Relating diagonalizable and nilpotent operators

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Algebra-Seminar Jena

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Linear algebra freshman stuff

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(equivalently, there is a vector v such that v, Nv, NNv, ..., $N^{n-1}v$ form a basis, and $N^n v = 0$).

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To a partition $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, $n_1 + \cdots + n_k = \dim(V)$ corresponds the class of

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A “diletant’s thought”: the Weyl group is in this case a symmetric group, and its conjugacy classes are also indexed by partitions! Coincidence or…?

Cyclic elements

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The centralizer

$$\mathfrak{z}_{\mathfrak{g}}(h) = \mathrm{Ker}(\mathrm{ad}_h) = \{x \in \mathfrak{g} \mid [h, x] = 0\}$$

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A semisimple element is regular if this centralizer does not contain anything else.

"Unusual" Cartan subalgebras

The "usual" Cartan subalgebra for $\mathfrak{sl}(7)$

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix}, \text{ with } \sum_i a_i = 0$$

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An “unusual” one (centralizer of a cyclic element)

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_6 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_6 & 0 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 & 0 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_6 & 0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 & a_1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \end{pmatrix}$$

“Unexpected” nilpotents

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$$N^3 = -(\lambda^2 + \mu^2)N$$

Евгений Борисович Дынкин

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1952 МАТЕМАТИЧЕСКИЙ СБОРНИК Т. 30 (72), № 2

Полупростые подалгебры полупростых алгебр Ли

Е. Б. Дынкин (Москва)

Изучение полупростых подалгебр полупростых алгебр Ли* (или, что равносильно, связных полупростых подгрупп полупростых групп Ли) важно как для алгебры, так и для геометрии. Как показал А. И. Мальцев [11], к этому вопросу сводится более общая задача изучения полупростых подалгебр в любых алгебрах Ли, задача о построении всех алгебр Ли с данным радикалом и др. С другой стороны, изучение транзитивных групп преобразований равносильно изучению пар «группа, стационарная подгруппа», откуда видно значение указанной задачи для геометрии.

Исследование полупростых подалгебр в произвольных полупростых алгебрах Ли легко сводится к исследованию полупростых подалгебр в простых алгебрах (см. [11]). Простые алгебры Ли исчерпываются четырьмя классическими сериями A_n, B_n, C_n, D_n** и пятью особыми алгебрами E_6, E_7, E_8, F_4, G_2. Изучение полупростых подалгебр алгебры A_n равносильно изучению всевозможных линейных представлений полупростых алгебр Ли. Основные результаты в этом направлении были получены Э. Картаном [16] и Г. Вейлем [21]. Описание полупростых подалгебр в алгебрах B_n, C_n и D_n было дано А. И. Мальцевым [11]. Что же касается особых алгебр, то среди них А. И. Мальцевым были изучены лишь простейшая алгебра G_2 и, частично, F_4***. Между тем, не говоря уже об общей теории, которая, таким образом, остается незавершенной, решение ряда важных вопросов, относящихся к классическим группам Ли, также зависит от построения полной классификации полупростых подгрупп особых групп.

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The Jacobson-Morozov theorem: for any nilpotent $e \in \mathfrak{g}$ one can find f and h as above!

This makes classification of nilpotents up to conjugacy equivalent to the classification of $\mathfrak{sl}(2)$-subalgebras.

The element h corresponding to e in the $\mathfrak{sl}(2)$-triple (e, f, h) is called the Dynkin characteristic of e. Eigenvalues of ad_h are integers, and one obtains the ad_h-eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{-d \leqslant k \leqslant d} \mathfrak{g}^{(k)},$$

with $e \in \mathfrak{g}^{(2)}$, $f \in \mathfrak{g}^{(-2)}$, $h \in \mathfrak{g}^{(0)}$, and d called depth of e.

Fixing a Cartan subalgebra $\mathfrak{t}$, and a system of positive roots such that e is a linear combination of positive root vectors, one can choose h from $\mathfrak{t}$ in such a way that values of all simple roots on it are nonnegative.

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Reminder

Brief recall — roots of a semisimple Lie algebra $\mathfrak{g}$ with chosen Cartan subalgebra $\mathfrak{t}$ are elements of the dual space $\mathfrak{t}^*$, i. e. linear forms α on $\mathfrak{t}$ such that there is an $x \in \mathfrak{g}$ with

$$\forall h \in \mathfrak{t}\ [h, x] = \alpha(h)x.$$

For each root α, the space $\mathfrak{g}_\alpha$ of all x as above is 1-dimensional, and its nonzero elements are called root vectors.

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The Killing form

$$\langle x, y\rangle = \operatorname{trace}(\mathrm{a}d_x \circ \mathrm{a}d_y)$$

pairs $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$, all other $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ being mutually orthogonal.

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Simple roots form a basis of $\mathfrak{t}^*$, and W is already generated by reflections in them only. Their Gram matrix with respect to $\langle -, - \rangle$ is encoded in the famous Dynkin diagram which determines the isomorphism type of $\mathfrak{g}$.

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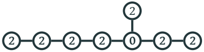
e. g. 

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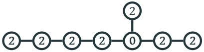
e. g. 

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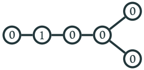
The "least general" (or "most degenerate") nilpotents are the single root vectors:

e. g. 

is the wDd of a root vector in $\mathfrak{so}(12)$ (type D_6).

THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By Bertram Kostant.

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1. Introduction.

1. Let $\mathfrak{g}$ be a complex simple Lie algebra and let G be the adjoint group of $\mathfrak{g}$. It is by now classical that the Poincaré polynomial $p_G(t)$ of G factors into the form,

$$p_G(t) = \prod_{i=1}^{l} (1 + t^{d_i}), \tag{1.1.1}$$

where l is the rank of $\mathfrak{g}$ and the d_i are odd integers. In this paper the integers m_i (elsewhere, sometimes $m_i + 1$) defined by $d_i = 2m_i + 1$ will be called the exponents of $\mathfrak{g}$. No doubt one of the reasons the problem of finding the exponents turned out to be as difficult as it was, is that there was no way known by which these numbers could be determined from a direct examination of the structure of $\mathfrak{g}$, particularly the root structure. The first procedure for extracting the exponents from the root structure of $\mathfrak{g}$ was found by R. Bott. The proof of the validity of this procedure depends upon Morse theory.[2] A second and much simpler way, which we shall presently describe, of "reading off" the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. (It is interesting that Shapiro discovered the procedure by misinterpreting the method of Bott.) However, even though one verifies

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A simple G has the same cohomology as a product of odd-dimensional spheres, but is not in general homotopy equivalent to it;

Bertram Kostant, 1959

Kostant was first to introduce and use cyclic elements. He showed that if e is regular nilpotent in a simple $\mathfrak{g}$ then $e + F$ is regular semisimple for any nonzero $F \in \mathfrak{g}^{(-d)}$.

Grading, as above, is with respect to the characteristic of e; in this case $\mathfrak{g}^{(-d)}$ is one-dimensional.

Studying the corresponding SL(2)-subgroup in the adjoint group G he found a transparent description of fundamental invariants for the adjoint action; in particular, of their degrees, which coincide with the Betti numbers of G.

A simple G has the same cohomology as a product of odd-dimensional spheres, but is not in general homotopy equivalent to it; note that SL(2) (over $\mathbb{C}$) is homotopy equivalent to the 3-dimensional sphere. Its maximal compact subgroup SU(2) *is* a 3-sphere.

Tonny A. Springer

Inventiones math. 25, 159–198 (1974)

Regular Elements of Finite Reflection Groups

T.A. Springer (Utrecht)

Introduction

If G is a finite reflection group in a finite dimensional vector space V then $v \in V$ is called regular if no nonidentity element of G fixes v. An element $g \in G$ is regular if it has a regular eigenvector (a familiar example of such an element is a Coxeter element in a Weyl group). The main theme of this paper is the study of the properties and applications of the regular elements. A review of the contents follows.

The space $\mathfrak{g}^{(-d)}$ is also one-dimensional for subregular nilpotents.

Springer proved that, and exhibited several further examples of *distinguished* nilpotents e with regular semisimple cyclic elements $e + F$.

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These might already have more than one-dimensional $\mathfrak{g}^{(-d)}$, in general.

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A nilpotent is distinguished if its centralizer in $\mathfrak{g}$ only contains nilpotent elements, no semisimple ones.

These might already have more than one-dimensional $\mathfrak{g}^{(-d)}$, in general.

Both Kostant and Springer provided a way to assign to a nilpotent a conjugacy class in the Weyl group W.

Эрнест Борисович Винберг

Эрнест Борисович Винберг, 1976

СЕРИЯ
МАТЕМАТИЧЕСКАЯ
ТОМ 40, № 3, 1976

УДК 519.4

Э. Б. ВИНБЕРГ

ГРУППА ВЕЙЛЯ ГРАДУИРОВАННОЙ АЛГЕБРЫ ЛИ

Роль группы Вейля в теории полупростых алгебр Ли состоит в том, что она описывает эквивалентность элементов картановской подалгебры относительно присоединенной группы. (Заметим, что априори неясно, что эта эквивалентность должна описываться какой-то группой, действующей в картановской подалгебре.) Группа Вейля полупростой алгебры Ли была впервые рассмотрена Г. Вейлем ([2]) в 1925 г. Э. Картан в последующих работах ([8]), ([9]) установил ее важнейшие свойства.

В 1927 г. Э. Картан ([10]) распространил понятие группы Вейля на симметрические пространства. Комплексное симметрическое пространство полупростой группы Ли G с локальной точки зрения есть не что иное, как алгебра Ли $\mathfrak{g}$, градуированная по модулю 2: $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ (где 0,1 — вычеты по модулю 2). Его картановская подалгебра есть подпространство $\mathfrak{c}\subset\mathfrak{g}_1$, состоящее из коммутирующих между собой полупростых элементов и обладающее тем свойством, что всякий полупростой элемент из $\mathfrak{g}_1$ эквивалентен элементу из $\mathfrak{c}$ относительно подгруппы $G_0\subset G$, соответствующей подалгебре $\mathfrak{g}_0\subset\mathfrak{g}$ и действующей в $\mathfrak{g}_1$ естественным образом. (Определяемая этим действием линейная группа есть «группа изотропии» симметрического пространства.) Группа Вейля симметрического пространства есть конечная линейная группа W, действующая в $\mathfrak{c}$ и обладающая тем свойством, что два элемента из $\mathfrak{c}$ эквивалентны относительно G_0 тогда и только тогда, когда они эквивалентны относительно W.

В настоящей работе понятия картановской подалгебры и группы Вейля распространяются на полупростые комплексные алгебры Ли, градуированные по любому модулю $m:\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\ \dots\ +\mathfrak{g}_{m-1}$ (0, 1, ...

Vinberg, 1976

The $\mathbb{Z}$-grading $(\mathfrak{g}^{(i)})_{-d\leqslant i\leqslant d}$ can be, for each natural m, wrapped around to obtain a $\mathbb{Z}/m\mathbb{Z}$-grading, with

$$\mathfrak{g}^{(i \bmod m)} = \bigoplus_{j\equiv i \bmod m} \mathfrak{g}^{(j)}.$$

Vinberg, 1976

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$$\mathfrak{g}^{(i \bmod m)} = \bigoplus_{j\equiv i \bmod m} \mathfrak{g}^{(j)}.$$

More generally, any cyclic grading defines an automorphism of finite order θ of $\mathfrak{g}$; such automorphisms have been classified by Kac, shortly before Vinberg's paper.

Vinberg investigated representations of the algebra $\mathfrak{g}^{(0 \bmod m)}$ and the corresponding group $G^{(0 \bmod m)}$ on the spaces $\mathfrak{g}^{(i \bmod m)}$.

He discovered that these representations share many of the pleasant properties with the adjoint representations. This is the content of his theory of theta-groups.

Vinberg, 1976

Vinberg in particular showed that for each $\mathfrak{g}^{(i \bmod m)}$ there is a Cartan subspace $\mathfrak{t}^{(i \bmod m)}$ and the small Weyl group $W^{(i \bmod m)}$ acting on it, obeying the analog of the Chevalley restriction theorem:

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$$\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W$$

provides an isomorphism between G-invariants of $\mathfrak{g}$ and W-invariants of $\mathfrak{t}$, the same is true for the theta-group

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The Cartan subspaces $\mathfrak{t}^{(i \bmod m)} \subseteq \mathfrak{g}^{(i \bmod m)}$ assemble together into a Cartan subalgebra of $\mathfrak{g}$, of the “unusual” kind I showed before for the $\mathfrak{sl}$ case.

Vinberg, 1976

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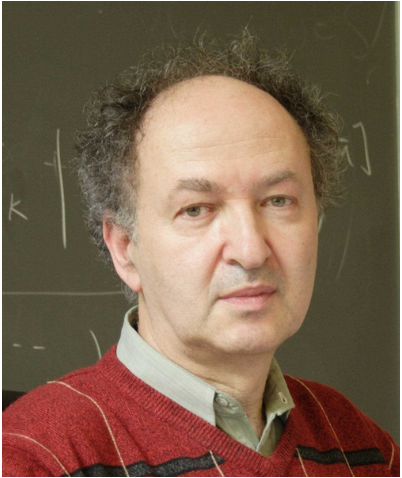
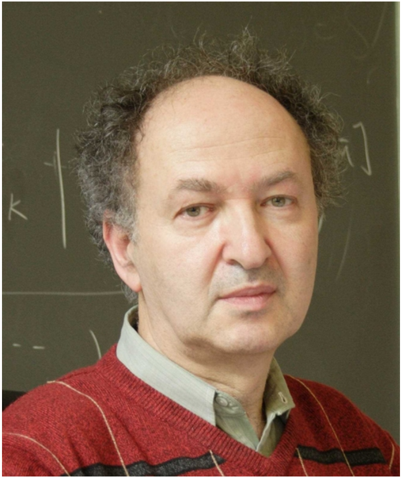
In "our" situation, if we choose to wrap modulo $m = d + 2$ so that $2 \bmod m = -d \bmod m$, then e and F fall into the same $\mathbb{Z}/m\mathbb{Z}$-graded piece; if $e + F$ is semisimple, it belongs to a Cartan subspace there.

David Kazhdan and George Lusztig



David Kazhdan and George Lusztig



David Kazhdan and George Lusztig, 1988

ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 2, 1988

FIXED POINT VARIETIES ON AFFINE FLAG MANIFOLDS

BY

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ABSTRACT

We study the space of Iwahori subalgebras containing a given element of a semisimple Lie algebra over $\mathbf{C}((\varepsilon))$. We also define and study a map from nilpotent orbits in a semisimple Lie algebra over $\mathbf{C}$ to conjugacy classes in the Weyl group.

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§0. Introduction

Let G be a semisimple, simply connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. We denote by $\mathscr{B}$ the variety of Borel subalgebras of $\mathfrak{g}$. For any nilpotent element $N_0 \in \mathfrak{g}$, we consider the closed subvariety $\mathscr{B}_{N_0}$ of $\mathscr{B}$ consisting

David Kazhdan and George Lusztig, 1988

Kazhdan and Lusztig assigned a conjugacy class in W to any nilpotent e.

(Source: video of a talk by Lusztig, 2010)

The case $\mathfrak{g} = \mathfrak{sl}(n)$:

For a nilpotent $e \in \mathfrak{g}$, generic $a_1, a_2, \ldots \in \mathfrak{g}$ and a scalar ε,

$$\tilde{e} := e + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

is a regular semisimple element of $\mathfrak{g}$.

Finding eigenvalues of $\mathrm{ad}_{\tilde{e}}$ one will encounter roots of various degrees $\sqrt[k]{\varepsilon}$.

Clearly one such degree k (LCM of all degrees encountered) will suffice.

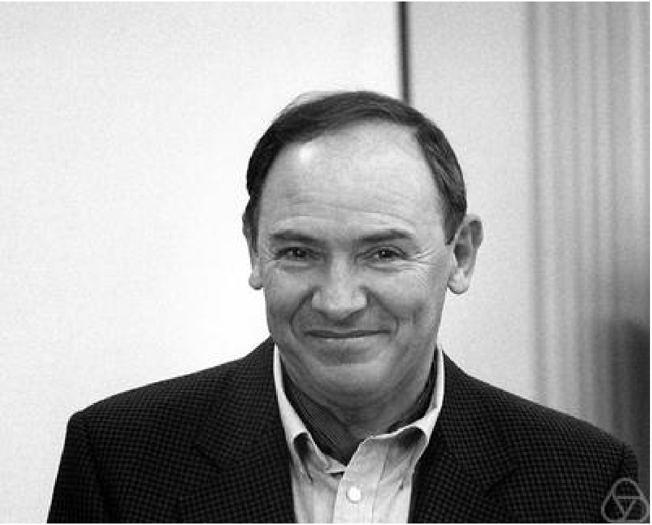
Eigenvalues of $\operatorname{ad}_{\tilde{e}}$ will be certain (rational) functions $\lambda_i(\sqrt[k]{\varepsilon})$ of this root.

If we would pick another one, i. e. multiply $\sqrt[k]{\varepsilon}$ by some kth root of unity $\zeta = \sqrt[k]{1}$, this would produce these λ_i in a different order, i. e. there is a permutation σ with

$$\lambda_i(\zeta\sqrt[k]{\varepsilon}) = \lambda_{\sigma(i)}(\sqrt[k]{\varepsilon}).$$

This σ determines the element of the Weyl group that Kazhdan and Lusztig assign to e.

Victor Kac



Elashvili - Kac - Vinberg, 2013

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CYCLIC ELEMENTS IN SEMISIMPLE LIE ALGEBRAS

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Abstract. We develop a theory of cyclic elements in semisimple Lie algebras. This notion was introduced by Kostant, who associated a cyclic element with the principal nilpotent and proved that it is regular semisimple. In particular, we classfiy all nilpotents giving rise to semisimple and regular semisimple cyclic elements. As an application, we obtain an explicit construction of all regular elements in Weyl groups.

0. Introduction

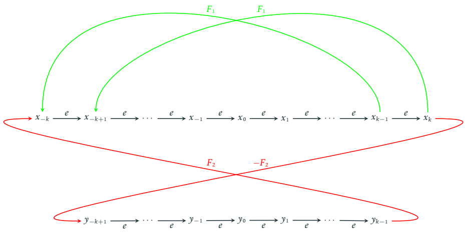
Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let e be a non-zero nilpotent element of $\mathfrak{g}$. By the Morozov–Jacobson theorem, the element e can be included in an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, h, f\}$, so that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Then the eigenspace decomposition of $\mathfrak{g}$ with respect to ad h is a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{j=-d}^{d} \mathfrak{g}_j, \tag{0.1}$$

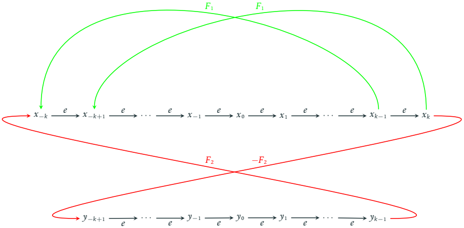
where $\mathfrak{g}_{\pm d} \neq 0$. The positive integer d is called the *depth* of this $\mathbb{Z}$-grading, and of the nilpotent element e. This notion was previously studied, e.g., in [P1].

An element of $\mathfrak{g}$ of the form $e+F$, where F is a non-zero element of $\mathfrak{g}_{-d}$, is called a *cyclic element*, associated to e. In [K1] Kostant proved that any cyclic element, associated to a principal (= regular) nilpotent element e, is regular semisimple, and in [S] Springer proved that any cyclic element, associated to a subregular nilpotent element of a simple exceptional Lie algebra, is regular semisimple as well, and, moreover, found two more distinguished nilpotent conjugacy classes in E_8 with the same property. Both Kostant and Springer use this property in order

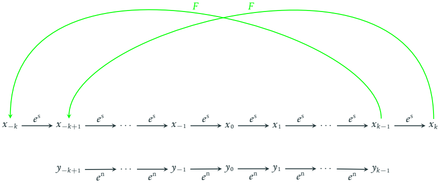
Cyclic elements – $\mathfrak{so}(4k)$, partition $[2k+1, 2k-1]$



Cyclic elements – $\mathfrak{so}(4k)$, partition $[2k+1, 2k-1]$



The cyclic element $e + \lambda_1 F_1 + \lambda_2 F_2$ loses semisimplicity when either $\lambda_2 = 0$



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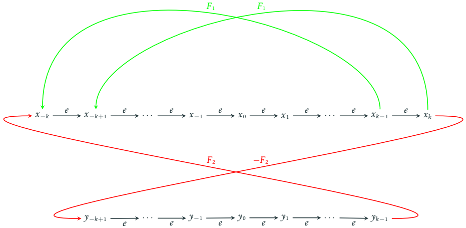
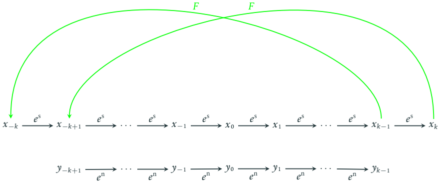
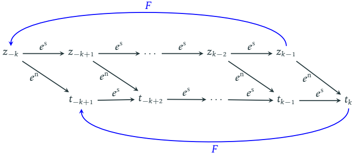
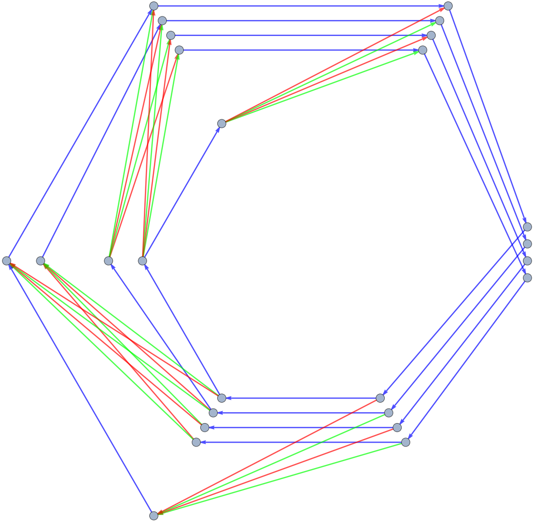


The cyclic element $e + \lambda_1 F_1 + \lambda_2 F_2$ loses semisimplicity when either $\lambda_2 = 0$ or $\lambda_2 = \pm\lambda_1$:



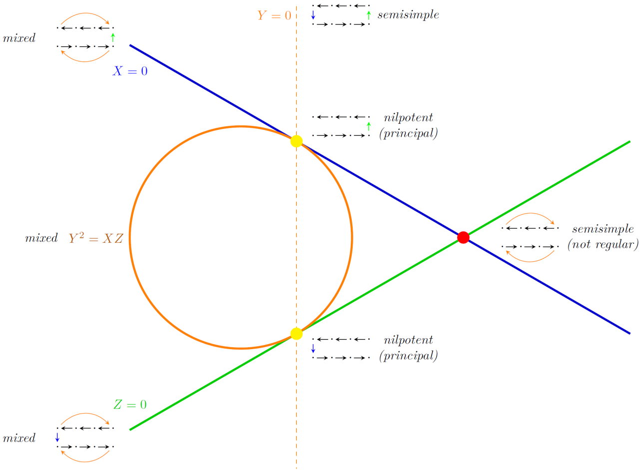


Cyclic elements – one of the F_4 cases



Cyclic elements – semisimplicity loss

THE CASE C_4, PARTITION $[4,4]$



Algebra structure on $\mathfrak{g}^{(-2k)}$

Given $x, y \in \mathfrak{g}^{(-2k)}$, define

$$x * y := [\mathrm{ad}_e^k(x), y].$$

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$$x * y := [\mathrm{ad}_e^k(x), y].$$

As it turns out, if there are semisimple cyclic elements at all, then d must be even, and then $e + F$ loses semisimplicity iff F belongs to some proper subalgebra of $(\mathfrak{g}^{(-d)}, *)$.

Irreducible nilpotent elements of semisimple type

For all irreducible nilpotent elements, $(\mathfrak{g}^{(-d)}, *)$ is a commutative algebra $\mathscr{C}_\lambda(n)$ generated by $p_1, ..., p_n$ with defining relations $p_i^2 = p_i$, $i = 1, \dots, n$, and

$$p_i p_j = \lambda(p_i + p_j), \qquad i \neq j.$$

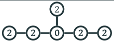
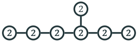
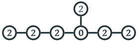
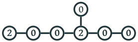
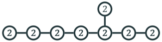
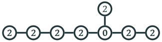
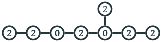
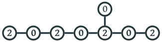
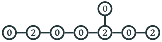
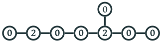
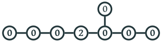
The singular set of $\mathfrak{g}^{(-d)}$ coincides with the union of all proper subalgebras of this algebra.

For classical algebras:

$\mathfrak{g}$	partition	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}^{(-d)}$	$(\mathfrak{g}^{(-d)}, *)$
$\mathfrak{sl}(2k+1)$	$[2k+1]$	$4k$	1	1	1
$\mathfrak{sp}(2k)$	$[2k]$	$4k-2$	1	1	1
$\mathfrak{so}(2k+1)$	$[2k+1]$	$4k-2$	1	1	1
$\mathfrak{so}(4k+4)$	$[2k+3, 2k+1]$	$4k+2$	2	1	$\mathscr{C}_{-k}(2)$

Irreducible nilpotent elements of semisimple type

For exceptional algebras:

$\mathfrak{g}$	weighted Dynkin diagram	label	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}^{(-d)}$	$(\mathfrak{g}^{(-d)}, *)$
E_6		$E_6(a_1)$	16	1	1	1
E_7		E_7	34	1	1	1
E_7		$E_7(a_1)$	26	1	1	1
E_7		$E_7(a_5)$	10	3	$\sigma_3 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{1}{3}}(3)$
E_8		E_8	58	1	1	1
E_8		$E_8(a_1)$	46	1	1	1
E_8		$E_8(a_2)$	38	1	1	1
E_8		$E_8(a_4)$	28	1	1	1
E_8		$E_8(a_5)$	22	2	$\sigma_2 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{2}{7}}(2)$
E_8		$E_8(a_6)$	18	2	σ_3	$\mathscr{C}_{-1}(2)$
E_8		$E_8(a_7)$	10	4	σ_5	$\mathscr{C}_{-\frac{1}{3}}(4)$
F_4		F_4	22	1	1	1
F_4		$F_4(a_2)$	10	2	$\sigma_2 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{1}{3}}(2)$
G_2		G_2	10	1	1	1

Irreducible nilpotent elements of semisimple type – examples

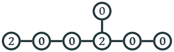
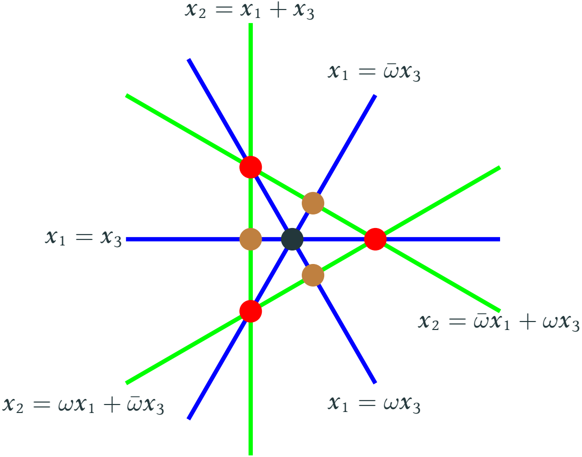
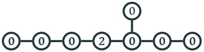
E_7, diagram  , $\dim(\mathfrak{g}^{(-d)}) = 3$, algebra $\mathscr{C}_{-\frac{1}{3}}(3)$ has six 2-dimensional subalgebras.

Image of the singular set in the projective plane: union of six lines.



Irreducible nilpotent elements of semisimple type – examples

E_8, diagram 

, $\dim(\mathfrak{g}^{(-d)}) = 4$, algebra $\mathscr{C}_{-\frac{1}{3}}(4)$ has ten 3-dimensional subalgebras.

Irreducible nilpotent elements of semisimple type – examples

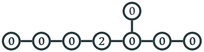
E_8, diagram 

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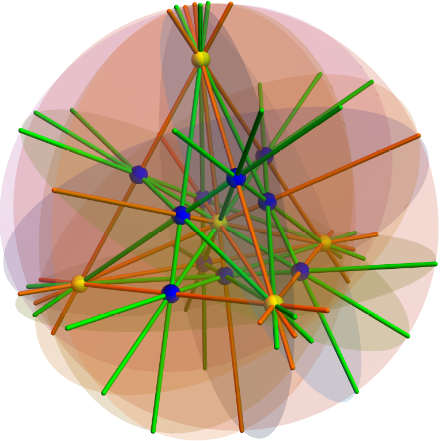
Image of the singular set in the projective 3-space: union of ten planes.



Irreducible nilpotent elements of semisimple type – examples

E_8, diagram $\begin{smallmatrix} & & & & 0 & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{smallmatrix}$, $\dim(\mathfrak{g}^{(-d)}) = 4$, algebra $\mathscr{C}_{-\frac{1}{3}}(4)$ has ten 3-dimensional subalgebras.

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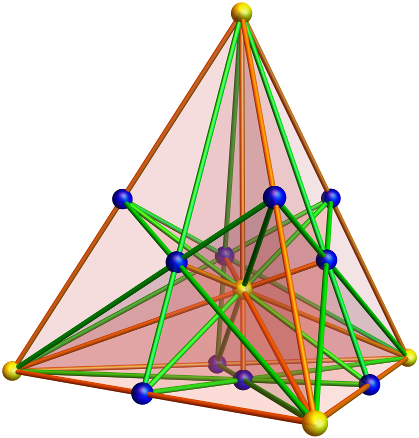

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is satisfied. It implies

$$[L_a, L_b]L_c + [L_b, L_c]L_a + [L_c, L_a]L_b = 0$$

where $L_a(b) := a * b$. Whereas the Jordan identity is

$$[L_a, L_{bc}] + [L_b, L_{ca}] + [L_c, L_{ab}] = 0.$$

Our map vs. Kazhdan-Lusztig

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On each stage, e^{s} is irreducible in its $[\mathfrak{q}, \mathfrak{q}]$, and we know that assigning to it the conjugacy class of W determined by its cyclic element agrees with the Kazhdan-Lusztig map.

Our map vs. Kazhdan-Lusztig: the E_7 case

Orbit					Spaltenstein		our map		Lusztig map
Bala-Carter	(Dynkin)	d	normal form	embedding	Carter	char. pol.	Carter	char. pol.	fibre
A_1		2	C_1	regular	A_1	$\phi_2\phi_1^6$	“———”	“———”	
$2A_1$		2	$2C_1$	regular	$2A_1$	$\phi_2^2\phi_1^5$	“———”	“———”	
$(3A_1)''$		2	$(3C_1)''$	regular	$(3A_1)'$	$\phi_2^3\phi_1^4$	“———”	“———”	
$(3A_1)'$		3	$(3C_1)'$	regular	$(4A_1)'$	$\phi_2^4\phi_1^3$	$(3A_1)''$	$\phi_2^3\phi_1^4$	$(4A_1)''$, $(3A_1)''$
$4A_1$		3	$4C_1$	regular	$[7A_1]$	$[\phi_2^7]$	$(4A_1)'$	$\phi_2^4\phi_1^3$	$7A_1$, $6A_1$, $5A_1$, $(4A_1)'$
A_2		4	A_2	regular	A_2	$\phi_3\phi_1^5$	“———”	“———”	
A_2+A_1			$+C_1$	regular	A_2+A_1	$\phi_3\phi_2\phi_1^4$	“———”	“———”	
A_2+2A_1			$+2C_1$	regular	A_2+2A_1	$\phi_3\phi_2^2\phi_1^3$	“———”	“———”	
A_2+3A_1			$+3C_1$	regular	A_2+3A_1	$\phi_3\phi_2^3\phi_1^2$	“———”	“———”	
$2A_2$		4	$2A_2$	regular	$2A_2$	$\phi_3^2\phi_1^3$	“———”	“———”	
$2A_2+A_1$		5	$2A_2$ $+C_1$	regular	$3A_2$	$\phi_3^3\phi_1$	$2A_2+A_1$	$\phi_3^2\phi_2\phi_1^2$	$3A_2$, $2A_2+A_1$
A_3		6	C_2	folding of A_3	A_3	$\phi_4\phi_2\phi_1^4$	“———”	“———”	
$(A_3+A_1)'$			$+C_1$	+ regular	$(A_3+2A_1)''$	$\phi_4\phi_2^3\phi_1^2$	$(A_3+A_1)''$	$\phi_4\phi_2^2\phi_1^3$	$(A_3+2A_1)''$, $(A_3+A_1)''$
$(A_3+A_1)''$			$+C_1$	+ regular	$(A_3+A_1)'$	$\phi_4\phi_2^2\phi_1^3$	“———”	“———”	$(A_3+A_1)'$
A_3+2A_1			$+2C_1$	+ regular	A_3+3A_1	$\phi_4\phi_2^4\phi_1$	$(A_3+2A_1)'$	$\phi_4\phi_2^3\phi_1^2$	A_3+3A_1, $(A_3+2A_1)'$
$D_4(a_1)$		6	$D_4(a_1)$	regular	$D_4(a_1)$	$\phi_4^2\phi_1^3$	“———”	“———”	
$D_4(a_1)+A_1$			$+C_1$	regular	$D_4(a_1)+A_1$	$\phi_4^2\phi_2\phi_1^2$	“———”	“———”	
A_3+A_2	$(=D_4(a_1)+2A_1)$		$+2C_1$	regular	$2A_3$	$\phi_4^2\phi_2^2\phi_1$	“———”	“———”	$D_4(a_1)+2A_1$, A_3+A_2
$A_3+A_2+A_1$	$(=D_4(a_1)+3A_1)$		$+(3C_1)''$	regular	$2A_3+A_1$	$\phi_4^2\phi_2^3$	“———”	“———”	$2A_3+A_1$, $A_3+A_2+A_1$
A_4		8	A_4	regular	A_4	$\phi_5\phi_1^3$	“———”	“———”	
A_4+A_1			$+C_1$	regular	A_4+A_1	$\phi_5\phi_2\phi_1^2$	“———”	“———”	
A_4+A_2			$+A_2$	regular	A_4+A_2	$\phi_5\phi_3\phi_1$	“———”	“———”	
D_4		10	G_2	folding of D_4	D_4	$\phi_6\phi_2^2\phi_1^3$	“———”	“———”	
D_4+A_1			$+C_1$	+ regular	$[D_4+3A_1]$	$[\phi_6\phi_2^5]$	D_4+A_1	$\phi_6\phi_2^3\phi_1^2$	D_4+3A_1, D_4+2A_1, D_4+A_1
$D_5(a_1)$	$(=D_4+2A_1)$		$+2C_1$	+ regular	$D_5(a_1)$	$\phi_6\phi_4\phi_2\phi_1^2$	D_4+2A_1	$\phi_6\phi_2^4\phi_1$	$D_5(a_1)$
$D_5(a_1)+A_1$	$(=D_4+3A_1)$		$+(3C_1)''$	+ regular	$D_5(a_1)+A_1$	$\phi_6\phi_4\phi_2^2\phi_1$	D_4+3A_1	$\phi_6\phi_2^5$	$D_5(a_1)+A_1$
A_5'		10	C_3	folding of A_5	$(A_5+A_1)''$	$\phi_6\phi_3\phi_2^2\phi_1$	$(A_5)''$	$\phi_6\phi_3\phi_2\phi_1^2$	$(A_5+A_1)''$, $(A_5)''$
A_5''		10	C_3	folding of A_5	$(A_5)'$	$\phi_6\phi_3\phi_2\phi_1^2$	“———”	“———”	
A_5+A_1	$(=(A_5+A_1)'')$		$+C_1$	+ regular	$[A_5+A_2]$	$[\phi_6\phi_3^2\phi_2]$	$(A_5+A_1)'$	$\phi_6\phi_3\phi_2^2\phi_1$	A_5+A_2, $(A_5+A_1)'$
$D_6(a_2)$		10	$D_6(a_2)$	regular	$D_6(a_2)+A_1$	$\phi_6^2\phi_2^3$	$D_6(a_2)$	$\phi_6^2\phi_2^2\phi_1$	$D_6(a_2)+A_1$, $D_6(a_2)$
$E_6(a_3)$	$(=(A_5+A_1)')$	10	$F_4(a_2)$	folding of E_6	$E_6(a_2)$	$\phi_6^2\phi_3\phi_1$	“———”	“———”	
$E_7(a_5)$	$(=D_6(a_2)+A_1)$	10	$E_7(a_5)$	regular	$E_7(a_4)$	$\phi_6^3\phi_2$	“———”	“———”	
A_6		12	A_6	regular	A_6	$\phi_7\phi_1$	“———”	“———”	
D_5		14	B_4	folding of D_5	D_5	$\phi_8\phi_2\phi_1^2$	“———”	“———”	
D_5+A_1			$+C_1$	+ regular	D_5+A_1	$\phi_8\phi_2^2\phi_1$	“———”	“———”	
$D_6(a_1)$			$+C_1'$	+ $IV_{B_4\subset E_7}$	$D_6(a_1)$	$\phi_8\phi_4\phi_1$	D_5	$\phi_8\phi_2\phi_1^2$	
$E_7(a_4)$	$(=D_6(a_1)+A_1)$		$+C_1'+C_1$	+ $IV_{B_4\subset E_7}$ + regular	A_7	$\phi_8\phi_4\phi_2$	D_5+A_1	$\phi_8\phi_2^2\phi_1$	
$E_6(a_1)$		16	$E_6(a_1)$	regular	$E_6(a_1)$	$\phi_9\phi_1$	“———”	“———”	
D_6		18	B_5	folding of D_6	D_6+A_1	$\phi_{10}\phi_2^3$	D_6	$\phi_{10}\phi_2^2\phi_1$	D_6+A_1, D_6
$E_7(a_3)$	$(=D_6+A_1)$		$+C_1$	+ regular	$E_7(a_3)$	$\phi_{10}\phi_6\phi_2$	D_6+A_1	$\phi_{10}\phi_2^3$	
E_6		22	F_4	folding of E_6	E_6	$\phi_{12}\phi_3\phi_1$	“———”	“———”	
$E_7(a_2)$			$+C_1$	+ $IV_{F_4\subset E_7}$	$E_7(a_2)$	$\phi_{12}\phi_6\phi_2$	“———”	“———”	
$E_7(a_1)$		26	$E_7(a_1)$	regular	$E_7(a_1)$	$\phi_{14}\phi_2$	“———”	“———”	
E_7		34	E_7	regular	E_7	$\phi_{18}\phi_2$	“———”	“———”	

Thank you for your patience!