Math for Economists, Calculus 1

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WEEK 3

1 Applications of Derivatives

1.1 Using the Derivative for Graphing

Theorem 1 *(a) If $f'(x_0) > 0$, then there is an open interval containing x_0 on which f is increasing.*

(b) If $f'(x_0) < 0$, then there is an open interval containing x_0 on which f is decreasing.

Proof. (a) By definition

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) > 0.$$

Thus if h is small enough, since $f'(x_0) > 0$, we have

$$\frac{f(x_0 + h) - f(x_0)}{h} > 0$$

too, and assuming h being positive we obtain

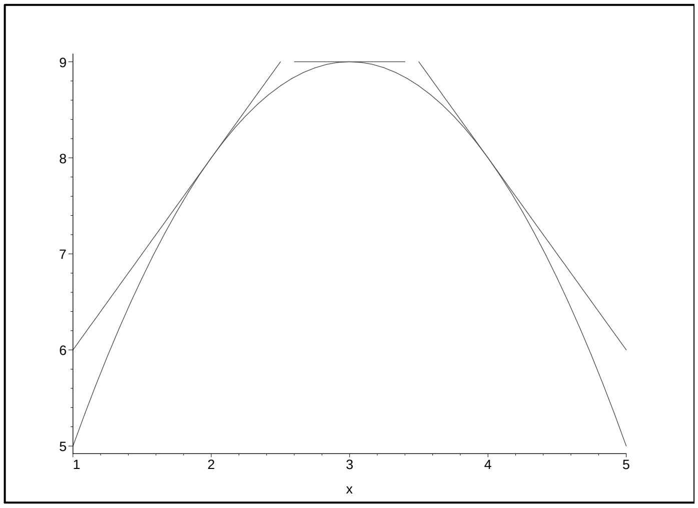
$$f(x_0 + h) - f(x_0) > 0$$

that is $f(x_0 + h) > f(x_0)$, i.e. f is increasing near x_0.

(b) Similarly, if $f'(x_0) < 0$ then $f(x_0+h) - f(x_0) < 0$, thus f is decreasing near x_0. **Q.E.D. (quod erat demonstrandum).**

Definition 1 *A point x_0 is called* ***critical point*** *of f if $f'(x_0) = 0$ or $f'(x_0)$ is not defined.*

A critical point is potential local minimum or local maximum point of f.



Examples

1. For $f(x) = x^2$ the point $x = 0$ is critical: $f'(0) = 2x|_{x=0} = 0$, and it is a point of minimum.

2. For $f(x) = -x^2$ the point $x = 0$ is critical: $f'(0) = -2x|_{x=0} = 0$, and it is a point of maximum.

3. For $f(x) = x^3$ the point $x = 0$ is critical: $f'(0) = 3x^2|_{x=0} = 0$, but this is neither minimum nor minimum.

4. For $f(x) = |x|$ the point $x = 0$ is critical: the derivative $f'(0)$ does not exist, and it is a point of minimum.

5. For $f(x) = \frac{1}{x}$ the point $x = 0$ is "critical": the derivative $f'(0)$ does not exist (as well as $f(0)$), but this is neither minimum nor maximum.

1.1.1 Graphing Algorithm "Sign Chart"

1. Find all critical points, say $x_1,\ x_2,\ ...\ ,\ x_n$.
2. Find (if possible) $f(x_1),\ f(x_2),\ ...\ ,\ f(x_n)$ and plot the corresponding points of the graph.
3. Check the sign of f' on each of intervals

$$(-\infty, x_1),\ (x_1, x_2),\ ...\ ,(x_{n-1}, x_n),\ (x_n, +\infty).$$

4. If $f' > 0$ on interval (x_i, x_{i+1}), draw the graph increasing connecting $f(x_i)$ and $f(x_{i+1})$. If $f' < 0$ on interval (x_i, x_{i+1}), draw the graph decreasing connecting $f(x_i)$ and $f(x_{i+1})$.

Example

Plot the graph of the function $f(x) = 2x^3 + 3x^2 - 12x$.

Solution.

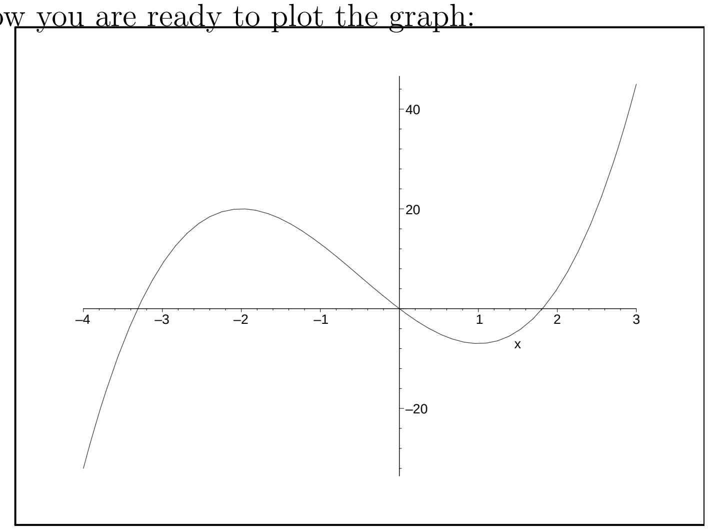
1. Derivative $f'(x) = 3x^2 + 6x - 12$.

2. Critical points $6x^2 + 6x - 12 = 0,\ x_1 = -2,\ x_2 = 1$.

3. Sign Chart

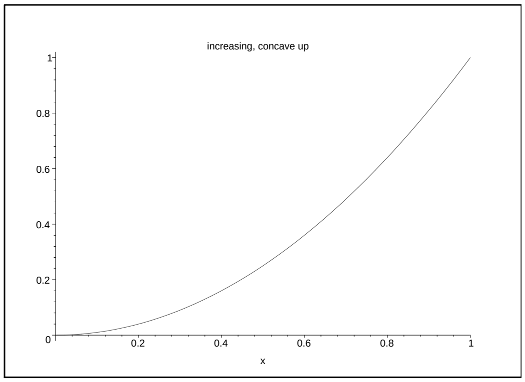
x	$-\infty,-2)$	-2	$(-2,1)$	1	$(1,+\infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	20	$\nearrow$	-7	$\searrow$

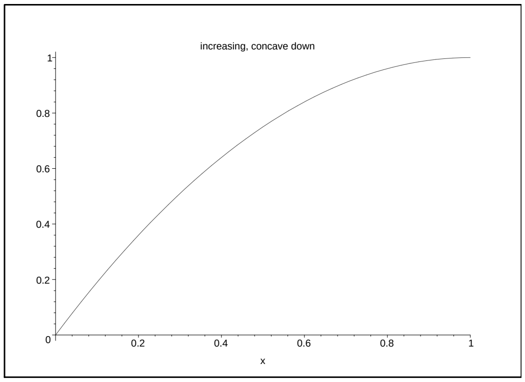
4. y-intercept $f(0) = 0$.
5. x-intercept $f(x) = 0$, $2x^3 + 3x^2 - 12 * x = 0$, $x(2x^2 - 3x - 12) = 0$
$x_1 = \frac{-1-\sqrt{105}}{4} \approx -3.3$, $x_2 = 0$, $x_3 = \frac{-1+\sqrt{105}}{4} \approx 1.8$.
Now you are ready to plot the graph:

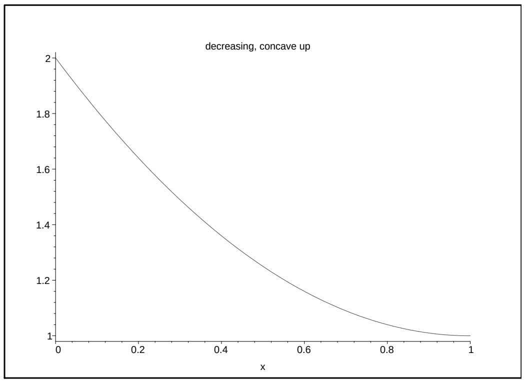


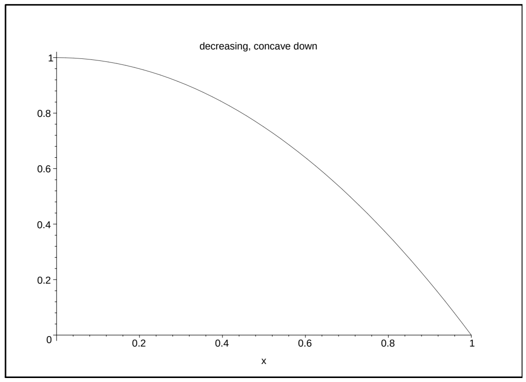
1.1.2 Second Derivatives and Convexity

Using calculus we can learn about the function more than where it is increasing or decreasing. For example where a function is *concave up* or *concave down*.

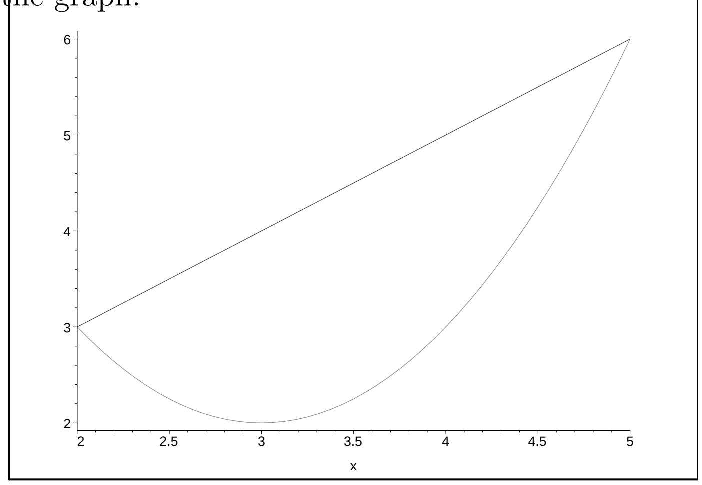




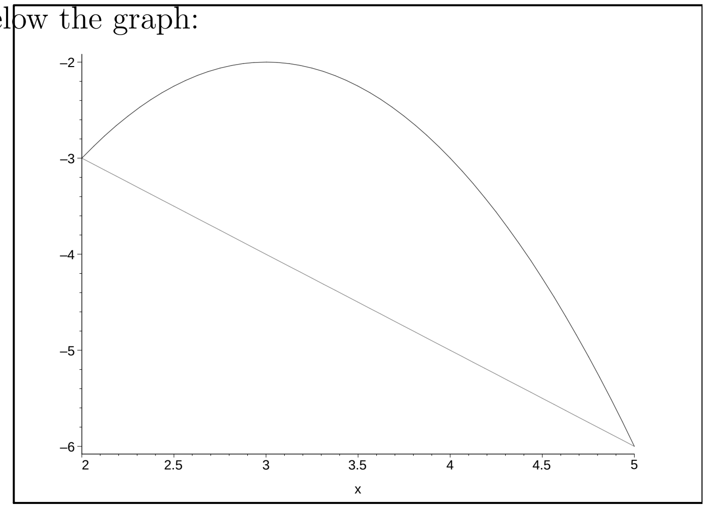




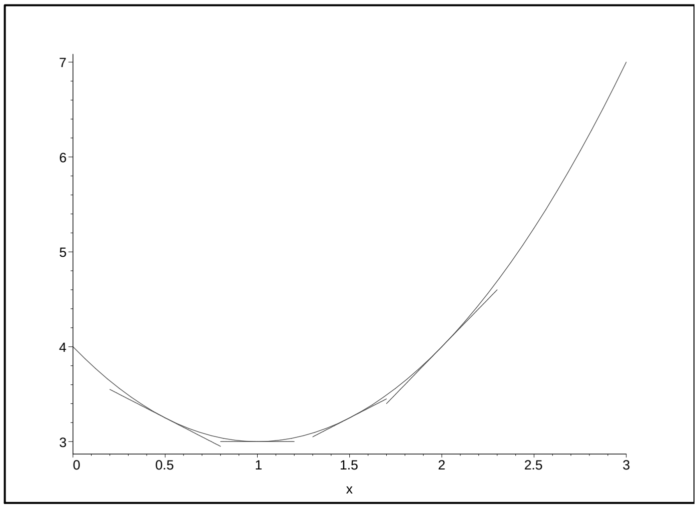
A function f is called **concave up** or simply convex if the secant line lies above the graph:



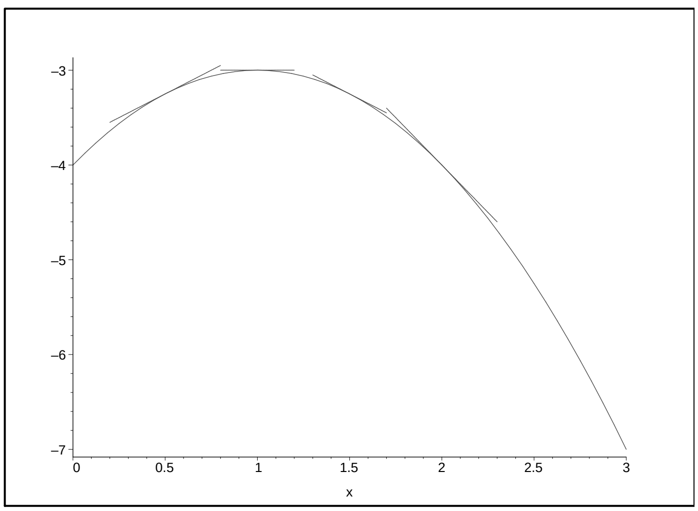
A function f is called **concave down** or simply concave if the secant line lies below the graph:



Second derivative test for concavity: f is concave up if $f'' > 0$, and f'' is concave down if $f'' < 0$.



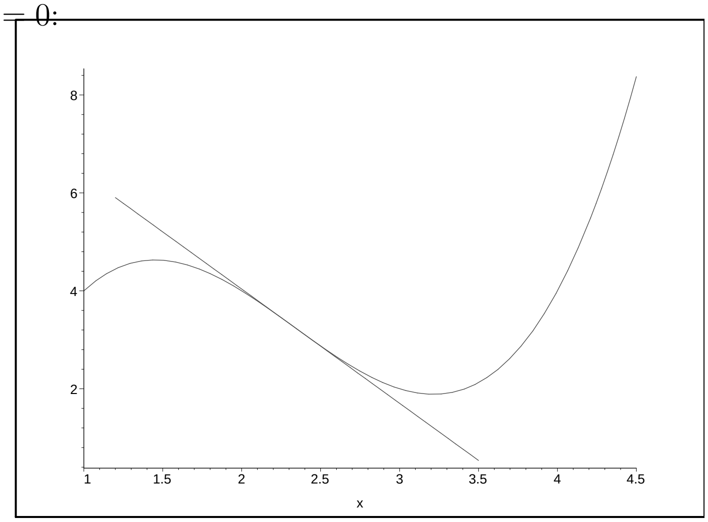
$$f(x) \text{ concave up} \Rightarrow f'(x) \text{ increases} \Rightarrow f''(x) > 0$$



$$f(x) \text{ concave down} \Rightarrow f'(x) \text{ decreases} \Rightarrow f''(x) < 0$$

A **second order critical point**, or **inflection point**, is a point where

$f''(x) = 0$:



2 Graphing Rational Functions

2.1 Vertical Asymptotes

A **rational function** is a ratio of two polynomials

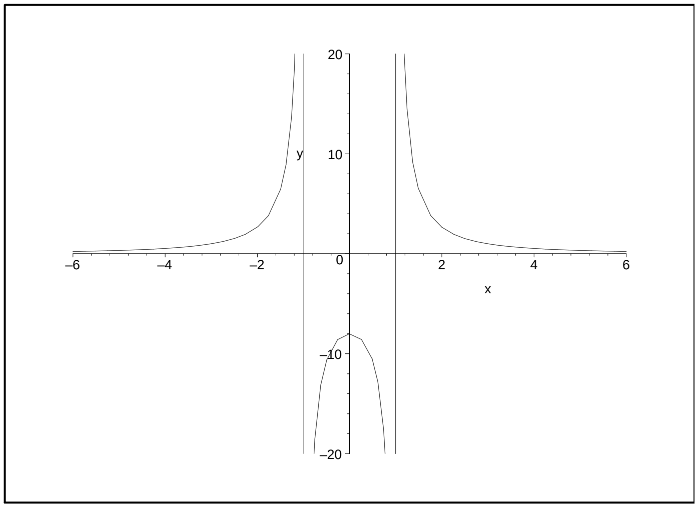
$$f(x) = \frac{P(x)}{Q(x)}.$$

Suppose x_0 is a root of denominator, i.e. $Q(x_0) = 0$. Then $f(x)$ is not defined for x_0 (that is x_0 *is not* in the domain of f), so the graph of f *can not* intersect the vertical line that crosses the x-axes at x_0. This vertical line is called **vertical asymptote** of f. Its equation is $x = x_0$.

On either side of vertical asymptote the graph goes to $+\infty$ or $-\infty$. The sign chart clarifies to find out which, but do not forget to include that x_0 (a zero of the denominator) to the list of critical points. Namely,

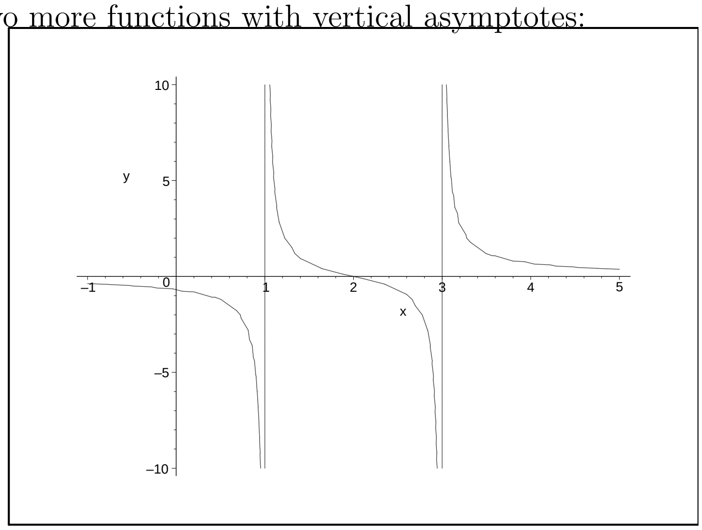
(left+) If $f' > 0$ just to the left of the asymptote, then f must go to $+\infty$ to the left of asymptote.
(left-) If $f' < 0$ just to the left of the asymptote, then f must go to $-\infty$ to the left (right) of asymptote.
(right+) If $f' < 0$ just to the right of the asymptote, then f must go to $+\infty$ to the right of asymptote.
(right-) If $f' > 0$ just to the right of the asymptote, then f must go to $-\infty$ to the right of asymptote.

All four cases are demonstrated on this graph:

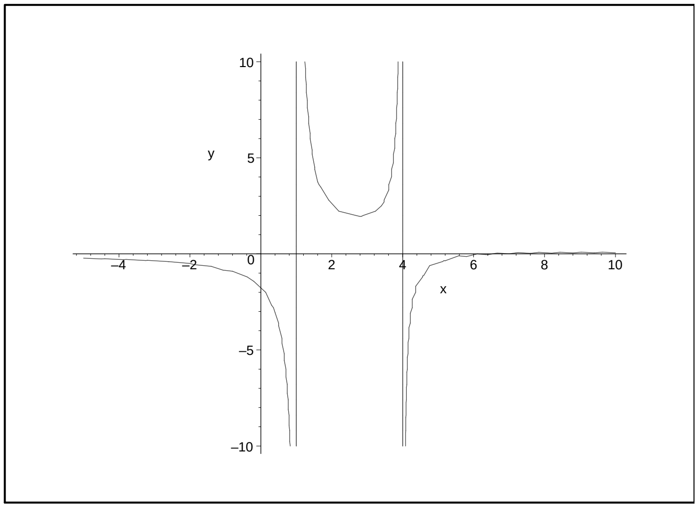


$$f(x) = \frac{8}{x^2-1}, \quad f'(x) = -\frac{16x}{(x^2-1)^2}$$

Two more functions with vertical asymptotes:



$$f(x) = \frac{x-2}{(x-1)(x-3)}, \; f'(x) = -\frac{x^2-4x+5}{(x-1)^2(x-3)^2}$$



$$f(x) = \frac{x-7}{(x-1)(x-4)} \quad f'(x) = -\frac{x^2-14x+31}{(x-1)^2(x-4)^2}$$

2.2 Tails and Horizontal Asymptotes

The "tail" of the graph is the shape of the graph for large positive and large negative values of x.

2.2.1 Tails of a monomial

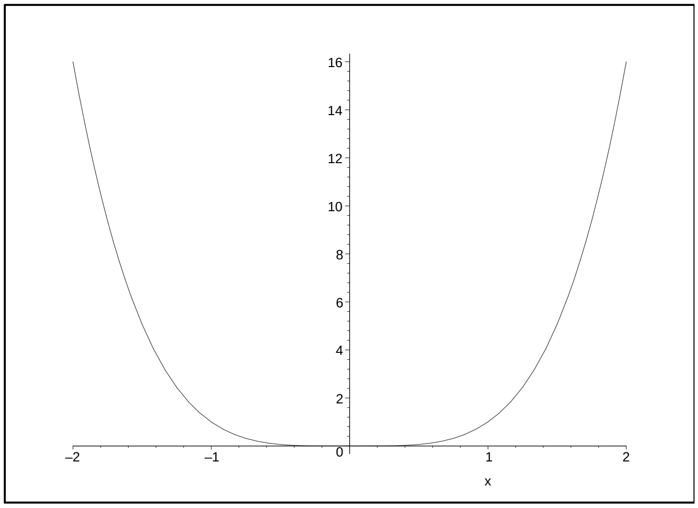
For a monomial $f(x) = ax^n$ with $a > 0$ we have:

When $x \to +\infty$ then $f(x) \to +\infty$, so the right tail goes to $+\infty$;

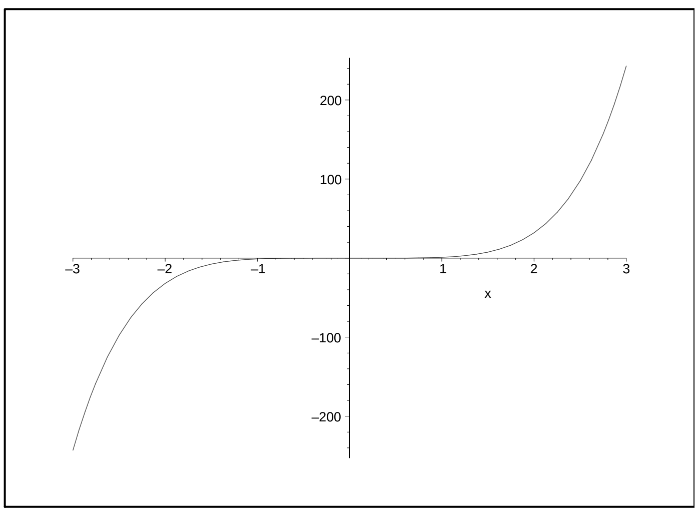
If n is even, when $x \to -\infty$ then $f(x) \to +\infty$, so the left tail goes to $+\infty$;

If n is odd, when $x \to -\infty$ then $f(x) \to -\infty$, so the left tail goes to $-\infty$.

For $a < 0$ the situation is symmetric to the above (observe it!).



$$f(x) = x^4$$



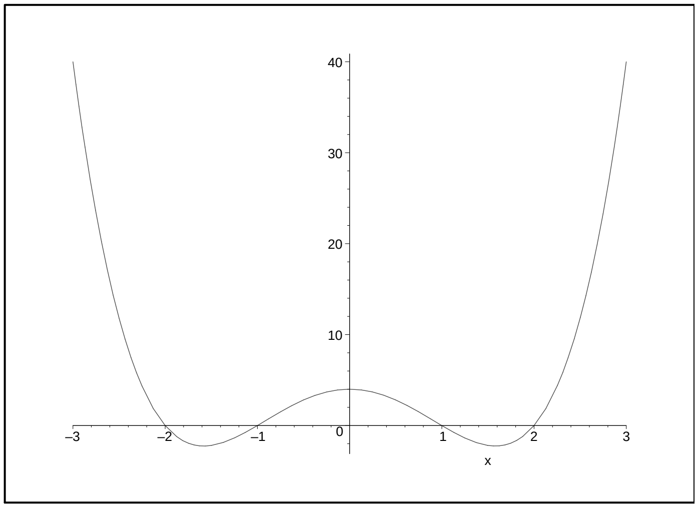
$$f(x) = x^5$$

2.2.2 Tails of a polynomial

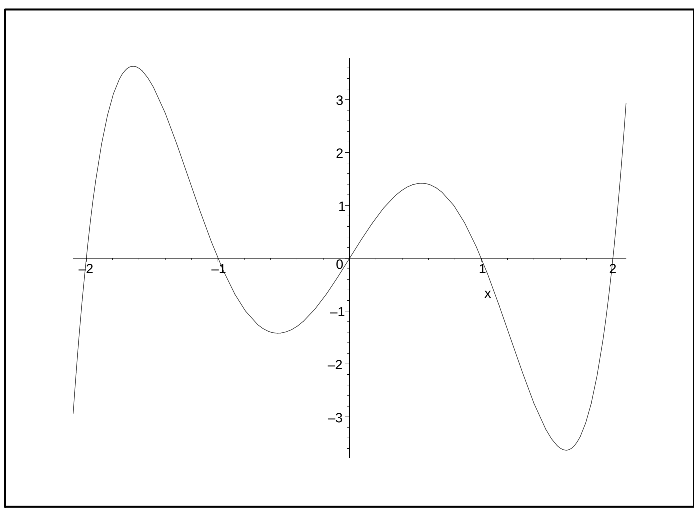
The shape of the tail of a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 \ ... \ + a_nx^n,$$

is the same as the shape of the **leading term** a_nx^n.



$$f(x) = x^4 - 5x^2 + 4$$



$$f(x) = x^5 - 5x^3 + 4x$$

2.2.3 Horizontal Asymptotes

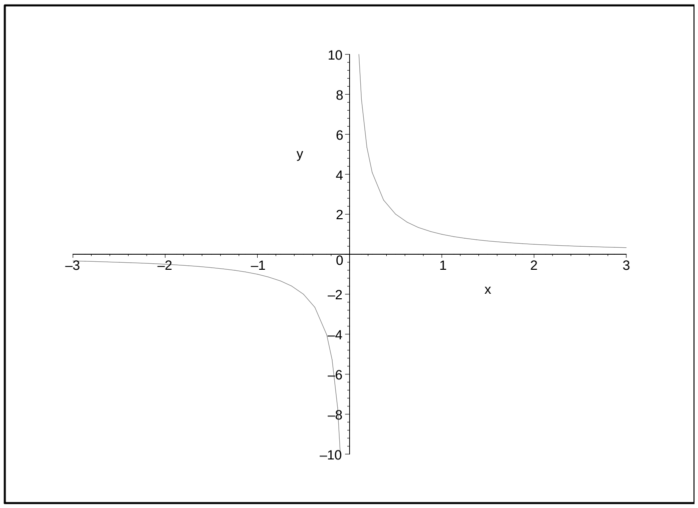
The line $y = b$ is a **horizontal asymptote** of f if either of following conditions hold:

$$\lim_{x \to -\infty} f(x) = b, \quad \lim_{x \to +\infty} f(x) = b.$$

Examples

1. The function $f(x) = \frac{1}{x}$ has horizontal asymptote $y = 0$, i.e. the x-axis:

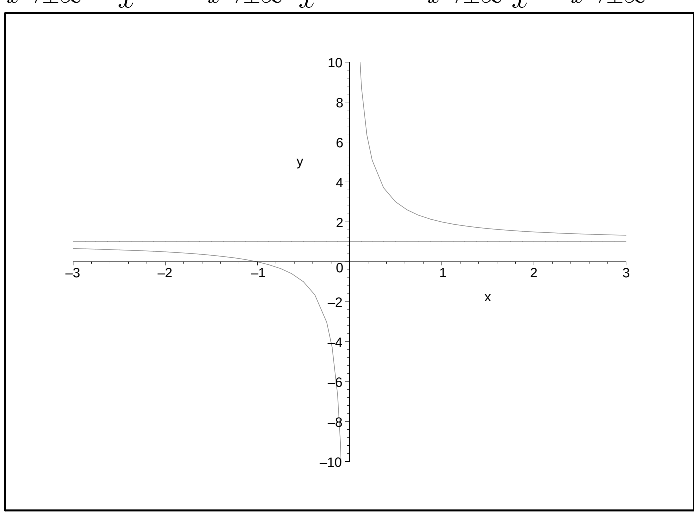
$$\lim_{x \to \pm\infty} \frac{1}{x} = 0.$$



$f(x) = \frac{1}{x}$

2. The function $f(x) = \frac{1+x}{x}$ has horizontal asymptote $y = 1$. Indeed, $\frac{1+x}{x} = \frac{1}{x} + 1$, so

$$\lim_{x\to\pm\infty} \frac{1+x}{x} = \lim_{x\to\pm\infty} (\frac{1}{x} + 1) = \lim_{x\to\pm\infty} \frac{1}{x} + \lim_{x\to\pm\infty} 1 = 0 + 1 = 1.$$

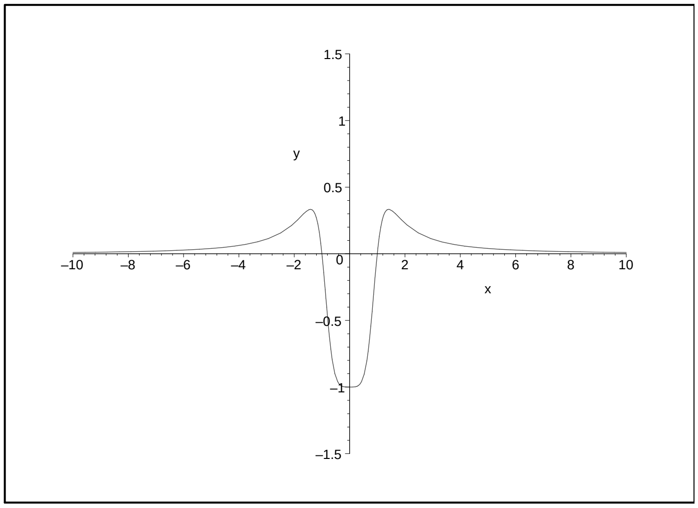


$f(x) = \frac{x+1}{x}$

3. The function $f(x) = \frac{x^4-1}{x^6+1}$ has horizontal asymptote $y = 0$, i.e. the x-axis:

$$\lim_{x\to\pm\infty} \frac{x^4-1}{x^6+1} = \lim_{x\to\pm\infty} \frac{\frac{x^4}{x^6} - \frac{1}{x^6}}{\frac{x^6}{x^6} + \frac{1}{x^6}} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^2} - \frac{1}{x^6}}{1 + \frac{1}{x^6}} = \frac{0}{1} = 0.$$

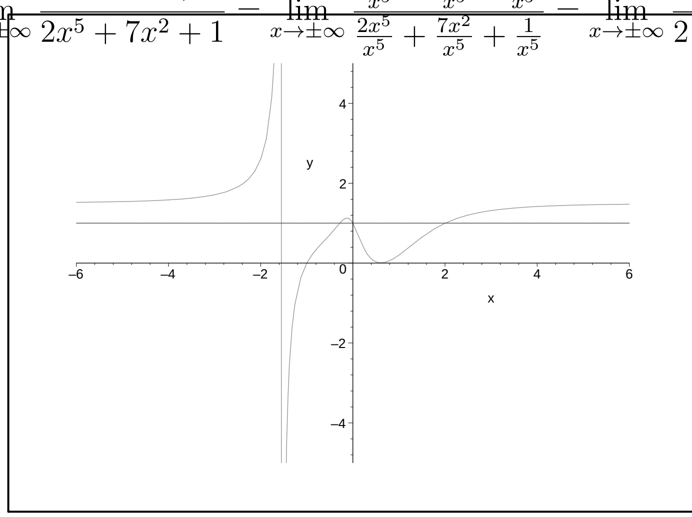
Pay attention that this function *intersects* its horizontal asymptote $y = 0$: $x = -1$ and $x = 1$ are the solutions of the equation $f(x) = 0$.



$f(x) = \frac{x^4-1}{x^6+1}$

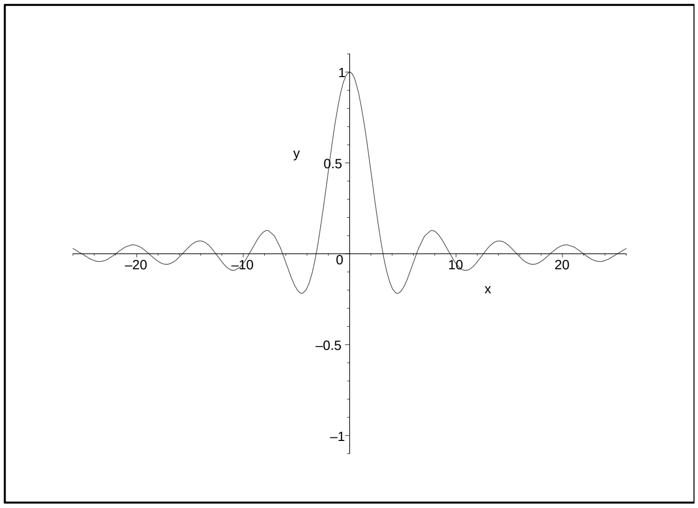
4. The function $f(x) = \frac{3x^5-2x+1}{2x^5+7x^2+1}$ has horizontal asymptote $y = \frac{3}{2}$:

$$\lim_{x\to\pm\infty} \frac{3x^5-2x+1}{2x^5+7x^2+1} = \lim_{x\to\pm\infty} \frac{\frac{3x^5}{x^5} - \frac{2x}{x^5} + \frac{1}{x^5}}{\frac{2x^5}{x^5} + \frac{7x^2}{x^5} + \frac{1}{x^5}} = \lim_{x\to\pm\infty} \frac{3 - \frac{2}{x^2} + \frac{1}{x^5}}{2 + \frac{7}{x^3} + \frac{1}{x^4}} = \frac{3}{2}.$$



$f(x) = \frac{3x^5-2x+1}{2x^5+7x^2+1}$

5. The function $y = \frac{1}{x} \cdot \sin x$ has a horizontal asymptote $y = 0$ and the graph of the function intercepts his asymptote infinitely many times at the points $x = \pi k + \pi/2$.



$$f(x) = \frac{1}{x} \cdot \sin x$$

Generally, the behavior of a rational function

$$f(x) = \frac{a_0 + a_1 \cdot x + \ \dots \ + a_m \cdot x^m}{b_0 + b_1 \cdot x + \ \dots \ + b_n \cdot x^n}$$

"ad infinitum" mirrors the behavior of the quotient of leading terms

$$l(x) = \frac{a_m \cdot x^m}{b_n \cdot x^n} = \frac{a_m}{b_n} x^{m-n}.$$

Case 1 $m > n$, in this case

$$lim_{x \to \infty} f(x) = lim_{x \to \infty} l(x) = lim_{x \to \infty} \frac{a_m}{b_n} x^{m-n} = +\infty,$$

so no horizontal asymptote in this case.

Case 2 $m = n$, in this case

$$lim_{x \to \infty} f(x) = lim_{x \to \infty} l(x) = lim_{x \to \infty} \frac{a_m}{b_m} x^{m-m} = \frac{a_m}{b_m},$$

so the horizontal asymptote in this case is the line $y = \frac{a_m}{b_m}$.

Case 3 $m < n$, in this case

$$lim_{x \to \infty} f(x) = lim_{x \to \infty} l(x) = lim_{x \to \infty} \frac{a_m}{b_n} \frac{1}{x^{n-m}} = 0,$$

so the horizontal asymptote in this case is the x-axes $y = 0$.

2.2.4 Oblique Asymptotes

The line $y = ax + b$ is an **oblique asymptote** of $f(x) = \frac{P(x)}{Q(x)}$ if

$$\lim_{x\to\pm\infty} (f(x) - (ax + b)) = 0.$$

Such asymptote exists if $deg\ P(x) = deg\ Q(x) + 1$. This is the linear function $y = ax + b$ which is the quotient of division $P(x) : Q(x)$.

Reminder. The *quotient* of division of $14 : 4$ is $q = 3$ and the *reminder* is $r = 2$, that is

$$\frac{14}{4} = 3 + \frac{2}{4}, \quad or \quad 14 = 3 \cdot 4 + 2,$$

notice that $r = 2 < 4$.

Generally, The quotient of division of $a : b$ is q and the reminder is r if

$$\frac{a}{b} = q + \frac{r}{b}, \quad a = b \cdot q + r,$$

and $0 \le r < b$.

If a and b are polynomials, then the quotient of division of $a : b$ is q and the reminder is r if $a = b \cdot q + r$ and $0 \le \deg r < \deg b$.

For example for $a = x^3 + 2x^2 + 3x$ and $b = x^2 - x + 1$ we have $q = x + 3$ and $r = 5x - 3$, indeed

$$b \cdot q + r = (x^2 - x + 1) \cdot (x + 3) + 5x - 3 =$$
$$x^3 - x^2 + x + 3x^2 - 3x + 3 + 5x - 3 = x^3 + 2x^2 + 3x = a.$$

Division of polynomials by MAPLE:

$> a := x^3 + 2 * x^2 + 3 * x;$

$$a := x^3 + 2 * x^2 + 3 * x$$

$> b := x^2 - x + 1;$

$$b := x^2 - x + 1$$

$> q := quo(a, b, x);$

$$q = x + 3$$

$> r := rem(a, b, x);$

$$r = 5x + 3$$

$> evala(b * q + r);$

$$x^3 + 2 * x^2 + 3 * x$$

Examples
1. Find the oblique asymptote of $\frac{x^3+1}{x^2-1}$.

Solution. Division gives

$$(x^3+1):(x^2-1) = x \; rem(x+1),$$

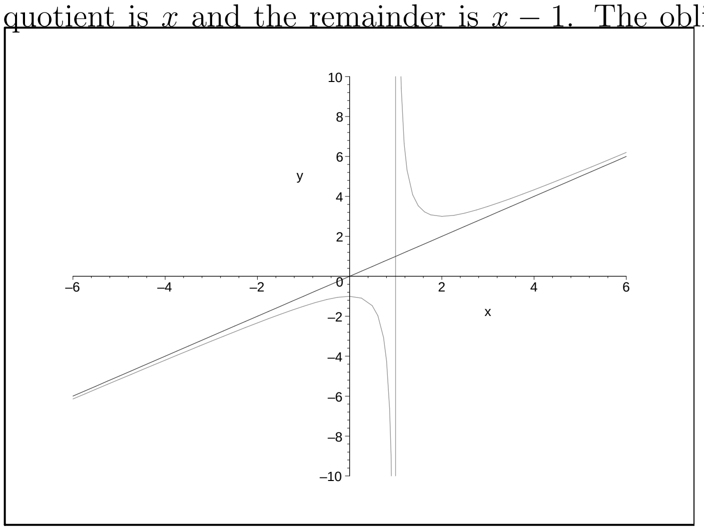
that is

$$\frac{x^3+1}{x^2-1} = x + \frac{x+1}{x^2-1}$$

or

$$x^3+1 = (x^2-1)\cdot x + (x+1),$$

so the quotient is x and the remainder is $x-1$. The oblique asymptote is $y=x$:



2. Find the oblique asymptote of $\frac{2x^3+4x^2-9}{-x^2+3}$.

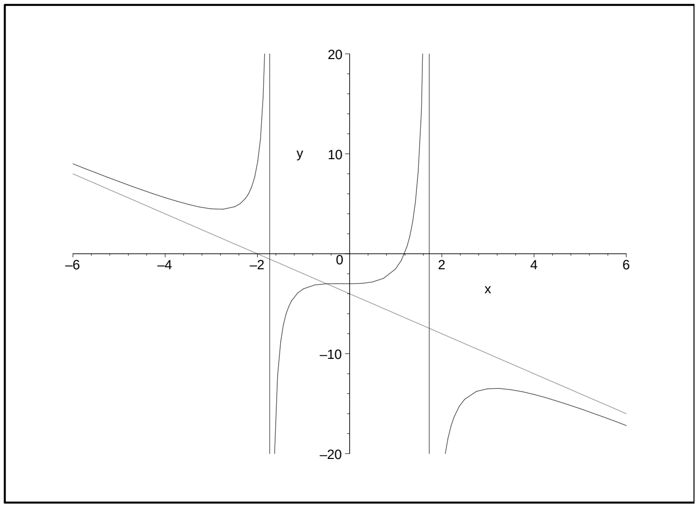
Solution. Division gives

$$(2x^3+4x^2-9):(-x^2+3) = (-2x-4) \; rem(6x+3),$$

that is

$$\frac{x^3+1}{x^2-1} = -2x-4 + \frac{6x+3}{-x^2+1}$$

so the quotient is $-2x-4$ and the remainder is $6x+3$. The oblique asymptote is $y=-2x-4$.



2.3 Summary: Asymptotes of rational Functions

A rational function

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 \cdot x + ... + a_n \cdot x^n}{b_0 + b_1 \cdot x + ... + b_m \cdot x^m}$$

has:

(a) Vertical asymptotes at zeros of denominator.

(b) If $deg\ P(x) > deg\ Q(x)$ then f has no Horizontal asymptotes.

(c) If $deg\ P(x) = deg\ Q(x) + 1$ then f has an oblique asymptote, which is the linear function $y = ax + b$, the quotient of division $P(x) : Q(x)$.

(d) If $deg\ P(x) = deg\ Q(x)$ then f has Horizontal asymptote $y = \frac{a_n}{b_m}$.

(e) If $deg\ P(x) < deg\ Q(x)$ then f has Horizontal asymptote $y = 0$.

2.4 Examples of Graphing

Example 1. Sketch the graph of $f(x) = \frac{8}{x^2-4}$.
Solution.

1. Intercepts. There are no x-intercepts, and the y-intercept is $f(0) = -2$.

2. Asymptotes.
Vertical: $x^2 - 4 = 0,\ x = -2,\ x = 2$. Horizontal: $y = 0$. Oblique: no.

3. Derivatives. $f'(x) = \frac{-16x}{(x^2-4)^2},\quad f''(x) = \frac{16(3x^2+4}{(x^2-4)^3)}$.

4. Critical points. $x = -2,\ x = 2$, and $f'(x) = \frac{-16x}{(x^2-4)^2} = 0, \quad x = 0$.

5. Increasing and decreasing intervals of f.

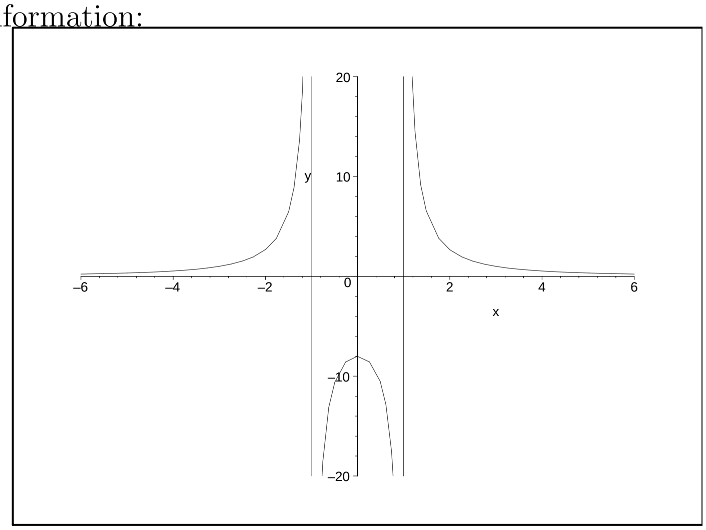
x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,+\infty)$
$f'(x)$	$+$	*no*	$+$	0	$-$	*no*	$-$
$f(x)$	$\nearrow$	*no*	$\nearrow$	-2	$\searrow$	*no*	$\searrow$

6. Inflection points. $x=-2,\ x=2$, and $f''(x) = \frac{16(3x^2+4)}{(x^2-4)^3} = 0$ has no solution.

7. Concavity.

x	$(-\infty,-2)$	-2	$(-2,2)$	2	$(2,+\infty)$
$f''(x)$	$+$	*no*	$-$	*no*	$+$
$f(x)$	*conc. up*	*no*	*conc. down*	*no*	*conc. up*

8. Sketch the graph. Now you are ready to sketch the graph using this information:



Example 2. Sketch the graph of $f(x) = \frac{x^2+4}{x}$.
Solution.

1. Intercepts. There are no x-intercepts, and no y-intercept.

2. Asymptotes.

Vertical: $x=0$. Horizontal: no. Oblique: yes, the division gives $f(x) = x + \frac{4}{x}$, f has the oblique asymptote $y = x$.

3. Derivatives. $f'(x) = \frac{x^2-4}{x^2}$, $\quad f''(x) = \frac{8}{x^3)}$.

4. Critical points. $x=-2,\ x=0\ x=2$.

5. Increasing and decreasing intervals of f.

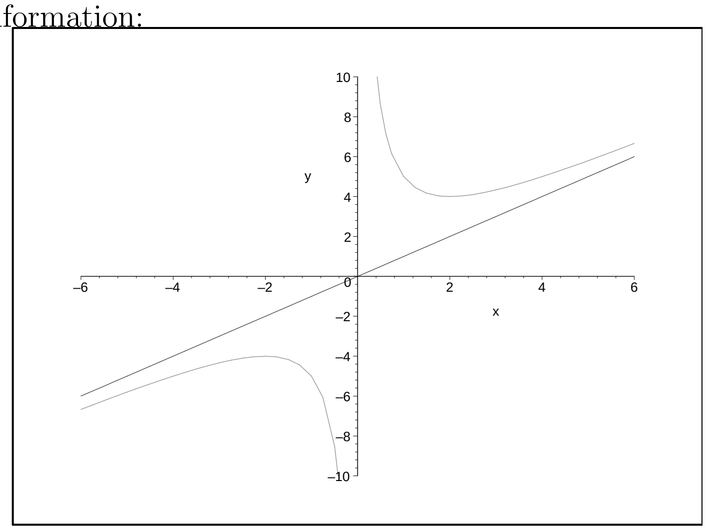
x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,+\infty)$
$f'(x)$	$+$	0	$-$	*no*	$-$	0	$+$
$f(x)$	$\nearrow$	-4	$\searrow$	*no*	$\searrow$	4	$\nearrow$

6. Inflection points. $f''(x) = \frac{8}{x^3} = 0$ has no solution, and $f''(0)$ does not exists, so $x = 0$ can be considered as an inflection point (where may be concavity changes).

7. Concavity.

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f''(x)$	$-$	*no*	$+$
$f(x)$	*conc. down*	*no*	*conc. up*

8. Sketch the graph. Now you are ready to sketch the graph using this information:



3 Maxima and Minima

A function f has a **local (or relative) interior maximum** at x_0 if $f(x) \leq f(x_0)$ for all x in *some open interval* containing x_0.

A function f has a global (or absolute) maximum at x_0 if $f(x) \leq f(x_0$ for all x in the domain of f.

A function f has a **local (or relative) interior minimum** at x_0 if $f(x) \geq f(x_0)$ for all x in some open interval containing x_0.

A function f has a global (or absolute) minimum at x_0 if $f(x) \geq f(x_0$ for all x in the domain of f.

A max or min can also occur at a boundary point of the domain of f. In this case it is called **boundary max** or **boundary min**.

3.1 First Order Conditions

Theorem 2 *If x_0 is an interior max or min of f then x_0 is a critical point.*

This means that the criticality is a *necessary* condition for optimality

So we must seek interior min or max points among critical points. But if x_0 is a critical point, how can we decide wether it is min, max or neither?

3.2 Second Order Condition

Theorem 3 *(a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a local max of f;*
(b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local min of f;
(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the second derivative test fails.

So the second order condition is *sufficient* for optimality.

3.3 Global Maxima and Minima

What conditions guarantee that a given critical point x_0 of f is a **global** max or min?

3.3.1 Only One Critical Point Case

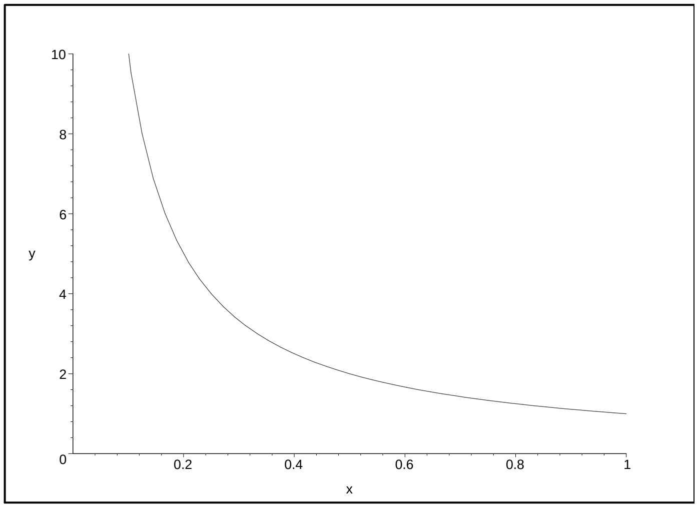
Suppose
(a) the domain of f is an open interval (finite or infinite) of R;
(b) x_0 is a local max (min) of f;
(c) x_0 is the only critical point of f
Then x_0 is the global max (min).

3.3.2 Nowhere Zero Second Derivative Case

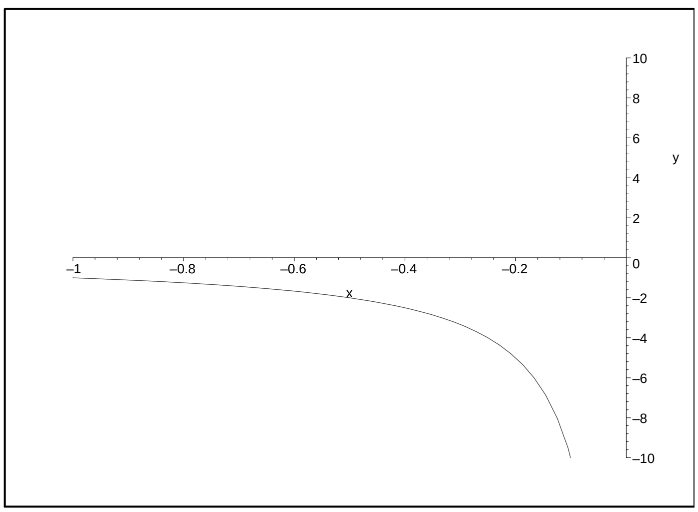
If the domain of f is an open interval (finite or infinite) I of R and $f''(x)$ is newer zero on I, then f has at most one critical point in I. This critical point is global maximum if $f'' < 0$ and global minimum if $f'' > 0$.

3.3.3 How to Find Global max and min

A function f defined on an open interval need not have a global min or max:



$f(x) = \frac{1}{x}$ does not have a global max on $(0, 1)$



$f(x) = \frac{1}{x}$ does not have a global min on $(-1, 0)$

However, a function f defined on a closed and bounded interval $[a, b]$ must have both a global min and global max.

How to find them?

(1) Find all critical points in (a, b);

(2) Evaluate f at these critical points and at the endpoints a and b;

(3) Choose the point from among these that gives the largest value of f (max) and smallest value of f (min).

Example

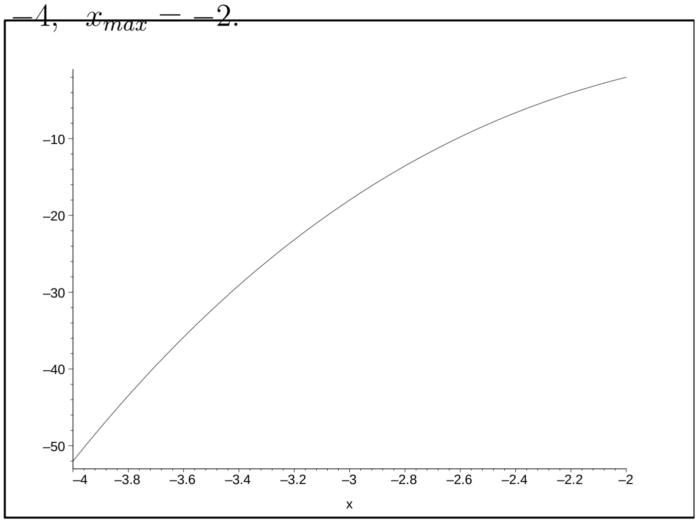
Find the global max and global min for $f(x) = x^3 - 3x$ on

$$(a) \quad D = [-4, -2], \qquad (b) \quad D_2 = (0, \infty).$$

Solution. Derivative $f'(x) = 3x^2 - 3$. Critical points

$$3x^2 - 3x = 0, \quad x^2 - 1 = 0, \quad (x-1)(x+1) = 0, \quad x_1 = -1, \; x_2 = 1.$$

$D_1 = [-4, -2]$**:** No critical points in this interval, so check just the endpoints $f(-4) = -4^3 - 12 = -64 + 12 = -52, \quad f(-2) = -2^3 + 6 = -2$, so $x_{min} = -4, \quad x_{max} = -2.$



$D_2 = (0, \infty)$**:** The critical point $x_2 = 1$ belongs to D_2, and it is a local min point: $f''(1) = 6x|_1 = 6 > 0$, besides, since $f''(x) = 6x > 0$ in whole interval $(0, \infty)$, it is global.