1 Constraint Optimization: Second Order Conditions

Reading [Simon], Chapter 19, p. 457-469.

The above described first order conditions are *necessary conditions* for constrained optimization.

Bellow we introduce appropriate second order sufficient conditions for constrained optimization problems in terms of *bordered* Hessian matrices.

1.1 Recall Nonconstrained case

In absolute (i.e. non constrained) optimization there are second order *sufficient conditions* in terms of Hessian matrix

$$D^2F = \begin{pmatrix} F_{x_1x_1} & F_{x_2x_1} & \dots & F_{x_nx_1} \\ F_{x_1x_2} & F_{x_2x_2} & \dots & F_{x_nx_2} \\ \dots & \dots & \dots & \dots \\ F_{x_1x_n} & F_{x_2x_n} & \dots & F_{x_nx_n} \end{pmatrix}.$$

1. Max. Suppose

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and n leading principal minors of $D^2F(x^*)$ alternate in sign

$$\begin{vmatrix} F_{x_1x_1} \end{vmatrix} < 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} < 0, \ \dots$$

at x^*. Then x^* is a strict local max.

Shortly

$$\left\{ \begin{array}{c} Df(x^*) = 0 \\ D^2f(x^*) < 0 \end{array} \right| \quad \Rightarrow \ x^* \quad max.$$

2. Min. Suppose

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and n leading principal minors of $D^2F(x^*)$ are positive

$$\begin{vmatrix} F_{x_1x_1} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} > 0, \ \dots$$

at x^*. Then x^* is a strict local min.

Shortly

$$\left\{\begin{array}{c} Df(x^*) = 0 \\ D^2 f(x^*) > 0 \end{array}\right| \Rightarrow x^* \;\; min.$$

3. Saddle. Suppose

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and some nonzero leading principal minors of $D^2F(x^*)$ violate previous two sign patterns. Then x^* is a saddle point.

1.2 Constrained Optimization of a Quadratic Form Subject of Linear Constraints

Recall one special case of constrained optimization, where the objective function is a quadratic form and all constraints are linear:

$$f(x_1, \; ... \; , x_n) = \sum_{i,j} a_{ij} x_i x_j,$$

$$\begin{array}{l} h_1(x_1, \; ... \; , x_n) = \sum_{i=1}^n B_{1i} x_i = 0 \\ ... \\ h_m(x_1, \; ... \; , x_n) = \sum_{i=1}^n B_{mi} x_i = 0. \end{array}$$

The sufficient condition in this particular case was formulated in terms of bordered matrix

$$H = \left(\begin{array}{ccccccc} 0 & ... & 0 & | & B_{11} & ... & B_{1n} \\ & ... & & | & & ... & \\ 0 & ... & 0 & | & B_{m1} & ... & B_{mn} \\ - & - & - & & - & - & - \\ B_{11} & ... & B_{m1} & | & a_{11} & ... & a_{1n} \\ & ... & & | & & ... & \\ B_{1n} & ... & B_{1n} & | & a_{1n} & ... & a_{nn} \end{array} \right) = \left(\begin{array}{cc} 0 & B \\ B^T & A \end{array} \right).$$

This $(m+n) \times (m+n)$ matrix has $m+n$ leading principal minors

$$M_1, \; M_2, \; ... \; , M_m, \; M_{m+1}, \; ... \; , M_{2m-1}, \; M_{2m}, \; M_{2m+1}, \; ... \; , M_{m+n} = H.$$

The first m matrices $M_1, \; ... \; , M_m$ are zero matrices.

Next $m-1$ matrices $M_{m+1}, \; ... \; , M_{2m-1}$ have zero determinant.

The determinant of the next minor M_{2m} is $\pm(det\ M')^2$ where M' is the left $m \times m$ minor of B, so $det\ M_{2m}$ does not contain information about f.

And only the determinants of last $n - m$ matrices $M_{2m+1},\ \dots\ , M_{m+n}$ carry information about both, the objective function f and the constraints h_i. Exactly these minors are essential for constraint optimization.

(i) If the determinant of $H = M_{m+n}$ has the sign $(-1)^n$ and the signs of determinants of last $m + n$ leading principal minors

$$M_{2m+1},\ \dots\ , M_{m+n}$$

alternate in sign, then Q is negative definite on the constraint set $Bx = 0$, so $x = 0$ is a strict global max of Q on the constraint set $Bx = 0$.

(ii) If the determinants of all last $m + n$ leading principal minors

$$M_{2m+1},\ \dots\ , M_{m+n}$$

have the same sign $(-1)^m$, then Q is positive definite on the constraint set $Bx = 0$, so $x = 0$ is a strict global min of Q on the constraint set $Bx = 0$.

(iii) If both conditions (i) and (ii) are violated by some from last $m + n$ leading principal minors

$$M_{2m+1},\ \dots\ , M_{m+n}$$

then Q is indefinite on the constraint set $Bx = 0$, so $x = 0$ is neither max nor min of Q on the constraint set $Bx = 0$.

This table describes the above sign patterns:

	M_{m+m+1}	M_{m+m+2}	...	M_{m+n-1}	M_{m+n}
negative	$(-1)^{m+1}$	$(-1)^{m+2}$	...	$(-1)^{n-1}$	$(-1)^n$
positive	$(-1)^m$	$(-1)^m$	...	$(-1)^m$	$(-1)^m$

1.3 Sufficient Condition for Constrained Optimization

Consider now the problem of maximizing $f(x_1, ..., x_n)$ on the constraint set

$$C_h = \{x \in R^n :\ h_i(x) = c_i,\ i = 1, ..., k\}.$$

As usual we consider the Lagrangian

$$L(x_1, ..., x_n, \mu_1, ..., \mu_k) = f(x_1, ..., x_n) - \sum_{i=1}^{k} \mu_i(h_i(x_1, ..., x_n) - c_i),$$

and the following bordered Hessian matrix

$$H = \begin{pmatrix} 0 & \dots & 0 & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \frac{\partial h_k}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \dots & \frac{\partial^2 L}{\partial x_n \partial x_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial h_1}{\partial x_n} & \dots & \frac{\partial h_k}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}.$$

This $(k+n) \times (k+n)$ matrix has $k+n$ leading principal minors

$$H_1,\ H_2,\ \dots,H_k,\ H_{k+1},\ \dots,H_{2k-1},\ H_{2k},\ H_{2k+1},\ \dots,H_{k+n} = H.$$

The first m matrices $H_1,\ \dots,H_k$ are zero matrices.

Next $k-1$ matrices $H_{k+1},\ \dots,H_{2k-1}$ have zero determinant.

The determinant of the next minor H_{2k} is $\pm(det\ H')^2$ where H' is the upper $k \times k$ minor of H after block of zeros, so $det\ H_{2k}$ does not contain information about f.

And only the determinants of last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{2k+(n-k)=k+n} = H$$

carry information about both, the objective function f and the constraints h_i.

Exactly these minors are essential for the following sufficient condition for constraint optimization.

Theorem 1 *Suppose that $x^* = (x_1^*, ..., x_n^*) \in R^n$ satisfies the conditions*

(a) $x^* \in C_h$;

(b) there exists $\mu^* = (\mu_1^*, ..., \mu_k^*) \in R^k$ such that $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ is a critical point of L;

(c) for the bordered Hessian matrix H the last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ alternate in sign where the last minor $H_{n+k} = H$ has the sign as $(-1)^n$.

Then x^* is a local max in C_h.

If instead of (c) we have the condition

(c') For the bordered hessian H all the last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ have the same sign as $(-1)^k$, then x^* is a local min on C_h.

This table describes the above sign patterns:

	H_{k+k+1}	H_{k+k+2}	...	H_{k+n-1}	H_{k+n}
max	$(-1)^{k+1}$	$(-1)^{k+2}$	...	$(-1)^{n-1}$	$(-1)^{n}$
min	$(-1)^{k}$	$(-1)^{k}$	...	$(-1)^{k}$	$(-1)^{k}$

Example 1. Find extremum of $F(x,y) = xy$ subject of $h(x,y) = x + y = 6$.

Solution. The Lagrangian here is

$$L(x,y) = xy - \mu(x + y - 6).$$

The first order conditions give the solution

$$x = 3, \; y = 3, \; \mu = 3$$

which needs to be tested against second order conditions before we can tell whether it is maximum, minimum or neither.

The bordered Hessian of our problem looks as

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here $n = 2, \; k = 1$ so we have to check just $n - k = 2 - 1 = 1$ last leading principal minors, so just H itself. Calculation shows that $det\ H = 2 > 0$ has the sign $(-1)^2 = (-1)^n$ so our critical point $(x = 3, y = 3)$ is max.

Example 2. Find extremum of $F(x,y,z) = x^2+y^2+z^2$ subject of $h_1(x,y,z) = 3x + y + z = 5, \quad h_2(x,y,z) = x + y + z = 1$.

Solution. The lagrangian here is

$$L(x,y,\mu_1,\mu_2) = x^2 + y^2 + z^2 - \mu_1(3x + y + z - 5) - \mu_2(x + y + z - 1).$$

The first order conditions give the solution

$$x = 2, \; y = -\frac{1}{2}, \; z = -\frac{1}{2}, \; \mu_1 = \frac{5}{2}, \; \mu_2 = -\frac{7}{2}.$$

Now it is time to switch to bordered hessian in order to tell whether it is maximum, minimum or neither

$$H = \begin{pmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Here $n = 3,\ k = 2$ so we have to check just $n - k = 3 - 2 = 1$ leading principal minors, so just H itself. Calculation shows that $det\ H = 16 > 0$ has the sign as $(-1)^k = (-1)^2 = +1$, so our critical point is min.

Example 3. Find extremum of $F(x, y) = x + y$ subject of $h(x, y) = x^2 + y^2 = 2$.

Solution. The lagrangian here is

$$L(x, y) = x + y - \mu(x^2 + y^2 - 2).$$

The first order conditions give two solutions

$$x = 1,\ y = 1, \mu = 0.5 \ \ and \ \ x = -1,\ y = -1, \mu = -0.5$$

Now it is time to switch to bordered hessian

$$H = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -2\mu & 0 \\ 2y & 0 & -2\mu \end{pmatrix}.$$

Here $n = 2,\ k = 1$ so we have to check just $n - k = 3 - 2 = 1$ leading principal minor $H_2 = H$.

Checking H for $(x = 1,\ y = 1,\ \mu = 0.5)$ we obtain $H = 4 > 0$, that is it has the sign of $(-1)^n = (-1)^2$, so this point is max.

Checking H for $(x = -1,\ y = -1,\ \mu = -0.5)$ we obtain $H = -4 < 0$, that is it has the sign of $(-1)^k = (-1)^1$, so this point is min.

1.4 Second Order Conditions for Mixed Constraints

Problem: maximize $f(x_1, ..., x_n)$ subject to k inequality and m equality constraints

$$g_1(x_1, ..., x_n) \leq b_1,\ \ldots\ , g_k(x_1, ..., x_n) \leq b_k,$$
$$h_1(x_1, ..., x_n) = c_1,\ \ldots\ , h_m(x_1, ..., x_n) = c_m.$$

Lagrangian:

$$\begin{aligned} &L(x_1, ..., x_n, \lambda_1, ..., \lambda_k)) = f(x_1, ..., x_n) + \\ &-\lambda_1[g_1(x_1, ..., x_n) - b_1] - \ \ldots\ - \lambda_k[g_(x_1, ..., x_n) - b_1] + \\ &-\mu_1[h_1(x_1, ..., x_n) - c_1] - \ \ldots\ - \mu_m[h_(x_1, ..., x_n) - c_1]. \end{aligned}$$

Theorem 2 *Suppose we have*

$$x^* = (x_1^*, ..., x_n^*),\ \ \lambda^* = (\lambda_1^*,\ \ldots\ , \lambda_k^*),\ \ \mu^* = (\mu_1^*,\ \ldots\ , \mu_m^*)$$

such that the first order conditions are satisfied, that is

$$\begin{array}{ll}(a) & \frac{\partial L}{\partial x_1}(x^*,\lambda^*)=0,\ \dots,\ \frac{\partial L}{\partial x_n}(x^*,\lambda^*)=0,\\ (b) & \lambda_1^*[g_1(x^*)-b_1]=0,\ \dots,\lambda_k^*[g_k(x^*)-b_1]=0,\\ (c) & h_1(x^*)=c_1,\ \dots,h_m(x^*)=c_m,\\ (d) & \lambda_1^*\geq 0,\ \dots,\lambda_k^*\geq 0,\\ (e) & g_1(x^*)\leq b_1,\ \dots,g_1(x^*)\leq b_1.\end{array}$$

Furthermore, suppose among inequality constraints

$$g_1,\ g_2,\ \cdots,g_b$$

are binding at x^* and others are not binding. Consider the following bordered Hessian

$$H=\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial g_b}{\partial x_1} & \dots & \frac{\partial g_b}{\partial x_n}\\ 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial h_m}{\partial x_1} & \dots & \frac{\partial h_m}{\partial x_n}\\ \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_b}{\partial x_1} & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \dots & \frac{\partial^2 L}{\partial x_n \partial x_1}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \frac{\partial g_1}{\partial x_n} & \dots & \frac{\partial g_b}{\partial x_n} & \frac{\partial h_1}{\partial x_n} & \dots & \frac{\partial h_m}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 L}{\partial x_n^2}\end{pmatrix}.$$

Then if last $n-(b+m)$ minors of this Hessian evaluated at (x^*,λ^*,μ^*) alternate in sign and the sign of largest minor (H) -s that of $(-1)^n$ then x^* is maximizer.

1.5 Again About the Meaning of Multiplier

Consider the problem

Maximize $f(x_1,...,x_n)$ subject to $h_i(x_1,...,x_n)=a_i,\ i=1,...,k.$

Let $(x^*,\mu^*)=(x_1^*,...,x_n^*,\mu_1^*,...,\mu_k^*)$ be the optimal solution of this problem. This solution depends on "budget" constraints $a=(a_1,...,a_k)$, so we can assume that x^* and μ^* are functions of a:

$$(x^*(a),\mu^*(a))=(x_1^*(a),...,x_n^*(a),\mu_1^*(a),...,\mu_k^*(a)).$$

The optimal value $f(x^*,\mu^*)$ also can be considered as a function of a:

$$f(x^*(a))=f(x_1^*(a),...,x_n^*(a)).$$

Calculation similar to one used in two variable case shows that

$$\frac{\partial}{\partial a_j}f(x_1^*(a),...,x_n^*(a))=\mu_j^*(a).$$

This formula has the following meaning:

-μ_j^* measures the sensitivity of optimal value $f(x^*(a))$ to changing the constraint a_j.

-In other words μ_j^* measures how the optimal value is affected by relaxation of j-th constraint a_j.

-One more interpretation: μ_j^* measures how the additional dollar invested in j-th input changes the optimal value.

That is why the Lagrange multiplier sometimes is called shadow price, internal value, marginal productivity of money.

Example 4. Consider the problem

Maximize $f(x,y) = x^2y$ on the constraint set $h(x,y) = 2x^2 + y^2 = 3$.
The first order condition gives the solution

$$x^*(3) = 1, \quad y^*(3) = 1, \quad \mu^*(3) = 0.5.$$

The second order condition allows to check that this is maximizer. The optimal value is $f(1,1) = 1$.

Now let us change the constraint to

$$h(x,y) = 2x^2 + y^2 = 3.3.$$

The first order condition gives new stationary point

$$x^*(3.3) = 1.048808848, \; y^*(3.3) = 1.048808848, \; \mu^*(3.3) = 0.5244044241$$

and the new optimal value is 1.153689733. So increasing the budget $a = 3$ to $a + \Delta a = 3.3$ increases the optimal value by $1.153689733 - 1 = 0.153689733$.
Now estimate the same increasing of optimal value using shadow price:

$$f(1.048808848, 1.048808848) - f(1,1) \approx f(1,1) + \mu^* \cdot 0.3 = 1 + 0.5 \cdot 0.3 = 1.15,$$

this is good approximation of 1.153689733.

1.5.1 Income Expansion Path

Back to a problem

Maximize $f(x_1, x_2)$ subject to $h(x_1, x_2) = a$.

The shadow price formula here looks as

$$\frac{\partial}{\partial a} f(x_1^*(a), x_2^*(a)) = \mu^*(a).$$

The curve $R \to R^n$ given by $a \to x^*(a) = (x_1^*(a), x_2^*(a))$ is called *income expansion path*. This is the path on which moves the optimal solution $x^*(a)$ when the constraint a changes.

Theorem 3 *If the objective function $f(x_1, x_2)$ is homogenous, and the constraint function $h(x_1, x_2) = p_1x_1 + p_2x_2$ is linear, then the income expansion path is a ray from origin.*

Proof. Try!

1.6 Envelop Theorems*

The above theorems about the meaning of multiplier are particular cases of so called Envelope Theorems.

1.6.1 Version with Maximizers

Let $f(x, c)$, $x \in R^n$, $c \in R^p$ be a function of variables $x = (x_1, \ \dots \ , x_n)$ and parameters $c = (c_1, \ \dots \ , c_p)$. Let us fix c^* and suppose $x^*(c^*)$ be the maximizer of $f(x, c^*)$ with c^* fixed. Then the maximal value $f(x^*(c^*), c^*)$ can be considered as a function of c:

$$F(c^*) = f(x^*(c^*), c^*).$$

Note that since of first order condition we have

$$D_x f(x^*(c^*), c^*) = 0$$

The Envelope Theorem gives an easy way to calculate the gradient of $F(c^*)$.

Theorem 4

$$D_c F(c^*) = D_c f(x^*(c^*), c^*).$$

Proof. By chain rule and the above mentioned first order condition we have

$$\begin{gathered} D_c F(c^*) = D_x f(x^*(c^*), c^*) \cdot D_c(x^*(c^*)) + D_c f(x^*(c^*), c^*) \\ = 0 \cdot D_c(x^*(c^*)) + D_c f(x^*(c^*), c^*) = D_c f(x^*(c^*), c^*). \end{gathered}$$

Example 6. Consider the maximization problem

$$max \quad f(x) = -x^2 + 2ax + 4a^2$$

which depends on the parameter a. What will be the effect of a unit increase of a on the maximal value of $f(x, a)$?

1. **Direct solution.** Just find a critical point

$$\begin{gathered} f'(x, a) = -2x + 2a = 0, \quad x^* = a, \quad F(a) = f(x^*(a), a) = \\ f(a, a) = -a^2 + 2a^2 + 4a^2 = 5a^2, \end{gathered}$$

so the rate of change of maximal value is $10a$.

2. Solution using the Envelope Theorem.

$$F'(a) = \frac{\partial}{\partial a} f(x,a)|_{x=a} = (2x + 8a)|_{x=a} = 10a.$$

1.6.2 Version with Constrained Maximum

Consider the problem

$$max \quad f(x,a) \quad s.t. \quad h(x,a) = 0,$$

depending on the parameter a. Let $(x^*(a), \mu^*(a))$ be the optimal solution for the fixed choice of parameter a, so $F(a) = f(x^*(a), a)$ is a maximal value as a function of a.

Theorem 5 *The rate of change of maximal value is*

$$F'(a) = \frac{d}{da} f(x^*(a), a) = \frac{\partial}{\partial a} L(x^*(a), \mu^*(a), a).$$

Remark. Actually when $f(x,a) = f(x)$, that is the objective function *does not* depend on the parameter a, and $h(x,a) = \bar{h}(x) - a$, that is the parameter a is the budget restriction, then we have our old result $F'(a) = \mu_i$, indeed,

$$F'(a) = \frac{\partial}{\partial a} L(x, \mu, a)|_{(x=x^*(a), \mu=\mu^*(a), a)} =$$
$$\frac{\partial}{\partial a}[f(x) - \mu(\bar{h}(x) - a)]|_{(x=x^*(a), \mu=\mu^*(a), a)} = \mu|_{(\mu=\mu^*(a), a)} = \mu^*(a).$$

Example 7. Consider the problem discussed above

$$max \quad f(x,y) = x^2 y \quad s.t. \quad h(x,y) = 2x^2 + y^2 = 3.$$

As it was calculated the first order condition gives the solution

$$x^*(3) = 1, \quad y^*(3) = 1, \quad \mu^*(3) = 0.5,$$

the second order condition allows to check that this is maximizer. The optimal value is $f(1,1) = 1$.

Now try to estimate how the optimal value will change if we replace the constraint by

$$h(x,y) = 2x^2 + 1.1y^2 = 3.$$

Instead of solving the problem

$$max \quad f(x,y) = x^2 y \quad s.t. \quad h(x,y) = 2x^2 + 1.1y^2 - 3 = 0$$

let us use the Envelope Theorem for

$$h(x,y,a) = 2x^2 + ay^2 - 3.$$

The Lagrangian in this case is

$$L(x, y, a) = x^2 y - \mu(2x^2 + ay^2 - 3),$$

then

$$F'(a) = \frac{d}{da} L(x, y, a) = -\mu y^2,$$

thus

$$F'(1) = \frac{d}{da} L(x, y, a)|_{(x=1, y=1, \mu=0.5)} = -\mu y^2|_{(x=1, y=1, \mu=0.5)} = -0.5 \cdot 1^2 = -0.5,$$

and

$$F(1.1) = F(1) + F'(1) \cdot 0.1 = 1 + (-0.5) \cdot 0.1 = 1 - 0.05 = 0.95.$$

Exercises

1. Write out the bordered Hessian for a constrained optimization problem with four choice variables and two constraints. Then state specifically the second-order sufficient condition for a maximum and for a minimum respectively.

2. For the following problems
(i) find stationary values,
(ii) ascertain whether they are min or max,
(iii) find whether relaxation of the constraint (say increasing of budget) will increase or decrease the optimal value?
(iv) At what rate?
(a) $f(x, y) = xy, \quad h(x, y) = x + 2y = 2$;
(b) $f(x, y) = x(y + 4), \quad h(x, y) = x + y = 8$;
(c) $f(x, y) = x - 3y - xy, \quad h(x, y) = x + y = 6$;
(d) $f(x, y) = 7 - y + x^2, \quad h(x, y) = x + y = 0$.

3. Find all the stationary points of $f(x, y, z) = x^2y^2$ subject of $x^2 + y^2 = 2$ and check the second order conditions.

4. Find all the stationary points of $f(x, y, z) = x^2y^2z^2$ subject of $x^2 + y^2 + z^2 = 3$ and check the second order conditions.

5. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ on the constraint set $h(x, y) = x^2 + xy + y^2 = 3$. Redo the problem, this time using the constraint $h(x, y) = 3.3$.

Now estimate the shadow price of increasing of constraint by 0.3 units and compare with previous result.

*And now estimate the change of optimal value when the objective function is changed to $f(x, y) = x^2 + 1.2y^2$ keeping $h = x^2 + xy + y^2 = 3$.

6. Find all the stationary points of $f(x, y, z) = x + y + z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $y = 0$. Check the second order conditions and classify minimums and maximums.

Homework

1. Exercise 1.
2. Exercise 2d.
3. Exercise 4.
4. Exercise 6.
5. Exercise 19.3 from [Simon].