1 Implicit Functions

Reading [Simon], Chapter 15, p. 334-360.

1.1 Examples

So far we were dealing with *explicitly* given functions

$$y = f(x_1, ..., x_n),$$

like $y = x^2$ or $y = x_1^2 x_2^3$.

But frequently the dependence of *endogenous* variable y on *exogenous* variables $(x_1, ..., x_n)$ can be given in a form

$$G(x_1, ..., x_n, y) = c.$$

If for each $(x_1, ..., x_n)$ this equation determines a corresponding value of y, we say that the endogenous variable y is an *implicit function* of exogenous variables $(x_1, ..., x_n)$.

Example. Suppose $G(x, y) = 4x + 2y - 5$. Then the equation

$$4x + 2y - 5 = 0$$

expresses y as an implicit function of x. This implicit function can be written explicitly as

$$y = 2.5 - 2x.$$

Example. Suppose $G(x, y) = xy^2 - 3y - e^x$. Then the equation

$$xy^2 - 3y - e^x = 0$$

yields an explicit function

$$y = \frac{1}{2x}(3 + \sqrt{9 + 4xe^x}).$$

By the way, there is another one

$$y = \frac{1}{2x}(3 + \sqrt{9 - 4xe^x}).$$

Example. Suppose $G(x, y) = y^5 - 5xy + 4x^2$. Then the equation

$$y^5 - 5xy + 4x^2 = 0$$

yields an implicit function $y = y(x)$ which can not be written in a form of explicit formula because there is no "general formula" for equations of order 5.

Example. Suppose $G(x, y) = x^2 + y^2 - 1$. The corresponding equation

$$x^2 + y^2 - 1 = 0$$

determines, as we know, the unit circle.

It is evident that

$$(x_1 = 0,\ y_1 = 1),\quad (x_2 = 0,\ y_2 = -1),\ (x_3 = 1,\ y_3 = 0),\quad (x_4 = -1,\ y_4 = 0)$$

all lay on this circle, that is all four are particular solutions of this equations.

1. Around $(x_1 = 0,\ y_1 = 1)$ this equation determines the explicit function

$$y = \sqrt{1 - x^2},$$

whose domain can be enlarged to $x \in (-1, 1)$.

2. Around $(x_2 = 0,\ y_2 = -1)$ this equation determines the explicit function

$$y = -\sqrt{1 - x^2},$$

whose domain can be enlarged to $x \in (-1, 1)$.

3. Around $(x_3 = 1,\ y_3 = 0)$ this equation determines the explicit function

$$x = \sqrt{1 - y^2},$$

whose domain can be enlarged to $y \in (-1, 1)$.

4. Around $(x_4 = 1,\ y_4 = 0)$ this equation determines the explicit function

$$x = -\sqrt{1 - y^2},$$

whose domain can be enlarged to $y \in (-1, 1)$.

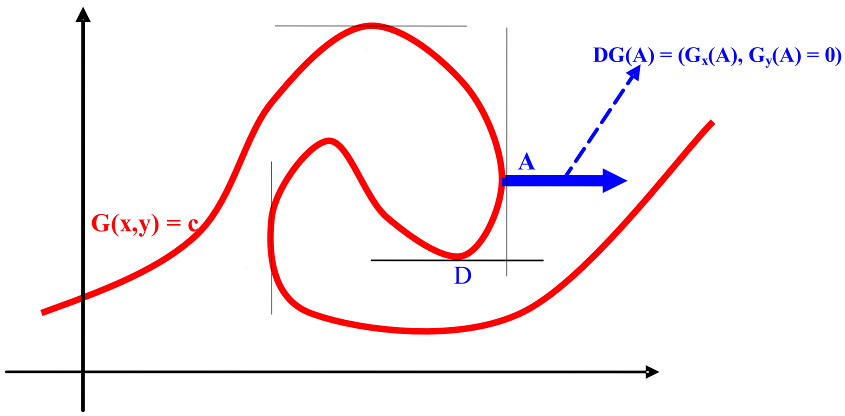
1.2 Implicit Function Theorem for R^2

So our question is: Suppose a function $G(x, y)$ is given. Consider the equation $G(x_0, y_0) = c$.

Does there exists a function $y = y(x)$ defined on some interval $(x_0 - \epsilon, x_0 + \epsilon)$ such that $G(x, y(x)) \equiv c$?
Or

Does there exists a function $x = x(y)$ defined in an interval $(y_0 - \epsilon, y_0 + \epsilon)$ such that $G(x(y), y) \equiv c$?

Such a function $y = y(x)$ (or $x = x(y)$) is called *implicit function* defined by the equation $G(x, y) = c$ around the point (*solution*) (x_0, y_0). The graph of implicit function must be a locus the level curve $G(x, y) = c$.



This picture shows that $y(x)$ *does not exist* around the point A of the level curve $G(x,y) = c$ (note that $x = x(y)$ does not exist around D).

Why so? What is wrong, with A? Because around A the level curve $G(x,y) = c$ can not pass the *vertical line test* (the *horizontal line test* for D). Note that the *tangent line* at A is *vertical*, and this means that the *gradient* at A is *horizontal*, and this means that $G_y(A) = 0$. This is wrong with A!

And what is wrong with D?

Theorem 1 *Suppose a point* $(x^*, y^*) \in R^2$ *is a particular solution of* $G(x^*, y^*) = c$ *and* $\frac{\partial G}{\partial y}(x^*, y^*) \neq 0$. *Then the equation*

$$G(x,y) = c$$

determines the function $y = y(x)$ *defined on some interval* $I = (x^* - \epsilon, x^* + \epsilon)$ *about the point* x^* *such that*

(a) $G(x, y(x)) \equiv c$ *for all* $x \in I$*;*

(b) $y(x^*) = y^*$*;*

(c) $y'(x^*) = -\frac{\frac{\partial G}{\partial x}(x^*,y^*)}{\frac{\partial G}{\partial y}(x^*,y^*)}$.

Proof. We prove the implication (a), (b) $\Rightarrow$ (c). So, suppose we have $y(x)$ such that $y(x^*) = y^*$ and $G(x, y(x)) = c$ for all $x \in I$. Differentiating $G(x, y(x)) = c$ with respect to x at x^* we obtain

$$\frac{\partial G}{\partial x}(x^*, y(x^*)) \cdot \frac{dx}{dx} + \frac{\partial G}{\partial y}(x^*, y(x^*)) \cdot \frac{dy}{dx}(x^*) = 0,$$

or

$$\frac{\partial G}{\partial x}(x^*, y^*) + \frac{\partial G}{\partial y}(x^*, y^*) \cdot y'(x^*) = 0,$$

and this gives (c).

In fact this Theorem states that may be y can not be solved as an function of x *explicitly* but it is possible to find the derivative $y'(x^*)$

and *approximate* the value of the unknown function $y(x)$ around small interval of x^*:

$$y(x^* + \Delta x) = y(x^*) + y'(x^*) \cdot \Delta x = y(x^*) - \frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)} \cdot \Delta x.$$

Example. Consider the equation

$$G(x, y) = x^2 - 3xy + y^3 - 7 = 0,$$

one solution of this equation is $(x^* = 4, y^* = 3)$ (check this!). Estimate the solution corresponding to $x_1 = 4.3$.
Solution. Since

$$\frac{\partial G}{\partial y}(4, 3) = (-3x + 3y^2)|_{(4,3)} = -12 + 27 = 15 \neq 0$$

the Theorem tells us that there exists a solution $y(x)$ around $x^* = 4$ s.t. $y(4) = 3$ and

$$y'(4) = -\frac{\frac{\partial G}{\partial x}(4, 3))}{\frac{\partial G}{\partial y}(4, 3))} = \frac{1}{15}.$$

Then, by linear approximation we obtain

$$y_1 = y(x_1) \approx y^* + y'(x^*) \cdot \Delta x = 3 + \frac{1}{15} \cdot 0.3 = 3.02.$$

By the way, $G(4.3, 3.2) = 0.0756 \neq 0$, that is our point but our point$(x_1, y_1) = (4.3, 3.2)$ does not lay exactly on the level curve $G(x, y) = 0$, but $y(4.3) \approx 3.2$ is just an approximation!

1.3 Implicit Function Theorem for Several variables

Theorem 2 *Suppose a point* $(x_1^*, ..., x_k^*, y^*) \in R^{k+1}$ *is a particular solution of*

$$G(x_1^*, ..., x_k^*, y^*) = c$$

and $\frac{\partial G}{\partial y}(x_1^*, ..., x_k^*, y^*) \neq 0$. *Then the equation*

$$G(x_1, ..., x_k, y) = c$$

determines the function $y = y(x_1, ..., x_k)$ *defined on an open ball* $B_\epsilon(x^*)$ *about the point* $x^* = (x_1^*, ..., x_k^*)$ *such that*

(a) $G(x_1, ..., x_k, y(x_1, ..., x_k)) \equiv c$ *for all* $x \in B_\epsilon x^*$;
(b) $y(x_1^*, ..., x_k^*) = y^*$;
(c) $\frac{\partial y}{\partial x_i} = -\frac{\frac{\partial G}{\partial x_i}(x_1^*, ..., x_k^*, y^*)}{\frac{\partial G}{\partial y}(x_1^*, ..., x_k^*, y^*)}$.

Proof. We prove the implication (a), (b) $\Rightarrow$ (c). So, suppose we have $y(x_1, ..., x_k)$ such that $G(x_1, ..., x_k, y(x_1, ..., x_k)) \equiv c$ for all $x \in B_\epsilon x^*$ and $y(x_1^*, ..., x_k^*) = y^*$. Differentiating $G(x_1, ..., x_k, y) = c$ with respect to x_i at x^* we obtain

$$\frac{\partial G}{\partial x_i}(x^*, y(x^*)) \cdot + \frac{\partial G}{\partial y}(x^*, y(x^*)) \cdot \frac{\partial y}{\partial x_i}(x^*) = 0,$$

solving from this equation $\frac{\partial y}{\partial x_i}(x^*)$ we obtain (c).

Example. Consider the equation

$$G(x, y, z) = x^3 + 3y^2 + 4xz^2 - 3z^2 y - 1 = 0.$$

Does this equation define $z = z(x, y)$ as a function of x and y
(a) in a neighborhood of $A = (x = 1,\ y = 1)$?
(b) in a neighborhood of $B = (x = 1,\ y = 0)$?
(c) in a neighborhood of $C = (x = 0.5,\ y = 0)$?

If so, compute z_x and z_y at this point. Besides, if x increases to 0.6 and y decreases to -0.2, estimate the corresponding change in z.

Solution. The partial derivative G_z is

$$G_z(x, y, z) = 8xz - 6yz.$$

(a) At point $A = (1, 1)$**:** the given equation in this case looks as

$$1 + 3 + 4z^2 - 3z^2 - 1 = 0,\ 3 + z^2 = 0,$$

so no solution (x_0, y_0, z_0) in this case.

(b) At point $B = (1, 0)$**:** the given equation in this case looks as

$$1 + 0 + 4z^2 - 0 - 1 = 0,\ z^2 = 0,\ z = 0,$$

so the solution in this case is $(x_0, y_0, z_0) = (1, 0, 0)$. But what about the implicit function $z = z(x, y)$ around B?

Let as check $G_z = \frac{\partial(x^3+3y^2+4xz^2-3z^2y-1)}{\partial z} = 8xz - 6yz$ at this point:

$$G_z(1, 0, 0) = (8xz - 6yz)|_{1,0,0)} = 0,$$

so the Implicit Function Theorem does not work at B.

(c) At point $C = (0.5, 0)$**:** the given equation in this case looks as

$$0.5^3 + 0 + 2z^2 - 0 - 1 = 0,\ 2z^2 = 1 - 0.5^3,\quad z^2 = 0.4375000000,$$
$$z = 0.6614378278, z' = -0.6614378278,$$

so we have two solutions in this case is $(x_0, y_0, z_0) = (0.5, 0, 0.6614378278)$ and $(x_0, y_0, z_0) = (0.5, 0, -0.6614378278)$.

Let as check $G_z = 8xz - 6yz$

$$G_z(0.5, 0, 0.6614378278) = 2.645751311 \neq 0,$$
$$G_z(0.5, 0, -0.6614378278) = -2.645751311 \neq 0,$$

so it both cases - it does.

Now we compute z_x and z_y at this point:

$$z_x(0.5, 0, 0.6614378278) = -\frac{G_x}{G_z}(0.5, 0, 0.6614378278),$$
$$-\frac{G_x(0.5,0,0.6614378278)}{G_z(0.5,0,0.6614378278)} = -\frac{2.500000000}{-2.645751311} = 0.9449111825.$$

and

$$z_y(0.5, 0, 0.6614378278) = -\frac{G_y}{G_z}(0.5, 0, 0.6614378278),$$
$$-\frac{G_y(0.5,0,0.6614378278)}{G_z(0.5,0,0.6614378278)} = -\frac{-2.500000000}{-1.312500000} = -0.4960783708.$$

Now we are ready to estimate the corresponding change in z if x increases to 0.6 and y decreases to -0.2:

$$\Delta z = z_x \cdot \Delta x + z_y \cdot \Delta y =$$
$$0.944911182 \cdot 0.1 + (-0.4960783708) \cdot (-0.2) = 0.1937067924.$$

Similarly for another solution $(0.5, 0, -0.6614378278)$.

Phhhhh!

1.4 Regular Points

For a given smooth enough function $G(x, y)$ the equation $G(x, y) = c$ defines the smooth curve, the level curve. Suppose a point (x^*, y^*) lays on this curve, i.e. is a solution of this equation.

1. If for (x^*, y^*) one has
$$\frac{\partial G}{\partial y}(x^*, y^*) \neq 0$$
then the locus of level curve $G(x, y) = c$ around (x^*, y^*) can be thought of as the graph of a function $y = y(x)$, and the slope of this curve is
$$-\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)}.$$

2. If for (x^*, y^*) one has
$$\frac{\partial G}{\partial x}(x^*, y^*) \neq 0$$
then the locus of level curve $G(x, y) = c$ around (x^*, y^*) can be thought of as the graph of a function $x = x(y)$, and the slope of this curve is
$$-\frac{\frac{\partial G}{\partial y}(x^*, y^*)}{\frac{\partial G}{\partial x}}(x^*, y^*).$$

3. If for (x^*, y^*) one has

$$\frac{\partial G}{\partial y}(x^*, y^*) \neq 0$$

and

$$\frac{\partial G}{\partial x}(x^*, y^*) \neq 0$$

then the locus of level curve $G(x, y) = c$ around (x^*, y^*) can be thought of as the graph of a function $y = y(x)$ and $y = y(x)$, and the slope of this curve at (x^*, y^*) with respect to x axis is:

$$-\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)}$$

and the slope of this curve with respect to y axis is:

$$-\frac{\frac{\partial G}{\partial y}(x^*, y^*)}{\frac{\partial G}{\partial x}(x^*, y^*)}.$$

(What? We have two different slopes for one curve? Explain).

Definition 1 *A point (x^*, y^*) is called* ***regular point*** *of $G(x, y)$ if*

$$\frac{\partial G}{\partial x}(x^*, y^*) \neq 0, \quad or \quad \frac{\partial G}{\partial y}(x^*, y^*) \neq 0.$$

If every point of $G(x, y) = c$ is regular, then the level set $G(x, y) = c$ is called a **regular curve**.

So the Implicit Function Theorem states that at each point of **regular curve** we can consider y as a function of x or x as a function of y.

Example. Consider $G(x, y) = x^2$, you know the graph of this function. Its level curve $G(x, y) = 0$ is just the y-axes: $G(x, y) = x^2 = 0 \Rightarrow \ x = 0$. Each point of this level curve $(0, y)$ is irregular: $G_x(0, y) = (2x)|_{0,y)} = 0$, $G_y(0, y) = 0|_{(0,y)} = 0$ (nevertheless this curve determines implicit function $x(y) = 0$).

1.5 Tangent of the Level Curve

Theorem 3 *The tangent vector to the level curve $G(x^*, y^*) = c$ at a regular point (x^*, y^*) is*

$$(G_y(x^*, y^*), -G_x(x^*, y^*)).$$

Proof. Recall that the tangent vector of a curve $(x(t), y(t))$ is $(x'(t), y'(t))$.

Suppose $G(x^*, y^*) = c$ and $G_y(x^*, y^*) \neq 0$, then there exists implicit function $y = y(x)$ around x^*, i.e. $G(x, y(x)) = c$. Thus this level curve is given by $(x, y(x))$ locally around x^*. Then its tangent vector at (x^*, y^*) is given by

$$(x', y'(x)) = (1, -\frac{G_x(x^*, y^*)}{G_y(x^*, y^*)}),$$

this vector is parallel to $(G_y(x^*, y^*), -G_x(x^*, y^*))$.

Corollary 1 *At a regular point the gradient is orthogonal to level curve.*

Proof.

$$\nabla G(x^*, y^*) \cdot (G_y(x^*, y^*), -G_x(x^*, y^*) =$$
$$(G_x(x^*, y^*), G_y(x^*, y^*) \cdot (G_y(x^*, y^*), -G_x(x^*, y^*) =$$
$$G_x(x^*, y^*) \cdot G_y(x^*, y^*) - G_y(x^*, y^*) \cdot G_x(x^*, y^*) = 0.$$

Important Example. Let $F(x, y) = x^2 + y^2$ and $G(x, y) = x \cdot y$. Find a point (x^*, y^*) on the level curve $G(x, y) = 1$ and c such that the curves $G(x, y) = 1$ and $F(x, y) = c$ touch each other at the point (x^*, y^*).

Solution. The slope of tangent line of $G(x, y) = 1$ at (x^*, y^*) is

$$-\frac{G_x(x^*, y^*)}{G_y(x^*, y^*)}$$

and the slope of tangent line of $F(x, y) = c$ at (x^*, y^*) is

$$-\frac{F_x(x^*, y^*)}{F_y(x^*, y^*)}.$$

So we can find (x^*, y^*) from the system of equations

$$\begin{cases} \frac{G_x(x^*, y^*)}{G_y(x^*, y^*)} = \frac{F_x(x^*, y^*)}{F_y(x^*, y^*)} \\ \\ G(x^*, y^*) = 1 \end{cases}$$

which in our case looks as

$$\begin{cases} \frac{y^*}{x^*} = \frac{2x^*}{2y^*} \\ \\ x^* \cdot y^* = 1 \end{cases}.$$

The solution gives $(x^* = 1, y^* = 1)$ and $(x^* = -1, y^* = 1)$. In both cases

$$c = F(1, 1) = F(-1, -1) = 2.$$

1.6 Implicit Function Theorem and Marginal Rate of Substitution

Suppose $G(x^*, y^*) = c$, and let us step from x^* to $x^* + \Delta x$. How shall we change y^* in order to stay on the same level set? That is for which Δy we shall have $G(x^* + \Delta x, y^* + \Delta y) = c$?

In other words, what combinations of linear movements Δx and Δy from (x^*, y^*) lead to *no change* in G?

If we have the locus of level curve $G(x, y) = c$ explicitly as $y = y(x)$ the, it's clear, $\Delta y = y(x^* + \Delta x) - y^*$.

But what if we have not explicit $y(x)$?

By linear approximation

$$G(x^* + \Delta x, y^* + \Delta y) - G(x^*, y^*) \approx \frac{\partial G}{\partial x}(x^*, y^*)\Delta x + \frac{\partial G}{\partial y}(x^*, y^*))\Delta y.$$

So for *no change*, i.e. for $G(x^* + \Delta x, y^* + \Delta y) - G(x^*, y^*) \approx 0$ we need

$$\frac{\partial G}{\partial x}(x^*, y^*)\Delta x + \frac{\partial G}{\partial y}(x^*, y^*)\Delta y \approx 0.$$

Thus

$$\frac{\Delta y}{\Delta x} \approx -\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)},$$

so the direction of *no change* of G at (x^*, y^*) is the direction of tangent line of the level curve. Besides, we can answer the above question: if we change x^* by Δx what should be the change Δy of y^* in order to keep

$$G(x^* + \Delta x, y^* + \Delta y) = G(x^*, y^*)?$$

As we see we have

$$\Delta y \approx -\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)}\Delta x.$$

In economics the slope of the tangent line to the level curve $G(x, y) = c$ at (x^*, y^*) is called *marginal rate of substitution MRS*, it is given by the derivative of implicit function $y = y(x)$

$$MRS = -\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)}.$$

MRS measures, in a marginal sense, how much should one increase y to compensate the loss of one unit of x to keep the same level of G.

That is it, but if you need, here is

Economical Interpretation. This use of the Implicit Function Theorem is the natural approach when studying the slope of an indifference curve of a utility function and the slope of an isoquant of a production function, since in these situations we are interested in which directions to move to keep the function constant.

The level curve of a utility function $U(x, y)$ is called an indifference curve of U.

Its slope at (x^*, y^*) is called the marginal rate of substitution (MRS) of U at (x^*, y^*) since it measures, in a marginal sense, how much more of good y the consumer would require to compensate for the loss of one unit of good x to keep the same level of satisfaction.

By the Implicit Function Theorem, the MRS at (x^*, y^*) is:

$$-\frac{\frac{\partial U}{\partial x}(x^*, y^*)}{\frac{\partial U}{\partial y}(x^*, y^*)}$$

Similarly, if $Q = F(K, L)$ is a production function, its level curves are called isoquants and the slope - F_K/F_L of an isoquant at (Ko, Lo) is called the marginal rate of technical substitution (MRTS). It measures how much of one input would be needed to compensate for a one-unit loss of the other unit while keeping production at the same level.

Example. Consider a function $f(x, y) = x^2 e^y$.

(a) What is the slope of the level set at $x = 2,\ y = 0$?

(b) In what direction should one move from the point $(2, 0)$ in order to increase f most quickly? Express your answer as a vector of length 1.

(c) Suppose at $(2, 0)$ the variable x is changed to 2.5. Estimate the corresponding change of $y = 0$ which substitutes this change of x, that is the output remains the same.

Solution.

$$\frac{\partial f}{\partial x}(x, y) = 2xe^y, \quad \frac{\partial f}{\partial y}(x, y) = x^2 e^y,$$

thus

$$\frac{\partial f}{\partial x}(2, 0) = 4, \quad \frac{\partial f}{\partial y}(2, 0) = 4.$$

(a) So the slope of the level curve at $(2, 0)$ is -1.

(b) The function f increases most rapidly in the direction of gradient

$$\nabla f(2, 0) = (4, 4).$$

The suitable vector of the length 1 is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(c) $f(2 + \Delta x, 0 + \Delta y) - f(2, 0) \approx \frac{\partial f}{\partial x}(2, 0) \cdot \Delta x + \frac{\partial f}{\partial y}(2, 0) \cdot \Delta y$
$= 4 \cdot 0.5 + 4 \cdot \Delta y = 0,\ \Delta y = -0.5$.

The same using MRS:

$$\frac{\Delta y}{\Delta x} \approx MRS = -\frac{f_x(2, 0)}{f_y(2, 0)} = -\frac{4}{4} = -1,$$

thus $\Delta y = MRS \cdot \Delta x = -1 \cdot \Delta x = -0.5$

1.7 System of Implicit Functions

First warming up. Start with a system of linear equations

$$\left\{\begin{array}{ccccccccccccc} a_{11}x_1 & + & ... & + & a_{1n}x_n & + & b_{11}y_1 & + & ... & + & b_{1m}y_m & = & c_1 \\ a_{21}x_1 & + & ... & + & a_{2n}x_n & + & b_{21}y_1 & + & ... & + & b_{2m}y_m & = & c_2 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{m1}x_1 & + & ... & + & a_{mn}x_n & + & b_{m1}y_1 & + & ... & + & b_{mm}y_m & = & c_m \end{array}\right.$$

Is it possible to express the (endogenous) variables $y_1, y_2, \ ... \ , y_m$ in terms of (exogenous) variables $x_1, x_2, \ ... \ x_n$? Answer is "yes" if

$$det \begin{pmatrix} b_{11} & ... & b_{1m} \\ ... & ... & ... \\ b_{m1} & ... & b_{mm} \end{pmatrix} \neq 0.$$

Now turn to general problem. Suppose m functions $F_i(x_1, ..., x_n, y_1, ..., y_m), \ i = 1, ..., m,$ (i.e. $F_i : R^{n+m} \to R, \ i = 1, ..., m$) are given. We consider a system of equations

$$\left\{\begin{array}{l} F_1(x_1, ..., x_n, y_1, ..., y_m) = c_1 \\ .... \\ F_m(x_1, ..., x_n, y_1, ..., y_m) = c_m \end{array}\right. , \qquad (1)$$

and suppose a point $(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*) \in R^{n+m}$ is a solution.

(a) Does there exist functions $y_1(x_1, ..., x_n), \ ... \ , y_m(x_1, ..., x_n)$ in some neighborhood of $x^* = (x_1^*, ..., x_n^*)$ such that

$$F_i(x_1, ..., x_n, y_1(x_1, ..., x_n), ..., y_m(x_1, ..., x_n)) \equiv c_i, \ i = 1, ..., m,$$

and $y_i(x_1^*, ..., x_n^*) = y_i^*, \ i = 1, ..., m$?

(b) How to compute partial derivatives $\frac{\partial y_i}{\partial x_j}(x_1^*, ..., x_n^*)$?

Theorem 4 *If the determinant of Jacobian matrix*

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & ... & \frac{\partial F_1}{\partial y_m} \\ ... & ... & ... \\ \frac{\partial F_m}{\partial y_1} & ... & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$$

evaluated at $(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*)$ *is nonzero, then there exist functions*

$$y_1(x_1, ..., x_n), \ ... \ , y_m(x_1, ..., x_n)$$

defined on a ball about $(x_1^*, ..., x_n^*)$ *satisfying the conditions*

$$F_i(x_1, ..., x_n, y_1(x_1, ..., x_n), ..., y_m(x_1, ..., x_n)) \equiv c_i, \ i = 1, ..., m,$$

and $y_i(x_1^*, ..., x_n^*) = y_i^*, \ i = 1, ..., m.$

Furthermore, the derivatives $\frac{\partial y_i}{\partial x_k}$ *can be solved from the system of linear equations*

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial y_1}{\partial x_k} \\ \cdots \\ \frac{\partial y_m}{\partial x_k} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x_k} \\ \cdots \\ \frac{\partial F_m}{\partial x_k} \end{pmatrix}. \qquad (2)$$

The solution of this system can be computed by

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_k} \\ \cdots \\ \frac{\partial y_m}{\partial x_k} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial F_1}{\partial x_k} \\ \cdots \\ \frac{\partial F_m}{\partial x_k} \end{pmatrix}$$

or by Cramer's rule (Here all the matrices evaluated at $(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*)$).

Sketch of Proof. Let us differentiate our system (1) by x_k:

$$\begin{cases} \frac{\partial F_1}{\partial x_k} + \frac{\partial F_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_k} + \cdots + \frac{\partial F_1}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_k} = 0 \\ \frac{\partial F_2}{\partial x_k} + \frac{\partial F_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_k} + \cdots + \frac{\partial F_2}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_k} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \frac{\partial F_m}{\partial x_k} + \frac{\partial F_m}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_k} + \cdots + \frac{\partial F_m}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_k} = 0 \end{cases}$$

and this system is exactly (2).

Example. One solution of the system

$$\begin{aligned} x^3 y - z &= 1 \\ x + y^2 + z^3 &= 6 \end{aligned}$$

is $(x = 1, y = 2, z = 1)$. Estimate the corresponding x and y when $z = 1.1$.

Solution. Take $F_1(x, y, z) = x^3 y - z,\ F_2(x, y, z) = x + y^2 + z^3$.

Evaluating the whole Jacobian at $(x = 1, y = 2, z = 1)$ we obtain

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 3x^2 y & x^3 & -1 \\ 1 & 2y & 3z^2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & -1 \\ 1 & 4 & 1 \end{pmatrix}.$$

The determinant

$$\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 6 & 1 \\ 1 & 4 \end{vmatrix} = 23 \neq 0,$$

so the Implicit Function Theorem asserts the existence of a solution $x(z)$ and $y(z)$ as functions of exogenous variable z.

Now calculate derivatives $x'(1) = x_z(1)$ and $y'(1) = y_z(1)$:

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_1}{\partial z} \\ -\frac{\partial F_2}{\partial z} \end{pmatrix},$$

evaluating we obtain a system of linear equations

$$\begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$

and the solution gives $x'(1) = \frac{7}{23}$, $y'(1) = \frac{-19}{23}$.

Now we are ready to estimate $x(1.1)$ and $y(1.1)$ by linear approximation:

$$x(1.1) \approx x(1) + x'(1) \cdot 0.1 = 1 + \frac{7}{23}0.1 = 1.03,$$

$$y(1.1) \approx y(1) + y'(1) \cdot 0.1 = 2 + \frac{-19}{23}0.1 = 1.91.$$

Finally we obtain the estimation of new solution $(x = 1.03, y = 1.91, z = 1.1)$.

Example. For the system

$$xz^3 + y^2v^4 = 2, \quad xz + yvz^2 = 2$$

$(x = 1, y = 1, z = 1, v = 1)$ is a solution.

(a) Can you estimate a new solution which correspond to $y = 1.1$ and $v = 1.2$?

(b) Can you estimate a new solution which correspond to $x = 1.1$ and $y = 1.2$?

Solution. Take $F_1(x, y, z) = xz^3 + y^2v^4$, $F_2(x, y, z) = xz + yvz^2$.

Evaluating the whole Jacobian at $(x = 1, y = 1, z = 1, v = 1)$ we obtain

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v} \end{pmatrix}_{(1,1,1,1)} = \begin{pmatrix} z^3 & 2yv^4 & 3xz^2 & 4y^2v^3 \\ z & vz^2 & x + 2yvz & yz^2 \end{pmatrix}_{(1,1,1,1)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix}.$$

(a) The determinant

$$\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix}_{(1,1,1,1)} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0,$$

so the Implicit Function Theorem does not alow to express x and z as functions of Y and v.

(b) The determinant

$$\begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v} \end{vmatrix}_{(1,1,1,1)} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9 \neq 0,$$

so the Implicit Function Theorem asserts the existence of solutions $z(x, y)$ and $v(x, y)$ as functions of exogenous variables x and y.

Now we calculate derivatives z_x, z_y, v_x, v_y at $(1, 1, 1, 1)$.

For z_x and v_x we have the system

$$\begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} z_x(1,1) \\ v_x(1,1) \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_1}{\partial x} \\ -\frac{\partial F_2}{\partial x} \end{pmatrix},$$

evaluating we obtain a system of linear equations

$$\begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_x(1,1) \\ v_x(1,1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

and the solution gives $z_x(1,1) = \frac{-1}{3}, \;\; v_x(1,1) = 0.$

Similarly, for z_y and v_y we have the system

$$\begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} z_y(1,1) \\ v_y(1,1) \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_1}{\partial y} \\ -\frac{\partial F_2}{\partial y} \end{pmatrix},$$

evaluating we obtain a system of linear equations

$$\begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_y(1,1) \\ v_y(1,1) \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$

and the solution gives $z_y(1,1) = \frac{-2}{9}, \;\; v_x(1,1) = -1.$

2 Inverse Function*

Suppose $f : X \to Y$ is a function from X to Y. This function is called *invertible* if there exists the *inverse* function $g : Y \to X$ such that

$$g(f(x)) = x \quad and \quad f(g(y)) = y$$

for each $x \in X$ and $y \in Y$. In other words the composition $g \circ f$ coincides with the identity map $id_X : X \to X$ and the composition $f \circ g$ coincides with the identity map $id_Y : Y \to Y$.

Examples

1. The map $f : R \to R$ given by $f(x) = 2x + 4$ is invertible and it's inverse is the map $g : R \to R$ given by $g(y) = 0.5y - 2$. Indeed,

$$g(f(x)) = g(2x + 4) = 0.5(2x + 4) - 2 = x,$$

and

$$f(g(y)) = f(0.5y - 2) = 2(0.5y - 2) + 4 = y.$$

2. The map $f : R \to R_+ = (0, +\infty)$ given by $f(x) = e^x$ is invertible and it's inverse is the map $g : R_+ \to R$ given by $g(y) = ln\ y$. Indeed,

$$g(f(x)) = g(e^x) = ln\ e^x = x, \quad and \quad f(g(y)) = f(ln\ y) = e^{ln\ y} = y.$$

A map $f : X \to Y$ is called *surjective* if for each $y \in Y$ there exists $x \in X$ s.t. $f(x) = y$.

A map $f : X \to Y$ is called *injective* if for $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

A map $f : X \to Y$ is called *bijective* if it is surjective and injective simultaneously.

Let us interpret these notions in terms of equation

$$f(x) = y$$

where x is considered as an unknown.

A map $f : X \to Y$ is surjective iff the equation $f(x) = y$ has a solution for each $y \in Y$.

A map $f : X \to Y$ is injective iff the equation $f(x) = y$ has either no solution or unique solution for each $y \in Y$.

A map $f : X \to Y$ is bijective iff the equation $f(x) = y$ has unique solution for each $y \in Y$.

Theorem 5 *A map $f : X \to Y$ is invertible if and only if it is a bijection.*

2.1 Invertible Linear maps

A linear map $F : R^n \to R^n$ in fact is given by a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

and $F(v) = A \cdot v$ for each vector $v \in R^n$.

Theorem 6 *A linear map $F : R^n \to R^n$ is invertible if and only if it's matrix A is nondegenerate, that is det $A \neq 0$. In this case the inverse map $G : R^n \to R^n$ is given by $G(w) = A^{-1} \cdot w$.*

2.2 Inverse Function Theorem

Suppose now $F : R^n \to R^n$ is function. As we know such a function is a collection of functions

$$\begin{array}{c} f_1 : R^n \to R \\ f_2 : R^n \to R \\ ... \\ f_n : R^n \to R, \end{array}$$

so that

$$F(x_1, ..., x_n) = (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n), ..., f_n(x_1, ..., x_n)).$$

The matrix which consists of partial derivatives of these functions

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} \\ ... & ... & ... \\ \frac{\partial f_n}{\partial x_1} & ... & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is called *Jacobian* of F. By $DF(x^*)$ is denoted the *numerical* matrix obtained by evaluation of Jacobian DF at a vector $x^* = (x_1^*, ..., x_n^*)$.

Theorem 7 *Suppose $F(x^*) = y^*$ and the Jacobian $DF(x^*)$ is nondegenerate matrix, then there exists an open ball $B_r(x^*)$ about x and an open set $U \subset R^n$ about y^* such that the restriction of the map F*

$$F : B_r(x^*) \to U$$

is invertible (this is called **locally invertible***). Furthermore, the jacobian of inverse map*

$$G = F^{-1} : U \to B_r(x^*)$$

is the inverse matrix of the Jacobian of F:

$$DG(y^*) = (DF(x^*))^{-1}.$$

Example. Consider the function $F : R^2 \to R^2$ given by

$$F(x, y) = (x^2 - y^2, 2xy).$$

It' Jacobian looks as

$$DF = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

and its determinant is $detDF(x, y) = 4(x^2 + y^2)$. By the Inverse Function Theorem, F is locally invertible at every point except $(0, 0)$.

Example. Show that the map $F(x, y) = (x + e^y, y + e^{-x})$ is everywhere locally invertible.

Solution. The Jacobian looks as

$$DF = \begin{pmatrix} 1 & e^y \\ -e^{-x} & 1 \end{pmatrix}$$

so it's determinant is $1 + e^{y-x}$. This expression is nonzero at any (x, y).

Exercises

1. Consider the equation $x^3 + 3y^2 + 4xz^2 - 3z^2y = 1$. Does this equation define z as a function of x and y:

(a) In a neighborhood of $x = 1,\ y = 1$?

(b) In a neighborhood of $x = 1,\ y = 0$?

(c) In a neighborhood of $x = 0.5,\ y = 0$?

If so, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at that point.

2. Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

(a) If $x_1 = 6$ and $x_2 = 3$, find a y which satisfies $F(x_1, x_2, y) = 0$.

(b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1 = 6,\ x_2 = 3$?

(c) If so, compute $\frac{\partial y}{\partial x_1}(6, 3)$ and $\frac{\partial y}{\partial x_2}(6, 3)$

(d) If x_1 increases to 6.2 and x_2 decreases to 2.9, estimate the corresponding change of y.

3. Show that if for functions $f(x, y)$ and $g(x, y)$ one has $f_x = g_y$ and $f_y = -g_x$, then level curves of f and g intersect orthogonally.

4. One solution of the system

$$2x^2 + 3xyz - 4uv = 16,$$
$$x + y + 3z + u - v = 10$$

is $x = 1, y = 2, z = 3, u = 0, v = 1$. If one varies u and v near their original values and plugs these new values into this system, can one find unique values of x, y and z that still satisfy this system? Explain.

5. Check that $x = 1, y = 4, u = 1, v = -1$ is a solution of the system

$$y^2 + 2u^2 + v^2 - xy = 15, \quad 2y^2 + u^2 + v^2 + xy = 38.$$

If y increases to 4.02 and x stays fixed, does there exist a (u, v) near $(1, -1)$ which solves this system? If not, why not? If yes, estimate the new u and v.

6. The economy of Northern Saskatchewan is in equilibrium when the system of equations

$$2xz + xy + z - 2\sqrt{z} = 11, \qquad xyz = 6$$

is satisfied. One solution of this set of equations is $x = 3, y = 2, z = 1$, and Northern Saskatchewan is in equilibrium at this point. Suppose that the prime minister discovers that the variable z (output of beaver pelts) can be controlled by simple decree.

a) If the prime minister raises z to 1.1, use calculus to estimate the change in x and y.

b) If x were in the control of the prime minister and not y or z, explain why you cannot use this method to estimate the effect of reducing x from 3 to 2.95.

7. Consider the system of equations

$$x + 2y + z = 5, \quad 3x^2yz = 12$$

as defining some endogenous variables in terms of some exogenous variables.

a) Divide the three variables into exogenous ones and endogenous ones in a neighborhood of $x = 2, y = 1, z = 1$ so that the Implicit Function Theorem applies.

b) If each of the exogenous variables in your answer to a) increases by 0.25, use calculus to estimate how each of the endogenous variables will change.

8. Consider the system of two equations in three unknowns: $x + 2y + z = 5, 3x^2yz = 12$.

a) At the point $x = 2, Y = 1, z = 1$, why can we treat z as an exogenous variable and x and y are the dependent variables?

b) If z rises to 1.2, use calculus to estimate the corresponding x and y.

Exercises 15.1-15.25 from [SB].

Homework

Exercise 15.6 from [SB], Exercise 15.9 from [SB], Exercise 15.13 from [SB], Exercise 15.22 from [SB], Exercise 15.24 from [SB].

Short Summary
Implicit Function

Implicit Function Theorem in R^1 If $G(x^*, y^*) = c$ and $\frac{\partial G}{\partial y}(x^*, y^*) \neq 0$, then $\exists \;\; y = y(x)$ on $(x^* - \epsilon, x^* + \epsilon)$ s.t. $G(x, y(x)) \equiv c, \;\; y(x^*) = y^*$ and $y'(x^*) = -\frac{\frac{\partial G}{\partial x}(x^*,y^*)}{\frac{\partial G}{\partial y}(x^*,y^*)}$.

Implicit Function Theorem for R^n**.** If $G(x_1^*, ..., x_k^*, y^*) = c$ and

$$\frac{\partial G}{\partial y}(x_1^*, ..., x_k^*, y^*) \neq 0$$

then $\exists \;\; y = y(x_1, ..., x_n)$ on $B_\epsilon(x^*)$ s.t. $G(x_1, ..., x_k, y(x_1, ..., x_k)) \equiv c$, $y(x_1^*, ..., x_k^*) = y^*$ and $\frac{\partial y}{\partial x_i} = -\frac{\frac{\partial G}{\partial x_i}(x_1^*,...,x_k^*,y^*)}{\frac{\partial G}{\partial y}(x_1^*,...,x_k^*,y^*)}$.

Implicit Function Theorem for System. If

$$\left\{ \begin{array}{l} F_1(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*) = c_1 \\ .... \\ F_m(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*) = c_m \end{array} \right) \;\; and \;\; \left| \begin{array}{ccc} \frac{\partial F_1}{\partial y_1} & ... & \frac{\partial F_1}{\partial y_m} \\ ... & ... & ... \\ \frac{\partial F_m}{\partial y_1} & ... & \frac{\partial F_m}{\partial y_m} \end{array} \right|_{(x_1^*,...,x_n^*,y_1^*,...,y_m^*)} \neq 0,$$

then $\exists \;\; y_1(x_1, ..., x_n), \; ... \; , y_m(x_1, ..., x_n)$ on $B_\epsilon(x^*, Y^*)$ s.t. for all $i = 1, ..., m$

$$F_i(x_1, ..., x_n, y_1(x_1, ..., x_n), ..., y_m(x_1, ..., x_n)) \equiv c_i, \quad y_i(x_1^*, ..., x_n^*) = y_i^*$$

and $\frac{\partial y_i}{\partial x_k}$ can be solved from

$$\left(\begin{array}{ccc} \frac{\partial F_1}{\partial y_1} & ... & \frac{\partial F_1}{\partial y_m} \\ ... & ... & ... \\ \frac{\partial F_m}{\partial y_1} & ... & \frac{\partial F_1}{\partial y_m} \end{array} \right) \cdot \left(\begin{array}{c} \frac{\partial y_1}{\partial x_k} \\ ... \\ \frac{\partial y_m}{\partial x_k} \end{array} \right) = - \left(\begin{array}{c} \frac{\partial F_1}{\partial x_k} \\ ... \\ \frac{\partial F_m}{\partial x_k} \end{array} \right).$$

Regular point of $G(x, y)$: $DG(x^*, y^*) \neq (0, 0)$.

Tangent vector of the level curve $G(x, y) = c$ at a regular point (x^*, y^*): $(G_y(x^*, y^*), -G_x(x^*, y^*))$.

At a regular point **gradient** is orthogonal to level curve.

Marginal Rate of Substitution $\Delta y \approx MRS \cdot \Delta x = -\frac{\frac{\partial G}{\partial x}(x^*,y^*)}{\frac{\partial G}{\partial y}(x^*,y^*)} \Delta x$.