ik} $(i = 1, \dots, m; k = 1, \dots, n)$,

$$|X| = (|\xi_{ik}|)_{i,k=1}^{m,n}, \qquad \|X\| = \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \xi_{ik}^2 \right)^{1/2}; \tag{1.4}$$

(6) X^* is the transposed matrix of the matrix X;

(7) E_n is the unit $n \times n$ matrix;

(8) $\det(X)$ is the determinant of the matrix X;

(9) $r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n \times n}$;

(10) if $X \in \mathbb{R}^{n \times n}$, then $\lambda_0(X)$ is a minimal eigenvalue of the matrix $(1/2)(X + X^*)$.

Inequalities between the matrices and the vectors are understood componentwise. Throughout the paper, it will be assumed that

$$\mathscr{P}_{ik}(t + \omega) = \Lambda_i \mathscr{P}_{ik}(t) \Lambda_k^{-1}, \quad q_i(t + \omega) = \Lambda_i q_i(t) \quad \text{for } t \in \mathbb{R} \ (i, k = 1, 2). \tag{1.5}$$

For each $i \in \{1, 2\}$, consider the differential system

$$\frac{dx}{dt} = \mathscr{P}_{ii}(t) x \tag{1.6}$$

and denote by X_i its fundamental matrix satisfying the initial condition

$$X_i(0) = E_{n_i}. \tag{1.7}$$

If, however, the matrix $\Lambda_i - X_i(\omega)$ is nonsingular, then it is assumed that

$$G_i(t, \tau) = X_i(t) \left(X_i^{-1}(\omega) \Lambda_i - E_{n_i} \right)^{-1} X_i^{-1}(\tau). \tag{1.8}$$

For each $i \in \{1, 2\}$, we define a matrix function $\Lambda_{i0} : [0, 3\omega] \to \mathbb{R}^{n_i \times n_i}$ in the following manner:

$$\Lambda_{i0}(s) = E_{n_i} \quad \text{for } 0 \le s \le \omega, \tag{1.9}$$

$$\Lambda_{i0}(s) = |\Lambda_i^k| \quad \text{for } k\omega < s \le (k+1)\omega \ (k = 1, 2). \tag{1.10}$$

THEOREM 1.1. *Let*

$$\det(\Lambda_i - X_i(\omega)) \ne 0 \quad (i = 1, 2), \tag{1.11}$$

and there exists a nonnegative matrix $A \in \mathbb{R}^{n_1 \times n_1}$ such that $r(A) < 1$, and

$$\int_t^{t+\omega} \int_\tau^{\tau+\omega} |G_1(t,\tau) \mathscr{P}_{12}(\tau) G_2(\tau, s) \mathscr{P}_{12}(s)| \Lambda_{10}(s) \, ds \, d\tau \le A \quad \text{for } 0 \le t \le \omega. \tag{1.12}$$

Then problem (1.1), (1.2) has a unique solution.

Example 1.2. Let $n_1 = n_2 = 1$, $\Lambda_1 = \Lambda_2 = 1$, $q_i(t) \equiv 0$, $\mathcal{P}_{i1}(t) \equiv p_i(t)$, and $\mathcal{P}_{i2}(t) \equiv -p_i(t)$, where $p_i : \mathbb{R} \to]0, +\infty[$ $(i = 1, 2)$ are the integrable on $[0, \omega]$ ω-periodic functions. Then conditions (1.5), (1.11), and (1.12), where $A = 1$, are fulfilled. On the other hand, in the considered case, system (1.1) has the form

$$\frac{dx_i}{dt} = p_i(t)(x_1 - x_2) \quad (i = 1, 2) \tag{1.13}$$

and therefore problem (1.1), (1.2) has an infinite set of solutions

$$\{(x_1, x_2) : x_1(t) \equiv x_2(t) \equiv c, \; c \in \mathbb{R}\}. \tag{1.14}$$

This example shows that the condition $r(A) < 1$ in Theorem 1.1 is nonimprovable and it cannot be replaced by the condition $r(A) \leq 1$.

Theorem 1.3. *Let*

$$X_i(\omega) = \Lambda_1, \qquad \det\left(\Lambda_2 - X_2(\omega)\right) \neq 0, \tag{1.15}$$

$$\det\left(Q_0\right) \neq 0, \tag{1.16}$$

where

$$Q_0 = \int_0^\omega X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)d\tau,$$
$$Q(t) = \int_t^{t+\omega} G_2(t,s)\mathcal{P}_{21}(s)X_1(s)ds. \tag{1.17}$$

Let, further, there exist a nonnegative matrix $A \in \mathbb{R}^{n_2 \times n_2}$ such that $r(A) < 1$, and

$$\int_t^{t+\omega} \left[H(t,\tau) + \int_\tau^{\tau+\omega} \left| Q(t)Q_0^{-1}X_1^{-1}(\tau)\mathcal{P}_{12}(\tau) \right| H(\tau,s)ds \right] d\tau \leq A \quad \text{for } 0 \leq t \leq \omega, \tag{1.18}$$

where

$$H(t,\tau) = \int_0^\tau \left| G_2(t,\tau)\mathcal{P}_{21}(\tau)X_1(\tau)X_1^{-1}(s)\mathcal{P}_{12}(s) \right| \Lambda_{20}(s)ds. \tag{1.19}$$

Then problem (1.1), (1.2) has a unique solution.

Example 1.4. Consider the problem

$$\frac{dx_1}{dt} = B_1 x_2, \qquad \frac{dx_2}{dt} = \varepsilon B_2 x_1 + B x_2,$$
$$x_i(t + \omega) = x_i(t) \quad \text{for } t \in \mathbb{R} \; (i = 1, 2), \tag{1.20}$$

where ε is a positive constant, $B_1 \in \mathbb{R}^{n_1 \times n_2}$, $B_2 \in \mathbb{R}^{n_2 \times n_1}$, $B \in \mathbb{R}^{n_2 \times n_2}$, and $\det(B) \neq 0$. This problem is obtained from problem (1.1), (1.2) when $\Lambda_i = E_{n_i}$ $(i = 1, 2)$, $\mathcal{P}_1$ is a zero matrix, $\mathcal{P}_{12}(t) \equiv B_1$, $\mathcal{P}_{21}(t) \equiv \varepsilon B_2$, $\mathcal{P}_{22}(t) \equiv B$, and $q(t) \equiv 0$. It is obvious that conditions

(1.5) and (1.15) are fulfilled for this problem. On the other hand, by virtue of (1.17) and (1.19), we have

$$Q(t) \equiv \varepsilon B^{-1} B_2, \qquad Q_0 = \varepsilon \omega B_1 B^{-1} B_2,$$
$$H(t,\tau) = \tau \varepsilon \, | \, (\exp(-\omega B) - E_{n_2})^{-1} \exp((t-\tau)B) B_2 B_1 \, |. \tag{1.21}$$

Therefore, condition (1.16) is fulfilled if and only if

$$\det(B_1 B^{-1} B_2) \neq 0. \tag{1.22}$$

If the latter inequality is fulfilled, then, by Theorem 1.3, there exists $\varepsilon_0 > 0$ such that, for arbitrary $\varepsilon \in \,]0, \varepsilon_0[$, problem (1.20) has only a trivial solution. If $\det(B_1 B^{-1} B_2) = 0$, then, for arbitrary ε, problem (1.20) has an infinite set of solutions

$$\{(x_1, x_2) : x_1(t) \equiv c x_{10}, \, x_2(t) = c x_{20}, \, c \in \mathbb{R}\}, \tag{1.23}$$

where $x_{10} \in \mathbb{R}^{n_1}$ is the eigenvector of the matrix $B_1 B^{-1} B_2$ corresponding to the zero eigenvalue and $x_{20} = -\varepsilon B^{-1} B_2 x_{10}$.

Example 1.4 shows that condition (1.16) is essential and cannot be omitted.

THEOREM 1.5. *Let there exist a matrix $A \in \mathbb{R}^{n_1 \times n_2}$, symmetric matrices $A_i \in \mathbb{R}^{n_i \times n_i}$ $(i = 1, 2)$, and an integrable function $\delta : [0, \omega] \to [0, +\infty[$ such that*

$$\Lambda_2^* A \Lambda_1 = A, \qquad \Lambda_i^* A_i \Lambda_i = A_i \quad (i = 1, 2) \tag{1.24}$$

and the following inequalities are fulfilled almost everywhere on $[0, \omega]$:

$$\lambda_0 (A_1 \mathcal{P}_{11}(t) + A^* \mathcal{P}_{21}(t)) \geq \delta(t), \qquad \lambda_0 (A_2 \mathcal{P}_{22}(t) + A \mathcal{P}_{12}(t)) \geq \delta(t), \tag{1.25}$$
$$\delta(t) \geq p(t), \tag{1.26}$$

where

$$p(t) = \frac{1}{2} (\|A_1 \mathcal{P}_{12}(t) + A^* \mathcal{P}_{22}(t)\| + \|A_2 \mathcal{P}_{21}(t) + A \mathcal{P}_{11}(t)\|). \tag{1.27}$$

If, moreover,

$$\int_0^\omega (\delta(t) - p(t)) \, dt > 0, \tag{1.28}$$

then problem (1.1), (1.2) has a unique solution.

Example 1.2 shows that conditions (1.5), (1.24), (1.25), and (1.26) do not guarantee the unique solvability of problem (1.1), (1.2). Therefore, condition (1.28) in Theorem 1.5 is essential and cannot be omitted.

2. Auxiliary propositions

In this section, we consider the problem

$$\frac{dx}{dt} = \mathscr{P}(t)x + q(t), \tag{2.1}$$

$$x(t + \omega) = \Lambda x(t) \quad \text{for } t \in \mathbb{R}, \tag{2.2}$$

assuming that $\Lambda \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, and $\mathscr{P} : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $q : \mathbb{R} \to \mathbb{R}^n$ are matrix and vector functions with components Lebesgue integrable on $[0, \omega]$ and satisfying the conditions

$$\mathscr{P}(t + \omega) = \Lambda \mathscr{P}(t)\Lambda^{-1} \quad \text{for } t \in \mathbb{R}, \tag{2.3}$$

$$q(t + \omega) = \Lambda q(t) \quad \text{for } t \in \mathbb{R}. \tag{2.4}$$

We denote by X the fundamental matrix of the homogeneous differential system

$$\frac{dx}{dt} = \mathscr{P}(t)x, \tag{2.5}$$

satisfying the initial condition

$$X(0) = E_n. \tag{2.6}$$

Condition (2.3) immediately implies the following lemma.

LEMMA 2.1. *The matrix function X satisfies the identity*

$$X(t + \omega) = \Lambda X(t)\Lambda^{-1}X(\omega) \quad \text{for } t \in \mathbb{R}. \tag{2.7}$$

LEMMA 2.2. *Problem (2.5), (2.2) has only a trivial solution if and only if*

$$\det\left(\Lambda - X(\omega)\right) \neq 0. \tag{2.8}$$

Proof. Let x be an arbitrary solution of system (2.5). Then

$$x(t) = X(t)c \quad \text{for } t \in \mathbb{R}, \tag{2.9}$$

where $c \in \mathbb{R}^n$. Hence, by Lemma 2.1, it follows that x is a solution of problem (2.5), (2.2) if and only if

$$\left(\Lambda X(t) - \Lambda X(t)\Lambda^{-1}X(\omega)\right)c = 0 \quad \text{for } t \in \mathbb{R}. \tag{2.10}$$

However, for the latter identity to be fulfilled, it is necessary and sufficient that c be a solution of the system of algebraic equations

$$\left(\Lambda - X(\omega)\right)c = 0. \tag{2.11}$$

Therefore, problem (2.5), (2.2) has only a trivial solution if and only if the latter system has only a trivial solution, that is, if (2.8) is fulfilled. $\square$

LEMMA 2.3. *Problem (2.1), (2.2) is uniquely solvable if and only if the corresponding homogeneous problem (2.5), (2.2) has only a trivial solution, that is, if inequality (2.8) is fulfilled. Moreover, if (2.8) is fulfilled, then the solution of problem (2.1), (2.2) admits the representation*

$$x(t) = \int_t^{t+\omega} G(t,s)q(s)ds \quad \text{for } t \in \mathbb{R},$$

(2.12)

where

$$G(t,s) = X(t)\left(X^{-1}(\omega)\Lambda - E_n\right)^{-1}X^{-1}(s).$$

(2.13)

Proof. By Lemma 2.2, to prove Lemma 2.3, it is sufficient to establish that if inequality (2.8) is fulfilled, then the vector function x given by equality (2.12) is a solution of problem (2.1), (2.2).

According to (2.7) and (2.13), we have

$$\frac{\partial G(t,s)}{\partial t} = \mathcal{P}(t)G(t,s) \quad \text{for } s \in \mathbb{R} \text{ and almost all } t \in \mathbb{R},$$

$$G(t,t+\omega)\Lambda - G(t,t) = X(t)\left(X^{-1}(\omega)\Lambda - E_n\right)^{-1}\left(X^{-1}(t+\omega)\Lambda - X^{-1}(t)\right)$$

$$= X(t)\left(X^{-1}(\omega)\Lambda - E_n\right)^{-1}\left(X^{-1}(\omega)\Lambda - E_n\right)X^{-1}(t)$$

$$= E_n \quad \text{for } t \in \mathbb{R},$$

(2.14)

$$G(t+\omega,s+\omega) = \Lambda X(t)\Lambda^{-1}X(\omega)\left(X^{-1}(\omega)\Lambda - E_n\right)^{-1}$$

$$\times \left(\Lambda^{-1}X(\omega)\right)^{-1}X^{-1}(t)\Lambda^{-1}$$

$$= \Lambda G(t,s)\Lambda^{-1} \quad \text{for } s \in \mathbb{R}, \ t \in \mathbb{R}.$$

If, along with these identities, we also take into consideration condition (2.4), then, from (2.12), we obtain

$$\frac{dx(t)}{dt} = \mathcal{P}(t)x(t) + G(t,t+\omega)q(t+\omega) - G(t,t)q(t)$$

$$= \mathcal{P}(t)x(t) + \left(G(t,t+\omega)\Lambda - G(t,t)\right)q(t)$$

$$= \mathcal{P}(t)x(t) + q(t) \quad \text{for almost all } t \in \mathbb{R}^n,$$

$$x(t+\omega) = \int_{t+\omega}^{t+2\omega} G(t+\omega,s)q(s)ds = \int_t^{t+\omega} G(t+\omega,s+\omega)q(s+\omega)ds$$

(2.15)

$$= \Lambda \int_t^{t+\omega} G(t,s)q(s)ds = \Lambda x(t) \quad \text{for } t \in \mathbb{R}.$$

Thus x is a solution of problem (2.1), (2.2).   □

3. Proofs of the main results

Proof of Theorem 1.1. By Lemma 2.3, it is sufficient to show that the homogeneous problem

$$\frac{dx_i}{dt} = \mathscr{P}_{i1}(t)x_1 + \mathscr{P}_{i2}(t)x_2, \tag{3.1}$$

$$x_i(t+\omega) = \Lambda_i x_i(t) \quad \text{for } t \in \mathbb{R} \ (i = 1,2) \tag{3.2}$$

has only a trivial solution.

Let (x_1, x_2) be an arbitrary solution of this problem. By virtue of Lemma 2.3, condition (1.11) and the equalities

$$\mathscr{P}_{12}(t+\omega)x_2(t+\omega) = \Lambda_1 \mathscr{P}_{12}(t)x_2(t),$$
$$\mathscr{P}_{21}(t+\omega)x_1(t+\omega) = \Lambda_2 \mathscr{P}_{21}(t)x_1(t) \quad \text{for almost all } t \in \mathbb{R} \tag{3.3}$$

guarantee the validity of the representations

$$x_1(t) = \int_t^{t+\omega} G_1(t,s)\mathscr{P}_{12}(s)x_2(s)ds,$$
$$x_2(t) = \int_t^{t+\omega} G_2(t,s)\mathscr{P}_{21}(s)x_1(s)ds. \tag{3.4}$$

Therefore,

$$x_1(t) = \int_t^{t+\omega} \int_\tau^{\tau+\omega} G_1(t,\tau)\mathscr{P}_{12}(\tau)G_2(\tau,s)\mathscr{P}_{21}(s)x_1(s)ds. \tag{3.5}$$

Let

$$x_1(t) = \left(x_{1k}(t)\right)_{k=1}^{n_1},$$
$$\rho_k = \max\left\{\left|x_{1k}(t)\right| : 0 \le t \le \omega\right\} \quad (k = 1,\ldots,n_1), \rho = (\rho_k)_{k=1}^{n_1}. \tag{3.6}$$

Then by (1.9), (1.10) for $i = 1$, we have

$$\left|x_1(s)\right| \le \Lambda_{10}(s)\rho \quad \text{for } 0 \le s \le 3\omega. \tag{3.7}$$

If, along with this, we also take into consideration inequality (1.12), then, from representation (3.5), we obtain

$$\left|x_1(t)\right| \le A\rho \quad \text{for } 0 \le t \le \omega. \tag{3.8}$$

Hence $\rho \le A\rho$ and, therefore,

$$\left(E_{n_1} - A\right)\rho \le 0. \tag{3.9}$$

According to the condition $r(A) < 1$ and the nonnegativeness of the matrix A, the matrix $E_{n_1} - A$ is nonsingular and $(E_{n_1} - A)^{-1}$ is nonnegative. Hence the multiplication of the

latter vector inequality by $(E_{n_1} - A)^{-1}$ gives $\rho \leq 0$. Therefore, $\rho = 0$, that is,

$$x_1(t) = 0 \quad \text{for } 0 \leq t \leq \omega. \tag{3.10}$$

By virtue of this equality, from (3.4), it follows that $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1, 2)$.   □

Proof of Theorem 1.3. Let (x_1, x_2) be an arbitrary solution of problem (3.1), (3.2). Then by the Cauchy formula, we have

$$x_1(t) = X_1(t)c + \int_0^t X_1(t)X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)x_2(\tau)d\tau, \tag{3.11}$$

where $c \in \mathbb{R}^{n_1}$. On the other hand, by Lemma 2.3, the nonsingularity of the matrix $\Lambda_2 - X_2(\omega)$ and the equality

$$\mathscr{P}_{21}(t + \omega)x_1(t + \omega) = \Lambda_2 \mathscr{P}_{21}(t)x_1(t) \quad \text{for almost all } t \in \mathbb{R} \tag{3.12}$$

guarantee the validity of the representation

$$x_2(t) = \int_t^{t+\omega} G_2(t, \tau)\mathscr{P}_{21}(\tau)x_1(\tau)d\tau. \tag{3.13}$$

Hence, by virtue of equalities (1.17) and (3.11), it follows that

$$x_2(t) = Q(t)c + \int_t^{t+\omega} z(t, \tau)d\tau, \tag{3.14}$$

where

$$z(t, \tau) = \int_0^\tau G_2(t, \tau)\mathscr{P}_{21}(\tau)X_1(\tau)X_1^{-1}(s)\mathscr{P}_{12}(s)x_2(s)ds. \tag{3.15}$$

By Lemma 2.1 and the equality $X_1(\omega) = \Lambda_1$, we have

$$X_1(t + \omega) = \Lambda_1 X(t) \quad \text{for } t \in \mathbb{R}. \tag{3.16}$$

Therefore, from (3.11), we find

$$x_1(t + \omega) = \Lambda_1 X_1(t)c + \Lambda_1 \int_0^{t+\omega} X_1(t)X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)x_2(\tau)d\tau. \tag{3.17}$$

Hence, by (3.2), it follows that

$$x_1(t) = X_1(t)c + \int_0^{t+\omega} X_1(t)X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)x_2(\tau)d\tau. \tag{3.18}$$

If now we again apply representation (3.11), then it becomes clear that the identity

$$\int_t^{t+\omega} X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)x_2(\tau)d\tau = 0 \quad \text{for } t \in \mathbb{R} \tag{3.19}$$

is valid.

Using (3.14), from the latter identity, we find

$$\tilde{Q}(t)c = -\int_t^{t+\omega}\int_\tau^{\tau+\omega} X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)z(\tau,s)ds\,d\tau, \tag{3.20}$$

where

$$\tilde{Q}(t) = \int_t^{t+\omega} X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)Q(\tau)d\tau. \tag{3.21}$$

By Lemma 2.1,

$$G_2(t+\omega,s+\omega) = \Lambda_2 G_2(t,s)\Lambda_2^{-1}. \tag{3.22}$$

If, along with this identity, we also take into account identities (1.5) and (3.16), then we obtain

$$Q(t+\omega) = \int_t^{t+\omega} G_2(t+\omega,s+\omega)\mathscr{P}_{21}(s+\omega)X_1(s+\omega)ds = \Lambda_2 Q(t). \tag{3.23}$$

Therefore, from (1.17) and (3.21), we have

$$\begin{aligned}
\tilde{Q}(t) &= \int_t^\omega X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)Q(\tau)d\tau \\
&\quad + \int_0^t X_1^{-1}(\tau+\omega)\mathscr{P}_{12}(\tau+\omega)Q(\tau+\omega)d\tau \\
&= \int_t^\omega X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)Q(\tau)d\tau \\
&\quad + \int_0^t X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)Q(\tau)d\tau = Q_0 \quad \text{for } t \in \mathbb{R}.
\end{aligned} \tag{3.24}$$

By virtue of this fact and condition (1.16), from (3.11), (3.14), and (3.20), we get

$$\begin{aligned}
x_1(t) &= \int_0^t X_1(t)X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)x_2(\tau)d\tau \\
&\quad - X_1(t)\int_0^\omega\int_\tau^{\tau+\omega} Q_0^{-1}X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)z(\tau,s)ds\,d\tau,
\end{aligned} \tag{3.25}$$

$$x_2(t) = \int_t^{t+\omega}\left(z(t,\tau) - \int_\tau^{\tau+\omega} Q(t)Q_0^{-1}X_1^{-1}(\tau)\mathscr{P}_{12}(\tau)z(\tau,s)ds\right)d\tau. \tag{3.26}$$

Let $x_2(t) = (x_{2k}(t))_{k=1}^{n_2}$,

$$\rho_k = \max\{|x_{2k}(t)| : 0 \le t \le \omega\} \quad (k = 1,\ldots,n_2), \ \rho = (\rho_k)_{k=1}^{n_2}. \tag{3.27}$$

Then, by (1.9), (1.10) for $i = 2$, we have

$$|x_2(s)| \le \Lambda_{20}(s)\rho \quad \text{for } 0 \le s \le 3\omega. \tag{3.28}$$

By this inequality and the notation (1.19) and (3.15), we have

$$|z(t,\tau)| \le H(t,\tau)\rho \quad \text{for } t \in \mathbb{R}, \ 0 \le \tau \le 3\omega. \tag{3.29}$$

Due to this estimate and inequality (1.18), from (3.26), we find

$$|x_2(t)| \leq A\rho \quad \text{for } 0 \leq t \leq \omega. \tag{3.30}$$

Hence it is clear that $\rho \leq A\rho$ and, therefore,

$$(E_{n_2} - A)\rho \leq 0. \tag{3.31}$$

By virtue of the condition $r(A) < 1$ and the nonnegativeness of the matrix A, the latter inequality implies $\rho = 0$. Therefore,

$$x_2(t) = 0, \quad z(t,\tau) = 0 \quad \text{for } 0 \leq t \leq \omega,\ 0 \leq \tau \leq 3\omega, \tag{3.32}$$

due to which we find from (3.2) and (3.25) that $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1,2)$. Thus problem (3.1), (3.2) has only a trivial solution. By Lemma 2.3, this fact guarantee the unique solvability of problem (1.1), (1.2). □

Proof of Theorem 1.5. By virtue of Lemma 2.3, it is sufficient to establish that problem (3.1), (3.2) has only a trivial solution.

Let (x_1, x_2) be an arbitrary solution of problem (3.1), (3.2) and

$$u(t) = \frac{1}{2}(A_1 x_1(t) \cdot x_1(t) + A_2 x_2(t) \cdot x_2(t)) + A x_1(t) \cdot x_2(t). \tag{3.33}$$

Then

$$
\begin{aligned}
u'(t) &= A_1 x_1'(t) \cdot x_1(t) + A_2 x_2'(t) \cdot x_2(t) + A x_1'(t) \cdot x_2(t) + A^* x_2'(t) \cdot x_1(t) \\
&= (A_1 \mathscr{P}_{11}(t) + A^* \mathscr{P}_{21}(t)) x_1(t) \cdot x_1(t) \\
&\quad + (A_2 \mathscr{P}_{22}(t) + A \mathscr{P}_{12}(t)) x_2(t) \cdot x_2(t) + (A_1 \mathscr{P}_{12}(t) + A^* \mathscr{P}_{22}(t)) x_2(t) \cdot x_1(t) \\
&\quad + (A_2 \mathscr{P}_{21}(t) + A \mathscr{P}_{11}(t)) x_1(t) \cdot x_2(t) \quad \text{for almost all } t \in \mathbb{R}.
\end{aligned}
\tag{3.34}
$$

However, by conditions (1.25) and the Schwartz inequality, for almost all $t \in [0,\omega]$, we have

$$
\begin{aligned}
(A_1 \mathscr{P}_{11}(t) &+ A^* \mathscr{P}_{21}(t)) x_1(t) \cdot x_1(t) \geq \delta(t) \|x_1(t)\|^2, \\
(A_2 \mathscr{P}_{22}(t) &+ A \mathscr{P}_{12}(t)) x_2(t) \cdot x_2(t) \geq \delta(t) \|x_2(t)\|^2, \\
(A_1 \mathscr{P}_{12}(t) &+ A^* \mathscr{P}_{22}(t)) x_2(t) \cdot x_1(t) + (A_2 \mathscr{P}_{21}(t) + A \mathscr{P}_{11}(t)) x_1(t) \cdot x_2(t) \\
&\leq 2p(t) \|x_1(t)\| \|x_2(t)\| \leq p(t) \left(\|x_1(t)\|^2 + \|x_2(t)\|^2 \right),
\end{aligned}
\tag{3.35}
$$

where p is the function given by equality (1.27). Therefore,

$$u'(t) \geq (\delta(t) - p(t)) \left(\|x_1(t)\|^2 + \|x_2(t)\|^2 \right) \quad \text{for almost all } t \in [0,\omega]. \tag{3.36}$$

On the other hand, by virtue of (1.24) and (3.2), we have

$$
\begin{aligned}
u(\omega) &= \frac{1}{2}\big(A_1\Lambda_1 x_1(0) \cdot \Lambda_1 x_1(0) + A_2\Lambda_2 x_2(0) \cdot \Lambda_2 x_2(0)\big) \\
&\quad + A\Lambda_1 x_1(0) \cdot \Lambda_2 x_2(0) \\
&= \frac{1}{2}\big(\Lambda_1^* A_1\Lambda_1 x_1(0) \cdot x_1(0) + \Lambda_2^* A_2\Lambda_2 x_2(0) \cdot x_2(0)\big) \\
&\quad + \Lambda_2^* A\Lambda_1 x_1(0) \cdot x_2(0) = u(0).
\end{aligned}
\tag{3.37}
$$

Thus

$$
0 = \int_0^\omega u'(t)dt \geq \int_0^\omega \big(\delta(t) - p(t)\big)\Big(\big\|x_1(t)\big\|^2 + \big\|x_2(t)\big\|^2\Big)dt.
\tag{3.38}
$$

Hence, by virtue of conditions (1.26) and (1.28), it follows that there exists $t_0 \in [0,\omega]$ such that

$$
x_i(t_0) = 0 \quad (i = 1, 2).
\tag{3.39}
$$

Therefore, $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1, 2)$ since system (3.1) with the zero initial conditions has only a trivial solution. $\qquad\square$

Acknowledgment

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I. Kiguradze: A. Razmadze Mathematical Institute, Georgian Academy of Sciences, Tbilisi 0193, Georgia

E-mail address: kig@rmi.acnet.ge