ISET MATH III Midterm

Problem 1. Find all values of k for which the 2 variable quadratic form $x^2 + 2kxy + y^2$ is
(a) positive definite; (b) negative definite; (c) positive semidefinite; (d) negative semidefinite; (e) indefinite. For each of your answer write corresponding matrix and verify definiteness using minors.

Answer
$pos.\ (-1,1),\ neg.\ \ no,\ pos.\ semi\ \ [-1,1],\ \ neg.\ semi\ no,$
$indef.\ (-\infty,-1)\bigcup(1,+\infty)$

Problem 2. Consider the following constrained quadratic

$$Q(x, y, z) = x^2 - z^2 + 4xy \quad subject \;\; to \quad 5x + k \cdot y - k \cdot z = 0.$$

(a) For warming up: Determine the definiteness of this constrained quadratic for $k = 1$.

Find all values of k for which the constrained quadratic is

(b) negative definite;

(c) positive definite;

(d) indefinite.

Answer

$$B = \begin{pmatrix} 0 & 5 & k & -k \\ 5 & 1 & 2 & 0 \\ k & 2 & 0 & 0 \\ -k & 0 & 0 & -1 \end{pmatrix}.$$

$$D_3 = -t^2 + 20t, \quad D_4 = 5t^2 - 20t.$$

(a) for $k = 1$ $D_3 = 19, \quad D_4 = -15$ so - negative definite.

(b) Our quadratic is negative definite if

$$\left\{ \begin{array}{l} D_3 > 0 \\ D_4 < 0 \end{array} \right. \left| \begin{array}{c} -t^2 + 20t > 0 \\ 5t^2 - 20t < 0 \end{array} \right| \begin{array}{c} 0 < k < 20 \\ 0 < k < 4 \end{array} \Bigg| \; 0 < k < 4.$$

(c) Our quadratic is positive definite if

$$\left\{ \begin{array}{l} D_3 < 0 \\ D_4 < 0 \end{array} \right. \left| \begin{array}{c} -t^2 + 20t < 0 \\ 5t^2 - 20t < 0 \end{array} \right| \begin{array}{c} k < 0, \quad 20 < k \\ 0 < k < 4 \end{array} \Bigg| \; \emptyset.$$

(d) Our quadratic is indefinite if $D4 > 0, \quad 5k^2 - 20k > 0, \quad k < 0 \; and \; 4 < k.$

Problem 3. Let $Q(x, y)$ be a 2-variable quadratic form and $Ax + By = 0$ be a linear constraint.

(i) Show that if $Q(x, y)$ is positive definite, then its restriction on any linear constraint $Ax + By = 0$ is also positive definite.

(ii) Show that for indefinite quadratic form $Q(x, y)$ the restriction on a linear constraint $Ax + By = 0$ can be (a) positive, (b) negative, (c) semipositive, (d) seminegative. Namely give an example of one indefinite quadratic form $Q(x, y)$ and indicate various linear constraints $Ax + By = 0$ which realize the possibilities (a), (b), (c), (d).

Answer

(i) Consider the problem

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = (x_1, x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to the linear constraint

$$Ax_1 + Bx_2 = 0.$$

Assume $Q > 0$, this means $a > 0, ac - b^2 > 0$, i.e. $ac > b^2$, thus $\sqrt{ac} > b$.

The definiteness of constrained form depends only on the sign if determinant of the (unique minor) bordered matrix

$$det\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = -(aB^2 - 2bAB + cA^2).$$

Now look:

$$aB^2 - 2bAB + cA^2 > aB^2 - 2\sqrt{ac}AB + cA^2 = (\sqrt{a}B - \sqrt{c}A)^2 > 0.$$

Thus $detH$ is negative, i.e. it has sign of $(-1)^1 = (-1)^m$, this gives the positivity.

(ii) Take, for example $Q(x, y) = x^2 - y^2$ (indefinite).

(a) Its restriction on the constraint $0 \cdot x + 1 \cdot y = 0$, i.e. on x-axes is $Q(x, 0) = x^2$ positive definite.

(b) The restriction on the constraint $1 \cdot x + 0 \cdot y = 0$, i.e. on the y-axes is $Q(0, y) = -y^2$ negative definite.

(c and d) The restriction on the constraint $1 \cdot x + y \cdot y = 0$, as well as on the constraint $1 \cdot x - y \cdot y = 0$ is $Q(0, y) = (x - y)(x + y) = 0$ identically zero, thus it is simultaneously positive and negative definite.

Problem 4. Formulate a criteria for positive definiteness and negative definiteness of a quadratic form with n variables and $n-1$ constraints.

Answer Here $m = n - 1$, so the following minors

$$M_{2m+1=2(n-1)+1=2n-1}, \ \dots \ , M_{m+n=n-1+n=2n-1},$$

are essential. So only the last minor $H = M_{2n-1}$. Thus, if the sign of $detH$ is $(-1)^n$ then the constrained form is negative definite, if the sign of $detH$ is $(-1)^{m=n-1}$ then positive definite.

Problem 5.

(i) Consider the curve (t^3, t) in R^2. (a) Plot this curve for $-1 \leq t \leq 1$; (b) Identify all the points (if any) where the tangent vector is vertical; (c) Identify all the points (if any) where the tangent vector is horizontal.

(ii) Let $F : R^3 \to R$ be a function given by $F(x, y, z) = xy + yz + xz$ and $\phi : R \to R^3$ be a curve given by $\phi(t) = (x(t) = t^2, y(t) = 1 - t^2, z(t) = 1 - t)$. Calculate $\frac{dF}{dt}$.

Answer
(i) Vertical at $t = 0$, horizontal - no.
(ii) $2t - 4t^3 - 1$.

Problem 6.

(i) Let $F : R^2 \to R^2$ be the function given by $F(x, y) = \begin{pmatrix} x^2y + y^2 \\ y^3x \end{pmatrix}$. Find the value of Jacobian DF at $(1, 2)$.

(ii) Let $f(x, y) = \sqrt{x \cdot y}$. Use gradient to approximate $f(1.01, 3.99)$.

(iii) Consider the function $f(x, y) = y^2e^{3x}$. In which direction should one move from the point $(0, 3)$ to increase most rapidly?

Answer
(i) $\begin{pmatrix} 4 & 5 \\ 8 & 12 \end{pmatrix}$
(ii) 0.0075.
(iii) $Df(x, y) = (3y^2e^{3x}, 2ye^{3x}),\ Df(0, 3) = (27, 6)$.

Problem 7. Sketch the level curves $f(x, y) = 1$ for the following functions

(a) $f(x, y) = max(x, y)$
(b) $f(x, y) = min(x, y)$
(c) $f(x, y) = x \cdot y + 1$
(d) $f(x, y) = x^2 - y^2 + 1$
(e) $f(x, y) = xy - x - y + 2.$

Problem 8. A bat named Bob moves along a path such that his position at time t is $(2t, t^2, 1 + t^2)$ from $t = 0$ until time $t = 2$. At time $t = 2$, he leaves this pat and flies along the tangent line to this path, maintaining the speed he had at time $t = 2$. What will Bob's position be at time $t = 5$?

Answer

Position after t seconds $x(t) = (2t, t^2, 1 + t^2)$.

Position after 2 seconds $x(2) = (4, 4, 5)$.

Speed at this point $x'(2) = (2, 2t, 2t)|_{t=2} = (2, 4, 4)$.

Position after s seconds of free moving $g(s) = (4, 4, 5) + s(2, 4, 4)$.

Position after 3 seconds of free moving $g(3) = (4, 4, 5) + 3(2, 4, 4) = (4, 4, 5) + (6, 12, 12) = (10, 16, 17)$.