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On periodic solutions of systems of differential equations with deviating arguments

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1. Formulation of the main results

Let us consider the differential system

$$\frac{dx_i(t)}{dt} = f_i(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))) \quad (i = 1, \dots, n), \tag{1.1}$$

where the functions $f_i : R \times R^n \to R$ $(i = 1, \dots, n)$ satisfy the local Carathéodory conditions and are periodic with respect to the first argument with period $\omega > 0$, i.e., the equality

$$f_i(t + \omega, x_1, \dots, x_n) = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{1.2}$$

holds for almost all $t \in R$ and all $(x_i)_{i=1}^n \in R^n$; $\tau_{ik} : R \to R$ $(i, k = 1, \dots, n)$ are measurable functions such that

$$[\tau_{ik}(t + \omega) - \tau_{ik}(t)]/\omega \quad (i, k = 1, \dots, n) \quad \text{are integer numbers} \tag{1.3}$$

for almost all $t \in R$.

The problem on ω-periodic solutions of systems of form (1.1) was investigated by many authors (see, for instance, [1–23] and the references cited therein). In this paper, new and optimal in a certain sense conditions of the existence, nonexistence and uniqueness of an ω-periodic solution of the above-mentioned system are established

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using the improved method of a priori estimation of periodic solutions of systems of one-sided functional-differential inequalities proposed in [4, 9].

Before starting the formulation of the main results, we introduce the notation which will be used throughout the paper. R is the set of real numbers; R^n is an n-dimensional real Euclidean space; $x=(x_i)_{i=1}^n$ is an n-dimensional column-vector with components $x_1,\dots,x_n$, $X=(x_{ik})_{i,k=1}^n$ is an $n\times n$ matrix with components x_{ik} $(i,k=1,\dots,n)$ and the norm

$$\|X\|=\sum_{i,k=1}^n |x_{ik}|;$$

and $r(X)$ is the spectral radius of the $n\times n$ matrix X.

If $p:[0,\omega]\to R$ is a summable function and $\int_0^\omega p(\xi)\,\mathrm{d}\xi\neq 0$, then

$$\Delta(p)=\left[1-\exp\left(\int_0^\omega p(\xi)\,\mathrm{d}\xi\right)\right]^{-1},$$

$$g(p)(t,s)=|\Delta(p)|\exp\left(\int_s^t p(\xi)\,\mathrm{d}\xi\right) \quad \text{for } 0\le s\le t, \tag{1.4}$$

$$g(p)(t,s)=|\Delta(p)-1|\exp\left(\int_s^t p(\xi)\,\mathrm{d}\xi\right) \quad \text{for } t<s\le \omega.$$

If $i\in\{1,\dots,n\}$, then

$$I_i=\{t\in[0,\omega]:\ \tau_{ii}^0(t)\neq t\}; \tag{1.5}$$

$$\nu_{ik}(t) \text{ is the integer part of } \tau_{ik}(t)/\omega,\ \tau_{ik}^0(t)=\tau_{ik}(t)-\nu_{ik}(t)\omega; \tag{1.6}$$

$$f_i^*(t,\rho_1,\dots,\rho_n) = \max\{|f_i(t,x_1,\dots,x_n)|:\ |x_1|\le\rho_1,\dots,|x_n|\le\rho_n\}. \tag{1.7}$$

Theorem 1.1. *Let for each* $i\in\{1,\dots,n\}$ *the condition*

$$f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}\,(\sigma_i x_i)\le p_i(t)|x_i|+\sum_{k=1}^n p_{ik}(t)|x_k|+q(t) \tag{1.8}$$

hold on the set $[0,\omega]\times R^n$, *and the conditions*

$$\left|\int_t^{\tau_{ii}^0(t)}|p_i(s)|\,\mathrm{d}s\right|\le p_{ii}^*(t),\ \left|\int_t^{\tau_{ii}^0(t)}|f_i^*(s,|x_1|,\dots,|x_n|)\,\mathrm{d}s\right| \le\sum_{k=1}^n p_{ik}^*(t)|x_k|+q^*, \tag{1.9}$$

$$|f_i(t,x_1,\dots,x_i,\dots,x_n)-f_i(t,x_1,\dots,\bar{x}_i,\dots,x_n)|\le l_i(t)|x_i-\bar{x}_i| \tag{1.10}$$

hold on the set $I_i \times R^n$. *Here* $p_i : [0,\omega] \to R$, p_{ik}, q, *and* $l_i : [0,\omega] \to [0,+\infty[(i,k=1,\ldots,n)$ *are summable functions*, $p^*_{ik} : [0,\omega] \to [0,+\infty[$ $(i,k=1,\ldots,n)$ *are essentially bounded functions*, $\sigma_i \in \{-1,1\}$, *and* q^* *is a nonnegative number. Moreover, let*

$$\int_0^\omega p_i(t)\,dt < 0 \quad (i=1,\ldots,n) \tag{1.11}$$

and there exist a constant nonnegative matrix $A=(a_{ik})_{i,k=1}^n$ *such that* $r(A)<1$ *and*

$$\int_0^\omega g(\sigma_i p_i)(t,s)[p_{ik}(s) + l_i(s)p^*_{ik}(s)]\,ds \le a_{ik} \quad \text{for } 0 \le t \le \omega \quad (i,k=1,\ldots,n). \tag{1.12}$$

Then system (1.1) *has at least one* ω*-periodic solution.*

Remark 1.1. *If* $p_i(t) \le 0$ *for* $0 \le t \le \omega$ *and* $\int_0^\omega p_i(t)\,dt < 0$ $(i=1,\ldots,n)$, *then by Eq.* (1.4) *we have* $\int_0^\omega g(\sigma_i p_i)(t,s)|p_i(s)|\,ds = 1$ $(i=1,\ldots,n)$. *Now for condition* (1.12) *to be fulfilled it is sufficient that the inequalities*

$$p_{ik}(t) + l_i(t)p^*_{ik}(t) \le a_{ik}|p_i(t)| \quad (i,k=1,\ldots,n)$$

hold almost everywhere on $[0,\omega]$.

Remark 1.2. *It is easy to show that the nonnegative matrix* $A=(a_{ik})_{i,k=1}^n$ *satisfies the condition* $r(A)<1$ *iff and only iff when the real parts of the eigenvalues of the matrix* $(a_{ik} - \delta_{ik})_{i,k=1}^n$, *where* δ_{ik} *is the Kronecker symbol, are negative* (*see, for instance,* [11, Lemma 6.7]).

If $\tau_{ii}(t) \equiv t$, then $I_i = \emptyset$ and conditions (1.9) and (1.10) in Theorem 1.1 become unnecessary. Therefore, this theorem immediately implies

Corollary 1.1. *Let* $\tau_{ii}(t) \equiv t$ $(i=1,\ldots,n)$ *and for each* $i \in \{1,\ldots,n\}$ *on the set* $[0,\omega] \times R^n$ *the condition* (1.8) *be fulfilled, where* $p_i : [0,\omega] \to R$, $p_{ik}, q : [0,\omega] \to [0,+\infty[$ *are summable functions*, $\sigma_i \in \{-1,1\}$. *Moreover, let inequalities* (1.11) *be fulfilled and there exist a constant non-negative matrix* $A=(a_{ik})_{i,k=1}^n$ *such that* $r(A)<1$ *and*

$$\int_0^\omega g(\sigma_i p_i)(t,s)p_{ik}(s)\,ds \le a_{ik} \qquad \text{for } 0 \le t \le \omega \quad (i,k=1,\ldots,n). \tag{1.13}$$

Then system (1.1) *has at least one* ω*-periodic solution.*

Theorem 1.2. *Let* $\tau_{ii}(t) \equiv t$ $(i=1,\ldots,n)$ *and for each* $i \in \{1,\ldots,n\}$ *on the set* $[0,\omega] \times R^n$ *the conditions*

$$\sigma_{0i} f_i(t,x_1,\ldots,x_n) \le f_{0i}(t,x_1,\ldots,x_n)x_i - l_i(t)\sum_{k=1}^n a_{ik}|x_k| - q_i(t), \tag{1.14}$$

$$|f_{0i}(t,x_1,\ldots,x_n)| \le l_i(t) \tag{1.15}$$

be fulfilled, where $f_{0i}:[0,\omega]\times R^n\to R$ *is the function satisfying the local Carathéodory conditions,* l_i *and* $q_i:[0,\omega]\to[0,+\infty[$ *are summable functions different from zero on sets of positive measure,* $\sigma_{0i}\in\{-1,1\}$, *and* $A=(a_{ik})_{i,k=1}^n$ *is a nonnegative constant matrix such that* $r(A)\geq 1$. *Then system* (1.1) *has no* ω*-periodic solution.*

Corollary 1.2. *Let* $\tau_{ii}(t)\equiv t$ $(i=1,\dots,n)$ *and for each* $i\in\{1,\dots,n\}$ *on the set* $[0,\omega]\times R^n$ *the inequality*

$$-q_{0i}(t)\leq\sigma_{0i}f_i(t,x_1,\dots,x_n)+\sigma_{1i}l_i(t)x_i+l_i(t)\sum_{k=1}^n a_{ik}|x_k|\leq -q_i(t) \tag{1.16}$$

hold, where l_i, q_i *and* $q_{0i}:[0,\omega]\to[0,+\infty[$ *are summable functions different from zero on the sets of positive measure,* σ_{0i}, $\sigma_{1i}\in\{-1,1\}$, *and* $A=(a_{ik})_{i,k=1}^n$ *is a constant nonnegative matrix. Then for system* (1.1) *to have an* ω*-periodic solution it is necessary and sufficient that* $r(A)<1$.

Theorem 1.3. *Let for almost all* $t\in[0,\omega]$ *the functions* $f_i(t,\cdot,\dots,\cdot):R^n\to R$ $(i=1,\dots,n)$ *have continuous partial derivatives with respect to the last n arguments and for each* $i\in\{1,\dots,n\}$ *on the set* $[0,\omega]\times R^n$ *the inequalities*

$$\sigma_i\frac{\partial f_i(t,x_1,\dots,x_n)}{\partial x_i}\leq p_i(t),\qquad\left|\frac{\partial f_i(t,x_1,\dots,x_n)}{\partial x_k}\right|\leq l_{ik}(t)\quad(k=1,\dots,n) \tag{1.17}$$

hold, where $p_i:[0,\omega]\to R$ *and* $l_{ik}:[0,\omega]\to[0,+\infty[$ *are summable functions and* $\sigma_i\in\{-1,1\}$. *Moreover, let inequalities* (1.11) *be fulfilled and there exist a constant nonnegative matrix* $A=(a_{ik})_{i,k=1}^n$ *such that* $r(A)<1$ *and*

$$\int_0^\omega g(\sigma_i p_i)(t,s)\left[(1-\delta_{ik})l_{ik}(s)+l_{ii}(s)\left|\int_s^{\tau_{ii}^0(s)}l_{ik}(\xi)\,\mathrm{d}\xi\right|\right]\mathrm{d}s$$
$$\leq a_{ik}\quad\text{for } 0\leq t\leq\omega\quad(i,k=1,\dots,n). \tag{1.18}$$

Then system (1.1) *has one and only one* ω*-periodic solution.*

Theorem 1.3′. *Let* $\tau_{ii}(t)\equiv t(i=1,\dots,n)$ *and for each* $i\in\{1,\dots,n\}$ *on the set* $[0,\omega]\times R^n$ *the condition*

$$[f_i(t,x_1,\dots,x_n)-f_i(t,\bar{x}_1,\dots,\bar{x}_n)]\,\mathrm{sgn}\,(\sigma_i(x_i-\bar{x}_i))$$
$$\leq p_i(t)(x_i-\bar{x}_i)+\sum_{k=1}^n p_{ik}(t)|x_k-\bar{x}_k| \tag{1.19}$$

hold, where $p_i:[0,\omega]\to R$ *and* $p_{ik}:[0,\omega]\to[0,+\infty[$ *are summable functions and* $\sigma_i\in\{-1,1\}$. *Moreover, let inequalities* (1.11) *and* (1.13) *be fulfilled, where* $A=(a_{ik})_{i,k=1}^n$ *is a constant nonnegative matrix such that* $r(A)<1$. *Then system* (1.1) *has one and only one* ω*-periodic solution.*

2. Auxiliary propositions

By conditions (1.3) and (1.6) the problem on ω-periodic solutions of system (1.1) is equivalent to the periodic boundary value problem

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(t, x_1(\tau_{i1}^0(t)), \dots, x_n(\tau_{in}^0(t))) \quad (i = 1, \dots, n), \tag{2.1}$$

$$x_i(\omega) = x_i(0) \quad (i = 1, \dots, n), \tag{2.2}$$

i.e., if system (1.1) has an ω-periodic solution, then its restriction on $[0, \omega]$ is a solution of problem (2.1), (2.2), and vice versa if problem (2.1), (2.2) is solvable, then the periodic extension on R of its arbitrary solution is an ω-periodic solution of system (1.1). This fact and the principle of a priori boundedness proved in [14] readily imply Lemma 2.1 on the existence of an ω-periodic solution of system (1.1).

In this section, along with problem (2.1), (2.2) we shall also consider the system of functional-differential inequalities

$$[x_i'(t) - h_i(t)x_i(t)] \operatorname{sgn}(\sigma_i x_i(t)) \le \sum_{k=1}^{n} h_{ik}(t)\|x_k\|_C + h_0(t) \quad (i = 1, \dots, n) \tag{2.3}$$

with boundary conditions (2.2), where $h_i : [0, \omega] \to R$, h_0 and $h_{ik} : [0, \omega] \to [0, +\infty[$ $(i, k = 1, \dots, n)$ are summable functions, $\sigma_i \in \{-1, 1\}$ and

$$\|x_k\|_C = \max\{|x_k(t)| \colon 0 \le t \le \omega\}.$$

By a solution of system (2.3) we shall understand an absolutely continuous vector function $(x_i)_{i=1}^n : [0, \omega] \to R^n$ which satisfies this system almost everywhere on $[0, \omega]$. A solution of system (2.3) satisfying the boundary conditions (2.2) will be called a solution of problem (2.3), (2.2).

Lemma 2.2 proved in this section contains the conditions of a priori boundedness of solutions of problem (2.3), (2.2).

Lemma 2.1. *Let there exist summable functions* $h_i : [0, \omega] \to R$ $(i = 1, \dots, n)$ *and a positive number* ρ *such that*

$$\int_0^{\omega} h_i(t)\,\mathrm{d}t \ne 0 \quad (i = 1, \dots, n) \tag{2.4}$$

and for any $\lambda \in\,]0, 1[$ *an arbitrary solution* $(x_i)_{i=1}^n$ *of the differential system*

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = (1 - \lambda)h_i(t)x_i(t) + \lambda f_i(t, x_1(\tau_{i1}^0(t)), \dots, x_n(\tau_{in}^0(t))) \quad (i = 1, \dots, n) \tag{2.5}$$

satisfying the boundary conditions (2.2), *admits the estimate*

$$\sum_{i=1}^{n} \|x_i\|_C < \rho. \tag{2.6}$$

Then system (1.1) *has at least one* ω*-periodic solution.*

Proof. By condition (2.4) the differential system

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = h_i(t)x_i(t) \quad (i = 1,\dots,n)$$

with boundary conditions (2.2) has only the trivial solution. Hence Corollary 2 from [14] implies that under the conditions of Lemma 2.1 problem (2.1), (2.2) is solvable. But, as mentioned above, the solvability of problem (2.1), (2.2) guarantees the existence of an ω-periodic solution of system (1.1). □

Lemma 2.2. *Let*

$$\sigma_i \int_0^\omega h_i(t)\,\mathrm{d}t < 0 \quad (i = 1,\dots,n) \tag{2.7}$$

and there exist a constant nonnegative matrix $A = (a_{ik})_{i,k=1}^n$ *such that* $r(A) < 1$ *and*

$$\int_0^\omega g(h_i)(t,s)h_{ik}(s)\,\mathrm{d}s \le a_{ik} \quad \textit{for } 0 \le t \le \omega \quad (i,k = 1,\dots,n). \tag{2.8}$$

Then any solution $(x_i)_{i=1}^n$ *of problem* (2.3), (2.2) admits the estimate

$$\sum_{i=1}^n \|x_i\|_C \le \rho_0 \int_0^\omega h_0(t)\,\mathrm{d}t, \tag{2.9}$$

where

$$\rho_0 = \|(E - A)^{-1}\| \sum_{i=1}^n \sup\{g(h_i)(t,s)\colon\ 0 \le t,s \le \omega\} \tag{2.10}$$

and E is the unit matrix.

Proof. Let $(x_i)_{i=1}^n$ be some solution of problem (2.3), (2.2). Assume that

$$y_i(t) = |x_i(t)|, \quad h_{0i}(t) = y_i'(t) - h_i(t)y_i(t) \quad (i = 1,\dots,n).$$

Then for each $i \in \{1,\dots,n\}$ the function y_i is a solution of the boundary value problem

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = h_i(t)y_i(t) + h_{0i}(t), \quad y_i(\omega) = y_i(0),$$

and the function h_{0i} satisfies the inequality

$$\sigma_i h_{0i}(t) \le \sum_{k=1}^n h_{ik}(t)\|y_k\|_C + h_0(t) \tag{2.11}$$

almost everywhere on $[0,\omega]$.

On the other hand, by condition (2.7) the homogeneous problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = h_i(t)u(t), \quad u(\omega) = u(0) \tag{2.12}$$

has only the trivial solution. Denoting its Green function by g_i, we have

$$y_i(t) = \int_0^\omega g_i(t,s)h_{0i}(s)\,\mathrm{d}s. \tag{2.13}$$

By conditions (1.4) and (2.7)

$$\sigma_i g_i(t,s) = g(h_i)(t,s) > 0 \quad \text{for } 0 \le s,t \le \omega \quad (i=1,\ldots,n).$$

Taking into account this and inequality (2.11), from (2.13) we obtain the inequality

$$y_i(t) \le \sum_{k=1}^n \left[\int_0^\omega g(h_i)(t,s)h_{ik}(s)\,\mathrm{d}s\right] \|y_k\|_C + \int_0^\omega g(h_i)(t,s)h_0(s)\,\mathrm{d}s \quad \text{for } 0 \le t \le \omega \quad (i=1,\ldots,n)$$

which by virtue of condition (2.8) and nonnegative functions y_i implies

$$\|y_i\|_C \le \sum_{k=1}^n a_{ik}\|y_k\| + \eta_i \quad (i=1,\ldots,n),$$

i.e.,

$$(E-A)(\|y_i\|_C)_{i=1}^n \le \eta, \tag{2.14}$$

where

$$\eta_i = \sup\{g(h_i)(t,s)\colon\ 0 \le t,s \le \omega\} \int_0^\omega h_0(s)\,\mathrm{d}s, \quad \eta = (\eta_i)_{i=1}^n. \tag{2.15}$$

By the nonnegativity of the matrix A and the inequalities $r(A) < 1$, the matrix $E-A$ has the nonnegative inverse $(E-A)^{-1}$. Therefore (2.14) implies

$$(\|y_i\|_c)_{i=1}^n \le (E-A)^{-1}\eta.$$

Hence by equalities (2.15) we obtain estimate (2.9), where ρ_0 is the number given by equality (2.10). □

3. Proofs of the main results

Proof of Theorem 1.1. Set

$$\begin{aligned} h_0(t) &= q(t) + q^* \sum_{i=1}^n l_i(t), \quad h_i(t) = \sigma_i p_i(t), \\ h_{ik}(t) &= p_{ik}(t) + l_i(t)p_{ik}^*(t) \quad (i,k=1,\ldots,n). \end{aligned} \tag{3.1}$$

Then inequalities (2.7) and (2.8) are fulfilled by virtue of conditions (1.11) and (1.12).

Let ρ_0 be the number given by equality (2.10) and

$$\rho = \rho_0 \int_0^{\omega} h_0(t)\,\mathrm{d}t. \tag{3.2}$$

By Lemma 2.1, to prove the theorem it is sufficient to establish that every solution of problem (2.5), (2.2) for any $\lambda \in]0,1[$ admits estimate (2.6).

Let $(x_i)_{i=1}^n$ be a solution of problem (2.5), (2.2) for some $\lambda \in]0,1[$. Then

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = (1-\lambda)h_i(t)x_i(t) + \lambda f_i(t, x_1(\tau_{i1}^0(t)), \ldots, x_i(t), \ldots, x_n(\tau_{in}^0(t)))$$
$$+ \lambda\delta_i(t) \quad (i = 1, \ldots, n), \tag{3.3}$$

where

$$\delta_i(t) = f_i(t, x_1(\tau_{i1}^0(t)), \ldots, x_i(\tau_{ii}^0(t)), \ldots, x_n(\tau_{in}^0(t)))$$
$$- f_i(t, x_1(\tau_{i1}^0(t)), \ldots, x_i(t), \ldots, x_n(\tau_{in}^0(t))).$$

Let $i \in \{1, \ldots, n\}$ be fixed arbitrarily. If $t \in I_i$, then by conditions (1.9), (1.10) and (2.5) we have

$$|\delta_i(t)| \le l_i(t)|x_i(\tau_{ii}^0(t)) - x_i(t)| = l_i(t)\left|\int_t^{\tau_{ii}^0(t)} x_i'(s)\,\mathrm{d}s\right|$$
$$= l_i(t)\left|\int_t^{\tau_{ii}^0(t)} [(1-\lambda)p_i(s)x_i(s) + \lambda f_i(s, x_1(\tau_{i1}^0(s)), \ldots, x_i(\tau_{in}^0(s)))]\,\mathrm{d}s\right|$$
$$\le l_i(t)\left[(1-\lambda)\left|\int_t^{\tau_{ii}^0(t)} |p_i(s)|\,\mathrm{d}s\right| \|x_i\|_C\right.$$
$$\left. + \lambda\left|\int_t^{\tau_{ii}^0(t)} f_i^*(s, \|x_1\|_C, \ldots, \|x_n\|_C)\,\mathrm{d}s\right|\right]$$
$$\le l_i(t)\left[(1-\lambda)p_{ii}^*(t)\|x_i\|_C + \lambda\sum_{k=1}^n p_{ik}^*(t)\|x_k\|_C + q^*\right]$$
$$\le l_i(t)\left[\sum_{k=1}^n p_{ik}^*(t)\|x_k\|_C + q^*\right]. \tag{3.4}$$

If $t \in [0,\omega]\backslash I_i$, then $\delta_i(t) = 0$. Therefore inequality (3.4) holds throughout $[0,\omega]$.

Taking into account conditions (1.8), (3.1) and (3.4), we find from system (3.3)

$$
\begin{aligned}
[x_i'(t) - h_i(t)x_i(t)]\,\mathrm{sgn}(\sigma_i x_i(t)) &= -\lambda p_i(t)|x_i(t)| + \lambda f_i(t, x_1(\tau_{i1}^0(t)),\ldots,\\
&\quad \times x_i(t),\ldots,x_n(\tau_{in}^0(t)))\mathrm{sgn}(\sigma_i x_i(t))\\
&\quad + \lambda\delta_i(t)\mathrm{sgn}(\sigma_i x_i(t))\\
&\le \lambda p_{ii}(t)|x_i(t)| + \lambda\sum_{k=1}^{n}(1-\delta_{ik})p_{ik}(t)|x_k(\tau_{ik}^0(t))|\\
&\quad + q(t) + \lambda l_i(t)\left[\sum_{k=1}^{n} p_{ik}^*(t)\|x_k\|_C + q^*\right]\\
&\le \sum_{k=1}^{n}[p_{ik}(t) + l_i(t)p_{ik}^*(t)]\,\|x_k\|_C + q(t) + q^* l_i(t)\\
&\le \sum_{k=1}^{n} h_{ik}(t)\|x_k\|_C + h_0(t).
\end{aligned}
$$

Thus we have proved that $(x_i)_{i=1}^n$ is a solution of problem (2.3), (2.2). On the other hand, since all conditions of Lemma 2.2 are fulfilled for this problem, estimate (2.6) is valid, where ρ is the number given by equalities (2.10) and (3.2). □

Proof of Theorem 1.2. By the constraints imposed on the functions l_i and q_i $(i=1,\ldots,n)$ and notation (1.4)

$$\int_0^\omega l_i(s)\,\mathrm{d}s > 0 \quad (i=1,\ldots,n), \tag{3.5}$$

$$\int_0^\omega q_i(s)\,\mathrm{d}s > 0 \quad (i=1,\ldots,n) \tag{3.6}$$

and there exists $\delta>0$ such that

$$g(l_i)(t,s) \ge \delta, \quad g(-l_i)(t,s) \ge \delta \quad \text{for } 0 \le t,s \le \omega \ (i=1,\ldots,n). \tag{3.7}$$

Assume now that the theorem is not true and system (1.1) has an ω-periodic solution $(x_i)_{i=1}^n$. Set

$$h_i(t) = \sigma_{0i} f_{0i}(t, x_1(\tau_{i1}^0(t)),\ldots,x_n(\tau_{in}^0(t))), \quad h_{0i}(t) = x_i'(t) - h_i(t)x_i(t),$$

$$\mu_i = \min\{|x_i(t)|\colon 0 \le t \le \omega\}.$$

Then for each $i \in \{1,\ldots,n\}$ the restriction of x_i on $[0,\omega]$ is a solution of the boundary value problem

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = h_i(t)x_i(t) + h_{0i}(t), \quad x_i(\omega) = x_i(0). \tag{3.8}$$

On the other hand, by the virtue of conditions (1.14) and (1.15) the inequalities

$$-\sigma_{0i}h_{0i}(t) \geq l_i(t)\sum_{k=1}^{n} a_{ik}\mu_k + q_i(t), \tag{3.9}$$

$$|h_i(t)| \leq l_i(t) \tag{3.10}$$

are fulfilled almost everywhere on $[0,\omega]$.

According to equalities (3.8)

$$x_i(0) = \exp\left(\int_0^\omega h_i(s)\,\mathrm{d}s\right)x_i(0) + \int_0^\omega \exp\left(\int_s^\omega h_i(\xi)\,\mathrm{d}\xi\right)h_{0i}(s)\,\mathrm{d}s.$$

Hence, with inequalities (3.6) and (3.9) taken into account, we find

$$\sigma_{0i}\left[\exp\left(\int_0^\omega h_i(s)\,\mathrm{d}s\right) - 1\right]x_i(0) \geq \int_0^\omega \exp\left(\int_s^\omega h_i(\xi)\,\mathrm{d}\xi\right)q_i(s)\,\mathrm{d}s > 0.$$

Therefore,

$$\int_0^\omega h_i(s)\,\mathrm{d}s \neq 0$$

and thus the homogeneous problem (2.12) has only the trivial solution.

Let g_i be the Green function of problem (2.12) and the number $\sigma_i \in \{-1,1\}$ be such that

$$\sigma_i\int_0^\omega h_i(s)\,\mathrm{d}s > 0. \tag{3.11}$$

Then

$$g_i(t,s) = -\sigma_i g(h_i)(t,s)$$

and

$$x_i(t) = -\sigma_i\int_0^\omega g(h_i)(t,s)h_{0i}(s)\,\mathrm{d}s. \tag{3.12}$$

On the other hand, by conditions (3.5), (3.10) and (3.11)

$$g(h_i)(t,s) \geq g(\sigma_i l_i)(t,s) \quad \text{for } 0 \leq t,s \leq \omega.$$

Taking into account this inequality and inequalities (3.7) and (3.9), we find from equality (3.12)

$$|x_i(t)| \geq \left[\int_0^\omega g(\sigma_i l_i)(t,s)l_i(s)\,\mathrm{d}s\right]\sum_{k=1}^{n} a_{ik}\mu_k + \eta_i = \sum_{k=1}^{n} a_{ik}\mu_k + \eta_i \quad \text{for } 0 \leq t \leq \omega,$$

where

$$\eta_i = \delta\int_0^\omega q_i(s)\,\mathrm{d}s > 0.$$

Therefore,

$$\mu_i \geq \sum_{k=1}^{n} a_{ik}\mu_k + \eta_i \quad (i=1,\ldots,n),$$

i.e.,

$$\mu \geq A\mu + \eta,$$

where $\mu=(\mu_i)_{i=1}^n$, $\eta=(\eta_i)_{i=1}^n$.

The last inequality implies

$$\mu \geq \left(\sum_{j=0}^{k} A^j\right)\eta \quad (k=0,1,2,\ldots)$$

and

$$\left\|\sum_{j=0}^{k} A^j\right\| \leq \gamma_0 \quad (k=0,1,2,\ldots),$$

where

$$\gamma_0 = \sum_{i=1}^{n} \frac{\mu_i}{\eta_0}, \qquad \eta_0 = \min\{\eta_1,\ldots,\eta_n\}.$$

Therefore,

$$\lim_{k\to+\infty} \left\|\sum_{j=0}^{k} A^j\right\| < +\infty.$$

But this is impossible, since $r(A) \geq 1$. The obtained contradiction proves the theorem. □

Proof of Corollary 1.2. Condition (1.16) implies, on the one hand, inequality (1.8) and, on the other, inequalities (1.14) and (1.15), where $\sigma_i = \sigma_{0i}\sigma_{1i}$,

$$p_i(t) = -\, l_i(t), \quad p_{ik}(t) = a_{ik}\, l_i(t), \quad q(t) = \max\{q_1(t),\ldots,q_n(t), q_{01}(t),\ldots,q_{0n}(t)\},$$

$$f_{0i}(t,x_1,\ldots,x_n) = -\, \sigma_{1i}\, l_i(t).$$

Hence by Corollary 1.1 and Remark 1.1 (by Theorem 1.2) it follows that if $r(A) < 1$ (if $r(A) \geq 1$), then system (1.1) has at least one ω-periodic solution (has no ω-periodic solution). □

Proof of Theorem 1.3. For arbitrary i and $k \in \{1,\dots,n\}$ we introduce the functions

$$f_{ik}(t,x_1,\dots,x_n) = \frac{\partial f_i(t,x_1,\dots,x_n)}{\partial x_k},$$

$$\varphi_{ik}(t,x_1,\dots,x_n,\bar{x}_1,\dots,\bar{x}_n) = \int_0^1 f_{ik}(t,sx_1+(1-s)\bar{x}_1,\dots,sx_n+(1-s)\bar{x}_n)\,\mathrm{d}s.$$

Then for each $i \in \{1,\dots,n\}$ we shall have

$$f_i(t,x_1,\dots,x_n) = f_i(t,0,\dots,0) + \sum_{k=1}^n \varphi_{ik}(t,x_1,\dots,x_n,0,\dots,0)x_k \tag{3.13}$$

and

$$f_i(t,x_1,\dots,x_n) - f_i(t,\bar{x}_1,\dots,\bar{x}_n) = \sum_{k=1}^n \varphi_{ik}(t,x_1,\dots,x_n,\bar{x}_1,\dots,\bar{x}_n)(x_k-\bar{x}_k). \tag{3.14}$$

On the other hand, according to condition (1.17) the inequalities

$$\sigma_i\varphi_{ii}(t,x_1,\dots,x_n,\bar{x}_1,\dots,\bar{x}_n) \le p_i(t),$$
$$|\varphi_{ik}(t,x_1,\dots,x_n,\bar{x}_1,\dots,\bar{x}_n)| \le l_{ik}(t) \quad (i,k=1,\dots,n) \tag{3.15}$$

hold on the set $[0,\omega]\times R^{2n}$. Moreover, it can be assumed without loss of generality that

$$|p_i(t)| \le l_{ii}(t). \tag{3.16}$$

Conditions (3.13)–(3.16) immediately imply conditions (1.8)–(1.10), where

$$p_{ik}(t) = (1-\delta_{ik})l_{ik}(t), \quad l_i(t) = l_{ii}(t), \quad p^*_{ik}(t) = \left|\int_t^{\tau^0_{ii}(t)} l_{ik}(s)\,\mathrm{d}s\right| \quad (k=1,\dots,n),$$

$$q(t) = \max\{|f_1(t,0,\dots,0)|,\dots,|f_n(t,0,\dots,0)|\}, \quad q^* = \int_0^\omega q(s)\,\mathrm{d}s.$$

By condition (1.18) for such p_{ik}, p^*_{ik} and l_i $(i,k=1,\dots,n)$ inequalities (1.12) hold and besides $r(A)<1$. Therefore, all conditions of Theorem 1.1 are fulfilled, which guarantees that system (1.1) has at least one ω-periodic solution.

To complete the proof, it remains for us to show that system (1.1) has at most one ω-periodic solution.

Let $(x_i)_{i=1}^n$ and $(\bar{x})_{i=1}^n$ be two arbitrary ω-periodic solutions of system (1.1). We set

$$y_i(t) = x_i(t) - \bar{x}_i(t) \quad (i=1,\dots,n),$$
$$\psi_{ik}(t) = \varphi_{ik}(t,x_1(\tau^0_{i1}(t)),\dots,x_n(\tau^0_{in}(t)),\bar{x}_1(\tau^0_{i1}(t)),\dots,\bar{x}_n(\tau^0_{in}(t))). \tag{3.17}$$

Then

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t}=\sum_{k=1}^{n}\psi_{ik}(t)y_k(\tau_{ik}^0(t)),\quad y_i(\omega)=y_i(0)\quad(i=1,\dots,n)$$

and therefore,

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t}=\psi_{ii}(t)y_i(t)+\sum_{k=1}^{n}(1-\delta_{ik})\psi_{ik}(t)y_k(\tau_{ik}^0(t))$$
$$+\psi_{ii}(t)\sum_{k=1}^{n}\int_t^{\tau_{ii}^0(t)}\psi_{ik}(s)y_k(\tau_{ik}^0(s))\,\mathrm{d}s\quad(i=1,\dots,n).$$

Hence by conditions (3.15)–(3.17) it follows that the restriction of $(y_i)_{i=1}^n$ on $[0,\omega]$ is a solution of the system of functional-differential inequalities

$$[y_i'(t)-h_i(t)y_i(t)]\,\mathrm{sgn}(\sigma_i y_i(t))\le\sum_{k=1}^{n}h_{ik}(t)\|y_k\|_C\quad(i=1,\dots,n)\tag{3.18}$$

under the periodic boundary conditions, where

$$h_i(t)=\sigma_i p_i(t),\quad h_{ik}(t)=(1-\delta_{ik})l_{ik}(t)+l_{ii}(t)\left|\int_t^{\tau_{ii}^0(t)}l_{ik}(s)\,\mathrm{d}s\right|.$$

But by inequalities (1.18) the functions h_i and h_{ik} $(i,k=1,\dots,n)$ satisfy condition (2.8). Since, in addition, we have $r(A)<1$, Lemma 2.2 implies $y_i(t)\equiv 0$ $(i=1,\dots,n)$. Therefore $x_i(t)\equiv\bar{x}_i(t)$ $(i=1,\dots,n)$. □

Proof of Theorem 1.3′. Note first that condition (1.19) implies condition (1.8), where

$$q(t)=\max\{|f_1(t,0,\dots,0)|,\dots,|f_n(t,0,\dots,0)|\}.$$

Therefore all conditions of Corollary 1.1 are fulfilled, which guarantees that system (1.1) has at least one ω-periodic solution.

Let us now prove the uniqueness. Let $(x_i)_{i=1}^n$ and $(\bar{x}_i)_{i=1}^n$ be arbitrary ω-periodic solutions of system (1.1) and $y_i(t)=x_i(t)-\bar{x}_i(t)$ $(i=1,\dots,n)$. Then by condition (1.19) the restriction of $(y_i)_{i=1}^n$ on $[0,\omega]$ is a solution of system (3.18) under the periodic boundary conditions, where

$$h_i(t)=\sigma_i p_i(t),\quad h_{ik}(t)=p_{ik}(t)$$

with h_i and h_{ik} $(i,k=1,\dots,n)$ satisfying the conditions of Lemma 2.2. Therefore $y_i(t)\equiv 0$ $(i=1,\dots,n)$, i.e., $x_i(t)\equiv\bar{x}_i(t)$ $(i=1,\dots,n)$. □

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