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ORDINARY
DIFFERENTIAL EQUATIONS

Oscillatory Properties of Higher-Order Advance Functional-Differential Equations

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1. MAIN RESULTS

Let C_{loc} be the space of continuous functions $x : [0, +\infty[\to \mathbf{R}$ with the topology of uniform convergence on each closed interval in $[0, +\infty[$, let L_{loc} be the space of locally Lebesgue integrable functions with the topology of convergence in mean on each closed interval in $[0, +\infty[$, and let $f : C_{\text{loc}} \to L_{\text{loc}}$ be a continuous operator. Consider the functional-differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1.1}$$

of order $n \geq 2$. Numerous papers (e.g., see [1–17] and the bibliography therein) deal with oscillatory properties of such equations. Nevertheless, these properties are little studied for the case in which f is an advance operator. In the present paper, we try to fill this gap. Some special cases of the theorems proved below were announced in [18, 19].

We introduce the following notions.

Definition 1.1. An operator $g : C_{\text{loc}} \to L_{\text{loc}}$ is called an *advance operator* if $g(u)(t) = g(v)(t)$ for almost all $t \in]0, +\infty[$ and for arbitrary functions $u, v \in C_{\text{loc}}$ satisfying the relation $u(s) = v(s)$ for $s \geq t$.

Definition 1.2. An operator $g : C_{\text{loc}} \to L_{\text{loc}}$ is said to be *nondecreasing* if $g(u)(t) \geq g(v)(t)$ for almost all $t \in]0, +\infty[$ and for any $u, v \in C_{\text{loc}}$ such that $u(s) \geq v(s)$ for $s \geq 0$.

We study the oscillatory properties of Eq. (1.1) under the assumption that

$$f : C_{\text{loc}} \to L_{\text{loc}} \quad \text{is a continuous odd advance operator} \tag{1.2}$$

and

$$(-1)^k f \quad \text{is nondecreasing} \tag{1.3$_k$}$$

for some $k \in \{1, 2\}$.

A solution of Eq. (1.1) on an interval $[a, +\infty[\subset [0, +\infty[$ is defined as a function $u : [a, +\infty[\to \mathbf{R}$ that, together with its first $n - 1$ derivatives, is absolutely continuous on each closed interval in $[a, +\infty[$ and satisfies Eq. (1.1) with $u(t) = u(a)$, $0 \leq t \leq a$, almost everywhere on $[a, +\infty[$.

A solution u of Eq. (1.1) defined on some interval $[a, +\infty[\subset [0, +\infty[$ is said to be *proper* if it does not vanish identically in an arbitrary neighborhood of $+\infty$.

If a proper solution has a sequence of zeros converging to $+\infty$, then it is said to be *oscillatory*, and otherwise it is said to be *nonoscillatory*.

Following [1, 2], we say that Eq. (1.1) has Property A if one of the following conditions holds: (i) each proper solution is oscillatory (for even n); (ii) each proper solution is oscillatory or satisfies the condition

$$\lim_{t \to +\infty} u^{(i)}(t) = 0 \qquad (i = 0, \dots, n - 1) \tag{1.4}$$

(for odd n).

Equation (1.1) has Property B if one of the following conditions holds: (i) each proper solution either is oscillatory, or satisfies condition (1.4), or satisfies the condition

$$\lim_{t\to+\infty} \left|u^{(i)}(t)\right| = +\infty \qquad (i = 0, \ldots, n-1) \tag{1.5}$$

(for even n); (ii) each proper solution either is oscillatory or satisfies condition (1.5) (for odd n).

Let $m \in \{0, \ldots, n-2\}$. By Definition 10.5 in [9], Eq. (1.1) has Property A_m (respectively, B_m) if each proper solution either is oscillating or satisfies the condition

$$\lim_{t\to+\infty} u^{(i)}(t) = 0 \qquad (i = m, \ldots, n-1) \tag{1.6}$$

[respectively, either is oscillatory, or satisfies condition (1.5), or satisfies condition (1.6)].

Proposition 1.1. *Let conditions* (1.2) *and* (1.3_1) [*respectively, conditions* (1.2) *and* (1.3_2)] *be satisfied. Equation* (1.1) *has Property* A (*respectively,* B) *if and only if it has Property* A_0 (*respectively,* B_0).

For each $\ell \in \{1, \ldots, n\}$, we set

$$h_\ell(t,x) = t^{\ell-1}x, \qquad f_\ell(t,x) = t^{n-\ell}\left|f\left(h_\ell(\cdot,x)\right)(t)\right|, \tag{1.7_ℓ}$$

and for arbitrary $a \geq 0$ and $c > 0$, we consider the initial value problem

$$v'(t) = \frac{1}{(n-1)!} f_\ell(t, v(t)), \qquad v(a) = c. \tag{1.8_ℓ}$$

Theorem 1.1. *Suppose that conditions* (1.2) *and* (1.3_k) *are satisfied and there exists an* $m \in \{0, \ldots, n-2\}$ *such that*

$$\int_0^{+\infty} f_{m+1}(t,\delta)dt = +\infty \quad \text{for} \quad \delta > 0 \tag{1.9}$$

and for arbitrary $a \geq 0$ *and* $c > 0$ *problem* (1.8_{ℓ_k}), *where*

$$\ell_k = m + 1 + 2^{-1}\left(1 + (-1)^{n-m+k}\right), \tag{1.10}$$

has no upper solution defined on $[a, +\infty[$. *If* $k = 1$ (*respectively,* $k = 2$), *then Eq.* (1.1) *has Property* A_m (*respectively,* B_m).

Theorem 1.2. *Suppose that conditions* (1.2) *and* (1.3_k) *are satisfied and there exist numbers* $m \in \{0, \ldots, n-2\}$ *and* $\delta_0 > 0$ *and a continuous function* $\omega :\ ]0, +\infty[\to]0, +\infty[$ *such that*

$$\int_1^{+\infty} \frac{dx}{\omega(x)} < +\infty, \tag{1.11}$$

$$f_{m+1}(t,x) \geq f_{m+1}(t,\delta_0)\,\omega(x) \quad \text{for} \quad t \geq 0, \quad x > 0. \tag{1.12}$$

If $k = 1$ (*respectively,* $k = 2$), *then condition* (1.9) *is necessary and sufficient for Eq.* (1.1) *to have Property* A_m (*respectively,* B_m).

Theorem 1.3. *Suppose that* $m \in \{0, \ldots, n-2\}$, $n - m$ *is odd* (*respectively, even*), *and conditions* (1.2) *and* (1.3_1) [*respectively,* (1.2) *and* (1.3_2)] *are satisfied. Moreover, suppose that there exists a locally integrable function* $g : [0, +\infty[\to [0, +\infty[$ *and a continuous function* $\omega :]0, +\infty[\to]0, +\infty[$ *such that*

$$f_{m+2}(t,x) \geq g(t)\omega(x) \quad \text{for} \quad t \geq 0, \quad x > 0 \tag{1.13}$$

and, in addition to (1.11),

$$\int_0^{+\infty} g(t)dt = +\infty. \tag{1.14}$$

Then condition (1.9) *is necessary and sufficient for Eq.* (1.1) *to have Property* A_m (*respectively,* B_m).

For arbitrary $\ell \in \{1, \dots, n\}$ and $a \in [0, +\infty[$, by $v_{a,\ell}$ we denote an upper solution, maximally extended to the right, of the problem

$$v'(t) = \frac{1}{\ell!(n-\ell)!} f_\ell(t, v(t)), \qquad v(a) = 1. \tag{1.15$_\ell$}$$

Theorems 1.1–1.3 deal with the case in which the intervals on which the functions $v_{a,\ell}$ ($\ell = m+1, \dots, n$) are defined are finite. In what follows, we consider the case in which the above-mentioned intervals coincide with $[a, +\infty[$.

For arbitrary $n_0 \in \{0, \dots, n-2\}$ and $k \in \{1, 2\}$, by $\mathscr{N}^{(k)}_{n_0,n}$ we denote the set of $\ell \in \{n_0, \dots, n\}$ such that $\ell + n + k$ is even.

Theorem 1.4. *Suppose that* $m \in \{0, \dots, n-2\}$, *and, in addition to* (1.2) *and* (1.3_k), *the conditions*

$$\int_0^{+\infty} f_\ell(t, \delta)dt = +\infty \quad \text{for} \quad \delta > 0 \quad (\ell = m+1, \dots, n) \tag{1.16}$$

are satisfied. Suppose that for arbitrary $a \geq 0$ *and* $\ell \in \{m+1, \dots, n-1\} \cap \mathscr{N}^{(k)}_{m+1,n}$, *problem* ($1.15_\ell$) *has an upper solution* $v_{a,\ell}$ *defined on* $[a, +\infty[$, *and*[1]

$$\int_a^{+\infty} t^{n-\ell-1} \left| f\left(w_{a,\ell}\right)(t)\right| dt = +\infty, \qquad w_{a,\ell}(t) = t^{\ell-1} v_{a,\ell}(t). \tag{1.17}$$

Then for $k = 1$ (*respectively,* $k = 2$), *Eq.* (1.1) *has Property* A_m (*respectively,* B_m).

Theorem 1.5. *Suppose that* $m \in \{0, \dots, n-2\}$, $n - m$ *is odd* (*respectively, even*), *and conditions* (1.2) *and* (1.3_1) [*respectively,* (1.2) *and* (1.3_2)] *are satisfied. Moreover, suppose that for each* $\delta > 0$, *there exist positive numbers* a, γ, *and* η *such that*

$$f_\ell(t, \delta) \geq \gamma t f_{m+1}(t, \eta) \quad \text{for} \quad t \geq a \quad (\ell = m+2, \dots, n). \tag{1.18}$$

Then condition (1.9) *is necessary and sufficient for Eq.* (1.1) *to have Property* A_m (*respectively,* B_m).

By way of example, we consider the functional-differential equation

$$u^{(n)}(t) = (-1)^k \sum_{i=1}^{j} \int_{\tau_0(t)}^{\tau(t)} |u(s)|^{\lambda_i} \operatorname{sgn} u(s) d_s p_i(s, t), \tag{1.19$_k$}$$

$$u^{(n)}(t) = (-1)^k \int_{\tau_0(t)}^{\tau(t)} u(s) d_s p(s, t), \tag{1.20$_k$}$$

where $k \in \{1, 2\}$, $j \geq 1$, and $\lambda_i > 0$ ($i = 1, \dots, j$). Moreover, throughout the following, we assume that $\tau, \tau_0 : [0, +\infty[\to [0, +\infty[$ are continuous functions and $p, p_i : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$

[1] If $k = 2$ and $m = n - 2$, then either condition (1.17) is omitted or $\mathscr{N}^{(2)}_{n-1,n} = \{n\}$.

$(i = 1, \dots, j)$ are nondecreasing functions of the first argument satisfying the conditions $\tau(t) > \tau_0(t) \geq t$ for $t \geq 0$, $p(s, \cdot) \in L_{\text{loc}}$ and $p_i(s, \cdot) \in L_{\text{loc}}$ $(i = 1, \dots, j)$ for $s \geq 0$.

Theorems 1.1–1.5 imply the following assertions.

Corollary 1.1. *Let* $j \geq 2$, $j_0 \in \{1, \dots, j-1\}$, $m \in \{0, \dots, n-2\}$, *and* $\lambda_i > 1$ $(i = j_0+1, \dots, j)$. *Then the condition*

$$\int_0^{+\infty} t^{n-m-1} \left[\sum_{i=j_0+1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{m\lambda_i} d_s p_i(s,t) \right] dt = +\infty \tag{1.21}$$

is sufficient for Eq. (1.19_1) [*respectively, Eq.* (1.19_2)] *to have Property* A_m (*respectively,* B_m). *If, moreover,*

$$\int_0^{+\infty} t^{n-m-1} \left[\sum_{i=1}^{j_0} \int_{\tau_0(t)}^{\tau(t)} s^{m\lambda_i} d_s p_i(s,t) \right] dt < +\infty, \tag{1.22}$$

then condition (1.21) *is also necessary.*

Corollary 1.2. *Let* $j \geq 2$, $j_0 \in \{1, \dots, j-1\}$, $\lambda_i > 1$ $(i = j_0+1, \dots, j)$, $m \in \{0, \dots, n-2\}$, $n - m$ *be odd* (*respectively, even*), *and*

$$\int_0^{+\infty} t^{n-m-2} \left[\sum_{i=j_0+1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{(m+1)\lambda_i} d_s p_i(s,t) \right] dt = +\infty. \tag{1.23}$$

Then the condition

$$\int_0^{+\infty} t^{n-m-1} \left[\sum_{i=1}^{j_0} \int_{\tau_0(t)}^{\tau(t)} s^{m\lambda_i} d_s p_i(s,t) \right] dt = +\infty \tag{1.24}$$

is sufficient for Eq. (1.19_1) [*respectively, Eq.* (1.19_2)] *to have Property* A_m (*respectively,* B_m). *If, moreover,*

$$\int_0^{+\infty} t^{n-m-1} \left[\sum_{i=j_0+1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{m\lambda_i} d_s p_i(s,t) \right] dt < +\infty, \tag{1.25}$$

then condition (1.24) *is also necessary.*

Corollary 1.3. *Let* $0 < \lambda_1 \leq \lambda_i < 1$ $(i = 1, \dots, j)$, $m \in \{0, \dots, n-2\}$, *and*

$$\int_0^{+\infty} g_\ell(t) dt = +\infty \qquad (\ell = m+1, \dots, n), \tag{1.26}$$

where

$$g_\ell(t) = t^{n-\ell} \sum_{i=1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{(\ell-1)\lambda_i} d_s p_i(s,t), \tag{1.27_ℓ}$$

and suppose that the condition

$$\int_0^{+\infty} t^{n-\ell-1} \left[\sum_{i=1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{(\ell-1)\lambda_i} \left(\int_0^s g_\ell(\xi) d\xi \right)^{\lambda_i/(1-\lambda_1)} d_s p_i(s,t) \right] dt = +\infty \tag{1.28}$$

is satisfied for each $\ell \in \{m+1,\dots,n-1\} \cap \mathscr{N}^{(1)}_{m+1,n}$ [*respectively,* $\ell \in \{m+1,\dots,n-1\} \cap \mathscr{N}^{(2)}_{m+1,n}$)]. *Then Eq.* (1.19_1) [*respectively,* (1.19_2)] *has Property* A_m (*respectively,* B_m).

Corollary 1.4. *Let* $0 < \lambda_1 \le \lambda_i < 1$ $(i = 1,\dots,j)$, $m \in \{0,\dots,n-2\}$, *let* $n-m$ *be odd* (*respectively, even*), *and let*

$$\liminf_{t\to+\infty} \left(t^{-2/\lambda_1}\tau_0(t)\right) > 0. \tag{1.29}$$

Then Eq. (1.9_1) [*respectively, Eq.* (1.19_2)] *has Property* A_m (*respectively,* B_m) *if and only if*

$$\int\limits_0^{+\infty} t^{n-m-1} \left[\sum_{i=1}^{j} \int\limits_{\tau_0(t)}^{\tau(t)} s^{m\lambda_i} d_s p_i(s,t)\right] dt = +\infty. \tag{1.30}$$

Corollary 1.5. *Let* $m \in \{0,\dots,n-2\}$ *and*

$$\int\limits_0^{+\infty} g_{m+1}(t)dt = +\infty,$$

$$\int\limits_0^{+\infty} t^{n-\ell_k-1} \left[\int\limits_{\tau_0(t)}^{\tau(t)} s^{\ell_k-1} \exp\left(\frac{1}{\ell_k!(n-\ell_k)!} \int\limits_0^s g_{\ell_k}(\xi)d\xi\right) d_s p(s,t)\right] dt = +\infty,$$

where ℓ_k *is the number given by* (1.10) *and* $g_\ell(t) = t^{n-\ell} \int_{\tau_0(t)}^{\tau(t)} s^{\ell-1} d_s p(s,t)$. *If* $k = 1$ (*respectively,* $k = 2$), *then Eq.* (1.20_k) *has Property* A_m (*respectively,* B_m).

Corollary 1.6. *Let* $m \in \{0,\dots,n-2\}$, *let* $n-m$ *be odd* (*respectively, even*), *and let*

$$\liminf_{t\to+\infty} \left(t^{-2}\tau_0(t)\right) > 0.$$

Then the condition

$$\int\limits_0^{+\infty} t^{n-m-1} \left[\int\limits_{\tau_0(t)}^{\tau(t)} s^m d_s p(s,t)\right] dt = +\infty$$

is necessary and sufficient for Eq. (1.20_1) [*respectively, Eq.* (1.20_2)] *to have Property* A_m (*respectively,* B_m).

Corollaries 1.3 and 1.4 are generalizations of Theorems 1.1 and 1.2 in [13], which deal with oscillatory properties of the Emden–Fowler advance differential equation of order n.

2. AUXILIARY ASSERTIONS

By $\tilde{C}^{n-1}_{\mathrm{loc}}([a_0,+\infty[)$, we denote the set of functions $u : [a_0,+\infty[\to \mathbf{R}$ that, together with their $(n-1)$st derivatives, are absolutely continuous on each finite closed subinterval of the interval $[a_0,+\infty[$. Lemmas 1.1–1.3 in [9] imply the following assertion.

Lemma 2.1. *Suppose that* $u \in \tilde{C}^{n-1}_{\mathrm{loc}}([a_0,+\infty[)$,

$$u(t) > 0 \quad \text{for} \quad t \ge a_0, \tag{2.1}$$

$$\operatorname{mes}\left\{s \in [t,+\infty[\ : \ u^{(n)}(s) \ne 0\right\} > 0 \quad \text{for} \quad t \ge a_0 \tag{2.2}$$

and the inequality

$$(-1)^k u^{(n)}(t) \ge 0 \tag{2.3}$$

is valid almost everywhere on $[a_0, +\infty[$ *for some* $k \in \{1, 2\}$. *Then there exists an* $a_1 \in [a_0, +\infty[$ *and an* $\ell \in \mathcal{N}_{0,n}^{(k)}$ *such that either* $\ell \le n-1$ *and*

$$u^{(i)}(t) > 0 \qquad (i = 0, \ldots, \ell), \qquad (-1)^{i-\ell} u^{(i)}(t) > 0 \qquad (i = \ell, \ldots, n-1) \quad \textit{for} \quad t \ge a_1, \tag{2.4}$$

$$\int_{a_1}^{+\infty} t^{n-\ell-1} \left| u^{(n)}(t) \right| dt < +\infty, \tag{2.5}$$

or $k = 2$, $\ell = n$, *and*

$$u^{(i)}(t) > 0 \qquad (i = 0, \ldots, \ell - 1) \quad \textit{for} \quad t \ge a_1. \tag{2.6}$$

Lemma 2.2. *Let a function* $u \in \tilde{C}_{\mathrm{loc}}^{n-1}([a_1, +\infty[)$ *satisfy inequality* (2.3) *almost everywhere on* $[a_1, +\infty[$, *where* $k \in \{1, 2\}$. *Moreover, let inequality* (2.4) *and the relation*

$$\int_{a_1}^{+\infty} t^{n-\ell} \left| u^{(n)}(t) \right| dt = +\infty \tag{2.7}$$

be valid for some $\ell \in \{1, \ldots, n-1\} \cap \mathcal{N}_{0,n}^{(k)}$. *Then there exists an* $a \in [a_1, +\infty[$ *such that*

$$u(t) \ge \frac{t^{\ell-1}}{\ell!} u^{(\ell-1)}(t) \quad \textit{for} \quad t \ge a, \tag{2.8}$$

$$u^{(\ell-1)}(t) > \ell! + \frac{1}{(n-\ell)!} \int_a^t s^{n-\ell} \left| u^{(n)}(s) \right| ds \quad \textit{for} \quad t \ge a. \tag{2.9}$$

Proof. By Lemma 1.3 in [9], the inequalities $(\ell - i) u^{(i)}(t) \ge t u^{(i+1)}(t)$ $(i = 0, \ldots, \ell - 1)$ are valid on $[a, +\infty[$ for a sufficiently large number $a \in [a_1, +\infty[$. This readily implies inequality (2.8).

Since $n - \ell - k$ is even, it follows from (2.3) that $(-1)^{n-\ell} s^{n-\ell} u^{(n)}(s) = s^{n-\ell} \left| u^{(n)}(s) \right|$ for almost all $s \in [a_1, +\infty[$. If we divide both sides of this identity by $(n - \ell)!$ and integrate the resulting relation from a_1 to t, then we obtain

$$\sum_{i=\ell-1}^{n-1} \frac{(-1)^{i-1-\ell}}{(i+1-\ell)!} t^{i+1-\ell} u^{(i)}(t) = c + \frac{1}{(n-\ell)!} \int_{a_1}^t s^{n-\ell} \left| u^{(n)}(s) \right| ds, \tag{2.10}$$

where $c = \sum_{i=\ell-1}^{n-1} \left((-1)^{i-1-\ell} / (i+1-\ell)! \right) a_1^{i+1-\ell} u^{(i)}(a_1)$. However, by (2.7),

$$c + ((n-\ell)!)^{-1} \int_{a_1}^{a} s^{n-\ell} \left| u^{(n)}(s) \right| ds > \ell!$$

for a sufficiently large $a \in [a_1, +\infty[$. This, together with condition (2.4) and relation (2.10), implies inequality (2.9) and completes the proof of the lemma.

In the sequel, we need the following obvious lemma on an integral inequality.

Lemma 2.3. *Let* $\varphi : [a, +\infty[\times [0, +\infty[\to [0, +\infty[$ *be a function locally integrable in the first argument and continuous and nondecreasing in the second argument. Moreover, suppose that there exists a positive number* c *and a continuous function* $y : [0, +\infty[\to]0, +\infty[$ *such that*

$$y(t) > c + \int_a^t \varphi(s, y(s)) ds \quad \textit{for} \quad t \ge a.$$

Then the problem $z'(t) = \varphi(t, z(t))$, $z(a) = c$ *on the interval* $[a, +\infty[$ *has an upper solution* z^*, *and* $y(t) > z^*(t) \ge c$ *for* $t \ge a$.

Lemma 2.4. *Let conditions* (1.2), (1.3_k), *and* (1.16) *be satisfied, where* $k = 1$ (*respectively,* $k = 2$) *and* $m \in \{0, \dots, n-2\}$, *and suppose that Eq.* (1.1) *does not have Property* A_m (*respectively,* B_m). *Then there exist* $a_1 \ge 0$, $a \ge a_1$, *and* $\ell \in \{m+1, \dots, n-1\} \cap \mathscr{N}_{m+1,n}^{(k)}$ *such that Eq.* (1.1) *has a solution* u *defined on* $[a_1, +\infty[$ *and satisfying conditions* (2.4) *and* (2.5), *problem* (1.15_ℓ) *has an upper solution* $v_{a,\ell}$ *defined on* $[a, +\infty[$, *and*

$$u(t) > t^{\ell-1} v_{a,\ell}(t) \ge t^{\ell-1} \quad \textit{for} \quad t \ge a. \tag{2.11}$$

Proof. First, we note that, by conditions (1.2) and (1.3_k) and relations (1.7_ℓ), for an arbitrary $\ell \in \{1, \dots, n\}$ we have $f_\ell(t,x) \equiv (-1)^k t^{n-\ell} f\left(h_\ell(\cdot, x)\right)(t) \operatorname{sgn} x$ and

$$f_\ell(t,y) \ge f_\ell(t,x) \ge 0 \quad \text{for} \quad t \ge 0, \quad y \ge x \ge 0. \tag{2.12}$$

Since Eq. (1.1) does not have Property A_m (respectively, B_m), it follows from conditions (1.2) and (1.3_k) that on some interval $[a_0, +\infty[$, there exists a solution u of this equation that satisfies relations (2.1) and (2.3) and the inequalities

$$\lim_{t \to +\infty} u^{(m)}(t) > 0, \tag{2.13}$$

$$\lim_{t \to +\infty} \left| u^{(n-1)}(t) \right| < +\infty. \tag{2.14}$$

Let us show that u satisfies condition (2.2) as well. Indeed, otherwise for some $a_1 \in [a_0, +\infty[$, we would have

$$u^{(n)}(t) = 0 \quad \text{for} \quad t \ge a_1 \tag{2.15}$$

and $u(t) = \sum_{i=1}^{\ell} c_i t^{i-1}$ for $t \ge a_1$, where $\ell \in \{m+1, \dots, n\}$ and $c_\ell > 0$. Therefore, there exists a $\delta > 0$ such that

$$u(t) > \delta t^{\ell-1} \quad \text{for} \quad t \ge a_1. \tag{2.16}$$

Taking account of this estimate, relations (1.7_ℓ) and (2.15), and conditions (1.2) and (1.3_k), we obtain $0 = t^{n-\ell} \left| u^{(n)}(t) \right| = t^{n-\ell} |f(u)(t)| \ge f_\ell(t, \delta)$ for $t \ge a_1$. This contradicts condition (1.16) and completes the proof of condition (2.2).

By Lemma 2.1 and condition (2.13), there exists an $a_1 \in [a_0, +\infty[$ and an $\ell \in \mathscr{N}_{m,n}^{(k)}$ such that either $\ell \le n-1$ and conditions (2.4) and (2.5) are satisfied, or $k = 2$, $\ell = n$, and inequalities (2.6) hold.

Let us first show that $\ell \le n-1$. Suppose the contrary: $\ell = n$. Then $k = 2$, and inequalities (2.3) and (2.6) are valid. Therefore, there exists a positive number δ such that $u(t) \ge t^{n-1} \delta$ for $t \ge a_1$. Taking account of conditions (1.2), (1.3_2), and (1.16) as well, we obtain

$$u^{(n-1)}(t) = u^{(n-1)}(a_1) + \int_{a_1}^{t} f(u)(s) ds > \int_{a_1}^{t} f_n(s, \delta) ds \to +\infty \quad \text{as} \quad t \to +\infty.$$

But this contradicts condition (2.14). We have thereby shown that $\ell \in \{m, \dots, n-1\} \cap \mathscr{N}_{m,n}^{(k)}$ and the function u satisfies conditions (2.4) and (2.5).

Suppose that $\ell = m$. Then, taking account of relation (2.13), we obtain $u(t) \ge \delta t^m$ for $t \ge a_1$, where δ is a positive constant. By this inequality and conditions (1.2), (1.3_k), and (1.16), one has

$$\int_{a_1}^{+\infty} t^{n-m-1} \left| u^{(n)}(t) \right| dt = \int_{a_1}^{+\infty} t^{n-m-1} |f(u)(t)| dt \ge \int_{a_1}^{+\infty} f_{m+1}(t, \delta) dt = +\infty,$$

which contradicts condition (2.5) and hence implies that $\ell \in \{m+1, \dots, n-1\} \cap \mathscr{N}_{m,n}^{(k)}$.

By (2.4), relations (2.16) are valid for some $\delta > 0$. Therefore,

$$\int\limits_{a_1}^{+\infty} t^{n-\ell} \left|u^{(n)}(t)\right| dt = \int\limits_{a_1}^{+\infty} t^{n-\ell} |f(u)(t)| dt \geq \int\limits_{a_1}^{+\infty} f_\ell(t,\delta) dt.$$

This, together with (1.6), implies (2.7).

By Lemma 2.2, there exists an $a \in [a_1, +\infty[$ such that the function u satisfies inequalities (2.8) and (2.9). Taking account of this fact and conditions (1.2) and (1.3_k) and setting $y(t) = u^{(\ell-1)}(t)/\ell!$, we obtain

$$\left|u^{(n)}(t)\right| = |f(u)(t)| \geq t^{\ell-n} f_\ell(t, y(t)) \qquad \text{for} \quad t \geq a,$$

$$y(t) > 1 + (\ell!(n-\ell)!)^{-1} \int\limits_a^t f_\ell(s, y(s)) ds \quad \text{for} \quad t \geq a.$$

This, together with Lemma 2.3 and condition (2.12), implies that problem (1.15_ℓ) has an upper solution $v_{a,\ell}$ defined on $[a, +\infty[$ and $u^{(\ell-1)}(t) > \ell! v_{a,\ell}(t) \geq \ell!$ for $t \geq a$. Consequently, the estimate (2.11) is valid, and the proof of the lemma is complete.

Lemma 2.5. *Let* $\ell_0 \in \{0, \ldots, n-2\}$, $k \in \{1, 2\}$, *and let conditions* (1.2) *and* (1.3_k) *be satisfied. Moreover, suppose that for arbitrary* $a \geq 0$ *and* $c > 0$, *problem* (1.8_{ℓ_0}) *has no upper solution defined on* $[a, +\infty[$. *Then*

$$\int\limits_0^{+\infty} f_\ell(t,\delta) dt = +\infty \quad \textit{for} \quad \delta > 0 \quad (\ell = \ell_0, \ldots, n), \tag{2.17}$$

and for arbitrary $a \geq 0$ *and* $\ell \in \{\ell_0, \ldots, n-1\}$, *problem* ($1.15_\ell$) *has no upper solution defined on* $[a, +\infty[$.

Proof. Suppose that for some $a_0 \geq 0$ and $\ell \in \{\ell_0, \ldots, n-1\}$, the problem

$$v'(t) = (\ell!(n-\ell)!)^{-1} f_\ell(t, v(t)), \qquad v(a_0) = 1,$$

has an upper solution v defined on $[a_0, +\infty[$. We set $w(t) = t^{\ell-\ell_0} v(t)$. Then, by (1.2), ($1.3_k$), and ($1.7_\ell$), we have $h_\ell(s, v(t)) \geq h_{\ell_0}(s, w(t)) > 0$ for $s \geq t \geq a_0$ and

$$f_\ell(t, v(t)) = (-1)^k t^{n-\ell} f\left(h_\ell(\cdot, v(t))\right)(t) \geq (-1)^k t^{n-\ell} f\left(h_{\ell_0}(\cdot, w(t))\right)(t) = t^{\ell_0-\ell} f_{\ell_0}(t, w(t))$$

for $t \geq a_0$. Therefore,

$$w(t) = t^{\ell-\ell_0} + \frac{1}{(n-\ell)!\ell!} t^{\ell-\ell_0} \int\limits_{a_0}^t f_\ell(s, v(s)) ds > c + \frac{1}{(n-1)!} t^{\ell-\ell_0} \int\limits_a^t s^{\ell_0-\ell} f_{\ell_0}(s, w(s)) ds$$

$$\geq c + \frac{1}{(n-1)!} \int\limits_a^t f_{\ell_0}(s, w(s)) ds \quad \text{for} \quad t \geq a,$$

where $a = a_0 + 2$ and $c = 1$. Therefore, by Lemma 2.3 and condition (2.12), there exists an upper solution of problem (1.8_{ℓ_0}) defined on $[a, +\infty[$. This contradicts the assumptions of the lemma and hence implies that for arbitrary $a \geq 0$ and $\ell \in \{\ell_0, \ldots, n-1\}$, problem ($1.15_\ell$) has no upper solution defined on $[a, +\infty[$.

In a similar way, we show that for any $a \geq 0$ and $c > 0$, problem (1.8_ℓ) has no upper solution defined on $[a, +\infty[$ if $\ell \in \{\ell_0, \ldots, n-1\}$ as well as if $\ell = n$.

To complete the proof of the lemma, it remains to show that relation (2.17) is valid. Indeed, otherwise, there would exist $\ell \in \{\ell_0, \dots, n\}$, $\delta > 0$, $c \in]0, \delta[$, and $a > 0$ such that

$$\delta > c + ((n-1)!)^{-1} \int_a^t f_\ell(s, \delta) ds \quad \text{for} \quad t \geq a.$$

This, together with Lemma 2.3, implies that there exists an upper solution of problem (1.8_ℓ) defined on $[a, +\infty[$. On the other hand, as was mentioned above, problem (1.8_ℓ) has no upper solutions. This contradiction completes the proof of the lemma.

Let $a \geq 0$, $m \in \{0, \dots, n-1\}$, and $c_i \in \mathbf{R}$ $(i = 0, \dots, m)$. Consider the following problem: find a solution u of Eq. (1.1) defined on $[a, +\infty[$ and satisfying the condition

$$u^{(i)}(a) = c_i \qquad (i = 0, \dots, m-1), \qquad \lim_{t \to +\infty} u^{(m)}(t) = c_m. \tag{2.18_m}$$

If $m = 0$, then condition (2.18_m) is treated as the condition $\lim_{t\to+\infty} u(t) = c_0$.

Lemma 2.6. *Let $k \in \{1, 2\}$, $m \in \{0, \dots, n-1\}$, and let the operator f satisfy conditions* (1.2) *and* (1.3_k). *Then condition* (1.9) *is necessary for Eq.* (1.1) *to have Property A_m or B_m. Moreover, if condition* (1.9) *fails, then there exist positive constants a_0 and γ_i $(i = 0, \dots, m)$ such that for arbitrary $a \geq a_0$ and $c_i \in [-\gamma_i, \gamma_i]$ $(i = 0, \dots, m)$, problem* (1.1), (2.18_m) *has at least one solution.*

Proof. Let condition (1.9) fail. Then there exist $a_0 > 1$, $\delta_0 > 0$, and $\gamma_m \in]0, \delta_0[$ such that

$$a_0 \geq 2\left(\delta_0 - \gamma_m\right)^{-2}, \qquad \int_{a_0}^{+\infty} f_{m+1}\left(s, \delta_0\right) ds \leq \frac{\delta_0 - \gamma_m}{2}. \tag{2.19}$$

If $m \geq 1$, then we choose the numbers γ_i $(i = 0, \dots, m-1)$ so as to ensure that

$$(\delta_0 - \gamma_m) \sum_{i=0}^{m-1} \gamma_i \leq 1.$$

By $C([a, +\infty[\,)$ we denote the Banach space of continuous bounded functions $x : [a, +\infty[\to \mathbf{R}$ with the norm $\|x\|_C = \sup\{|x(t)| : t \geq a\}$. For an arbitrary $x \in C([a, +\infty[\,)$, we set $y_0(x)(t) = x(t)$ for $t \geq a$ and $y_0(x)(t) = x(a)$ for $0 \leq t \leq a$. For $m \geq 1$, we set

$$y_m(x)(t) = \sum_{i=0}^{m-1} \frac{c_i}{i!}(t-a)^i + \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} x(s) ds \quad \text{for} \quad t \geq a$$

and $y_m(x)(t) = y_m(x)(a)$ for $0 \leq t \leq a$.

Let $a \geq a_0$ and $c_i \in [-\gamma_i, \gamma_i]$ $(i = 0, \dots, m)$. In the ball

$$\mathscr{B} = \left\{x \in C([a, +\infty[\,) : \|x\|_C \leq (\gamma_m + \delta_0)/2\right\},$$

we consider the operator

$$g(x)(t) = c_m - \frac{1}{(n-m-1)!} \int_t^{+\infty} (t-s)^{n-m-1} f\left(y_m(x)\right)(s) ds. \tag{2.20}$$

Then, by conditions (1.2), (1.3_k), and (2.19), we have

$$|y_m(x)(t)| \le h_{m+1}(t,\delta_0),$$
$$|f(y_m(x))(t)| \le |f(h_{m+1}(\cdot,\delta_0))(t)| = t^{m+1-n} f_{m+1}(t,\delta_0),$$
$$|g(x)(t)| \le \gamma_m + \int\limits_t^{+\infty} f_{m+1}(s,\delta_0)\,ds \le \frac{\gamma_m+\delta_0}{2} \quad \text{for} \quad t \ge a.$$

These inequalities, together with (1.2), imply that the operator $g : \mathscr{B} \to \mathscr{B}$ is compact. By the Schauder principle, there exists an $x \in \mathscr{B}$ such that $x(t) = g(x)(t)$ for $t \ge a$. We set $u(t) = y_m(x)(t)$ for $t \ge a$. By (2.20), the function u is a solution of problem (1.1), (2.18_m).

Since problem (1.1), (2.18_m) is solvable for arbitrary $c_i \in [-\gamma_i, \gamma_i]$ $(i = 0, \dots, m)$, it follows that there exist infinitely many solutions of Eq. (1.1) such that

$$0 < \lim_{t\to+\infty} \left|u^{(m)}(t)\right| < +\infty.$$

Consequently, Eq. (1.1) has neither Property A_m nor Property B_m. The proof of the lemma is complete.

3. PROOF OF THE MAIN RESULTS

Proof of Proposition 1.1

To be definite, we assume that conditions (1.2) and (1.3_1) are satisfied; the case of conditions (1.2) and (1.3_2) can be treated in a similar way.

By the definitions of Properties A and A_0, we find that if Eq. (1.1) has Property A [respectively, n is odd and Eq. (1.1) has Property A_0], then it has Property A_0 (respectively, Property A) as well. Therefore, to prove Proposition 1.1, it suffices to show that if n is even and Eq. (1.1) has Property A_0, then it does not have any nonoscillatory proper solution. Suppose the contrary: Eq. (1.1) has a nonoscillatory proper solution u defined on some interval $[a_0, +\infty[$. Then, by conditions (1.2) and (1.3_1) and Property A_0 of Eq. (1.1), we can assume that u satisfies conditions (1.4) and (2.1)–(2.3), where $k = 1$. On the other hand, by Lemma 2.1, the function u satisfies inequalities (2.4) for some $\ell \in \{1, \dots, n-1\}$ and $a_1 \in [a_0, +\infty[$. But this contradicts condition (1.4) and hence completes the proof of Proposition 1.1.

Proof of Theorem 1.1

First, we note that if the assumptions of Theorem 1.1 are valid, then, by Lemma 2.5, inequalities (1.16) also hold and for arbitrary $a \ge 0$ and $\ell \in \{\ell_k, \dots, n-1\}$, problem ($1.15_\ell$) has no upper solution defined on $[a, +\infty[$.

Now we suppose that $k = 1$ (respectively, $k = 2$), and Eq. (1.1) does not have Property A_m (respectively, Property B_m). Then, by Lemma 2.4, for some $a \ge 0$ and $\ell \in \{\ell_k, \dots, n-1\}$, problem ($1.15_\ell$) has an upper solution defined on $[a, +\infty[$. On the other hand, by the above considerations, this problem has no upper solution. This contradiction completes the proof of the theorem.

Proof of Theorem 1.2

By Lemma 2.6, condition (1.9) is necessary for Property A_m or B_m of Eq. (1.1). Consequently, it remains to show that if $k = 1$ (respectively, $k = 2$) and condition (1.9) is satisfied, then Eq. (1.1) has Property A_m (respectively, B_m). First, we note that, by conditions (1.9), (1.11), and (1.12), problem (1.8_{m+1}) has no upper solution defined on $[a, +\infty[$ for arbitrary $a \ge 0$ and $c > 0$. This fact, together with Lemma 2.5, implies the validity of (1.16) and the absence of an upper solution of problem (1.15_ℓ) defined on $[a, +\infty[$ for arbitrary $a \ge 0$ and $\ell \in \{m+1, \dots, n-1\}$. Hence Eq. (1.1) has Property A_m (respectively, B_m), since otherwise, by Lemma 2.4, problem (1.15_ℓ) would have an upper solution defined on $[a, +\infty[$ for some $a \ge 0$ and $\ell \in \{m+1, \dots, n-1\}$. The proof of the theorem is complete.

Proof of Theorem 1.3

By Lemma 2.6, condition (1.9) is necessary for Eq. (1.1) to have Property A_m (respectively, Property B_m). Let us prove the sufficiency of this condition.

If $n-m$ is odd (respectively, even), then it follows from (1.10) that $\ell_1 = m+2$ (respectively, $\ell_2 = m+2$). This, together with conditions (1.11), (1.13), and (1.14), implies that problem (1.8_{ℓ_1}) [respectively, problem (1.8_{ℓ_2})] has no upper solution defined on $[a,+\infty[$ for any $a \ge 0$ and $c > 0$. Now if we use Theorem 1.1, then we find that Eq. (1.1) has Property A_m (respectively, Property B_m).

Proof of Theorem 1.4

Suppose the contrary: $k=1$ (respectively, $k=2$), and Eq. (1.1) does not have Property A_m (respectively, B_m). By Lemma 2.4, there exist $a_1 \ge 0$, $a \ge a_1$, and $\ell \in \{m+1,\dots,n-1\} \cap \mathscr{N}^{(k)}_{m+1,n}$ such that Eq. (1.1) has an upper solution u defined on $[a_1,+\infty[$ and satisfying conditions (2.5) and (2.11). On the other hand, it follows from conditions (1.3_k), (2.5), and (2.11) that

$$\int\limits_a^{+\infty} t^{n-\ell-1}\left|f\left(w_{a,\ell}\right)(t)\right| dt \le \int\limits_a^{+\infty} t^{n-\ell-1}|f(u)(t)|dt = \int\limits_a^{+\infty} t^{n-\ell-1}\left|u^{(n)}(t)\right| dt < +\infty,$$

which contradicts condition (1.17) and completes the proof of the theorem.

Proof of Theorem 1.5

The necessity of condition (1.9) for Eq. (1.1) to have Property A_m (respectively, B_m) follows from Lemma 2.6.

Before proving the sufficiency, we note that conditions (1.9) and (1.18) provide the validity of relations (1.16).

Now we suppose that condition (1.9) is satisfied and nevertheless Eq. (1.1) does not have Property A_m (respectively, B_m). Then, by Lemma 2.4, there exist $a_1 \ge 0$, $a \ge a_1$, and

$$\ell \in \{m+1,\dots,n-1\} \cap \mathscr{N}^{(1)}_{m+1,n}$$

[respectively, $\ell \in \{m+1,\dots,n-1\} \cap \mathscr{N}^{(2)}_{m+1,n}$)] such that Eq. (1.1) has an upper solution u defined on $[a,+\infty[$ and satisfying conditions (2.5) and (2.11). Moreover, since $n-m$ is odd (respectively, even), we have $\ell \ge m+2$.

For $\delta = 1$, we choose positive numbers a, γ, and η so as to satisfy (1.18). Then, by relations (1.7_ℓ) and conditions (1.2), (1.3_k), and (2.11), we have

$$t^{n-\ell-1}\left|u^{(n)}(t)\right| = t^{n-\ell-1}|f(u)(t)| \ge t^{-1} f_\ell(t,1) \ge \gamma f_{m+1}(t,\eta) \quad \text{for} \quad t \ge a.$$

This, together with (2.5), implies that $\int_a^{+\infty} f_{m+1}(t,\eta)dt < +\infty$. This contradicts condition (1.9) and completes the proof of the theorem.

Equations (1.19_k) and (1.20_k) considered in Corollaries 1.1–1.6 follow from Eq. (1.1) in the cases

$$f(u)(t) = (-1)^k \sum_{i=1}^{j} \int\limits_{\tau_0(t)}^{\tau(t)} |u(s)|^{\lambda_i} \operatorname{sgn} u(s) d_s p_i(s,t), \tag{3.1}$$

$$f(u)(t) = (-1)^k \int\limits_{\tau_0(t)}^{\tau(t)} u(s) d_s p(s,t), \tag{3.2}$$

respectively. By the restrictions imposed on the functions τ_0, τ, p_i $(i = 1, \dots, n)$, and p, in both cases, the operator f satisfies conditions (1.2) and (1.3_k). On the other hand, if f admits the representation (3.1), then, by (1.7_ℓ), we obtain

$$f_\ell(t,x) = t^{n-\ell} \sum_{i=1}^{j} \left(\int_{\tau_0(t)}^{\tau(t)} s^{(\ell-1)\lambda_i} d_s p_i(s,t) \right) |x|^{\lambda_i}; \qquad (3.1_\ell)$$

but if f admits (3.2), then

$$f_\ell(t,x) = t^{n-\ell} \left(\int_{\tau_0(t)}^{\tau(t)} s^{\ell-1} d_s p(s,t) \right) |x|. \qquad (3.2_\ell)$$

Proof of Corollary 1.1

For each $\ell \in \{1, \dots, n\}$, from (3.1_ℓ), we obtain the inequality

$$f_\ell(t,x) \ge g_\ell(t) x^{\lambda(x)} \quad \text{for} \quad t \ge 0, \quad x \ge 0, \qquad (3.3_\ell)$$

where

$$g_\ell(t) = t^{n-\ell} \sum_{i=j_0+1}^{j} \int_{\tau_0(t)}^{\tau(t)} s^{(\ell-1)\lambda_i} d_s p_i(s,t), \qquad (3.4_\ell)$$

$\lambda(x) = \min\{\lambda_i : i = j_0 + 1, \dots, j\} > 1$ for $x \ge 1$, and $\lambda(x) = \max\{\lambda_i : i = j_0 + 1, \dots, j\}$ for $0 \le x < 1$. Moreover,

$$g_\ell(t) \ge g_{m+1}(t) \quad \text{for} \quad t \ge 1 \quad (\ell = m+1, \dots, n), \qquad (3.5)$$

since $\tau_0(t) \ge t$ and $\lambda_i > 1$ $(i = j_0 + 1, \dots, j)$.

First, we suppose that relation (1.21) is valid. Then, by (3.3_ℓ), (3.4_ℓ), and (3.5), relations (1.16) are valid, and for arbitrary $a \ge 0$, $c > 0$, and $\ell \in \{m+1, \dots, n\}$, problem ($1.8_\ell$) has no upper solution defined on $[a, +\infty[$. This, together with Theorem 1.1, implies that Eq. (1.19_1) [respectively, Eq. (1.19_2)] has Property A_m (respectively, B_m).

Let us proceed to the case in which relation (1.21) fails and condition (1.22) is satisfied. Then, by (3.1_{m+1}), condition (1.9) fails, and it follows from Lemma 2.6 that Eq. (1.9_1) [respectively, Eq. (1.19_2)] does not have Property A_m (respectively, B_m). The proof of the corollary is complete.

Proof of Corollary 1.2

We set $\omega(x) = x^{\lambda(x)}$ and $g(t) = g_{m+2}(t)$. Then, by (1.23), (3.3_{m+2}), and (3.4_{m+2}), conditions (1.11), (1.13), and (1.14) are satisfied. On the other hand, if condition (1.25) is satisfied, then, by the representation (3.1_{m+1}), relation (1.9) holds if and only if relation (1.24) is valid. If now we use Theorem 1.3, then the validity of Corollary 1.2 becomes obvious.

Proof of Corollary 1.3

Since $0 < \lambda_1 \le \lambda_i < 1$ $(i = 1, \dots, j)$, it follows from (3.1_ℓ) for each $\ell \in \{1, \dots, n\}$ that inequality (3.3_ℓ) is valid, where g_ℓ is the function given by (1.27_ℓ), $\lambda(x) = 1$ for $0 \le x < 1$, and $\lambda(x) = \lambda_1$ for $x \ge 1$.

On the one hand, relations (1.26) and (3.3_ℓ) imply (1.16); on the other hand, they imply that

$$v_{a,\ell}(t) > \left[\frac{1-\lambda_1}{\ell!(n-\ell)!} \int_a^t g_\ell(s) ds \right]^{1/(1-\lambda_1)} \qquad \text{for} \quad t \ge a \quad (\ell = 1, \dots, n-1), \qquad (3.6)$$

where $v_{a,\ell}$ is an upper solution of problem (1.15_ℓ). By these estimates and condition (1.28), it follows from (3.1) that relation (1.17) is valid for arbitrary $a \geq 0$ and $\ell \in \{m+1, \ldots, n-1\} \cap \mathcal{N}_{m+1,n}^{(k)}$. Consequently, all assumptions of Theorem 1.4 are satisfied; therefore, if $k = 1$ (respectively, $k = 2$), then Eq. (1.19_k) has Property A_m (respectively, B_m).

Proof of Corollary 1.4

By (1.27_ℓ) and (1.29), there exist $a > 1$ and $\gamma_0 > 0$ such that

$$g_\ell(t) \geq t^{m+1-\ell} \left[\tau_0(t)\right]^{(\ell-1-m)\lambda_1} g_{m+1}(t) \geq \gamma_0 t g_{m+1}(t)$$

for $t \geq a$ ($\ell = m+2, \ldots, n$). These estimates, together with inequalities (3.3_ℓ), $\ell = m+2, \ldots, n$, imply that inequalities (1.18) are valid for arbitrary $\delta > 0$, where $\gamma = \gamma_0 \delta^{\lambda(\delta)}$ and $\eta = 1$. On the other hand, by (3.1_ℓ), relation (1.9) holds if and only if relation (1.30) is valid. If we now use Theorem 1.5, then the validity of Corollary 1.4 becomes obvious.

Corollaries 1.5 and 1.6 can be proved by analogy with Corollaries 1.3 and 1.4. The only difference is that the representations (3.1) and (3.1_ℓ) are replaced by the representations (3.2) and (3.2_ℓ); thus, relation (3.6) is replaced by $v_{a,\ell}(t) \geq \exp\left((\ell!(n-\ell)!)^{-1} \int_a^t g_\ell(s)ds\right)$ for $t \geq a$ ($\ell = 1, \ldots, n-1$).

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