Some computational experience with Hesselink strata

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Null-cones

Let a reductive algebraic group G act linearly on a (say, complex) vector space V.

By the null-cone of this action is meant the set of common zeros of all homogeneous invariants of positive degree for this action.

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It will be denoted $\mathfrak{N}_G(V)$ (sometimes just $\mathfrak{N}(V)$, if G is clear from the context).

An element is in the null-cone if and only if the Zariski closure of its orbit contains zero.

The Hilbert-Mumford criterion

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To say what this means precisely, we need a norm on $\mathfrak{g}$. For that, let us choose an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$ that takes rational values and is positive definite on the rational vector space $\mathfrak{t}(\mathbb{Q})$ of elements with rational eigenvalues of the Lie algebra $\mathfrak{t}$ for some maximal torus $T \subset G$.

Characteristics

Having such a norm, for a nilpotent element $v \in \mathfrak{N}_G(V)$ we may consider the set $\Lambda(v)$ of all 1-parametric subgroups of G such that the corresponding 1-dimensional torus $S \subset G$ satisfies $\overline{Sv} \ni 0$ and moreover its infinitesimal generator $h \in \mathfrak{g}$, which is a semisimple element of $\mathfrak{g}$ with rational eigenvalues, has smallest possible norm among all semisimple elements of $\mathfrak{g}$ with the same properties. Such h is called a characteristic of the nilpotent element v.

Characteristics and strata

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Following Hesselink, we can now consider an equivalence relation on $\mathfrak{N}_G(V)$: say that $x \sim y$ if $\Lambda(x) = \Lambda(gy)$ for some $g \in G$.

Equivalence classes are the Hesselink strata of $\mathfrak{N}_G(V)$, they form a finite partition of $\mathfrak{N}_G(V)$.

See his “Desingularizations of varieties of nullforms”, *Invent. math.* **55** (1979), 141–163.

The adjoint case

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For representations "close" to the adjoint representation strata also consist of single orbits. Namely, by the theory of theta-groups of Vinberg, this is so for the representation of the algebraic group with Lie algebra $\mathfrak{g} = \mathfrak{a}^{(0)}$ on $V = \mathfrak{a}^{(1)}$, for a cyclically graded Lie algebra

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By analogy, in the general case elements of V belonging to $\mathfrak{N}(V)$ are also called nilpotent.

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It seems to be an open question whether, in case $\mathfrak{N}_G(V)$ contains only finitely many G-orbits, one can choose a scalar product in such a way that each stratum would consist of a single orbit.

Structure of strata

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These actions integrate to actions of a parabolic subgroup $G_{\geqslant 0}(h) \subset G$, its Levi subgroup $G_0(h)$, and the reductive subgroup $\tilde{G}_0(h)$ of $G_0(h)$ respectively.

Structure of strata

Hesselink proved that each stratum H is an irreducible subvariety of $\mathfrak{N}_G(V)$, open in its Zariski closure $\overline{H}$, and there exists an isomorphism $\varphi : H \to E$ onto an invariant open subset of the total space of a homogeneous vector bundle E over G/P_H, where $P_H = G_{\geqslant 0}(h)$ for a characteristic h of an element of H.

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Moreover there exists a morphism $\pi : E \to \overline{H}$ which is resolution of singularities of the variety $\overline{H}$ such that $\pi|_{\pi^{-1}H} = \varphi^{-1}$. In particular this gives a resolution of singularities for each irreducible component of the null-cone $\mathfrak{N}_G(V)$.

Structure of strata – the gradient flow approach

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Let $\mathbf{P}V$ be the projective space of 1-dimensional subspaces of V. Each $\ell \in \mathbf{P}V$ gives rise to the function φ_ℓ on G given by

$$\varphi_\ell(g) = \|gv\|^2,$$

where $\| - \|$ is the norm determined by a nonsingular Hermitian form on V invariant under the action of a maximal compact subgroup $K \subset G$, and v is a unit vector in ℓ.

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The moment map $m : \mathbf{P}V \to \mathfrak{k}^*$ to the dual space of the tangent algebra $\mathfrak{k}$ of K is in our case given by $m(\ell)(k) = -\frac{1}{4} d_e \varphi_\ell(ik)$

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Linda Ness in "A Stratification of the Null Cone Via the Moment Map" (*Amer. J. Math.* **106** (1984), 1281–1329) studied the gradient flow of the map

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where now $|| - ||$ is induced on $\mathfrak{k}^*$ by our chosen norm on $\mathfrak{g}$.

She showed that those critical points of $||m||^2$ which are not minima all lie in $\mathfrak{N}_G(V)$, and the Hesselink stratification corresponds to the stratification of $\mathfrak{N}_G(V)$ into attraction basins of those critical points under the gradient flow of $||m||^2$.

Structure of strata – *nilpotent pieces* of Lusztig

The condition $0 \in \overline{Sv}$ above can be viewed as stating that certain morphism of schemes $f : \mathbf{G}_\mathrm{m} \to V$ extends to a morphism $\tilde{f} : \mathbf{G}_\mathrm{a} \to V$. The coordinate ring of the fiber of this morphism is $\mathbf{k}[t]/(t^m)$ (**k** the base field), and this m gives a refined information about elements of the nullcone.

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There is a description of strata in terms of unipotent pieces of Lusztig, which should work in positive characteristic too. This should be described in the talk by Geck.

Clarke & Premet, "The Hesselink stratification of nullcones and base change", *Invent. math.* **191** (2013), pp. 631–669.

The Popov algorithm

Popov in “The Cone of Hilbert Nullforms” (*Proc. Steklov Math. Inst.* **241** (2003), 177–194) solved the problem of a constructive description of strata.

He gave an algorithm producing a finite subset $\mathcal{X} \subset \mathfrak{t}$ of the Lie algebra of a maximal torus of G with the following property.

For each $h \in \mathcal{X}$ consider the open subset of $V_{\geqslant 2}(h)$,

$$V_{\geqslant 2}(h)^\circ := \pi_2^{-1}\left(V_2(h) \setminus \mathfrak{N}_{\tilde{G}_0(h)}(V_2(h))\right)$$

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consisting of vectors whose projection under $\pi_2 : V_{\geqslant 2}(h) \to V_2(h)$ does not belong to the null-cone for the action of $\tilde{G}_0(h)$ on $V_2(h)$.

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Then the sets $G \cdot V_{\geqslant 2}(h)^\circ$, with h running through $\mathcal{X}$, exhaust without repetition the collection of all Hesselink strata of the null-cone $\mathfrak{N}_G(V)$.

The Popov algorithm

$$
\begin{tikzcd}
\rho^{-1}(H) \arrow[r, "\cong"] \arrow[d, hook] & H = G \cdot V_{\geqslant 2}(h)^\circ \arrow[d, hook] \\
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$$V_{\geqslant 2}(h)^\circ = \left\{ v_2 + v_{>2} \in V_2(h) \oplus V_{>2}(h) \mid v_2 \notin \mathfrak{N}_{\tilde{G}_0(h)}(V_2(h)) \right\}$$

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For theta-groups and in some other cases "close" to them, each G-orbit has a representative in $V_2(h)$. In general this is not true.

The Popov algorithm

Given $h \in \mathcal{X}$, Popov computes dimension of the corresponding stratum $H = G \cdot V_{\geqslant 2}(h)^\circ$ as

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Popov's algorithm has been realized by Norbert A'Campo, at http://www.geometrie.ch/HNC/hnc.html. We successfully used his program to check the known cases for spinors up to dimension 14 and in dimension 16.

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The Popov algorithm

More recently Willem de Graaf kindly provided us with another implementation of the algorithm that uses the system GAP. Using his version we successfully calculated the spin_{15} case, and checked its agreement with the cases when we were able to run the A'Campo program.

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More recently Willem de Graaf kindly provided us with another implementation of the algorithm that uses the system GAP. Using his version we successfully calculated the spin_{15} case, and checked its agreement with the cases when we were able to run the A'Campo program.

We also used his program to compute practically all cases of classifications of nilpotent orbits that were known before, for ranks up to 8, and checked agreement with them.

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Willem's version works by listing all possible candidates h for characteristics up to the action of the Weyl group of G and then calling itself for each action of the smaller reductive group $\tilde{G}_0(h)$ on the smaller space $V_2(h)$.

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It is irreducible of dimension 2^k for $n = 2k + 1$, or direct sum of two irreducibles each of dimension 2^{k-1}, called half-spinors, for $n = 2k$. In what follows we will everywhere mean the latter.

Spinors

Classification of Spin(n)-orbits of the spin representation for $n \leqslant 12$ has been carried out by J.-I. Igusa (*Amer. J. Math.* **92** (1970), 997–1028).

A CLASSIFICATION OF SPINORS UP TO DIMENSION TWELVE.[1]

By JUN-ICHI IGUSA.

A spinor is an element of the vector space in which the spin representation takes place. The group G that is being represented is the two-sheeted covering group of the special orthogonal group called the spin group. We regard two spinors to be "congruent" or G-equivalent if there exists an element of G which transforms one to another via the spin representation. By a *classification of spinors*, we understand (1) *the decomposition of the space of spinors into equivalence classes or "orbits"* and (2) *the determining of the structure of the stabilizer subgroup of G for each orbit.*

The objective of this paper is to classify spinors up to dimension twelve.

Spinors

The case $n = 13$ was done by Gatti & Viniberghi (Kac and Vinberg, actually),

ADVANCES IN MATHEMATICS 30, 137–155 (1978)

Spinors of 13-dimensional Space

V. GATTI AND E. VINIBERGHI

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

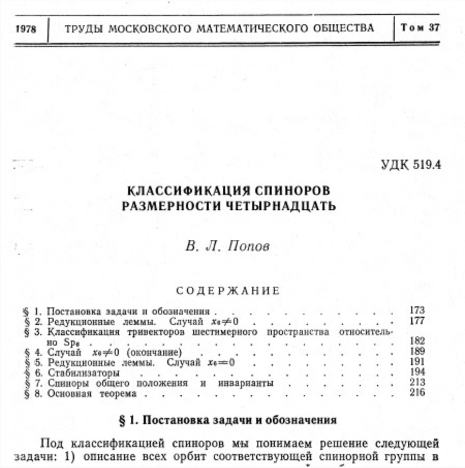
In the present article we develop a method of classification of orbits of an algebraic linear group. The method is demonstrated on the example of classification of spinors of the 13-dimensional space.

This method consists in reducing the orbit classification of a linear group to the classification of some special orbits of its complete regular subgroups (Sect. 1). (We recall that a complete regular subgroup is a subgroup of an algebraic group which is the centralizer of a semisimple element of the Lie algebra).

Let $G \mid V$ be an algebraic linear group. An element $x \in V$ is called *semisimple* if its orbit $G \cdot x$ is closed, and it is called *nilpotent* if $\overline{G \cdot x} \ni 0$. We say that there exists *Jordan decomposition* for $G \mid V$ if any element $x \in V$ can be represented in the form $x = x_s + x_n$, where x_s is a semisimple element, x_n is a nilpotent element for the stabilizer $G_{x_s} \mid V$ of x_s and $G_x \subset G_{x_s}$.

Spinors

and $n = 14$ by V. L. Popov, both in 1978.



1978 | ТРУДЫ МОСКОВСКОГО МАТЕМАТИЧЕСКОГО ОБЩЕСТВА | Том 37

УДК 519.4

КЛАССИФИКАЦИЯ СПИНОРОВ РАЗМЕРНОСТИ ЧЕТЫРНАДЦАТЬ

В. Л. Попов

СОДЕРЖАНИЕ

§ 1. Постановка задачи и обозначения

Под классификацией спиноров мы понимаем решение следующей задачи: 1) описание всех орбит соответствующей спинорной группы в

Spinors

Classification for $n = 16$ was done by Antonyan and Elashvili in 1982.

თბილისის მათემატიკის ინსტიტუტის შრომები, ტ. LXX, 1982
ТРУДЫ ТБИЛИССКОГО МАТЕМАТИЧЕСКОГО ИНСТИТУТА, т. LXX, 1982

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КЛАССИФИКАЦИЯ СПИНОРОВ РАЗМЕРНОСТИ ШЕСТНАДЦАТЬ

Изучение спиноров в размерности 16 представляет особый интерес. Дело в том, что уже в следующих размерностях возникают принципиальные трудности, в то время как спиноры меньших размерностей (за исключением, правда, размерности 15) классифицированы в работах разных авторов (см. по этому поводу [5]).

Основным объектом исследования в нашей работе является градуированная по модулю 2 простая алгебра Ли типа E_8, в которой спиноры размерности 16 могут быть реализованы. К ней мы применяем, предложенный

Spinors

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In fact, except for spin_{13} these are all theta-groups. As mentioned before, for theta-groups each stratum consists of a single orbit and has a representative in $V_2(h)$.

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This is so in particular for spin_n with $n \leqslant 14$ and $n = 16$. Actually, spin_{13} does not arise as a theta-group, but nevertheless it has this property, as computed by Kac and Vinberg.

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So the smallest dimension where infinite families of nilpotent orbits occur is $n = 15$.

Spinors – computations

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An important number associated to each stratum is its modality. According to “Modality of Representations, and Packets for θ-Groups” by Popov (*Lie Groups, Geometry, and Representation Theory*, 2018, 459–479), modality of a stratum is the codimension of orbits of maximal dimension in it.

Spinors – computations

There is one circumstance that is very helpful for our computations, stated concisely as inclusions $\mathrm{spin}_{15} \subset \mathrm{spin}_{16} \subset \mathrm{Ad}(E_8)$.

Note that the group Spin(15) can be realized as a subgroup of Spin(16), namely as the group of fixed points for the outer automorphism of Spin(16) of order two, in such a way that the spin representation of Spin(15) (of dimension 128) is the restriction of one of the irreducible half-spin representations of Spin(16).

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Moreover in its turn Spin(16) can be realized as the group of fixed points of an inner automorphism of order two of the simple Lie algebra of type E_8, acting on the subspace of the Lie algebra $\mathfrak{e}_8$ which is the eigenspace with eigenvalue -1 for the corresponding inner automorphism of $\mathfrak{e}_8$.

Computations

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For example, there are three strata having largest dimension of all, namely 113, with characteristics $(8, 4, 4, 4, 4, 4, 4)$, $(8, 8, 4, 4, 4, 4, 4)$ and $(4, 4, 4, 4, 4, 4, 8)$. Of these, only the first possesses orbits of maximal possible dimension, namely 105 (note that this is the dimension of the group $\mathrm{Spin}(15)$).

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This is actually the only stratum possessing 105-dimensional orbits. Moreover it turns out that if, using the above circumstance, we realize a representative of such an orbit as an element of the algebra $\mathfrak{e}_8$, we obtain a principal nilpotent element of $\mathfrak{e}_8$.

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This is actually the only stratum possessing 105-dimensional orbits. Moreover it turns out that if, using the above circumstance, we realize a representative of such an orbit as an element of the algebra $\mathfrak{e}_8$, we obtain a principal nilpotent element of $\mathfrak{e}_8$. The remaining 113-dimensional strata have maximal dimensions of orbits equal to 104; as nilpotents in $\mathfrak{e}_8$ these orbits have labels $\mathrm{E}_6 + \mathrm{A}_1$ and E_7 respectively.

Representatives with small support

There is also exactly one stratum with representatives of orbits of maximal dimension subregular in $\mathfrak{e}_8$. It has characteristic $(4,4,4,4,0,4,4)$ and maximal dimension of orbit 104.

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For each stratum we found representatives of orbits of maximal dimensions which are linear combinations of as few weight vectors as possible.

Realizing the corresponding weights in Spin(16) and E_8 may provide additional insight into the structure of the strata.

Representatives with small support

As it happens, the largest number of weight vectors needed to produce representatives of orbits of maximal dimension in each stratum is 8, except for the stratum with characteristic $(4, 0, 4, 0, 4, 4, 0)$ where we could not find a representative with less than 9 weight vectors.

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Also, to achieve $\leqslant 8$ weight vectors, in some cases we had to pick a representative with non-generic projection onto V_2.

Actually we do not know whether in each stratum there is a vector with orbit of maximal dimension that projects onto a vector in general position with respect to the action of $\tilde{G}_0(h)$ on V_2.

Distinguished nilpotents

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By analogy with the adjoint representation, it is interesting to study a generalization of this notion: those orbits in the nullcone with the property that stabilizers of their representatives have trivial reductive part.

Our computations show that among 169 strata of spin_{15} precisely 105 possess orbits of maximal dimension with this property.

It is interesting that of these 105 “distinguished strata” only 12 have an orbit open in it, for all others maximal dimensions of their orbits are strictly smaller.

Strata with Zariski dense orbits

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More generally, modalities of strata range from 0 to 9; numbers of strata with respective modalities are as follows:

modality	0	1	2	3	4	5	6	7	8	9
quantity	57	29	21	8	14	12	12	7	7	2

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Largest modality, 9, is achieved twice, on two of the three already mentioned strata of largest dimension 113, with maximal dimension of orbit 104. (The remaining 113-dimensional stratum, with orbits of maximal dimension 105, has modality 8.)

The rôle of V_2

Recall that for adjoint representations and, more generally, for Vinberg theta-groups, all strata consist of single orbits. Moreover in these cases each characteristic h has the property that $G_0(h)$ possesses an open orbit on $V_2(h)$.

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The number of strata having both a representative of an orbit of maximal dimension in $V_2(h)$ and an open $G_0(h)$-orbit on $V_2(h)$ is 48.

Some computation stages

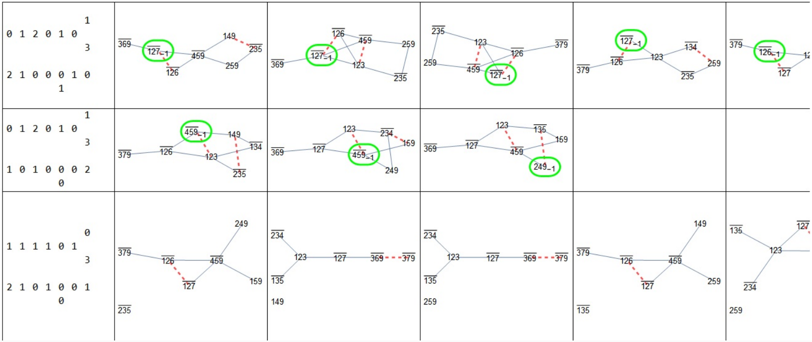
This is how we ensure that a vector from V_2 has G_0-orbit of maximal possible dimension.

```
*GAP* - GNU Emacs at DESKTOP-M4AP3T8
File  Edit  Options  Buffers  Tools  Complete  In/Out  Signals  Help
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> grach(g,V,hes[74],vecs2[74]);
     -72/5 :     0    ->     1 : 0
     -68/5 :     0    ->     1 : 0
     -64/5 :     0    ->     1 : 0
       -12 :     1    ->     2 : 1
     -56/5 :     1    ->     2 : 1
     -52/5 :     1    ->     2 : 1
     -48/5 :     1    ->     3 : 1
     -44/5 :     2    ->     4 : 2
        -8 :     3    ->     4 : 3
     -36/5 :     3    ->     5 : 3
     -32/5 :     3    ->     6 : 3
     -28/5 :     4    ->     6 : 4
     -24/5 :     4    ->     6 : 4
        -4 :     5    ->     7 : 5
     -16/5 :     5    ->     7 : 5
     -12/5 :     5    ->     7 : 5
      -8/5 :     5    ->     7 : 5
      -4/5 :     6    ->     7 : 6
         0 :     7    ->     7 : 7
       4/5 :     6    ->     6 : 6
       8/5 :     5    ->     6 : 5
      12/5 :     5    ->     6 : 5
      16/5 :     5    ->     5 : 4
         4 :     5    ->     4 : 4
      24/5 :     4    ->     4 : 4
      28/5 :     4    ->     3 : 3
      32/5 :     3    ->     2 : 2
      36/5 :     3    ->     2 : 2
         8 :     3    ->     2 : 2
      44/5 :     2    ->     1 : 1
      48/5 :     1    ->     1 : 1
      52/5 :     1    ->     1 : 1
      56/5 :     1    ->     0 : 0
        12 :     1    ->     0 : 0
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> 
U:**-  *GAP*          Bot (110,5)    (Inferior Gap:run)
```

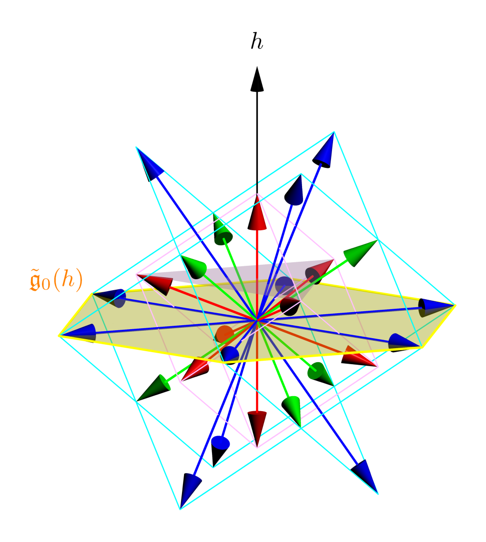
Some computation stages

These are the Dynkin schemes for some of the combinations of weight vectors that GAP proposes as orbit representatives.

```
In[9]:= Grid[
    Map[Function[p, Prepend[Map[diagre[#[[1]], #[[4]] /. shortnames, #[[5]]] &, Select[lnew, #[[{2, 3}]] == p &]],
      Grid[{{"", "", "", "", "", "", p[[1]][[7]]}, Take[p[[1]], 6], {"", "", "", "", "", "", p[[1]][[8]]}, ConstantArray["", 9],
        p[[2]][[{8, 7, 6, 5, 4, 3, 1}]], {"", "", "", "", p[[2]][[2]]}}]]], newcases], Alignment → Left, Frame → All, ItemSize → Full]
```



Possible simplifications?



One of the directions for spin_7

Possible simplifications?

What is the largest class of representations V with a *generalized Jacobson-Morozov theorem*: for each v in the nullcone, there is a semisimple h with $hv = 2v$ and v not in the $\tilde{\mathfrak{g}}_0(h)$-nullcone of $V_2(h)$?

As already said, this is true for theta-groups. It is also true for spin_{13}, as shown by Kac and Vinberg.

Possible simplifications?

Recently such representations have been investigated by Holweck and Oeding in `https://arxiv.org/abs/2206.13662` (their presentation few days ago: Jordan Decompositions of Tensors and Applications to Quantum Information at the Joint Mathematics Meeting in Boston). They observe examples, more general than theta-groups, of pairs of representations V, V' of a reductive $\mathfrak{g}$ admitting a $\mathfrak{g}$-equivariant map $\cdot : V \otimes V' \to \mathfrak{g}$ with certain properties

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In some cases one can take $V' = V$ – for example, action of the even part of a reductive (?) Lie superalgebra on its odd part. In some examples (including theta-groups) one needs $V' = V^*$, but there are others too.

Some more computational experiments

Using Willem's program we looked at the behaviour of strata for direct sums of several copies of the same representation V of a reductive group G.

In the work of Hesselink, Draisma, Le Bruyn and others, there is apparent certain kind of stabilization. After the direct sum is large enough that multiplicity of each irreducible summand of V exceeds the rank of G, the number of strata remains the same.

In all cases that we have been able to compute we saw this happen.

Actually, in most cases even numbers of strata of the same dimension stay the same.

Some more computational experiments

For example, here is the case of the irrep with highest weight $(1,0,1)$ for B_3:

3	4	5	6
[11, 1]	[12, 1]	[13, 1]	[14, 1]
[14, 2]	[16, 2]	[18, 2]	[20, 2]
[15, 1]	[18, 1]	[21, 1]	[24, 1]
[17, 1]	[21, 1]	[25, 1]	[29, 1]
[19, 1]	[23, 1]	[27, 1]	[31, 1]
[23, 2]	[28, 2]	[33, 2]	[38, 2]
[26, 2]	[32, 2]	[38, 2]	[44, 2]
[27, 1]	[33, 1]	[39, 1]	[45, 1]
[30, 1]	[37, 1]	[44, 1]	[51, 1]
[41, 1]	[52, 1]	[63, 1]	[74, 1]
[42, 1]	[54, 1]	[66, 1]	[78, 1]
[45, 1]	[57, 1]	[69, 1]	[81, 1]
[47, 1]	[60, 1]	[73, 1]	[86, 1]
[51, 1]	[65, 1]	[79, 1]	[93, 1]
[54, 1]	[69, 1]	[84, 1]	[99, 1]
[55, 1]	[71, 1]	[87, 1]	[103, 1]
[57, 1]	[73, 1]	[89, 1]	[105, 1]
[59, 1]	[76, 1]	[93, 1]	[110, 1]
[60, 1]	[77, 1]	[94, 1]	[111, 1]
[62, 1]	[80, 1]	[98, 1]	[116, 1]
[65, 1]	[84, 1]	[103, 1]	[122, 1]
[66, 1]	[85, 1]	[104, 1]	[123, 1]
[72, 1]	[93, 1]	[114, 1]	[135, 1]
[77, 1]	[101, 1]	[125, 1]	[149, 1]
[78, 1]	[102, 1]	[126, 1]	[150, 1]
[80, 3]	[104, 3]	[128, 3]	[152, 3]
[81, 2]	[105, 2]	[129, 2]	[153, 2]

But there are cases, e. g. $(1,1,1)$ for C_3, when, although the total number of strata remains the same (namely 1495), these multiplicities change.

Thank you!