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On the Kneser Problem for Two-Dimensional Differential Systems with Advanced Arguments

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For the differential system

$$u_i'(t) = f_i(t, u_1(\tau_{i1}(t)),\ u_2(\tau_{i2}(t)))\ (i = 1, 2)$$

with advanced arguments τ_{ik} $(i, k = 1, 2)$, sufficient conditions are established for the existence and uniqueness of a solution of the Kneser problem

$$\varphi(u_1(0),\ u_2(0)) = 0,\ u_1(t) \geq 0,\ u_2(t) \geq 0 \quad \text{for } t \geq 0,$$

and the asymptotic behaviour of this solution is studied.

Keywords: Two-dimensional system; Kneser problem

2000 Mathematics Subject Classifications: 34K10, 34K15

1. INTRODUCTION

In this paper we consider the problem on the existence of a solution $(u_1, u_2): R_+ \to R^2$ of the differential system

$$u_i'(t) = f_i(t, u_1(\tau_{i1}(t)),\ u_2(\tau_{i2}(t)))\ (i = 1, 2) \tag{1.1}$$

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satisfying the conditions

$$\varphi(u_1(0),\ u_2(0)) = 0,\ u_1(t) \geq 0,\ u_2(t) \geq 0 \quad \text{for } t \geq 0. \tag{1.2}$$

Throughout this paper, we will assume that R^k is the k-dimensional Euclidean spaces, $R_+ = [0, +\infty[$, the functions $f_i: R_+ \times R^2 \to R$ $(i = 1,2)$ satisfy the local Carathéodory conditions, while $\varphi: R^2 \to R$ and $\tau_{ik}: R_+ \to R_+$ $(i,k = 1,2)$ are continuous functions.

By a solution of system (1.1) on R_+ is understood a vector function $(u_1, u_2): R_+ \to R^2$ which is absolutely continuous on each finite segment contained in R_+ and satisfies (1.1) almost everywhere on R_+.

We are especially interested in the case, where the functions f_i and τ_{ik} $(i,k = 1,2)$ satisfy the conditions

$$f_i(t,0,0) = 0,\ f_i(t,x,y) \leq 0 \quad \text{for } t \geq 0,\ x \geq 0,\ y \geq 0\ (i = 1,2), \tag{1.3}$$

$$\tau_{ik}(t) \geq t \quad \text{for } t \geq 0\ (i,k = 1,2), \tag{1.4}$$

while the function φ satisfies one of the following two conditions:

$$\varphi(0,0) < 0,\ \varphi(x,y) > 0 \quad \text{for } x > r,\ y \geq 0 \tag{1.5}$$

and

$$\varphi(0,0) < 0,\ \varphi(x,y) > 0 \quad \text{for } x \geq 0,\ y \geq 0,\ x + y > r, \tag{1.6}$$

where r is a positive constant.

If

$$f_1(t,x,y) = -y,\ f_2(t,x,y) = -f(t,x,-y),\ \tau_{ik}(t) \equiv t\ (i,k = 1,2),$$
$$\varphi(x,y) = x - r,$$

then (1.1), (1.2) is equivalent to the problem

$$u'' = f(t,u,u'), \tag{1.1'}$$

$$u(0) = r,\ u(t) \geq 0,\ u'(t) \leq 0 \quad \text{for } t \geq 0. \tag{1.2'}$$

In the case $f(t,x,y) \equiv f(t,x)$ problem (1.1′), (1.2′) first was posed and solved by Kneser [17]. The interest in this problem essentially enhanced after the appearance of the works of Thomas [22] and Fermi [3], dealing with the distribution of electrons in a heavy atom. In these works the considered physical problem is reduced to the Kneser problem for the differential equation

$$u'' = t^{-1/2}u^{3/2}$$

which subsequently became known as the Thomas-Fermi equation.

In the papers of Mambriani, Scorca-Dragoni, Lampariello, Tonelli (see [21] where the results of these authors are used), and Hartman and Wintner [4] problem (1.1′), (1.2′) is investigated in full detail when $f: R_+ \times R^2 \to R$ is a continuous function, while in [9] this problem is studied when the function f satisfies the local Carathéodory conditions.

In [6–8, 10–13, 15, 18] optimal, in a sense, conditions are established for the solvability and unique solvability of the Kneser problem for higher order differential equations and asymptotic properties of solutions of this problem are studied.

Nonlinear Kneser problems for differential systems of the type

$$\frac{dx_i(t)}{dt} = f_i(t, x_1(t), \dots, x_n(t)) \quad (i = 1, \dots, n)$$

are investigated in [1, 2, 5, 16, 19, 20].

As for problem (1.1), (1.2), in the case $\tau_{ik}(t) \not\equiv t$ $(i, k = 1, 2)$ it has remained practically unstudied.

In this paper, we make an attempt to fill to some extent the existing gap.

2. AUXILIARY STATEMENTS

2.1. Lemmas on the Solvability of Problem (1.1), (1.2)

For system (1.1) along with (1.2) we will consider the auxiliary boundary value problem

$$\varphi(u_1(0),\ u_2(0)) = 0,\ u_1(t) = u_1(a),\ u_2(t) = 0 \quad \text{for } t \geq a, \qquad (1.2_a)$$

where $a \in]0, +\infty[$.

LEMMA 2.1 *Let conditions* (1.3) *and* (1.5) *be fulfilled and let there exist positive numbers* a_0 *and* ρ_0 *such that for any* $a > a_0$ *problem* (1.1), (1.2_a) *has at least one nonnegative solution* (u_1, u_2) *whose second component admits the estimate*

$$u_2(0) \leq \rho_0. \tag{2.1}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof According to the condition of the lemma, for any natural m system (1.1) has a nonnegative solution (u_{1m}, u_{2m}) on $[0, a_0+m]$ satisfying the boundary conditions

$$\begin{gathered}\varphi(u_{1m}(0),\ u_{2m}(0)) = 0,\\ u_{1m}(t) = u_{1m}(a_0+m), \quad u_{2m}(t) = 0 \quad \text{for } t \geq a_0+m\end{gathered} \tag{2.2}$$

and the second component of this solution admits the estimate

$$u_{2m}(0) \leq \rho_0.$$

Hence in view of (1.3) and (1.5) it is obvious that the functions u_{1m} and u_{2m} are nonincreasing and

$$\begin{gathered}u_{1m}(t) \leq r,\ u_{2m}(t) \leq \rho_0 \quad \text{for } 0 \leq t \leq a_0+m,\\ |u'_{im}(t)| \leq f_i^*(t) \quad \text{for } 0 \leq t \leq a_0+m \ (i = 1, 2),\end{gathered} \tag{2.3}$$

where

$$f_i^*(t) = \max\{|f_i(t, x, y)|:\ 0 \leq x \leq r,\ 0 \leq y \leq \rho_0\} \ (i = 1, 2).$$

Consequently, the sequences $(u_{im})_{m=1}^{+\infty}$ $(i = 1, 2)$ are uniformly bounded and equicontinuous on each finite segment contained in R_+.[1] By the Arzela-Ascoli lemma, from these sequences we can choose subsequences $(u_{im_j})_{j=1}^{+\infty}$ $(i = 1, 2)$ converging uniformly on each finite segment contained in R_+. On the other hand,

$$\begin{gathered}u_{im_j}(t) = u_{im_j}(0) + \int_0^t f_i(s, u_{1m_j}(\tau_{i1}(s)),\ u_{2m_j}(\tau_{i2}(s)))\, ds\\ \text{for } 0 \leq t \leq a_0+m_j \ (i = 1, 2;\ j = 1, 2, \ldots).\end{gathered} \tag{2.4}$$

[1]The values of u_{1m} and u_{2m} on $[a_0+m, +\infty[$ are defined by boundary conditions (2.2).

Put

$$u_i(t) = \lim_{j\to+\infty} u_{im_j}(t) \quad \text{for } t \geq 0 \ (i = 1,2). \tag{2.5}$$

If now in equalities (2.4) we pass to the limit as $j \to +\infty$, then by virtue of the Lebesgue dominated convergence theorem we find

$$u_i(t) = u_i(0) + \int_0^t f_i(s, u_1(\tau_{i1}(s)),\ u_2(\tau_{i2}(s)))ds \quad \text{for } t \geq 0 \ (i = 1,2).$$

Consequently, (u_1, u_2) is a solution of system (1.1) on R_+. On the other hand, if along with (2.2) and (2.5) we take into account the fact that (u_{1m_j}, u_{2m_j}) $(j = 1, 2, \ldots)$ are nonnegative, then it becomes clear that (u_1, u_2) satisfies conditions (1.2). ■

LEMMA 2.2 *Let all the conditions of Lemma* 2.1 *be fulfilled and let, moreover,*

$$\int_0^{+\infty} f_2^*(t)\, dt < +\infty, \tag{2.6}$$

where

$$f_2^*(t) = max\{|f_2(t,x,y)|:\ 0 \leq x \leq r,\ 0 \leq y \leq \rho_0\}.$$

Then problem (1.1), (1.2) *has a solution* (u_1, u_2) *such that*

$$\lim_{t\to+\infty} u_2(t) = 0. \tag{2.7}$$

Proof Let $(u_{im})_{m=1}^{+\infty}$ and $(u_{im_j})_{j=1}^{+\infty}$ $(i = 1,2)$ be sequences appearing in the proof of Lemma 2.1 and let (u_1, u_2) be a vector function whose components are given by (2.5). As is shown above, (u_1, u_2) is a solution of problem (1.1), (1.2). On the other hand, according to (2.2) and (2.3) we have

$$u_{2m_j}(t) \leq \int_t^{a_0+m_j} f_2^*(s)ds \quad \text{for } 0 \leq t \leq a_0 + m_j \quad (j = 1, 2, \ldots),$$

whence on account of (2.5) and (2.6) we get

$$u_2(t) \leq \int_t^{+\infty} f_2^*(s)ds \quad \text{for } t \geq 0.$$

Consequently, u_2 satisfies condition (2.7). ■

2.2. Lemmas on *A Priori* Estimates

Consider the system of differential inequalities

$$u_1'(t) \leq -\delta(t, u_2(a_0)),\ u_2'(t) \geq -[h(t) + |u_1'(t)|]\omega(u_2(t)) \tag{2.8}$$

with the initial condition

$$u_1(0) \leq r, \tag{2.9}$$

where $\delta : [0, a_0] \times R_+ \to R_+$ is a continuous in the first and nondecreasing in the second argument function, $h : [0, a_0] \to R_+$ is a summable function and $\omega : R_+ \to]0, +\infty[$ is a nondecreasing continuous function.

Consider also on the segment $[0, a_0]$ the system of differential inequalities

$$u_1'(t) \leq -lt^\beta u_2^{\lambda_1}(t^\alpha),\ -h(t)(1 + u_2(\tau(t)))^{1+\lambda_2} \leq u_2'(t) \leq 0, \tag{2.10}$$

where $a_0 \in]0, 1]$, $0 < \alpha \leq 1$, $\beta > -1$, $l > 0$, $\lambda_1 > 0$, $\lambda_2 \geq 0$, while $h : [0, a_0] \to R_+$ and $\tau : [0, a_0] \to [0, a_0]$ are measurable functions.

A vector function (u_1, u_2) with the nonnegative components $u_i : [0, a_0] \to R_+$ $(i = 1, 2)$ is said to be a nonnegative solution of problem (2.8), (2.9) (of problem (2.10), (2.9)) if the functions u_1 and u_2 are absolutely continuous, the function u_1 satisfies the inequality (2.9), and the system of differential inequalities (2.8) (2.10) holds almost everywhere on $[0, a_0]$.

LEMMA 2.3 *Let*

$$\lim_{y \to +\infty} \int_0^{a_0} \delta(s, y) ds > r, \quad \int_0^{+\infty} \frac{dy}{\omega(y)} = +\infty.$$

Then there exists a positive number ρ_0 such that the second component of an arbitrary nonnegative solution (u_1, u_2) of the problem (2.8), (2.9) *admits the estimate*

$$u_2(t) \leq \rho_0 \quad \textit{for } 0 \leq t \leq a_0.$$

LEMMA 2.4 *Let $\tau(t) \geq t$ for $0 \leq t \leq a_0$ and*

$$\int_0^{a_0} [\tau(t)]^{-((1+\beta)\lambda_2/(\alpha\lambda_1))} h(t)dt < +\infty. \tag{2.11}$$

Then there exists a positive number ρ_0 such that the second component of an arbitrary nonnegative solution (u_1, u_2) of the problem (2.10), (2.9) *admits the estimate*

$$u_2(0) < \rho_0.$$

The proofs of Lemmas 2.3 and 2.4 are contained in [14, see Lemmas 2.5 and 2.6].

2.3. A Lemma on Nonnegative Solutions of Linear Homogeneous Differential Systems with Advanced Arguments

The following assertion is obvious.

LEMMA 2.5 *Let $\tilde{l}_{ik}: R_+ \to R_+$ $(i,k = 1,2)$ be locally summable functions and $\tau_{ik}: R_+ \to R_+$ $(i,k = 1,2)$ be continuous functions satisfying inequalities* (1.4). *Let, moreover, the differential system*

$$\frac{dv_i(t)}{dt} = -\sum_{k=1}^{2} \tilde{l}_{ik}(t) v_k(\tau_{ik}(t)) \quad (i = 1,2) \tag{2.12}$$

have a solution (v_1, v_2) satisfying the condition

$$v_i(t) \geq 0 \quad \text{for } t \geq t_0 \quad (i = 1,2), \tag{2.13}$$

where t_0 is a sufficiently large positive number. Then

$$v_i(t) \geq 0 \quad \text{for } 0 \leq t \leq t_0 \quad (i = 1,2).$$

If instead of (2.13) *the equalities*

$$v_i(t) = 0 \quad \text{for } t \geq t_0 \quad (i = 1,2)$$

are fulfilled, then

$$v_i(t) \equiv 0 \quad (i = 1,2).$$

3. EXISTENCE THEOREMS

THEOREM 3.1 *Let conditions* (1.3)–(1.5) *be fulfilled and let there exist numbers* $a_i > 0$ $(i = 1, 2)$ *and* $y_0 \geq 0$ *such that*

$$\tau_{12}(t) \leq a_2 \quad \textit{for } 0 \leq t \leq a_1, \tag{3.1}$$

$$f_1(t,x,y) \leq -\delta(t,y) \quad \textit{for } 0 \leq t \leq a_1, \quad 0 \leq x \leq r, \quad y \geq y_0, \tag{3.2}$$

and

$$\begin{aligned} f_2(t,x,y) \geq -[h(t) + |f_1(t,x,y)|]\omega(y) \quad \textit{for } 0 \leq t \leq a_2, \\ 0 \leq x \leq r, \quad y \geq y_0, \end{aligned} \tag{3.3}$$

where $\delta : [0, a_1] \times [y_0, +\infty[\to R_+$ *is a function summable in the first and nondecreasing in the second argument, while* $h : [0, a_2] \to R_+$ *and* $\omega : [y_0, +\infty[\to]0, +\infty[$ *are summable and nondecreasing continuous functions, respectively. Let, moreover,* $\tau_{1i}(t) \equiv \tau_{2i}(t)$ $(i = 1, 2)$,

$$\lim_{y \to +\infty} \int_0^{a_1} \delta(t,y)dt > r, \tag{3.4}$$

and

$$\int_{y_0}^{+\infty} \frac{dy}{\omega(y)} = +\infty. \tag{3.5}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof Conditions (1.4) and (3.1) imply $a_2 \geq a_1$. Suppose $a_2 = a_0$,

$$\omega(y) = \omega(y_0) \quad \text{for } 0 \leq y \leq y_0,$$

and

$$\begin{aligned} \delta(t,y) = 0 \quad \text{for } a_1 < t \leq a_0, \quad y \geq y_0, \\ \delta(t,y) = 0 \quad \text{for } 0 \leq t \leq a_0, \quad 0 \leq y \leq y_0. \end{aligned} \tag{3.6}$$

Moreover, without loss of generality we assume that the inequality

$$h(t) > \frac{1}{\omega(y_0)} \max\{|f_2(t,x,y)| \ : \ 0 \leq x \leq r, \quad 0 \leq y \leq y_0\}$$

holds on $[0, a_0]$. Then by virtue of (1.3) and (3.1)–(3.3) we have

$$\begin{aligned} &\tau_{12}(t) \leq a_0 \quad \text{for } 0 \leq t \leq a_1, \\ &f_1(t,x,y) \leq -\delta(t,y) \quad \text{for } 0 \leq t \leq a_0, \quad 0 \leq x \leq r, \quad y \geq 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &f_2(t,x,y) \geq -[h(t) + |f_1(t,x,y)|]\omega(y) \quad \text{for } 0 \leq t \leq a_0, \\ &0 \leq x \leq r, \quad y \geq 0. \end{aligned} \tag{3.8}$$

On the other hand, according to (3.4)–(3.6) the functions δ and ω satisfy the conditions of Lemma 2.3. Below under ρ_0 we will mean the number appearing in that lemma.

By virtue of Corollary 1.3 from [14] problem (1.1), (1.2_a) has a nonnegative solution (u_1, u_2) for an arbitrarily fixed $a \in]a_0, +\infty[$. By Lemma 2.1, to complete the proof we have to show that u_2 admits estimate (2.1).

In view of (1.3)–(1.5) the functions u_1 and u_2 are nonincreasing and satisfy the inequalities

$$\begin{aligned} &u_1(t) \leq u_1(0) \leq r \quad \text{for } 0 \leq t \leq a, \\ &u_2(\tau_{22}(t)) \leq u_2(t) \quad \text{for } 0 \leq t \leq a. \end{aligned} \tag{3.9}$$

If along with these conditions we take into account (3.6)–(3.8), then it will become evident that the restriction of (u_1, u_2) on $[0, a_0]$ is a solution of (2.8), (2.9). Hence in view of the choice of ρ_0 we get estimate (2.1). ■

Remark 3.1 Condition (3.4) in Theorem 3.1 is essential and it cannot be replaced by the condition

$$\lim_{y \to +\infty} \int_0^{a_1} \delta(t,y)\, dt = r. \tag{3.4'}$$

To convince ourselves that this is so, consider the case, where

$$f_1(t,x,y) = \begin{cases} -(ry/(1+y)) & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}, \tag{3.10}$$

$$f_2(t,x,y) \equiv -x(1+y), \quad \tau_{ik}(t) \equiv t \quad (i,k = 1,2), \tag{3.11}$$

and for system (1.1) consider the boundary value problem

$$u_1(0) = r, \quad u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \text{for } t \geq 0. \tag{3.12}$$

On account of (3.10) and (3.11) inequalities (3.1)–(3.3), where $a_1=a_2=1$, $y_0=0$, $\delta(t,y)=ry/(1+y)$, $h(t)=r$, $\omega(y)=1+y$, hold. Consequently, in this case all the conditions of Theorem 3.1, except (3.4), are fulfilled. Instead of (3.4) we have (3.4′). Let us now show that nevertheless, problem (1.1), (3.12) has no solution. Assume the contrary that this problem has a solution (u_1,u_2). Then in view of (3.10) and (3.11) we get

$$r-u_1(1)=r\int_0^1\frac{u_2(t)}{1+u_2(t)}dt<r, \tag{3.13}$$
$$u_1(t)=u_1(1) \quad \text{for } t\geq 1,$$

and

$$u_2(1)-u_2(t)=u_1(1)\int_1^t(1+u_2(s))ds\geq(t-1)u_1(1) \quad \text{for } t\geq 1.$$

The latter inequality results in $u_1(1)=0$. But this contradicts condition (3.13).

Remark 3.2 Condition (3.5) in Theorem 3.1 cannot be replaced by the condition

$$\int_{y_0}^{+\infty}\frac{y^{\varepsilon}\,dy}{\omega(y)}=+\infty \tag{3.5′}$$

no matter how small $\varepsilon>0$ would be. To convince ourselves that this is so, consider the boundary value problem

$$u_1'(t)=-u_2(t), \quad u_2'(t)=-\frac{2}{\varepsilon}u_1(t)(1+u_2^2(t))^{1+\varepsilon/2}, \tag{3.14}$$

$$u_1(0)=1, \quad u_1(t)\geq 0, \quad u_2(t)\geq 0 \quad \text{for } t\geq 0, \tag{3.15}$$

which is obtained from (1.1), (1.2) in the case, where

$$\varphi(x,y)=x-1, \quad r=1, \quad \tau_{ik}(t)\equiv t \ (i,k=1,2),$$
$$f_1(t,x,y)=-y, \quad f_2(t,x,y)=-\frac{2}{\varepsilon}x(1+y^2)^{1+\varepsilon/2}.$$

In view of these equalities, inequalities (3.1)–(3.3), where $a_1=a_2=1$, $\delta(t,y)\equiv y$, $h(t)=1$, $\omega(y)=(2/\varepsilon)(1+y)^{1+\varepsilon}$, are fulfilled. Consequently, for problem (3.14), (3.15) all the conditions of Theorem 3.1, except (3.5), hold. Instead of (3.5) condition (3.5′) is fulfilled. Let us now show that problem (3.14), (3.15) has no solution. Indeed, should this problem have a solution (u_1, u_2), the functions u_1 and u_2 would satisfy the conditions

$$\lim_{t\to+\infty} u_i(t) = 0 \ \ (i=1,2) \quad \text{and} \quad \frac{2u_2(t)u_2'(t)}{[1+u_2^2(t)]^{1+\varepsilon/2}} = \frac{4}{\varepsilon} u_1(t)u_1'(t).$$

Thus

$$-\int_0^{+\infty} \frac{du_2^2(t)}{[1+u_2^2(t)]^{1+\varepsilon/2}} = -\frac{2}{\varepsilon}\int_0^{+\infty} du_1^2(t)$$

and

$$\int_0^{u_2^2(0)} \frac{dy}{(1+y)^{1+\varepsilon/2}} = \frac{2}{\varepsilon}.$$

But this is impossible, since

$$\int_0^{u_2^2(0)} \frac{dy}{(1+y)^{1+\varepsilon/2}} < \int_0^{+\infty} \frac{dy}{(1+y)^{1+\varepsilon/2}} = \frac{2}{\varepsilon}.$$

If we replace condition (3.3) in Theorem 3.1 by the condition

$$f_2(t,x,y) \geq -h(t)\omega(y) \quad \text{for } 0 \leq t \leq a_2, \quad 0 \leq x \leq r, \quad y \geq y_0, \qquad (3.3')$$

then the requirement on the fulfillment of the identities $\tau_{1i}(t) \equiv \tau_{2i}(t)$ $(i=1,2)$ would be unnecessary. More precisely, the following theorem is valid.

THEOREM 3.1′ *Let conditions* (1.3)–(1.5) *be fulfilled and let there exist numbers* $a_i > 0$ $(i=1,2)$ *and* $y_0 \geq 0$ *such that along with* (3.1) *and* (3.2) *condition* (3.3′) *holds, where* $\delta: [0,a_1] \times [y_0, +\infty[\to R_+$ *is a summable in the first and nondecreasing in the second argument function satisfying inequality* (3.4), $h: [0,a_2] \to R_+$ *is a summable function, while* $\omega: [y_0, +\infty[\to]0, +\infty[$ *is a nondecreasing continuous function satisfying condition* (3.5). *Then problem* (1.1), (1.2) *has at least one solution.*

The proof of this theorem is omitted, since it is analogous to that of Theorem 3.1.

According to Remark 3.2, it is impossible, in general, to omit conditions (3.3) and (3.5) from Theorem 3.1. However it appears that these conditions can be replaced by a certain restriction imposed on the function τ_{22}. More precisely, the following theorem is valid.

THEOREM 3.2 *Let there exist numbers* $a_i > 0$ $(i = 1, 2)$ *and* $y_0 \geq 0$ *such that along with* (1.3)–(1.5) *conditions* (3.1) *and* (3.2) *are fulfilled, where* $\delta : [0, a_1] \times [y_0, +\infty[\to R_+$ *is a function summable in the first, nondecreasing in the second argument and satisfying inequality* (3.4). *Let, moreover,*

$$\tau_{22}(t) \geq a_2 \quad \text{for } t \geq 0. \tag{3.16}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof In view of (3.4) there exists $\rho_1 \in [y_0, +\infty[$ such that

$$\int_0^{a_1} \delta(t, \rho_1)dt > r. \tag{3.17}$$

Put

$$\begin{aligned} f_2^*(t) &= \max\{|f_2(t, x, y)| : \ 0 \leq x \leq r, \ 0 \leq y \leq \rho_1\}, \\ \rho_0 &= \rho_1 + \int_0^{a_2} f_2^*(t)dt. \end{aligned} \tag{3.18}$$

By virtue of Corollary 1.4 from [14] and conditions (1.3), (1.5), for an arbitrarily fixed $a \in]a_2, +\infty[$ problem (1.1), (1.2_a) has a nonnegative solution (u_1, u_2), the functions u_1 and u_2 do not increase and u_1 satisfies condition (3.9). By Lemma 2.1, to complete the proof it is sufficient to show that u_2 admits estimate (2.1).

First show that

$$u_2(a_2) < \rho_1. \tag{3.19}$$

Assume the contrary that $u_2(a_2) \geq \rho_1$. Then in view of (3.1), (3.2) and (3.9) we get

$$u_2(\tau_{12}(t)) \geq u_2(a_2) \geq \rho_1 \quad \text{for } 0 \leq t \leq a_1$$

and

$$r \geq u_1(0) - u_1(a_1) = -\int_0^{a_1} u_1'(s)ds \geq \int_0^{a_1} \delta(s, u_2(\tau_{12}(s)))\, ds \geq$$
$$\geq \int_0^{a_1} \delta(s, \rho_1)ds,$$

which contradicts inequality (3.17). So the validity of estimate (3.19) is proved. Hence according to (3.16) follows

$$u_2(\tau_{22}(t)) < \rho_1 \quad \text{for } t \geq 0.$$

If along with this we take into account conditions (3.9), (3.19) and equalities (3.18), then we obtain

$$u_2(0) = u_2(a_2) - \int_0^{a_2} f_2(t, u_1(\tau_{21}(t)), u_2(\tau_{22}(t)))dt <$$
$$< \rho_1 + \int_0^{a_2} f_2^*(t)dt = \rho_0.$$

■

COROLLARY 3.1 *Let conditions* (1.3)–(1.5) *and*

$$\tau_{22}(t) > \limsup_{s \to 0} \tau_{12}(s) \quad \text{for } t \geq 0 \tag{3.20}$$

be fulfilled. Let, moreover, there exist $a_0 \in]0, +\infty[$ *and a nondecreasing function* $\delta : R_+ \to R_+$ *such that*

$$f_1(t, x, y) \leq -\delta(y) \quad \text{for } 0 \leq t \leq a_0, \quad 0 \leq x \leq r, \quad y \geq 0 \tag{3.21}$$

and

$$\lim_{y \to +\infty} \delta(y) = +\infty. \tag{3.22}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof By virtue of (3.20) and (3.21) there exist $a_1 \in]0, a_0]$ and $a_2 \in [a_1, +\infty[$ such that the functions τ_{12}, f_1 and τ_{22} satisfy inequalities (3.1), (3.2) and (3.16), where $\delta(t, y) \equiv \delta(y)$ and $y_0 = 0$. On the other hand, according to (3.22) the function δ satisfies condition (3.4). Therefore all the conditions of Theorem 3.2 are fulfilled. ■

THEOREM 3.3 *Let conditions* (1.3)–(1.5) *be fulfilled and let for some* $a_0 \in]0, 1]$ *the inequalities*

$$\tau_{12}(t) \leq t^{\alpha} \quad for\ 0 \leq t \leq a_0, \tag{3.23}$$

$$f_1(t,x,y) \leq -lt^{\beta}y^{\lambda_1} \quad for\ 0 \leq t \leq a_0,\ \ 0 \leq x \leq r,\ \ y \geq 0, \tag{3.24}$$

and

$$f_2(t,x,y) \geq -h(t)(1+y)^{1+\lambda_2} \quad for\ 0 \leq t \leq a_0,\ \ 0 \leq x \leq r,\ \ y \geq 0 \tag{3.25}$$

hold, where $0 < \alpha \leq 1$, $\beta > -1$, $l > 0$, $\lambda_1 > 0$, $\lambda_2 \geq 0$, *and* $h : [0, a_0] \to R_+$ *is a measurable function satisfying the condition*

$$\int_0^{a_0} [\tau_{22}(t)]^{-((1+\beta)\lambda_2/(\alpha\lambda_1))} h(t)dt < +\infty. \tag{3.26}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof Suppose

$$\tau(t) = \min\{a_0, \tau_{22}(t)\}.$$

Then in view of (3.26) condition (2.11) holds.

Let ρ_0 be the positive constant appearing in Lemma 2.4. By virtue of Corollary 1.5 from [14] and conditions (1.3), (1.5), for an arbitrarily fixed $a \in]a_0^{\alpha}, +\infty[$ problem (1.1), (1.2_a) has a nonnegative solution (u_1, u_2), the functions u_1 and u_2 do not increase and u_1 satisfies (3.9). If along with this fact we take into account conditions (3.23)–(3.25), then it will become evident that the restriction of (u_1, u_2) on $[0, a_0]$ is a nonnegative solution of problem (2.10), (2.9). Hence in view of the choice of ρ_0 we obtain (2.1). By Lemma 2.1, estimate (2.1) guarantees the solvability of problem (1.1), (1.2), since ρ_0 does not depend on a. ∎

THEOREM 3.4 *If conditions* (1.3), (1.4) *and* (1.6) *are fulfilled, then problem* (1.1), (1.2) *has at least one solution.*

Proof First of all note that (1.5) follows from (1.6).

According to Theorem 1.2 from [14], for any $a>0$ problem (1.1), (1.2_a) has a nonnegative solution (u_1, u_2). On the other hand, by virtue of (1.6) it is clear that

$$u_2(0) \leq r.$$

By Lemma 2.1, this estimate guarantees the solvability of (1.1), (1.2). ■

As is obvious from the proofs of Theorems 3.1 – 3.4, if there hold the conditions of one of these theorems, then all the conditions of Lemma 2.1 are fulfilled. If along with this fact we take into consideration Lemma 2.2, then it becomes evident the validity of the following theorem.

THEOREM 3.5 *Let all the conditions of one of Theorems* 3.1 – 3.3, 3.1′ (*Theorem* 3.4) *be fulfilled and let*, *moreover*,

$$\int_0^{+\infty} f_2^*(t;\rho)dt < +\infty \quad for\ \rho>0 \ \left(\int_0^{+\infty} f_2^*(t;r)dt < +\infty\right),$$

where

$$f_2^*(t;\rho) = max\{|f_2(t,x,y)| : \ 0 \leq x \leq r, \quad 0 \leq y \leq \rho\}.$$

Then problem (1.1), (1.2) *has at least one solution* (u_1, u_2) *satisfying condition* (2.7).

Remark 3.3 If the conditions of Theorem 3.5 are fulfilled, then problem (1.1), (1.2) may have also a solution which doesn't satisfy (2.7). Indeed, for the problem

$$u_1'(t) = -\exp(-t)u_2(t), \quad u_2'(t) = 0,$$
$$u_1(0) = 1, \quad u_1(t) \geq 0, \ u_2(t) \geq 0 \quad for\ t \geq 0$$

all the conditions of Theorem 3.5 *hold, but nevertheless for any* $c \in [0, 1]$ *this problem has a solution* (u_1, u_2) *with the components*

$$u_1(t) = 1 - c + c\exp(-t), \quad u_2(t) = c.$$

4. BEHAVIOUR OF SOLUTIONS OF PROBLEM (1.1), (1.2) AS $t \to +\infty$

THEOREM 4.1 *Let conditions* (1.3) *and* (1.5) *be fulfilled and let the inequalities*

$$f_i(t,x,y) \le -\delta_i(t,x,y) \ (i=1,2) \tag{4.1}$$

hold on the set $R_+ \times [0,r] \times R_+$, where δ_i: $R_+ \times [0,r] \times R_+ \to R_+$ $(i=1,2)$ *are locally summable in the first and nondecreasing in the last two arguments functions satisfying*

$$\sum_{i=1}^{2} \int_0^{+\infty} \delta_i(t,x,y)dt = +\infty \quad for\ 0 \le x \le r, \quad y \ge 0, \ \ x+y>0. \tag{4.2}$$

Then every solution (u_1,u_2) *of problem* (1.1), (1.2) *satisfies the condition*

$$\lim_{t\to+\infty} u_i(t) = 0 \ (i=1,2). \tag{4.3}$$

Proof In view of (1.3) and (1.5) the functions u_1 and u_2 do not increase and

$$0 \le u_1(t) \le r \quad \text{for } t \ge 0. \tag{4.4}$$

Suppose

$$\lim_{t\to+\infty} u_i(t) = \eta_i \ (i=1,2).$$

Then on account of (4.1) and (4.4) we find

$$-u_i'(t) \ge \delta_i(t,\eta_1,\eta_2) \quad \text{for } t \ge 0 \ (i=1,2)$$

and

$$\sum_{i=1}^{2} \int_0^{+\infty} \delta_i(t,\eta_1,\eta_2)dt \le u_1(0)+u_2(0) < +\infty,$$

which, by virtue of (4.2), results in $\eta_1=\eta_2=0$. Consequently, equalities (4.3) hold. ■

Remark 4.1 From the proof of Theorem 4.1 it is clear that if instead of (4.2) the condition

$$\sum_{i=1}^{2}\int_{0}^{+\infty}\delta_i(t,0,y)dt=+\infty \quad \text{for } y>0$$

$$\left(\sum_{i=1}^{2}\int_{0}^{+\infty}\delta_i(t,x,0)dt=+\infty \quad \text{for } 0<x\le r\right)$$

is fulfilled, then an arbitrary solution (u_1,u_2) of problem (1.1), (1.2) would be satisfy the condition

$$\lim_{t\to+\infty}u_2(t)=0\ \left(\lim_{t\to+\infty}u_1(t)=0\right)$$

instead of (4.3).

THEOREM 4.2 *Let conditions* (1.3) *and* (1.5) *be fulfilled and let the inequalities*

$$f_1(t,x,y)\le -\delta_1(t,y),\quad f_2(t,x,y)\le -\delta_2(t,x) \tag{4.5}$$

hold on $R_+\times[0,r]\times R_+$, *where* $\delta_1:R_+\times R_+\to R_+$ *and* $\delta_2:R_+\times[0,r]\to R_+$ *are locally summable in the first and nondecreasing in the second argument. Let, moreover, either*

$$\int_{0}^{+\infty}\delta_i(t,x)dt=+\infty \quad \textit{for } 0<x\le r\ (i=1,2) \tag{4.6}$$

or there exist $k\in\{1,2\}$ *such that*

$$\begin{aligned}&\int_{0}^{+\infty}\delta_k(t,x)dt<+\infty,\\ &\int_{0}^{+\infty}\delta_{3-k}\left(t,\int_{\tau_{3-kk}(t)}^{+\infty}\delta_k(s,x)ds\right)dt=+\infty \quad \textit{for } 0<x\le r.\end{aligned} \tag{4.7}$$

Then an arbitrary solution (u_1,u_2) *of problem* (1.1), (1.2) *satisfies condition* (4.3).

Proof If (4.6) holds, then the validity of (4.3) follows from Theorem 4.1.

Consider the case where condition (4.7) is fulfilled for some $k \in \{1,2\}$. Then

$$\int_0^{+\infty} \delta_{3-k}(t,x)dt = +\infty \quad \text{for } 0 < x \leq r. \tag{4.8}$$

On the other hand, due to (1.3) and (1.5) the functions u_1 and u_2 do not increase and u_1 satisfies (4.4). By virtue of (4.4) and (4.5) we have

$$\delta_i(t, u_{3-i}(\tau_{i3-i}(t))) \leq -u_i'(t) \quad \text{for } t \geq 0 \ (i = 1,2) \tag{4.9}$$

and

$$\int_0^{+\infty} \delta_i(s, u_{3-i}(\tau_{i3-i}(s)))ds \leq u_i(0) < +\infty \quad (i = 1,2). \tag{4.10}$$

(4.8) and (4.10) imply

$$\lim_{t\to+\infty} u_k(t) = 0.$$

Therefore to complete the proof, it remains to show that

$$\lim_{t\to+\infty} u_{3-k}(t) = 0. \tag{4.11}$$

Assume the contrary that (4.11) is violated. Then there exists $\eta_0 \in]0,r[$ such that

$$u_{3-k}(t) \geq \eta_0 \quad \text{for } t \geq 0.$$

On account of this inequality from (4.9) and (4.10) we find

$$u_k(t) \geq \int_t^{+\infty} \delta_k(s,\eta_0)ds \quad \text{for } t \geq 0$$

and

$$\int_0^{+\infty} \delta_{3-k}\left(t, \int_{\tau_{3\ kk}(t)}^{+\infty} \delta_k(s,\eta_0)ds\right)dt < +\infty,$$

which contradicts (4.7). The contradiction obtained proves the validity of the theorem. ■

5. UNIQUENESS THEOREMS

THEOREM 5.1 *Let along with* (1.3) – (1.5) *the condition*

$$\varphi(x,y) < \varphi(\bar{x},\bar{y}) \quad \text{for } 0 \le x < \bar{x} \le r, \quad 0 \le y \le \bar{y} \tag{5.1}$$

be fulfilled and let the functions f_i $(i=1,2)$ *with respect to the last two arguments have partial derivatives satisfying the local Carathéodory conditions. Let, moreover, there exist locally summable functions* l_{ik}: $R_+ \to R_+ (i,k=1,2)$ *and* l_0: $R_+ \to R_+$ *such that*

$$\int_0^{+\infty} l_{2k}(t)dt < +\infty \quad (k=1,2), \tag{5.2}$$

$$\begin{gathered}\int_0^{+\infty} l_{12}(t) \int_{\tau_{12}(t)}^{+\infty} l_{21}(s)ds\, dt < +\infty,\\ \int_0^{+\infty} l_{11}(t) \int_t^{\tau_{11}(t)} \left[l_{12}(s) \int_{\tau_{12}(s)}^{+\infty} l_{21}(\xi)d\xi + l_{11}(s) \right] ds\, dt < +\infty,\end{gathered} \tag{5.3}$$

$$mes\{l_0(t) > 0 : t \in R_+\} > 0, \tag{5.4}$$

and the inequalities

$$-l_{i1}(t) \le \frac{\partial f_i(t,x,y)}{\partial x} \le 0, \quad -l_{i2}(t) \le \frac{\partial f_i(t,x,y)}{\partial y} \le 0 \ (i=1,2), \tag{5.5}$$

$$\frac{\partial f_1(t,x,y)}{\partial y} \le -l_0(t) \tag{5.6}$$

are fulfilled on $R_+ \times [0,r] \times R_+$. *Then problem* (1.1), (1.2), (2.7) *has one and only one solution.*

Proof By virtue of (1.3), (5.4) – (5.6) the inequalities

$$\begin{gathered} f_1(t,x,y) \le -l_0(t)y,\\ |f_2(t,x,y)| \le l_{21}(t)r + l_{22}(t)y \le (l_{21}(t)r + l_{22}(t))(1+y)\end{gathered} \tag{5.7}$$

hold on $R_+ \times [0,r] \times R_+$, and there exists $a_1 > 0$ such that

$$\int_0^{a_1} l_0(t)dt > 0.$$

Consequently, all the conditions of Theorem 3.1′, *i.e.*, conditions (3.1), (3.2), (3.3′), (3.4) and (3.5) are fulfilled, where $a_2 = \max\{\tau_{12}(t): 0 \le t \le a_1\}$, $\delta(t,y) = l_0(t)y$ and $\omega(y) = 1 + y$. On the other hand, according to (5.2)

$$\int_0^{+\infty} f_2^*(t;\rho)dt \le \int_0^{+\infty} [rl_{21}(t) + \rho l_{22}(t)]dt < +\infty \quad \text{for } \rho > 0,$$

where

$$f_2^*(t;\rho) = \max\{|f_2(t,x,y)| : 0 \le x \le r, 0 \le y \le \rho\}.$$

Hence by Theorem 3.5 follows the solvability of problem (1.1), (1.2), (2.7). To complete the proof, it remains to show that this problem has no more than one solution.

First of all note that in view of (5.2) and (5.3) the inequalities

$$\int_{t^*}^{+\infty} l_{22}(t)dt < \frac{1}{2}, \tag{5.8}$$

$$\int_{t^*}^{+\infty} l_{11}(t) \int_t^{\tau_{11}(t)} \left[l_{12}(s) \int_{\tau_{12}(s)}^{+\infty} l_{21}(\xi)d\xi + l_{11}(s) \right] ds\, dt + \\ + \int_{t^*}^{+\infty} l_{12}(t) \int_{\tau_{12}(t)}^{+\infty} l_{21}(s)ds\, dt < \frac{1}{4} \tag{5.9}$$

are fulfilled for some $t^* > 0$.

Let (u_1, u_2) and $(\bar{u}_1, \bar{u}_2)$ be arbitrary solutions of problem (1.1), (1.2), (2.7). Put

$$v_i(t) = \bar{u}_i(t) - u_i(t) \quad (i = 1, 2).$$

Then the vector function (v_1, v_2) is a solution of system (2.12) satisfying the conditions

$$\varphi(u_1(0) + v_1(0), u_2(0) + v_2(0)) = \varphi(u_1(0), u_2(0)) \tag{5.10}$$

and

$$\lim_{t \to +\infty} v_2(t) = 0, \tag{5.11}$$

where

$$\tilde{l}_{i1}(t)=\begin{cases}-[f_i(t,\bar{u}_1(\tau_{i1}(t)),u_2(\tau_{i2}(t)))-f_i(t,u_1(\tau_{i1}(t)),u_2(\tau_{i2}(t)))]/v_1(\tau_{i1}(t)) & \text{for } v_1(\tau_{i1}(t))\neq 0\\ l_{i1}(t) & \text{for } v_1(\tau_{i1}(t))=0\end{cases},$$

$$\tilde{l}_{i2}(t)=\begin{cases}-[f_i(t,\bar{u}_1(\tau_{i1}(t)),\bar{u}_2(\tau_{i2}(t)))-f_i(t,\bar{u}_1(\tau_{i1}(t)),u_2(\tau_{i2}(t)))]/v_2(\tau_{i2}(t)) & \text{for } v_2(\tau_{i2}(t))\neq 0\\ l_{i2}(t) & \text{for } v_2(\tau_{i2}(t))=0\end{cases}.$$

On the other hand, by virtue of (5.5),

$$0\le \tilde{l}_{ik}(t)\le l_{ik}(t)\quad \text{for } t\ge 0\ \ (i,k=1,2) \tag{5.12}$$

and the functions $\tilde{l}_{ik}:R_+\to R_+ (i,k=1,2)$ are locally summable.

Due to (5.11), from (2.12) we have

$$v_2(t)=\int_t^{+\infty}[\tilde{l}_{21}(s)v_1(\tau_{21}(s))+\tilde{l}_{22}(s)v_2(\tau_{22}(s))]ds. \tag{5.13}$$

If we now suppose

$$v^*(t)=\sup\{|v_2(s)|:s\ge t\},$$

then on account of (5.2), (5.8) and (5.12), from (5.13) we get

$$\begin{aligned} v^*(t)&\le \int_t^{+\infty}[\tilde{l}_{21}(s)|v_1(\tau_{21}(s))|+\tilde{l}_{22}(s)|v_2(\tau_{22}(s))|]ds\le\\ &\le \int_t^{+\infty}\tilde{l}_{21}(s)|v_1(\tau_{21}(s))|ds+v^*(t)\int_t^{+\infty}l_{22}(s)ds\le\\ &\le \int_t^{+\infty}\tilde{l}_{21}(s)|v_1(\tau_{21}(s))|ds+\frac{1}{2}v^*(t)\quad \text{for } t\ge t^*,\\ |v_2(t)|&\le v^*(t)\le 2\int_t^{+\infty}\tilde{l}_{21}(s)|v_1(\tau_{21}(s))|ds\le\\ &\le 2\int_t^{+\infty}l_{21}(s)|v_1(\tau_{21}(s))|ds\quad \text{for } t\ge t^*, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} v_2(t)&\ge \int_t^{+\infty}\tilde{l}_{21}(s)v_1(\tau_{21}(s))ds-\\ &\quad -2\int_t^{+\infty}l_{22}(s)ds\int_t^{+\infty}\tilde{l}_{21}(s)|v_1(\tau_{21}(s))|ds\ge\\ &\ge \int_t^{+\infty}\tilde{l}_{21}(s)[v_1(\tau_{21}(s))-|v_1(\tau_{21}(s))|]ds\quad \text{for } t\ge t^*. \end{aligned} \tag{5.15}$$

There are two possibilities: either

$$v_1(1) \neq 0 \quad \text{for } t \geq t^* \tag{5.16}$$

or there exists $t_0 \in [t^*, +\infty[$ such that

$$v_1(t_0) = 0. \tag{5.17}$$

Let inequality (5.16) hold. Then without loss of generality it can be assumed that

$$v_1(t) > 0 \quad \text{for } t \geq t^*.$$

Thus from (5.15) we have

$$v_2(t) \geq 0 \quad \text{for } t \geq t^*.$$

By Lemma 2.5 and condition (5.1), the last two inequalities result in

$$v_1(0) > 0, \quad v_2(0) \geq 0$$

and

$$\varphi(u_1(0) + v_1(0), u_2(0) + v_2(0)) > \varphi(u_1(0), u_2(0)),$$

which contradicts (5.10). The contradiction obtained proves that equality (5.17) holds for some $t_0 \in [t^*, +\infty[$.

From (2.12) and (5.17) we have

$$v_1'(t) = -\tilde{l}_{11}(t)v_1(t) - \tilde{l}_{11}(t)\int_t^{\tau_{11}(t)} v_1'(s)ds - \tilde{l}_{12}(t)v_2(\tau_{12}(t)) =$$
$$= -\tilde{l}_{11}(t)v_1(t) + q(t)$$

and

$$v_1(t) = \int_{t_0}^t \exp\left(-\int_s^t \tilde{l}_{11}(\xi)d\xi\right)q(s)ds, \tag{5.18}$$

where

$$q(t) = \tilde{l}_{11}(t)\int_t^{\tau_{11}(t)} [\tilde{l}_{11}(s)v_1(\tau_{11}(s)) + \tilde{l}_{12}(s)v_2(\tau_{12}(s))]ds -$$
$$- \tilde{l}_{12}(t)v_2(\tau_{12}(t)).$$

Suppose

$$v_0 = \sup\{|v_1(t)| : t \geq t_0\}.$$

Then by virtue of (5.12) and (5.14) we find

$$|q(t)| \leq 2v_0 l_{11}(t) \int_t^{\tau_{11}(t)} \left[l_{11}(s) + l_{12}(s) \int_{\tau_{12}(s)}^{+\infty} l_{21}(\xi) d\xi \right] ds + \\ + 2v_0 l_{12}(t) \int_{\tau_{12}(t)}^{+\infty} l_{21}(s) ds \quad \text{for } t \geq t_0.$$

If along with this inequality we take into account (5.9) and the fact that the function $\tilde{l}_{11}$ is nonnegative, then from (5.18) we obtain

$$v_0(t) \leq \int_{t_0}^{+\infty} |q(t)| dt \leq \frac{1}{2} v_0.$$

Hence it is clear that $v_0 = 0$ and, consequently, $v_1(t) = 0$ for $t \geq t_0$. Thus (5.14) implies $v_2(t) = 0$ for $t \geq t_0$. If we now apply Lemma 2.5, then it will become evident that $v_i(t) \equiv 0$ $(i = 1, 2)$, *i.e.*, $u_i(t) \equiv \bar{u}_i(t)$ $(i = 1, 2)$. ■

The example in Remark 3.3 shows that if all the conditions of Theorem 5.1 are fulfilled, then problem (1.1), (1.2) may have an infinite set of solutions which do not satisfy condition (2.7). To guarantee the uniqueness of a solution of problem (1.1), (1.2) it is sufficient to replace condition (5.4) by the condition

$$\int_0^{+\infty} l_0(t) dt = +\infty. \tag{5.19}$$

More precisely, the following theorem is valid.

THEOREM 5.2 *Let conditions* (1.3)–(1.5) *and* (5.1) *be fulfilled and let the functions* f_i $(i = 1, 2)$ *with respect to the last two arguments have partial derivatives satisfying the local Carathéodory conditions and inequalities* (5.5) *and* (5.6) *on* $R_+ \times [0, r] \times R_+$, *where* $l_{ik} : R_+ \to R_+$ $(i, k = 1, 2)$ *and* $l_0 : R_+ \to R_+$ *are locally summable functions satisfying conditions* (5.2), (5.3) *and* (5.19). *Then problem* (1.1), (1.2) *has one and only one solution.*

Proof As is shown when proving Theorem 5.1, the function f_1 satisfies (5.7). However, by virtue of Remark 4.1 from the fulfilment of (5.7) and (5.19) follows that an arbitrary solution (u_1, u_2) of problem (1.1), (1.2) satisfies condition (2.7). Consequently, in this case problems (1.1), (1.2) and (1.1), (1.2), (2.7) are equivalent. On the other hand, the fulfilment of the conditions of Theorem 5.2 guarantees the fulfilment of the conditions of Theorem 5.1, and thus problem (1.1), (1.2), (2.7) has one and only one solution. Therefore problem (1.1), (1.2) is uniquely solvable. ■

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