Math for Economists, Calculus 1

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WEEK 1

1 Introduction

1.1 Mathematical Model of Production

Data: x number of units produced, F the fixed cost, c the production cost of one unit, p the wished selling price of one unit, k the reduction coefficient.

The total cost of production of x units

$$C(x) = c \cdot x + F.$$

The selling price of one unit when x units are produced

$$p(x) = p - k \cdot x.$$

The total revenue

$$R(x) = x \cdot p(x) = -kx^2 + px.$$

The total profit

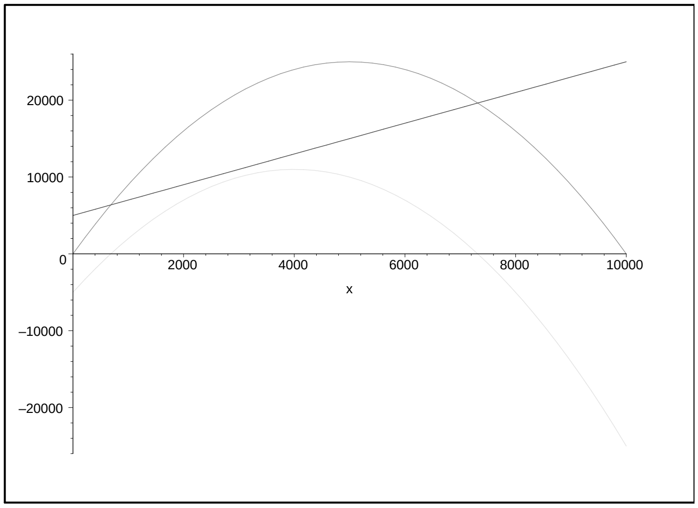
$$P(x) = R(x) - C(x) = -kx^2 + px - cx - F = \\ -kx^2 + (p - c)x - F.$$

Both functions $R(x)$ and $P(x)$ are concave quadratic functions, thus they both have maximum.

$> C(x) := 2 * x + 5000; p(x) := 10 - 0.001 * x;$

$> R(x) := x * p(x); P(x) := R(x) - C(x);$

$> plot(R(x), C(x), P(x), x = 0..10000);$



Exercises

1. The total cost of a company per month is given by $C(x) = 2x + 5000$ and the Price - demand function is given by $p(x) = 10 - 0.001x$. Thus the revenue and profit functions are given by

$$R(x) = xp(x) = -0.001x^2 + 10x,$$
$$P(x) = R(x) - C(x) = -0.001x^2 + 8x - 5000.$$

A) How many units should the company manufacture each month to maximize the revenue?

$R'(x) = -0.002x + 10 = 0, \quad x = 5000$

B) What is the maximal revenue?

R(5 000)=25 000

C) What is the selling price of one unit when the revenue is maximal?

$p(5000) = 5.$

D) How many units should the company manufacture each month to maximize the profit?

$P'(x) = -0.002x + 8 = 0, \quad x = 4000.$

E) What is the maximal profit?

$P(4000) = 11000.$

F) What is the selling price of one unit when the profit is maximal?

$p(4000) = 6$.

G) Find the break-even points.

$R(x) = C(x),\ P(x) = 0,\ -0.001x^2 + 8x - 5000 = 0,\ x_1 = 683,\ x_2 = 7317$.

2. The total cost of a company per month is given by $C(x) = 100x + 1000$ and the Price - demand function is given by $p(x) = 300 - 0.1x$.

A) How many TV sets should the company manufacture each month to maximize the revenue?

B) What is the maximal revenue?

C) What is the selling price of one TV set when the revenue is maximal?

D) How many TV sets should the company manufacture each month to maximize the profit?

E) What is the maximal profit?

F) What is the selling price of one TV set when the profit is maximal?

3. Suppose the government decides to introduce additional $ 10 tax for each TV set. Solve the previous problem in this case.

2 Functions

2.1 Vocabulary of Functions

2.1.1 The Notion of Function

A function (map, transformation) from the set X (domain) to the set Y (codomain, or target)

$$f : X \to Y$$

is a rule that assigns to each element $x \in X$ one element $f(x) \in Y$.

For example the function which assigns to any number its square is written as $f(x) = x^2$. To the number 2 it assigns the number $f(2) = 2^2 = 4$, and to the number -3 it assigns the number $f(-3) = (-3)^2 = 9$.

A function $y = f(x)$ has the input x which is called **independent variable** or **exogenous variable**, and the output y which is called **dependent variable** or **endogenous variable**.

2.1.2 Types of Functions

A **monomial** is a function of type $f(x) = a \cdot x^n$.

A **polynomial** is a sum of monomials:

$$p(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ \dots \ + a_{n-1} \cdot x^{n-1} + a_n \cdot x^n,$$

the **degree** of this polynomial is n, the numbers a_i are called **coefficients**.

A **rational function** is a ratio of two polynomials

$$f(x) = \frac{a_0 + a_1 \cdot x + \ \dots \ + a_m \cdot x^m}{b_0 + b_1 \cdot x + \ \dots \ + b_n \cdot x^n}.$$

The degree of rational function is defined as $m - n$.

An exponential function looks as $f(x) = a \cdot b^x$.

2.1.3 Graphs

The graph of a function $y = f(x)$ is the set of all points of the Cartesian plane whose coordinates are $(x, f(x))$. That is

$$\Gamma(f) = \{(x, y), \ y = f(x)\}.$$

2.1.4 Increasing and Decreasing Functions

A function $y = f(x)$ is called **increasing** if $x_1 < x_2$ implies $f(x_1) < f(x_2)$. That is

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

The graph of an increasing function goes upwards.

A function $y = f(x)$ is called **decreasing** if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

The graph of an decreasing function goes downwards.

A function which is either increasing or decreasing, is called **monotonic**.

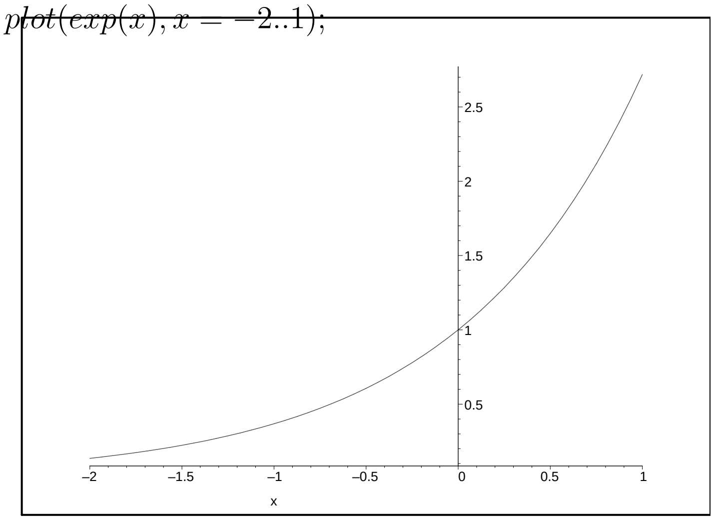
Examples

1. The function $f : R \to R$ given by $f(x) = e^x$ is increasing. Indeed, suppose $x_2 > x_1$, compare $f(x_2)$ and $f(x_1)$:

$$f(x_2) - f(x_1) = e^{x_2} - e^{x_1} = e^{x_1}(e^{x_2 - x_1} - 1) > 0$$

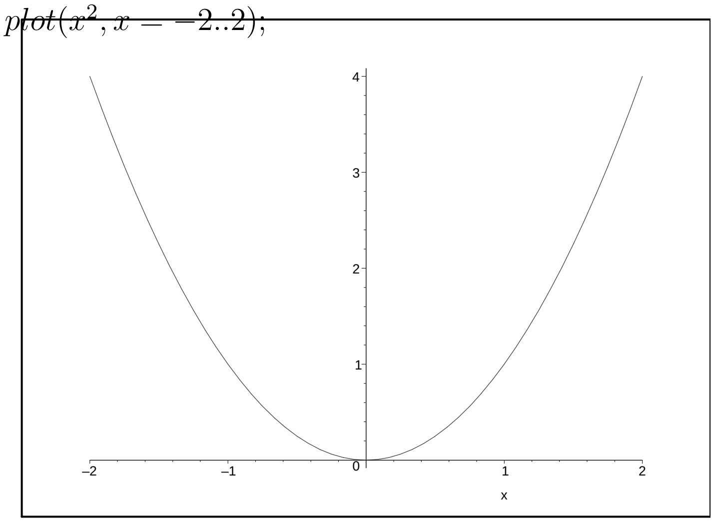
since $e^x > 0$ for arbitrary x, and $e^m > 1$ for $m > 0$.

$> plot(exp(x), x = -2..1);$



2. The function $f : R \to R$ given by $f(x) = x^2$ is not monotonic:
(a) it is not increasing: $1 > -2$ but $f(1)$ is not more than $f(-2)$, namely $f(1) = 1 < 4 = f(-2)$.
(b) it is not decreasing: $2 > 1$ but $f(2)$ is not less than $f(1)$, namely $f(2) = 4 > 1 = f(1)$.

$> plot(x^2, x = -2..2);$



Theorem. Suppose $f, g : R \to R$ are two increasing functions. Then the composition $f \circ g(x) = f(g(x))$ is increasing too.
Proof. Take $x_2 > x_1$, then $g(x_2) > g(x_1)$ since g is increasing, and $f(g(x_2)) > f(g(x_1))$ since f is increasing. Q.E.D. (quod erat demonstrandum).

Try to prove the
Theorem. If $f(x)$ is increasing, then $g(x) = -f(x)$ is decreasing.

Exercise

4. Is the function $f(x) = 1/x$ monotonic on whole its domain?

2.1.5 Minima and Maxima

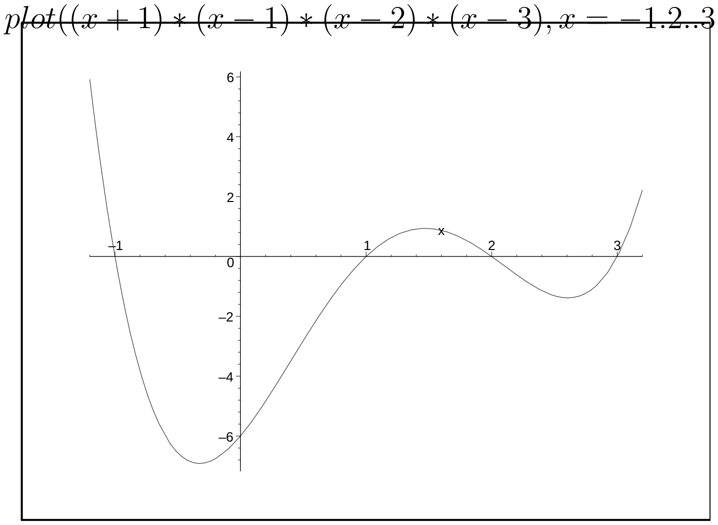
For a function $y = f(x)$ a point $(x_0, f(x_0))$ is called **local maximum** if $f(x_0) \geq f(x)$ for all x-s from *some* neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$. The function changes at this point from increasing to decreasing.

A point $(x_0, f(x_0))$ is a **global maximum** if $f(x_0) \geq f(x)$ for *all* x.

For a function $y = f(x)$ a point $(x_0, f(x_0))$ is called **local minimum** if $f(x_0) \leq f(x)$ for all x-s from *some* neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$. The function changes at this point from decreasing to increasing.

A point $(x_0, f(x_0))$ is a **global minimum** if $f(x_0) \leq f(x)$ for *all* x.

$> plot((x+1)*(x-1)*(x-2)*(x-3), x = -1.2..3.2);$



Exercises

5. Suppose x_0 is a point of minimum for $f(x)$. Then the same point x_0 for the function $g(x) = -f(x)$ is —— .

6. Find all local and global minimums and maximums for the function $f(x) = |x^2 - 4|$.

2.1.6 Domain and Range

For a function

$$f : X \to Y$$

the set X is called **domain**, and the set Y is called **target** or **codomain**.

A number x_0 belongs to the **domain** of a function $y = f(x)$ if $f(x_0)$ is defined.

Example. For the function $y = \frac{1}{x-3}$ the number $x_0 = 2$ belongs to the domain: $f(2) = \frac{1}{2-3} = \frac{1}{-1} = -1$ but $x_0 = 3$ does not: $f(3) = \frac{1}{3-3} = \frac{1}{0}$ is not defined.

The **image** or **range** of f is the set of all elements $y \in Y$ that correspond to some x:

$$Im\ f = \{y \in Y, y = f(x)\}.$$

For an element $y \in Y$ its **preimage** $f^{-1}(y)$ is the set of all elements $x \in X$ such that $f(x) = y$.

More generally, let V be a set of numbers. The preimage of V is defined as

$$f^{-1}(V) = \{x \in X,\ f(x) \in V\}.$$

Example. For the function $f : R \to R$ defined by $f(x) = x^2$

$$\begin{array}{cc} Im\ f = [0, +\infty), & f^{-1}(4) = \{-2, 2\}, \\ f^{-1}(0) = \{0\}, & f^{-1}(-9) = \emptyset, \end{array}$$

$$\begin{array}{c} f^{-1}([0, 9]) = [-3, +3], \\ f^{-1}((2, 9)) = (-3, -\sqrt{2}) \bigcup (\sqrt{2}, 3). \end{array}$$

2.1.7 Surjections, Injections, Bijections

A function $f : X \to Y$ is said to be **surjective** or *onto*, if its values span its whole codomain.

There are several equivalent ways to say the same:

- for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$, i.e. **(id est - that is)**

$$\forall y \in Y\ \exists\ x \in X\ s.t.\ f(x) = y;$$

- its range $f(X)$ is equal to its codomain Y, i.e. $Im\ f = Y$;
- for all $y \in Y$ the preimage $f^{-1}(y)$ is nonempty, i.e.

$$\forall y \in Y \quad f^{-1}(y) \neq \emptyset.$$

A surjective function is called *a surjection*.

Examples

1. The function $f : R \to R$ given by $f(x) = x^2$ is not a surjection: for example for $y = -4$ there exists no x s.t. $f(x) = y$.

2. But the same $f(x) = x^2$ considered as a function $f : R \to [0, \infty)$ is: take arbitrary $y \in [0, \infty)$, then $x = \sqrt{y}$ is its preimage.

3. The function $f : R \to R$ given by $f(x) = 2x + 4$ is a surjection: take arbitrary $y \in R$, then $x = \frac{y-4}{2}$ is its preimage.

Exercise

7. Which of the following functions $f : R \to R$ is surjective?

$$(a)\ \ f(x) = e^x,\ \ (b)\ \ f(x) = x^3,\ \ (c)\ \ f(x) = 4 - x^2.$$

A function $f : X \to Y$ is said to be **injective** or *one-to-one* function if distinct arguments are sent to distinct values.

There are several equivalent ways to say the same:

- $x_1 \neq x_2 \quad \Rightarrow f(_1) \neq f(x_2)$;
- for every $y \in Y$ the preimage $f^{-1}(y)$ consists of no more than one element, i.e. is either empty or one point.

Examples

1. The function $f : R \to R$ given by $f(x) = x^2$ is not an injection: $2 \neq -2$ but $f(2) = 4 = f(-2)$.

2. The function $f : R \to R$ given by $f(x) = 2x + 4$ is an injection: suppose $x_1 \neq x_2$, that is $x_1 - x_2 \neq 0$, then

$$f(x_1) - f(x_2) = (2x_1 + 4) - (2x_2 + 4) = 2(x_1 - x_2) \neq 0.$$

Exercise

8. Which of the following functions $f : R \to R$ is injective?

$$(a)\ \ f(x) = e^x,\ \ (b)\ \ f(x) = x^3,\ \ (c)\ \ f(x) = 4 - x^2.$$

A function $f : X \to Y$ is said to be **bijective** or *one-to-one correspondence* if it is simultaneously a surjection and an injection.

Examples

1. The function $f : R \to [0, \infty)$ given by $f(x) = x^2$ is not a bijection: it is a surjection but it is not an injection.

2. The function $f : R \to R$ given by $f(x) = 2x + 4$ is a bijection: it is a surjection and an injection as well.

Exercises

9. Which of the following functions $f : R \to R$ is bijective?

$$(a)\ \ f(x) = e^x,\ \ (b)\ \ f(x) = x^3,\ \ (c)\ \ f(x) = 4 - x^2.$$

10. Is the correspondence $f : X = \{0, 1, 2, 3, 4\} \to Y = \{0, 1, 2, 3\}$ given by the table

x	0	1	2	3	4
y	0	2	1	3	3

a surjection? an injection? a bijection?

2.1.8 Even and Odd Functions

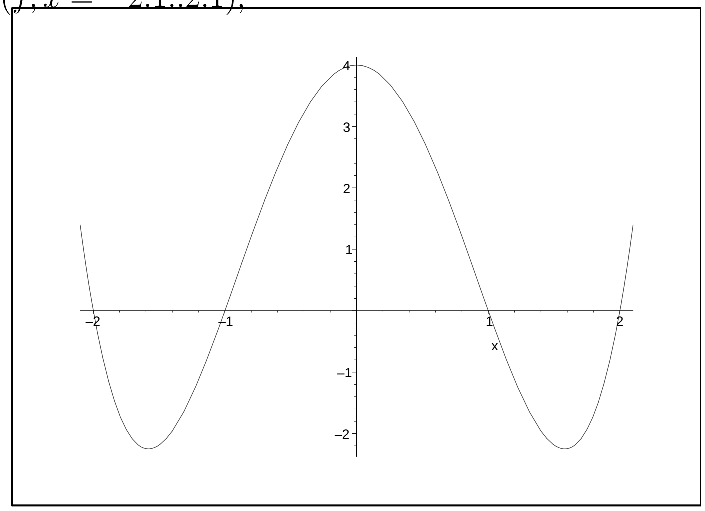
A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for all x in the domain of f.

The graph of an even function is symmetrical about y-axes:

$> f := simplify((x-2)*(x-1)*(x+1)*(x+2));$

$$f := (x^2 - 4)(x^2 - 1)$$

$> plot(f, x = -2.1..2.1);$



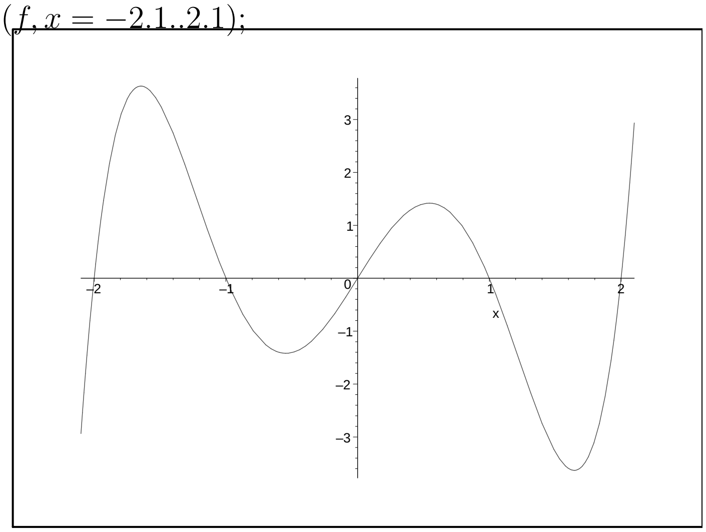
A function $y = f(x)$ is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

The graph of an odd function is symmetrical about the origin:

$> f := simplify((x-2)*(x-1)*x*(x+1)*(x+2));$

$$f := x(x^2 - 4)(x^2 - 1)$$

$> plot(f, x = -2.1..2.1);$



All monomials of even degree

$$y = c, \quad y = x^2, \quad y = x^4, \; ... \; , \quad y = x^{2k}, \; ...$$

are even functions. More examples $y = cosx, \quad y = 1/(x^2 - 1)$.

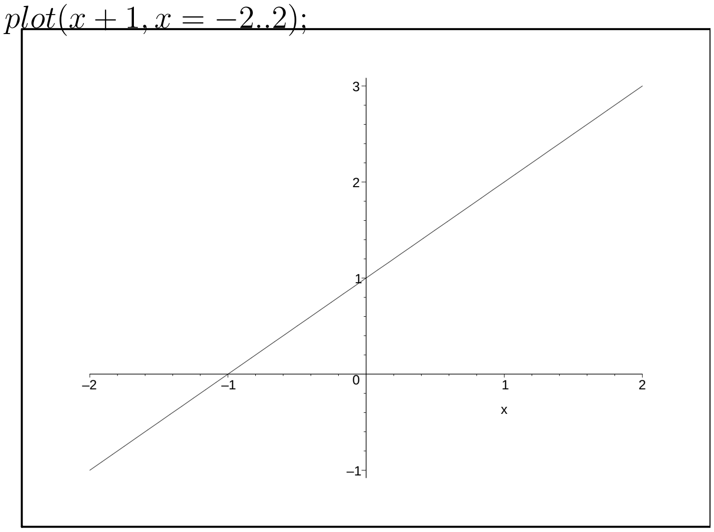
All monomials of odd degree

$$y = x, \quad y = x^3, \quad y = x^5, \; ... \; , \quad y = x^{2k+1}, \; ...$$

are odd functions. More examples $y = sinx, \quad y = 1/(x^3 - x)$.

The function $y = x + 1$ is neither even nor odd:

$> plot(x + 1, x = -2..2);$



for example $y(2) = 1 + 1 = 2$ and $y(-2) = -2 + 1 = -1$.

Examples

Determine whether the following functions are odd, even or neither.

(a) $f(x) = x^4 + 2 : \quad f(-x) = (-x)^4 + 2 = x^4 + 2 = f(x), \quad even.$

(b) $g(x) = x^3 + 3x : \quad g(-x) = (-x)^3 + 3(-x) = -x^3 - 3x = -(x^3 + 3x) = -g(x), \quad odd.$

(c) $h(x) = 2^x : \; h(-x) = 2^{-x} = (1/2)^x, \quad neither \; (just \; take \; x = 1).$

Exercises

Justify all your answers.

11. Suppose $O(x)$ is an odd function and $E(x)$ is an even function.

What can you say about **a.** $O(x) \cdot E(x)$? **b.** $O(x) + E(x)$? **c.** $|O(x)|$? **d.** $O(|x|)$? **e.** $(O(x))^2$? **f.** $(O(x))^3$? **g.** $(O(x))^2 \cdot E(x)$? **h.** $O(x) \cdot (E(x))^2$?

12. For a function $f : R \to R$ check wether the functions **a.** $|f(x)|$, **b.** $f(|x|)$ are even, odd or neither.

2.2 Linear Functions

A **linear function** is a polynomial of degree 1:

$$f(x) = a \cdot x + b,$$

(well, remember we called it quasi-linear, and the term linear was reserved for $y = ax$).

The graph of linear function is a straight line. Such a line is characterized by its **slope** and **y-intercept**.

If two points (x_0, y_0) and (x_1, y_1) belong to a line then its slope is

$$\frac{y_1 - y_0}{x_1 - x_0}.$$

The slope of a line is the change of y when x is increased by unit. In other words it is the tangens of the angel between the line and x-axes.

Theorem 1 *The slope of the graph of $f(x) = a \cdot x + b$ is a.*

Proof.

$$slope = \frac{f(x+1) - f(x)}{x} = \frac{(a \cdot (x+1) + b) - (a \cdot x + b)}{x} = ...$$

finish it yourself.

The graph of a linear function $y = ax + b$ (a straight line) has two remarkable points:
The **y-intercept**, which is a point where the line intersects the y-axes. This is the point $(x = 0, y = b)$.
And the **x-intercept**, which is a point where the line intersects the x-axes. This is the point $(x = \frac{-b}{a}, y = 0)$.

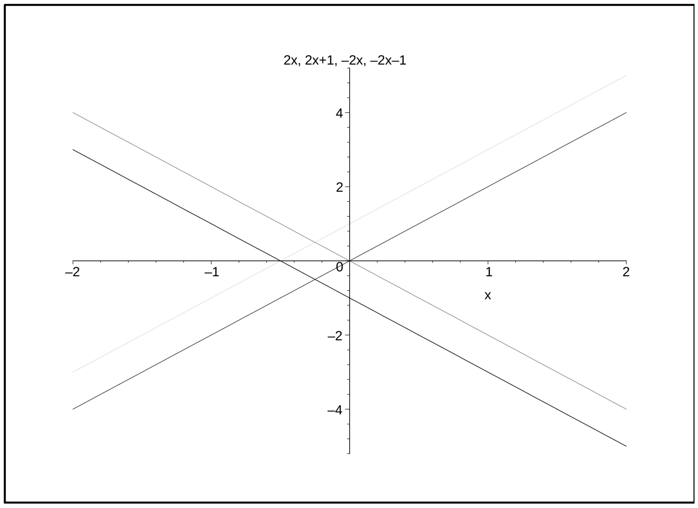
Try to prove yourself the following statements:

A linear function $y = ax + b$ is increasing if and only if (iff) the slope of its graph is positive, that is $a > 0$.

A linear function $y = ax + b$ is decreasing iff the slope of its graph is negative, that is $a < 0$.

Otherwise, if $a = 0$, the function is constant, its graph is a straight line parallel to x-axes.

$> plot(2*x, 2*x+1, -2*x, -2*x-1, x = -2..2, title = "2x, 2x+1, -2x, -2x-1");$



Two lines $y = ax + b$ and $y = cx + d$ are orthogonal if their slopes satisfy the condition $a \cdot c = -1$.

2.2.1 Interpreting the Slope

Recall that the slope measures how much y changes as x increases by unit. So the slope measures the **rate of change** of a function.

If a function $S(t) = v \cdot t$ describes the **movement**, then the slope v is the **speed or velocity**.

If a function $C(q) = c \cdot q$ describes the **cost** of manufacturing of q units of output then the slope measures the cost of the production of **one more unit**. In economics it is called **marginal cost**.

Exercises

13. Write the equation of a straight line which goes trough the points $(1, 5)$ and (3,9).

Solution. $y = ax + b, \quad a =?, \ b =?$. Substitution of $(x = 1, y = 5)$ gives $5 = x + b$. Substitution of $(9, 5)$ gives $9 = 3x + b$. Solving the system

$$\begin{cases} 5 = a + b \\ 9 = 3a + b \end{cases}$$

we obtain that the slope is $a = 2$ and the y-intercept is $b = 3$ so the equation is $y = 2x + 3$. By the way, the x-intercept is

$$2x + 3 = 0, \ x = -3/2.$$

14. Write the equation of a straight line whose x-intercept is -2 and the y-intercept is 1.

15. Find the intersection point of the lines from previous two exercises.

16. Write the equation of a line which passes trough the origin and is orthogonal to $y = x$.

17. Write the equation of a line which passes trough the origin and is orthogonal to $y = 2x$.

18. Write the equation of a line which is orthogonal to the line from exercise 1 and passes trough the point (1,3).