Reading
[SB], Ch. 16.1-16.3, p. 375-393

1 Quadratic Forms

A *quadratic function* $f : R \to R$ has the form $f(x) = a \cdot x^2$. Generalization of this notion to two variables is the *quadratic form*

$$Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2.$$

Here each term has degree 2 (the sum of exponents is 2 for all summands).

A quadratic form of three variables looks as

$$\begin{aligned} f(x_1, x_2, x_3) = \\ & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \\ & a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + \\ & a_{31}x_1x_3 + a_{32}x_3x_2 + a_{33}x_3^2. \end{aligned}$$

A general quadratic form of n variables is a real-valued function $Q : R^n \to R$ of the form

$$\begin{array}{rccccccc} Q(x_1, x_2, ..., x_n) = & a_{11}x_1^2 & + & a_{12}x_1x_2 & + & ... & + & a_{1n}x_1x_n + \\ & a_{21}x_2x_1 & + & a_{22}x_2^2 & + & ... & + & a_{2n}x_2x_n + \\ & ... & ... & ... & ... & & & \\ & a_{n1}x_nx_1 & + & a_{n2}x_nx_2 & + & ... & + & a_{nn}x_n^2 \end{array}$$

In short $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij}x_ix_j$.

As we see a quadratic form is determined by the matrix

$$A = \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix}.$$

1.1 Matrix Representation of Quadratic Forms

Let $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij}x_ix_j$ be a quadratic form with matrix A. Easy to see that

$$Q(x_1, ..., x_n) = (x_1, ..., x_n) \cdot \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix}.$$

Equivalently $Q(x) = x^T \cdot A \cdot x$.
Example. The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ whose symmetric matrix is $A = \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$ is the product of three matrices

$$(x_1, x_2, x_3) \cdot \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1.1.1 Symmetrization of matrix

The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ can be represented, for example, by the following 2×2 matrices

$$\begin{pmatrix} 5 & -2 \\ -8 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & -3 \\ -7 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$$

the last one is *symmetric*: $a_{ij} = a_{ji}$.

Theorem 1 *Any quadratic form can be represented by symmetric matrix.*

Indeed, if $a_{ij} \neq a_{ji}$ we replace them by new $a'_{ij} = a'_{ji} = \frac{a_{ij}+a_{ji}}{2}$, this does not change the corresponding quadratic form.

Generally, one can find symmetrization A' of a matrix A by $A' = \frac{A+A^T}{2}$.

1.2 Definiteness of Quadratic Forms

A quadratic form of one variable is just a quadratic function $Q(x) = a \cdot x^2$.

If $a > 0$ then $Q(x) > 0$ for each nonzero x.

If $a < 0$ then $Q(x) < 0$ for each nonzero x.

So the sign of the coefficient a determines the sign of *one variable* quadratic form.

The notion of *definiteness* described bellow generalizes this phenomenon for multivariable quadratic forms.

1.2.1 Generic Examples

The quadratic form $Q(x, y) = x^2 + y^2$ is *positive* for *all* nonzero (that is $(x, y) \neq (0, 0)$) arguments (x, y). Such forms are called *positive definite*.

The quadratic form $Q(x, y) = -x^2 - y^2$ is *negative* for *all* nonzero arguments (x, y). Such forms are called *negative definite*.

The quadratic form $Q(x, y) = (x - y)^2$ is *nonnegative*. This means that $Q(x, y) = (x - y)^2$ is either positive or zero for nonzero arguments. Such forms are called *positive semidefinite*.

The quadratic form $Q(x,y) = -(x-y)^2$ is *nonpositive*. This means that $Q(x,y) = (x-y)^2$ is either negative or zero for nonzero arguments. Such forms are called *negative semidefinite*.

The quadratic form $Q(x,y) = x^2 - y^2$ is called *indefinite* since it can take both positive and negative values, for example $Q(3,1) = 9-1 = 8 > 0,\ Q(1,3) = 1-9 = -8 < 0$.

1.2.2 Definiteness

Definition. A quadratic form $Q(x) = x^T \cdot A \cdot x$ (equivalently *a symmetric matrix* A) is
(a) **positive definite** if $Q(x) > 0$ for all $x \neq 0 \in R^n$;
(b) **positive semidefinite** if $Q(x) \geq 0$ for all $x \neq 0 \in R^n$;
(c) **negative definite** if $Q(x) < 0$ for all $x \neq 0 \in R^n$;
(d) **negative semidefinite** if $Q(x) \leq 0$ for all $x \neq 0 \in R^n$;
(e) **indefinite** if $Q(x) > 0$ for some x and $Q(x) < 0$ for some other x.

1.2.3 Definiteness and Optimality

Determining the definiteness of quadratic form Q is equivalent to determining wether $x = 0$ is max, min or neither. Particularly:
If Q is positive definite then $x = 0$ is global maximum;
If Q is negative definite then $x = 0$ is global minimum.

1.2.4 Definiteness of 2 Variable Quadratic Form

Let $Q(x_1,x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = (x_1,x_2) \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a 2 variable quadratic form.

Here $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is the symmetric matrix of the quadratic form. The determinant $\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$ is called *discriminant* of Q.

Easy to see that

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = a(x_1 + \frac{b}{a}x_2)^2 + \frac{ac-b^2}{a}x_2^2.$$

Let us use the notation $D_1 = a,\ D_2 = ac - b^2$. Actually D_1 and D_2 are *leading principal minors* of A. Note that there exists one more principal (non leading) minor (of degree 1) $D_1' = c$.

Then

$$Q(x_1,x_2) = D_1(x_1 + \frac{b}{a}x_2)^2 + \frac{D_2}{D_1}x_2^2.$$

From this expression we obtain:

1. If $D_1 > 0$ and $D_2 > 0$ then the form is of $x^2 + y^2$ type, so it is *positive definite*;

2. If $D_1 < 0$ and $D_2 > 0$ then the form is of $-x^2 - y^2$ type, so it is *negative definite*;

3. If $D_1 > 0$ and $D_2 < 0$ then the form is of $x^2 - y^2$ type, so it is *indefinite*;

If $D_1 < 0$ and $D_2 < 0$ then the form is of $-x^2 + y^2$ type, so it is also *indefinite*;

Thus if $D_2 < 0$ then the form is indefinite.

Semidefiniteness depends not only on leading principal minors $D_1,\ D_2$ but also on *all* principal minors, in this case on $D_1' = c$ too.

4. If $D_1 \geq 0,\ D_1' \geq 0$ and $D_2 \geq 0$ then the form is *positive semidefinite.*
Note that only $D_1 \geq 0$ and $D_2 \geq 0$ is not enough, the additional condition $D_1' \geq 0$ here is absolutely necessary: consider the form $Q(x_1, x_2) = -x_2^2$ with $a = 0,\ b = 0,\ c = -1$, here $D_1 = a \geq 0,\ D_2 = ac - b^2 \geq 0$, nevertheless the form is not positive semidiefinite.

5. If $D_1 \leq 0,\ D_1' \leq 0$ and $D_2 \geq 0$ then the form is *negative semidefinite.*
Note that only $D_1 \leq 0$ and $D_2 \geq 0$ is not enough,the additional condition $D_1' \leq 0$ again is absolutely necessary: consider the form $Q(x_1, x_2) = x_2^2$ with $a = 0,\ b = 0,\ c = 1$, here $D_1 = a \leq 0,\ D_2 = ac - b^2 \geq 0$, nevertheless the form is not negative semidiefinite.

1.2.5 Definiteness of 3 Variable Quadratic Form

Let us start with the following
Example. $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$. The symmetric matrix of this quadratic form is

$$\begin{pmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{pmatrix}.$$

The leading principal minors of this matrix are

$$|D_1| = \begin{vmatrix} 1 \end{vmatrix} = 1, \quad |D_2| = \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -2, \quad |D_3| = \begin{vmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{vmatrix} = -18.$$

Now look:

$$
\begin{aligned}
&Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 = \\
&x_1^2 - 4x_1x_2 + 8x_1x_3 + 2x_2^2 - 7x_3^2 = x_1^2 - 4x_1(x_2 - 2x_3) + 2x_2^2 - 7x_3^2 = \\
&[x_1^2 - 4x_1(x_2 - 2x_3) + 4(x_2 - 2x_3) - 4(x_2 - 2x_3)] + 2x_2^2 - 7x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2x_2^2 - 16x_2x_3 - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2(x_2^2 - 8x_2x_3) - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2^2 - 8x_2x_3 + 16x_3^2 - 16x_3^2] - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 - 16x_3^2) - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 + 32x_3^2 - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 + 9x_3^2 = \\
&|D_1| l_1^2 + \tfrac{D_2}{D_1} l^2 + \tfrac{D_3}{D_2} l_3^2,
\end{aligned}
$$

where

$$
\begin{array}{rrrr}
l_1 = x_1 & -2x_2 & +4x_3, \\
l_2 = & x_2 & -4x_3, \\
l_3 = & & x_3.
\end{array}
$$

That is (l_1, l_2, l_3) are linear combinations of (x_1, x_2, x_3). More precisely

$$
\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

where

$$
P = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}
$$

is a nonsingular matrix (changing variables).

Now turn to general 3 variable quadratic form

$$
Q(x_1, x_2, x_3) = (x_1, x_2, x_3) \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
$$

The following three determinants

$$
|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad |D_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

are *leading principal minors.*

It is possible to show that, as in 2 variable case, if $|D_1| \neq 0,\ |D_2| \neq 0$, then

$$
Q(x_1, x_2, x_3) = |D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \frac{|D_3|}{|D_2|} l_3^2
$$

where l_1, l_2, l_3 are some linear combinations of x_1, x_2, x_3 (this is called **Lagrange's reduction**).

This implies the following criteria:

1. The form is positive definite iff $|D_1| > 0$, $|D_2| > 0$, $|D_3| > 0$, that is all principal minors are positive.

2. The form is negative definite iff $|D_1| < 0$, $|D_2| > 0$, $|D_3| < 0$, that is principal minors alternate in sign starting with negative one.

Example. Determine the definiteness of the form $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$.
Solution. The matrix of our form is

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

The leading principal minors are

$$|D_1| = 3 > 0, \quad |D_2| = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} > 5, \quad |D_3| = \begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 18 > 0,$$

thus the form is positive definite.

1.2.6 Definiteness of n Variable Quadratic Form

Let $Q(x_1, ..., x_n) = (x_1, ..., x_n) \cdot \begin{pmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ be an n variable quadratic form.

The following n determinants

$$|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ ... \ , \ |D_n| = \begin{vmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{vmatrix}$$

are *leading principal minors.*

As in previous cases, it is possible to show that

$$Q(x_1, \ ... \ , x_3) = |D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \ ... \ + \frac{|D_n|}{|D_{n-1}|} l_n^2$$

where $(l_1, l_2, \ ... \ , l_n)$ are linear combinations of $(x_1, x_2, \ ... \ , x_n)$, more precisely

$$\begin{pmatrix} l_1 \\ ... \\ l_n \end{pmatrix} = \begin{pmatrix} p_{11} & ... & p_{1n} \\ . & . & . \\ p_{n1} & ... & p_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} p_{11} & \dots & p_{1n} \\ . & . & . \\ p_{n1} & \dots & p_{nn} \end{pmatrix}$$

is a nonsingular matrix (changing variables).

Theorem 2 *1. A quadratic form is positive definite if and only if*

$$|D_1| > 0, \quad |D_2| > 0, \quad \dots \quad , |D_n| > 0,$$

that is all principal minors are positive;

2. A quadratic form is negative definite if and only if

$$|D_1| < 0, \quad |D_2| > 0, \quad |D_3| < 0, \quad |D_4| > 0, \quad \dots \quad ,$$

that is principal minors alternate in sign starting with negative one.

3. If some kth order leading principal minor is nonzero but does not fit either of the above two sign patterns, then the form is indefinite.

The situation with semidefiniteness is more complicated, here are involved not only leading principal minors, but *all* principal minors.

Theorem 3 *1. A quadratic form is positive semidefinite if and only if all principal minors are* ≥ 0*;*

2. A quadratic form is negative semidefinite if and only if all principal minors of odd degree are ≤ 0*, and all principal minors of even degree are* ≥ 0*.*

1.3 Definiteness and Eigenvalues

As we know a symmetric $n \times n$ matrix has n real eigenvalues (maybe some multiple).

Theorem 4 *Given a quadratic form* $Q(x) = x^T Ax$ *and let* $\lambda_1, \dots, \lambda_n$ *be eigenvalues of* A*. Then* $Q(x)$ *is*

- *positive definite iff* $\lambda_i > 0, i = 1, \dots, n$*;*
- *negative definite iff* $\lambda_i < 0, i = 1, \dots, n$*;*
- *positive semidefinite iff* $\lambda_i \geq 0, i = 1, \dots, n$*;*
- *negative semidefinite iff* $\lambda_i \leq 0, i = 1, \dots, n$*;*

Proof. Just $x^T Ax > 0 \Rightarrow \forall \lambda_i > 0$. Let v be the normalized eigenvector of λ_i, that is $Av = \lambda_i v$. Then

$$0 < vTAv = \lambda_i vTv = \lambda_i.$$

1.4 Linear Constraints and Bordered Matrices

1.4.1 Two Variable Case

The quadratic form $Q(x_1, x_2) = x_1^2 - x_2^2$ is indefinite (why?).

But if we restrict Q to the subset (subspace) of R^2 determined by the constraint $x_2 = 0$ we obtain a one variable quadratic form $q(x) = Q(x, 0) = x^2$ which is definitely positive definite. Thus the restriction of Q on x_1 axis is positive definite.

Similarly, the constraint $x_1 = 0$ gives $q(x) = Q(0, x) = -x^2$ which is negative definite. Thus the restriction of Q on x_2 axis is positive definite.

Now let us consider the constraint $x_1 - 2x_2 = 0$. Solving x_1 from this constraint we obtain $x_1 = 2x_2$. Substituting in Q we obtain one variable quadratic form $q(x) = Q(2x, x) = 4x^2 - x^2 = 3x^2$ which is positive definite. Thus the restriction of Q on the line $x_1 + 2x_2 = 0$ is positive definite.

Let us repeat the last calculations for a general 2 variable quadratic form

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = (x_1, x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to the linear constraint

$$Ax_1 + Bx_2 = 0.$$

Let us, as above, solve x_1 from the constraint

$$x_1 = -\frac{B}{A}x_2$$

and substitute in Q:

$$Q(x_1, x_2) = Q(-\tfrac{B}{A}x_2, x_2) = a_{11}(-\tfrac{B}{A}x_2)^2 + 2a_{12}(-\tfrac{B}{A}x_2)x_2 + a_{22}x_2^2 =$$

$$= \tfrac{aB^2 - 2bAB + cA^2}{A^2}x_2^2.$$

So the definiteness of $Q(x_1, x_2)$ on the constraint set $Ax_1 + Bx_2 = 0$ depends on the sign of coefficient $\frac{aB^2-2bAB+cA^2}{A^2}$, more precisely on the nominator

$$aB^2 - 2bAB + cA^2,$$

which is nothing else than It is easy to see that

$$-det\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}.$$

This matrix

$$\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$$

is called *bordered* matrix of A.

Thus we have proved the

Theorem 5 *A two variable quadratic form $Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ restricted on the constrained set $Ax_1 + Bx_2 = 0$ is positive (resp. negative) if and only if the determinant of the bordering matrix*

$$det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$$

is negative (resp. positive).

1.4.2 n Variable m Constraint Case

Analogous result holds in general case.

Let

$$x = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & .. & a_{1n} \\ ... & & \\ a_{n1} & ... & a_{nn} \end{pmatrix}$$

and

$$Q(x_1, \ ... \ , x_n) = x^T A x$$

be an n variable quadratic form. Consider a constraint set

$$\begin{pmatrix} B_{11} & ... & B_{1n} \\ ... & & \\ B_{m1} & ... & B_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ ... \\ 0 \end{pmatrix}.$$

In fact this set is the null space of B, is not it?

The bordered matrix for this situation looks as

$$H = \begin{pmatrix} 0 & ... & 0 & | & B_{11} & ... & B_{1n} \\ & ... & & | & & ... & \\ 0 & ... & 0 & | & B_{m1} & ... & B_{mn} \\ - & - & - & & - & - & - \\ B_{11} & ... & B_{m1} & | & a_{11} & ... & a_{1n} \\ & ... & & | & & ... & \\ B_{1n} & ... & B_{mn} & | & a_{n1} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}.$$

This $(m+n) \times (m+n)$ matrix has $m+n$ leading principal minors

$$M_1, \ M_2, \ ... \ , M_m, \ M_{m+1}, \ ... \ , M_{2m-1}, \ M_{2m}, \ M_{2m+1}, \ ... \ , M_{m+n} = H.$$

The first m matrices $M_1, \ \dots \ , M_m$ are zero matrices.

Next $m-1$ matrices $M_{m+1}, \ \dots \ , M_{2m-1}$ have zero determinant.

The determinant of the next minor M_{2m} is $\pm det \ B'^2$ where B' is the left $m \times m$ minor of B, it does not carry any information about A.

And only the determinants of last $n-m$ matrices $M_{2m+1}, \ \dots \ , M_{m+n}$ carry information about the matrix A, i.e. about the quadratic form Q. Exactly these minors are essential for constraint definiteness.

Theorem 6 *(i) If the determinant of $H = M_{m+n}$ has the sign $(-1)^n$ and the signs of determinants of last $m+n$ leading principal minors*

$$M_{2m+1}, \ \dots \ , M_{m+n}$$

alternate in sign, then Q is negative definite on the constraint set $Bx = 0$, so $x = 0$ is a strict global max of Q on the constraint set $Bx = 0$.

(ii) If the determinants of all last $m+n$ leading principal minors

$$M_{2m+1}, \ \dots \ , M_{m+n}$$

have the same sign $(-1)^m$, then Q is positive definite on the constraint set $Bx = 0$, so $x = 0$ is a strict global min of Q on the constraint set $Bx = 0$.

(iii) If both conditions (i) and (ii) are violated by some nonzero minors from last $m+n$ leading principal minors

$$M_{2m+1}, \ \dots \ , M_{m+n}$$

then Q is indefinite on the constraint set $Bx = 0$, so $x = 0$ is neither max nor min of Q on the constraint set $Bx = 0$.

This table describes the above sign patterns:

	M_{m+m+1}	M_{m+m+2}	...	M_{m+n-1}	M_{m+n}
negative	$(-1)^{m+1}$	$(-1)^{m+2}$	...	$(-1)^{n-1}$	$(-1)^n$
positive	$(-1)^m$	$(-1)^m$	...	$(-1)^m$	$(-1)^m$

Example

Determine the definiteness of the following constrained quadratics

$$\begin{aligned} &Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2, \\ &\textit{subject to} \ \ x_1 + x_2 + x_3 = 0. \end{aligned}$$

Solution. Here $n = 3,\ m = 1$.

The bordered matrix here is

$$\begin{pmatrix} 0 & | & 1 & 1 & 1 \\ - & - & - & - & - \\ 1 & | & 1 & -1 & 2 \\ 1 & | & -1 & 1 & 0 \\ 1 & | & 2 & 0 & -1 \end{pmatrix}.$$

The leading principal minors are

$$|M_1| = 0, \quad |M_2| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1,$$

$$|M_3| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -4 < 0, \quad |M_4| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & -1 \end{vmatrix} = 16 > 0.$$

in this case $n = 3,\ m = 1$, so the essential minors are M_3 and M_4.
For negative definiteness the sign pattern must be

$$(-1)^{n-1} = (-1)^2 = " + ", \quad (-1)^n = (-1)^3 = " - "$$

And for positive definiteness the sign pattern must be

$$(-1)^m = (-1)^1 = " - ", \quad (-1)^m = (-1)^1 = " - ",$$

since we have $-4 < 0,\ 16 > 0$, which differs from both patterns, our constrained quadratic is indefinite.

1.5 Change of variables*

Let $Q = x^T Ax$ be a quadratic form of variable $x \in R^n$. Let us step to new variable $y \in R^n$ which is connected to x by $x = Py$ where P is some nonsingular matrix . Note that in this case $x^T = (Py)^T = y^T P^T$. Then

$$Q = x^T Ax = y^T P^T APy = y^T (P^T AP)y,$$

so the matrix of new quadratic form of variable y is $B = P^T AP$.

If A is symmetric, $B = P^T AP$ is symmetric too (prove it using the definition of symmetric matrix $A = A^T$).

Jacobi's Theorem states that any symmetric matrix A can be transformed to a diagonal matrix $\Lambda = P^T AP$ by an orthogonal matrix P. The elements of Λ are uniquely determined up to permutation.

If we allow P to be a nonsingular matrix, then A can be transformed to a diagonal matrix where each diagonal element is 1, -1 or 0.

Important remark: Let

$$D_1,\ D_2,\ \dots\ D_n$$

be the leading principal minors of A, and let Λ be corresponding diagonal matrix with 1,0,-1 on the diagonal, then the Λ-s leading principal minors have the same sign pattern as D_i-s.

Silvester's Law of inertia states that the number of 0-s n_0, the number of 1-s n_+ and the number of -1-s n_- are invariant in the sense that any other diagonalization gives the same n_0, n_+ and n_-.

The *signature* of a quadratic form is defined as $n_+ - n_-$, so the signature is an invariant of quadratic form too.

A quadratic form Q is positive definite if $n_+ = n$.

A quadratic form Q is negative definite if $n_- = n$.

A quadratic form Q is positive semi definite if $n_- = 0$.

A quadratic form Q is negative semi definite if $n_+ = 0$.

A quadratic form Q is indefinite if $n_+ > 0$ and $n_- > 0$.

1.6 Why Quadratic Forms?*

Recall Taylor formula

$$f(x_0+h) = f(x_0) + f'(x_0)\cdot h + \frac{f''(x_0)}{2}\cdot h^2 + \frac{f'''(x_0)}{3!}\cdot h^3 + \ \dots$$

and ask a question: is this function increasing at $x = x_0$? That is if $h > 0$ is

$$f(x_0+h) - f(x_0) = f'(x_0)\cdot h + \frac{f''(x_0)}{2}\cdot h^2 + \frac{f'''(x_0)}{3!}\cdot h^3 + \ \dots$$

positive?

Notice, that for small enough h the quadratic term $\frac{f''(x_0)}{2}\cdot h^2$, the cubical term $\frac{f'''(x_0)}{3!}\cdot h^3$ etc are smaller that the linear term $f'(x_0)\cdot h$, so the positivity of $f(x_0+h) - f(x_0)$ depends on positivity of the coefficient $f'(x_0)$ of that linear form $f'(x_0)\cdot h$, that is on the positivity of the derivative at x_0.

Well, from this easily follows, that a local extremum we can expect at a critical point $f'(x_0) = 0$.

Now ask a question: how to recognize, is a critical point x_0 min or max?

Address again to Taylor formula which now, since $f'(x_0) = 0$, looks as

$$f(x_0+h) = f(x_0) + \frac{f''(x_0)}{2}\cdot h^2 + \frac{f'''(x_0)}{3!}\cdot h^3 + \ \dots\ .$$

Consider the difference

$$f(x_0+h) - f(x_0) = \frac{f''(x_0)}{2}\cdot h^2 + \frac{f'''(x_0)}{3!}\cdot h^3 + \ \dots\ .$$

The point x_0 is min if this difference $f(x_0+h)-f(x_0)$ is positive for all small enough values of h.

For small enough h the cubical term $\frac{f'''(x_0)}{3!} \cdot h^3$, the term of degree 4 $\frac{f^{(4)}(x_0)}{4!} \cdot h^4$ etc are smaller that the quadratic term $\frac{f''(x_0)}{2} \cdot h^2$, so the positivity of the difference $f(x_0+h) - f(x_0)$ depends on positivity of the coefficient $\frac{f''(x_0)}{2}$ of that quadratic form $\frac{f''(x_0)}{2} \cdot h^2$, that is on the positivity of the second derivative at x_0.

So, at $h = 0$ this form has minimum (it is positive definite) if its coefficient $f''(x_0)$ is positive (our good old second order condition).

Now consider a function of two variables $F(x_1, x_2)$. Again there exists Taylor for two variables

$$
\begin{aligned}
&F(x_1+h_1, x_2+h_2) = \\
&F(x_1,x_2) + \tfrac{\partial F}{\partial x_1}(x_1,x_2)\cdot h_1 + \tfrac{\partial F}{\partial x_2}(x_1,x_2)\cdot h_2 + \\
&\tfrac{1}{2}\tfrac{\partial^2 F}{\partial x_1^2}(x_1,x_2)\cdot h_1^2 + \tfrac{\partial^2 F}{\partial x_1 \partial x_2}(x_1,x_2)\cdot h_1 h_2 + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x_2^2}(x_1,x_2)\cdot h_2^2 + \\
&\tfrac{1}{3}\tfrac{\partial^3 F}{\partial x_1^3}(x_1,x_2)\cdot h_1^3 + \ \ldots \ .
\end{aligned}
$$

As in one-variable case, an optimum (min or max) can be expected at a critical point, where the linear form $\frac{\partial F}{\partial x_1}(x_1,x_2)\cdot h_1 + \frac{\partial F}{\partial x_2}(x_1,x_2)\cdot h_2$ vanishes. And the minimality or maximality depends on the quadratic form

$$Q(h_1,h_2) = \frac{1}{2}\frac{\partial^2 F}{\partial x_1^2}(x_1,x_2)\cdot h_1^2 + \frac{\partial^2 F}{\partial x_1 \partial x_2}(x_1,x_2)\cdot h_1 h_2 + \frac{1}{2}\frac{\partial^2 F}{\partial x_2^2}(x_1,x_2)\cdot h_2^2.$$

Namely, if this form is *positive definite*, that is if $Q(h_1,h_2) > 0$ for all $(h_1,h_2) \neq (0,0)$, then (x_1,x_2) is a point of *minimum*, and if the form is negative definite, that is $Q(h_1,h_2) < 0$ for all $(h_1,h_2) \neq (0,0)$, then (x_1,x_2) is a point of maximum.

Exercises

1. Write the following quadratic forms in matrix form
(a) $Q(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2$.
(b) $Q(x_1, x_2) = 5x_1^2 - 10x_1x_2 + x_2^2$.
(c) $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$.

2. By direct matrix multiplication express each of the following matrix products as a quadratic form

$$(a)\begin{pmatrix} x_1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (b)\begin{pmatrix} u & v \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}.$$

3. Write the quadratic form corresponding to the matrix A and then find a symmetric matrix which determines the same quadratic form for

$$(a)\ A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}, \quad (b)\ A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 0 & 7 & 6 \end{pmatrix} \quad (c)\ A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 6 \end{pmatrix}.$$

4. Write an example of 2 variable quadratic form $Q(x_1, x_2)$ which is
(a) positive definite.
(b) negative definite.
(c) positive semidefinite.
(d) negative semidefinite.
(e) indefinite.

5. Write an example of 3 variable quadratic form $Q(x_1, x_2, x_3)$ which is
(a) positive definite.
(b) negative definite.
(c) positive semidefinite.
(d) negative semidefinite.
(e) indefinite.

6. Determine definiteness of the following symmetric matrices (quadratic forms)

$$(a)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad (b)\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}, \quad (c)\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}, \quad (d)\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix},$$

$$(e)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \quad (f)\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (g)\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}.$$

7. Formulate criteria for positive and negative definiteness of the quadratic form given by a diagonal matrix

$$\begin{pmatrix} a_1 & 0 & ... & 0 & 0 \\ 0 & a_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & a_{n-1} & 0 \\ 0 & 0 & ... & 0 & a_n \end{pmatrix}.$$

8. Determine the definiteness of the following constrained quadratics

(a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, *subject to* $x_1 + x_2 = 0$.
(b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, *subject to* $x_1 + x_2 = 0$.
(c) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, *subject to* $x_1 + x_2 + x_3 = 0$.
(d) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, *subject to* $x_1 + x_2 + x_3 = 0$.
(e) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, *subject to* $x_1 + x_2 - x_3 = 0$.
(f) $Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$, *subject to* $x_1 + x_2 - x_3 + x_4 = 0$. $x_1 - 9x_2 + x_4 = 0$.

Homework

5, 6(g), 7, 8(e), 8(f).

Essay (optional, please type)

1. **Lagrangian reduction.** If all leading minors $|D_k|$ of the symmetric matrix of a quadratic form are nonzero then

$$Q(x_1, \ ... \ , x_3) = |D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \ ... \ + \frac{|D_n|}{|D_{n-1}|} l_n^2$$

where $l_1, \ ... \ , l_n$ are some linear combinations of $x_1, \ ... \ , x_n$

We have shown it for $n = 2$ in general case and for some particular example for $n = 3$ (see above).

(a) Give a general proof for $n = 3$.

(b) Deduce the criterion for definiteness from this reduction.

2. **Linear restriction of a quadratic form.** We have actually proved the criterion for definiteness of restricted quadratic form in terms of bordered matrix for $n = 2$ and $m = 1$ (see above). Do the same for $n = 3$ and $m = 1$.

3. **Definiteness of restricted form.** Show that if a quadratic form $Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ is positive (negative) definite then its restriction on a linear constraint set $Ax_1 + Bx_2 = 0$ is also positive (negative) definite. Hint: Calculate the determinant of bordered matrix. Or just substitute in Q the value $x_2 = -\frac{A}{B}x_1$. But if a form is indefinite then the restricted form can be whatever you wish (give examples).

Short Summary
Quadratic Forms

$$Q(x_1, x_2, ..., x_n) = \sum_{i,j}^{n} a_{ij}x_ix_j = (x_1, ..., x_n)\cdot\begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix} = x^T\cdot A\cdot x$$

A is symmetric. If not, take its symmetrization $A' = \frac{A+A^T}{2}$.

Definiteness of $Q(x)$:
(a) positive definite if $Q(x) > 0$ for all $x \neq 0 \in R^n$;
(b) positive semidefinite if $Q(x) \geq 0$ for all $x \neq 0 \in R^n$;
(c) negative definite if $Q(x) < 0$ for all $x \neq 0 \in R^n$;
(d) negative semidefinite if $Q(x) \leq 0$ for all $x \neq 0 \in R^n$;
(e) indefinite if $Q(x) > 0$ for some x and $Q(x) < 0$ for some other x.

Definiteness and Optimality
If Q is positive definite then $x = 0$ is global maximum;
If Q is negative definite then $x = 0$ is global minimum.

Leading principal minors

$$|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \; ... \; , \; |D_n| = |A|.$$

A quadratic form $Q(x)$ is:
Positive definite iff $|D_1| > 0, \quad |D_2| > 0, \quad ... \quad , |D_n| > 0$.
Negative definite iff $|D_1| < 0, \quad |D_2| > 0, \quad |D_3| < 0, \quad |D_4| > 0, \quad ...$.
Indefinite iff some nonzero D_k violates above sign patterns.

Positive semidefinite iff all principal minors $M_k \geq 0$.
Negative semidefinite iff all $M_{2k+1} \leq 0$ and $M_{2k} \geq 0$.

Definiteness of $Q(x) = x^T \cdot A \cdot x$ **on the constrained set** $B \cdot x = 0$**:**

	M_{m+m+1}	M_{m+m+2}	...	M_{m+n-1}	M_{m+n}
negative	$(-1)^{m+1}$	$(-1)^{m+2}$	...	$(-1)^{n-1}$	$(-1)^n$
positive	$(-1)^m$	$(-1)^m$	...	$(-1)^m$	$(-1)^m$

where $M_{2m+1}, ..., M_{m+n}$ are last $n - m$ minors of the bordered matrix

$$\left(\begin{array}{ccccccc} 0 & ... & 0 & | & B_{11} & ... & B_{1n} \\ & ... & & | & & ... & \\ 0 & ... & 0 & | & B_{m1} & ... & B_{mn} \\ - & - & - & & - & - & - \\ B_{11} & ... & B_{m1} & | & a_{11} & ... & a_{1n} \\ & ... & & | & & ... & \\ B_{1n} & ... & B_{mn} & | & a_{n1} & ... & a_{nn} \end{array}\right).$$