Reading: [Simon], Chapter 21, p. 505-522.

1 Concave and convex functions

1.1 Convex Sets

Definition 1 *A set $X \subset R^n$ is called convex if given any two points x', $x'' \in X$ the line segment joining x' and x'' completely belongs to X, in other words for each $t \in [0,1]$ the point*

$$x^t = (1-t)x' + tx''$$

is also in X for every $t \in [0,1]$.

The intersection of convex sets is convex.
The union of convex sets is not necessarily convex.

Let $X \subset R^n$. The *convex hull* of X is defined as the smallest convex set that contain X.

The convex hull of X consists of all points which are *convex combinations* of some points of X

$$CH(X) = \{y \in R^n : y = \sum t_i x_i, \quad x_i \in X, \quad \sum t_i = 1\}.$$

1.2 Concave and Convex Function

A function f is concave if the line segment joining any two points on the graph is never above the graph. More precisely

Definition 2 *A function $f : S \subset R^n \to R$ defined on a convex set S is concave if given any two points $x', x'' \in S$ we have*

$$(1-t)f(x') + tf(x'') \leq f((1-t)x' + tx'')$$

for any $t \in [0,1]$.

f is called strictly concave if

$$(1-t)f(x') + tf(x'') < f((1-t)x' + tx'').$$

Definition 3 *A function $f : S \subset R^n \to R$ is convex if given any two points $x', x'' \in S$ we have*

$$(1-t)f(x') + tf(x'') \geq f((1-t)x' + tx'')$$

for any $t \in [0,1]$.

f is called strictly convex if

$$(1-t)f(x') + tf(x'') > f((1-t)x' + tx'').$$

Roughly speaking concavity of a function means that the **graph is above chord**.

It is clear that if f is *concave* then $-f$ is *convex* and vice versa.

Theorem 1 *A function $f : S \subset R^n \to R$ is concave (convex) if and only if its restriction to every line segment of R^n is concave (convex) function of one variable.*

Theorem 2 *If f is a concave (convex) function then a local maximizer (minimizer) is global.*

1.2.1 Characterization in Terms of Graphs

Given a function $f : S \subset R^n \to R$ defined on a convex set S.

The *hypograph* of f is defined as the set of points $(x, y) \in S \times R$ lying on or bellow the graph of the function:

$$hyp\ f = \{(x, y) :\ x \in S,\ y \leq f(x)\}.$$

Similarly, the *epigraph* of f is defined as the set of points $(x, y) \in S \times R$ lying on or above the graph of the function:

$$epi\ f = \{(x, y) :\ x \in S,\ y \geq f(x)\}.$$

Theorem 3 *(a) A function $f : S \subset R^n \to R$ defined on a convex set S is concave if and only if its hypograpf $hyp\ f$ is convex.*

(b) A function $f : S \subset R^n \to R$ defined on a convex set S is convex if and only if its epigraph $epi\ f$ is convex.

Proof of (a). Let $(x_1, y_1), (x_2, y_2) \in hyp\ f$, let us show that

$$(xt, yt) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in hyp\ f.$$

$$y_t = ty_1 + (1-t)y_2 \leq tf(x_1) + (1-t)f(x_2) \leq f(tx_1 + (1-t)x_2) = f(x_t).$$

1.2.2 Characterization in Terms of Level Sets

Given a function $f : S \subset R^n \to R$ defined on a convex set S.

Take any number $K \in R$.

The *upper contour set* U_K of f is defined as

$$U_K = \{x \in S,\ f(x) \geq K\}.$$

Similarly, the *lower contour set* L_K of f is defined as

$$L_K = \{x \in S,\ f(x) \leq K\}.$$

Theorem 4 *(a) Suppose a function $f : S \subset R^n \to R$ defined on a convex set S is concave. Then for every K the upper contour set U_K is either empty or a convex set.*

(b) If f is convex, then for every K the lover contour set L_K is either empty or a convex set.

Proof. Let us prove only (a).

Let $x_1, x_2 \in U_k$, let us show that $x_t = tx_1 + (1-t)x_2 \in U_K$:

$$f(x_t) = f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2) \geq tK + (1-t)K = K.$$

Remark. Notice that this is only necessary condition, not sufficient: consider the example $f(x) = e^x$ or $f(x) = x^3$.

1.2.3 Examples of Concave Functions

Theorem 5 *Suppose $f_1, ..., f_n$ are concave (convex) functions and $a_1 > 0, ..., a_n > 0$, then the linear combination*

$$F = a_1 f_1 + \ ... \ + a_n f_n$$

is concave (convex).

Proof.

$$F((1-t)x + ty) = \sum a_i f_i((1-t)x + ty) \geq \sum a_i[(1-t)f_i(x) + tf_i(y)] =$$
$$(1-t)\sum a_i f(x) + t\sum a_i f_i(y) = (1-t)F(x) + tF(y).$$

A function of the form $f(x) = f(x_1, x_2, ..., x_n) = a_0 + a_1 x_1 + a_2 x_2 + \ ... \ + a_n x_n$ is called *affine function* (if $a_0 = 0$, it is a linear function).

Theorem 6 *An affine function is both concave and convex.*

Proof. The theorem follows from previous theorem and following easy to prove statements:

(1) The function $f(x_1, ..., x_n) = x_i$ is concave and convex;
(2) The function $f(x_1, ..., x_n) = -x_i$ is concave and convex;
(3) The constant function $f(x_1, ..., x_n) = a$ is concave and convex.

Theorem 7 *A concave monotonic transformation of a concave function is itself concave.*

Proof. Let $f : R^n \to R$ be a concave function and $g : R \to R$ be concave and increasing, then

$$(g \circ f)(1-t)x + ty) =$$
$$g(f((1-t)x + ty)) \geq g((1-t)f(x) + tf(y)) \geq (1-t)g(f(x)) + tg(f(y)) =$$
$$(1-t)(g \circ f)(x)) + t(g \circ f)(y)),$$

here the first inequality holds since f is concave and g is increasing, and the second inequality holds since g is concave.

Remark. Note that just monotonic transformation of a concave function is not necessarily concave: consider, for example $f(x) = x$ and $g(z) = z^3$.

Thus the *concavity of a function is not ordinal*, it is cardinal property.

Economic Example

Suppose production function $f(x)$ is *concave* and the cost function $c(x)$ is *convex*. Suppose also p is the positive selling price. Then the profit function

$$\pi(x) = pf(x) + (-c(x))$$

is *concave* as a linear combination with positive coefficients of concave functions. Thus a local maximum of profit function is global in this case (see bellow).

1.3 Calculus Criteria for Concavity

For one variable functions we have the following statements

1. A C^1 function $f : U \subset R \to R$ is concave if and only if its first derivative $f'(x)$ is decreasing function.

2. A C^2 function $f : U \subset R \to R$ is concave if and only if its second derivative $f''(x)$ is ≤ 0.

In n-variable case usually instead of $f'(x)$ we consider the Jacobian (gradient) $Df(x)$ and instead of $f''(x)$ we consider the hessian $D^2f(x)$.

It is not clear how to generalize the above statements 1 and 2 to n-variable case since the statement "$Df(x)$ (which is a vector) is decreasing function" has no sense as well as "$D^2f(x)$ (which is a matrix) is positive".

Let us reformulate the statements 1 and 2 in the following forms:

1'. A C^1 function $f : U \subset R \to R$ is concave if and only if

$$f(y) - f(x) \leq f'(x)(y - x)$$

for all $x, y \in U$.

Hint: Observe that for concave $f(x)$ and $x < y$ one has

$$f'(x) \geq \frac{f(y) - f(x)}{y - x} \geq f'(y).$$

2'. A C^2 function $f : U \subset R \to R$ is concave if and only if the one variable quadratic form $Q(y) = f''(x) \cdot y^2$ is negative semidefinite for all $x \in U$.

Hint: Observe that the quadratic form $Q(y) = f''(x) \cdot y^2$ is negative semidefinite if and only if the coefficient $f''(x) \leq 0$.

Now we can formulate the multi-variable generalization of 1:

Theorem 8 *A C^1 function $f : U \subset R^n \to R$ is concave if and only if*

$$f(y) - f(x) \leq Df(x)(y - x),$$

for all $x, y \in U$, that is

$$f(y) - f(x) \leq \frac{\partial f}{\partial x_1}(x)(y_1 - x_1) + \ ... \ + \frac{\partial f}{\partial x_n}(x)(y_n - x_n).$$

Similarly f is convex if and only if

$$f(y) - f(x) \geq Df(x)(y - x).$$

Remember that concavity of a function means that the **graph is above chord**? Now we can say

Roughly speaking concavity of a function means that the **tangent is above graph**.

From this theorem follows

Corollary 1 *Suppose f is concave and for some $x_0, \ y \in U$ we have*

$$Df(x_0)(y - x_0) \leq 0,$$

then $f(y) \leq f(x_0)$ for THIS y.

Particularly, if directional derivative of f at x_0 in any feasible direction is nonpositive, i.e.

$$D_{y-x_0} f(x_0) = Df(x_0)(y - x_0) \leq 0$$

for ALL $y \in U$, then x_0 is GLOBAL max of f in U.

Indeed, since of concavity of f we have

$$f(y) - f(x_0) \leq Df(x_0)(y - x_0) \leq 0.$$

The following theorem is the generalization of 2:

Theorem 9 *A C^2 function $f : U \subset R^n \to R$ defined on a convex open set U is*

(a) concave if and only if the Hessian matrix $D^2 f(x)$ is negative semidefinite for all $x \in U$;

(b) strictly concave if the Hessian matrix $D^2 f(x)$ is negative definite for all $x \in U$;

(c) convex if and only if the Hessian matrix $D^2 f(x)$ is positive semidefinite for all $x \in U$;

(d) strictly convex if the Hessian matrix $D^2 f(x)$ is positive definite for all $x \in U$;

Remark. Note that the statement (b) (and (d) too) is not "only if": If f is strictly concave then the Hessian is not necessarily negative definite for ANY x. Analyze, for example $f(x) = -x^4$.

Let us recall criteria for definiteness of matrix in terms of principal minors:

(1) A matrix H is positive definite if and only if its n **leading principal minors** are > 0.

(2) A matrix H is negative definite if and only if its n **leading principal minors** alternate in sign so that all odd order ones are < 0 and all even order ones are > 0.

(3) A matrix H is positive semidefinite if and only if its $2^n - 1$ **principal minors** are all ≥ 0.

(4) A matrix H is negative semidefinite if and only if its $2^n - 1$ **principal minors** alternate in sign so that odd order minors are ≤ 0 and even order minors are ≥ 0.

Example. Let us determine the definiteness of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Its first order principal minors are

$$M_1 = 1, \quad M_1' = 0,$$

and the only second order principal minor is

$$M_2 = 0.$$

We are in the situation (3), so our matrix is positive semidefinite. Note that corresponding quadratic form is $Q(x, y) = y^2$.

Example. Let $f(x, y) = 2x - y - x^2 + 2xy - y^2$. Its Hessian is

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

which is constant (does not depend on (x, y)) and negative semidefinite. Thus f is concave.

Example. Consider the function $f(x, y) = 2xy$. Its Hessian is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the only second order principal minor is $-1 < 0$ the matrix ix indefinite, thus f is neither concave nor convex.

Example. Consider the Cobb-Douglas function $f(x,y) = cx^a y^b$ with $a,b,c > 0$ in the first orthant $x > 0,\ y > 0$.

Its hessian is

$$\begin{pmatrix} a(a-1)cx^{a-2}y^b & abcx^{a-1}y^{b-1} \\ abcx^{a-1}y^{b-1} & b(b-1)cx^a y^{b-2} \end{pmatrix}.$$

The principal minors of order 1 of this matrix are

$$M_1 = a(a-1)cx^{a-2}y^b, \quad M_1' = b(b-1)cx^a y^{b-2}$$

and the only principal minor of order 2 is

$$M_2 = abcx^{2a-2}y^{2b-2}(1-(a+b)).$$

When this function is concave? For this the Hessian must be negative semidefinite. This happens when all principal minors of degree 1 M_1 and M_1' are ≤ 0 and (only) principal minor of degree 2 M_2 is ≥ 0.

Recall that we work in the first orthant $x > 0,\ y > 0$, and $a,b,c > 0$.

If our $f(x,y) = cx^a y^b$ exhibits constant or decreasing return to scale (CRS or DRS), that is $a+b \leq 1$, then clearly $a \leq 0,\ b \leq 0$, and we have thus the Cobb-Douglas function is concave if and $M_1 \leq 0,\ M_1' \leq 0,\ M_2 \geq 0$, thus f is concave.

Remark. So we have shown that if a Cobb-Douglas function $f(x,y) = cx^a y^b$ is CRS or DRS, it is concave. But can it be convex?

1.4 Concave Functions and Optimization

Concavity of a function replaces the second derivative test to separate local max, min or saddle, moreover, for a concave function a critical point which is local max (min) is global:

Theorem 10 *Let $f : U \subset R^n \to R$ be concave (convex) function defined on a convex open set U. If x^* is a critical point, that is $Df(x^*) = 0$, then it is global maximizer (minimizer).*

Proof. Since $Df(x^*) = 0$ from the inequality

$$f(y) - f(x^*) \leq Df(x^*)(y - x^*) = 0$$

follows $f(y) \leq f(x^*)$ for all $y \in U$.

The next result is stronger, it allows to find maximizer also on the boundary of U if it is not assumed open:

Theorem 11 *Let $f : U \subset R^n \to R$ be concave function defined on a convex set U. If x^* is a point, which satisfies*

$$Df(x^*)(y - x^*) \leq 0$$

for each $y \in U$, then x^* is a global maximizer of f on U.

Similarly, if f is convex and

$$Df(x^*)(y - x^*) \geq 0$$

for each $y \in U$, then x^* is a global minimizer of f on U.

Proof. From

$$f(y) - f(x^*) \leq Df(x^*)(y - x^*) \leq 0$$

follows $f(y) \leq f(x^*)$ for all $y \in U$.

Remark. Here is an example of global maximizer which is not a critical point: Suppose $f : R \to R$ is an increasing and convex function on $[a, b]$. Then $f'(b)(x - b) \leq 0$ for all $x \in [a, b]$. Thus b is global maximizer of f on $[a, b]$.

Lagrange Case

Consider the problem

$$max \;\; f(x_1, ..., x_n) \;\; s.t. \;\; h_i(x) = c_i, \; i = 1, ..., k.$$

As we know if $x^* = (x_1^*, ..., x_n^*)$ is a maximizer, then there exist $\mu^* = (\mu_1^*, ..., \mu_k^*)$ such that (x^*, μ^*) satisfies Lagrange conditions $Df(x^*) - \mu^* \cdot Dh(x^*) = 0$ and $h_i(x^*) = c_i, \; i = 1, ..., k$.

This is the sufficient condition for a global maximum:

Theorem 12 *Suppose f is concave, each h_i is convex, (x^*, μ^*) satisfies Lagrange conditions and each $\mu_i \geq 0$. Then x^* is a global maximizer.*

KKT Case

Consider the problem

$$max \;\; f(x_1, ..., x_n) \;\; s.t. \;\; g_i(x) \leq c_i, \; i = 1, ..., k.$$

As we know if $x^* = (x_1^*, ..., x_n^*)$ is a maximizer, then there exist $\lambda^* = (\lambda_1^*, ..., \lambda_k^*)$ such that (x^*, λ^*) satisfies KKT conditions $Df(x^*) - \lambda^* \cdot Dg(x^*) =$, $\lambda_i \cdot (h_i(x^* - c_i) = 0, \; i = 1, ..., k, \; \lambda_i \geq 0, \;\; g_i(x^*) = c_i, \; i = 1, 2, ..., k$.

This is the sufficient condition for a global maximum:

Theorem 13 *Suppose f is concave, each g_i is convex, and (x^*, λ^*) satisfies KKT conditions. Then x^* is a global maximizer.*

Example. Consider a production function $y = g(x_1, ..., x_n)$, where y denotes output, $x = (x_1, ..., x_n)$ denotes the input bundle, p denotes the price of output and w_i is the cost per unit of input i. Then the cost function is

$$C(x) = w_1x_1 + ... + w_nx_n,$$

and the profit function is

$$\pi(x) = pg(x) - C(x).$$

Our first claim is that if g is concave, then π is concave too: $C(x)$, as a linear function, is convex, then $-C(x)$ is concave, besides $pg(x)$ is concave too since $p > 0$, then $\pi(x) = pg(x) + (-C(x))$ is concave.

The first order condition gives

$$\frac{\partial \pi(x)}{\partial x_i} = p\frac{\partial g(x)}{\partial x_i} - w_i = 0.$$

Since of concavity this condition is necessary and sufficient to be interior maximizer. This means that the maximizer of profit is the value of x where marginal revenue product $p\frac{\partial g(x)}{\partial x_i}$ equals to the factor price w_i for each input.

1.5 Quasiconcave Functions

Recall the property of a concave function f: for each K the *lower level set*

$$L_K = \{x,\ f(x) \leq K\}$$

is concave.

This property is taken as the definition of quasiconcave function:

Definition 1. A function $f(x)$ defined on a convex subset $U \subset R^n$ is quasiconcave if

$$L_K = \{x : f(x) \leq K\}$$

is a convex set for any constant K.

Similarly, f is quasiconvex if

$$U_K = \{x : f(x) \geq K\}$$

is a convex set for any constant K.

Definition 2. A function $f(x)$ defined on a convex subset $U \subset R^n$ is quasiconcave if

$$f(tx + (1-t)y) \geq min(f(x), f(y))$$

for each $x, y \in U$ and $t \in [0, 1]$.

Similarly, f is quasiconvex if

$$f(tx + (1-t)y) \leq max(f(x), f(y)).$$

Remark. Concavity implies, but is not implied by quasiconcavity. Indeed, the function $f(x) = x^3$ is quasiconcave (and quasiconvex) but not concave (and convex).

Remark. Besides f s quasiconcave f and only if $-f$ is quasiconvex.

Theorem 14 *Definition 1 and Definition 2 are equivalent.*

Proof. (a) Def. 1 $\Rightarrow$ Def. 2.
Given:

$$U_K = \{x,\ f(x) \geq K\}$$

is a convex set.
Prove:

$$f(tx + (1-t)y) \geq min(f(x), f(y)).$$

Indeed, take $K = min(f(x), f(y))$, suppose this min is $f(x)$. Then $K = f(x) \leq f(x)$, so $x \in U_K$, and $K = f(x) \leq f(y)$, so $y \in U_K$. Then, since of convexity of U_K we have $tx + (1-t)y \in U_K$, that is $K \leq f(tx + (1-t)y)$.

(b) Def. 2 $\Rightarrow$ Def. 1.
Given:

$$f(tx + (1-t)y) \geq min(f(x), f(y)).$$

Prove:

$$U_K = \{x,\ f(x) \geq K\}$$

is a convex set.

Indeed, suppose $x, y \in U_K$, that is $f(x) \geq K,\ f(y) \geq y$. We want to prove that $f(tx + (1-t)y) \in U_K$, i.e. $f(tx + (1-t)y) \geq K$. Indeed, assume $min(f(x), f(y)) = f(x)$, then

$$f(tx + (1-t)y \geq min(f(x), f(y)) = f(x) \geq K.$$

Theorem 15 *A monotonic transformation gf of a quasiconcave function f is itself quasiconcave.*

Proof. Take any $K \in R$. Since g is monotonic, there exists $K' \in R$ such that $K = g(K')$. Then

$$U_K(gf) = \{x,\ gf(x) \geq K\} = \{x,\ gf \geq g(K')\} = \{x,\ f(x) \geq K'\} = U_{K'}(f)$$

is a convex set.

Remark. Thus the quasiconcavity is ordinal property (recall, the concavity is cardinal: a monotonic transformation of concave is not necessarily concave, for example $f(x) = x$ is concave, $g(x) = x^3$ is monotonically increasing, but $g(f(x)) = x^3$ is not concave).

In particular a monotonic transformation of concave is quaziconcave. But there exists quaziconcave function which is not monotonic transformation of a concave function.

Example. Every Cobb-Douglas function $F(x_1, x_2) = Ax_1^p x_2^q,\ p, q > 0$ is quasiconcave:

(a) As we know an DRS (Decreasing Return to Scale) Cobb-Douglas function such as $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$ concave.

(b) An IRS (Increasing Return to Scale) Cobb-Douglas function, such as $x_1^{2/3} x_2^{2/3}$ is quasiconcave. Indeed, IRS Cobb-Douglas is monotonic transformation of DRS Cobb-Douglas:

$$x_1^{2/3} x_2^{2/3} = (x_1^{1/3} x_2^{1/3})^2,$$

so $x_1^{2/3} x_2^{2/3} = g(f(x_1, x_2)$ where $f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$ and $g(z) = z^2$.

Example. Any CES function $Q(x, y) = (ax^r + by^r)^{\frac{1}{r}},\ a, b > 0,\ 0 < r < 1$ is quasiconcave: $Q(x, y) = gq(x, y)$ where $q(x, y) = (ax^r + by^r)$ is a concave function because it is positive linear combination of concave functions, and $q(z) = z^{\frac{1}{r}}$ is monotonic transformation.

Example. Any increasing function $f : R \to R$ is quasiconcave (and quasiconvex):

$$U_K = \{x,\ f(x) \geq K\} = [f^{-1}K, +\infty)$$

is a convex set.

Example. Each function $f : R^1 \to R^1$ which monotonically rises until it reaches a global maximum and the monotonically decrease, such as $f(x) = -x^2$, is quasiconcave: U_K is convex.

1.5.1 Calculus Criterion for Quasiconcavity

F is quasiconcave if and only if

$$F(y) \geq F(x) \quad \Rightarrow \quad DF(x)(y - x) \geq 0.$$

F is quasiconvex if and only if

$$F(y) \leq F(x) \quad \Rightarrow \quad DF(x)(y - x) \geq 0.$$

Exercises

1. By drawing diagrams, determine which of the following sets is convex.

$(a)\ \{(x, y) : y = e^x\}$. $(b)\ \{(x, y) : y \geq e^x\}$. $(c)\ \{(x, y) : xy \geq 1, x > 0, y > 0\}$.

2. Determine the definiteness of the following symmetric matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

3. For each of the following functions, determine which, if any, of the following conditions the function satisfies: concavity, strict concavity, convexity, strict convexity. (Use whatever technique is most appropriate for each case.)

$(a)\ \ f(x, y) = x + y$
$(b)\ \ f(x, y) = x^2$
$(c)\ \ f(x, y) = x + y - e^x - e^{x+y}$
$(d)\ \ f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$
$(e)\ \ f(x, y) = 3e^x + 5x^4 - ln\ x$
$(f)\ \ f(x, y, z) = Ax^a y^b z^c,\ a, b, c > 0.$

4. Let $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1 - 2x_2 + 1$. Is f convex, concave, or neither?

5. Prove that any homogenous function on $(0, +\infty)$ is either concave or convex.

6. Suppose that a firm that uses 2 inputs has the production function $f(x_1, x_2) = 12x^{1/3}x^{1/2}$ and faces the input prices (p_1, p_2) and the output price q. Show that f is concave for $x_1 > 0$ and $x_2 > 0$, so that the firm's profit is concave.

7. Let $f(x_1, x_2) = x_1^3 + 2x_1^2 + 2x_1x_2 + (1/2)x_2^2 - 8x_1 - 2x_2 - 8$. Find the range of values of (x_1, x_2) for which f is convex, if any.

8. Determine the values of a (if any) for which the function

$$2x^2 + 2xz + 2ayz + 2z^2$$

is concave and the values for which it is convex.

9. Show that the function $f(w, x, y, z) = -w^2 + 2wx - x^2 - y^2 + 4yz - z^2$ is not concave.

Homework

Exercise 21.2c from [Simon], Exercise 21.12 from [Simon], Exercise 21.18 from [Simon], Exercise 3f, Exercise 6.

Short Summary
Concave and Convex

Convex set $X \subset R^n$: $x',\ x'' \in X \ \Rightarrow \ x^t = (1-t)x' + tx'' \in X$.

Convex hull $CH(X) = \{y \in R^n : y = \sum t_i x_i, \quad x_i \in X, \quad \sum t_i = 1\}$.

Convex function $f : S \subset R^n \to R$: $x', x'' \in S \ \Rightarrow \ (1-t)f(x') + tf(x'') \leq f((1-t)x' + tx'')$, i.e. **graph is above chord**.

Hypograph: $hyp\ f = \{(x,y) : \ x \in S,\ y \leq f(x)\}$. f is concave iff $hyp\ f$ is convex.

Epigraph: $epi\ f = \{(x,y) : \ x \in S,\ y \geq f(x)\}$. f is convex iff $epi\ f$ is convex.

Upper contour set: $U_K = \{x \in S,\ f(x) \geq K\}$. If f is concave then U_K is convex.

Lower contour set: $U_K = \{x \in S,\ f(x) \leq K\}$. If f is convex then U_L is convex.

Calculus Criteria

C^1 function $f : U \subset R^n \to R$ is concave iff $f(y) - f(x) \leq Df(x)(y-x)$.

C^2 function $f : U \subset R^n \to R$ is concave iff $D^2 f(x) \leq 0$.

Concavity and Optimization

If f is concave and $D(x^*) = 0$ then x^* is global max.

If f is concave and $Df(x^*)(y - x^*) \leq 0$ for $\forall y$ then x^* is global max.

Quaziconcavity

f **quasiconcave** if $U_K = \{x : f(x) \geq K\},\ \forall K$. Equivalently

$$f(tx + (1-t)y) \geq min(f(x), f(y)), \quad \forall\ x, y,\ t \in [0,1].$$

Concavity - cardinal, quasiconcavity - ordinal.

Calculus Criterion

F is quasiconcave iff

$$F(y) \geq F(x) \quad \Rightarrow \quad DF(x)(y-x) \geq 0.$$