Questions in combinatorics arising from the index of certain series of Lie algebras

Mamuka Jibladze - report on some ongoing work with A. Elashvili and D. Zagier

Oberseminar Algebra, Institut für Mathematik, Universität Jena, June 13, 2023

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If there is a φ such that $d\varphi$ is invertible (i. e. index of $\mathfrak{g}$ is zero), $\mathfrak{g}$ is called *Frobenius*.

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Then

$$r := \sum_{i=1}^{n} b_i \wedge (d\varphi)^{-1}(b^i) = \sum_{i,j} r_{ij} b_i \wedge b_j \in \Lambda^2 \mathfrak{g}$$

satisfies cYBe.

So what?

Very roughly, if r as above satisfies cYBe, then it is a linear term of an element $R \in U(\mathfrak{g})[\![t]\!] \otimes U(\mathfrak{g})[\![t]\!]$ of a formal power series over the tensor square of the universal enveloping algebra of $\mathfrak{g}$, which satisfies

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This in turn can be used to construct, among other things, invariants of knots and links with values in $\mathfrak{g}$-modules.

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And, if the composite $R \circ R$ is not identity, this can be used to "store information about ongoing place exchanges". That is, to upgrade representations of symmetric groups to representations of braid groups.

An unexplored generalization

Side note: $d\varphi$ is indeed the coboundary of φ, in the *Chevalley-Eilenberg complex*

$$C^\bullet(\mathfrak{g};\mathfrak{g}^*) = (\mathrm{Hom}(\Lambda^\bullet\mathfrak{g},\mathfrak{g}^*), d)$$

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As Drinfeld noted back in 80ies, all of the above works more generally with, in place of $d\varphi$, any d-cocycle in $\mathrm{Hom}(\mathfrak{g},\mathfrak{g}^*)$ which happens to be an invertible linear map. This is a generalization of the Frobenius condition, don't know if anybody has studied it.

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But centralizers of regular semisimple elements are precisely Cartan subalgebras, so that for semisimple $\mathfrak{g}$, index = rank.

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In fact, a Lie algebra $\mathfrak{g}$ with nonzero center also cannot be Frobenius, since the center is included in the null space of $d\varphi$ for any $\varphi \in \mathfrak{g}^*$.

The smallest Frobenius Lie algebra

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Form a matrix over the field of rational functions $\mathbb{C}(x_1, ..., x_n)$, with $\sum_k c_{ij}^k x_k$ at the (i,j)-th place.

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The index of $\mathfrak{g}$ is n minus rank of this matrix.

Example: $\mathfrak{sl}_2$

For the algebra $\mathfrak{sl}_2$ of traceless 2×2-matrices, choose this basis: e, f, h with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

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The corresponding matrix over $\mathbb{C}(x_e, x_f, x_h)$ is

	(e)	(f)	(h)
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Rank of this matrix is 2, so the index is $3 - 2 = 1$.

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To compute the index this way, we sort of need to already know *everything* about our algebra.

Random approach

Fact: those φ which achieve the index form a Zariski dense subset of $\mathfrak{g}^*$.

This means that if we pick some φ at random, then dimension of the null space of $d\varphi : \mathfrak{g} \to \mathfrak{g}^*$ will be equal to the index of $\mathfrak{g}$.

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But who knows? Besides, this is a typical black box, you cannot prove any theorems with it.

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This very fact makes it difficult to find at least one such explicitly described φ. What we actually need is a φ which is at the same time generic and very special (most economic) among all generic ones!

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The story seems to originate with an earlier result by Elashvili.

Frobenius Lie algebras — examples

The *maximal parabolic* subalgebras $\mathfrak{p}^+_{(p,q)}$, $\mathfrak{p}^-_{(p,q)}$ in $\mathfrak{sl}_n$, $p + q = n$, are made of traceless matrices with the zero lower left $p \times q$ (resp. upper right $q \times p$) corner.

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For example, $\mathfrak{p}^+_{(4,3)}$ consists of matrices of the form

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Elashvili computed the index of $\mathfrak{p}^{\pm}_{(p,q)}$, it is $\gcd(p,q) - 1$.

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We so get a Frobenius algebra when p and q are mutually coprime.

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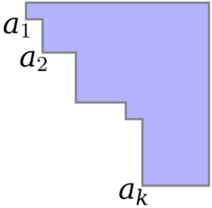

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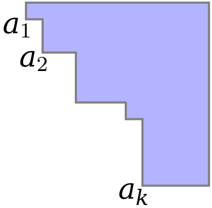

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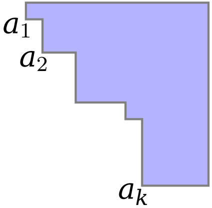
It turns out that $\gcd(a_1,a_2,\ldots,a_k)=1$ is necessary but not sufficient; while $\gcd(a_i,a_j)=1$, $1\leqslant i<j\leqslant k$ is neither necessary nor sufficient.



Theorem of Dergachyov and Kirillov

Dergachyov and Kirillov gave beautiful expression for the index of $\mathfrak{p}^{\pm}_{(a_1,\ldots,a_k)}$

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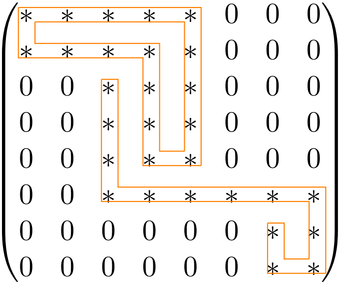

The obtained picture consists of 3 cycles
$\Rightarrow$ the algebra $\mathfrak{b}_{2,4,2;5,3}$ has index 3.

Proof idea — genericity paradox at work

It turns out that, for a form to be generic, it suffices for it to detect values at some crucial spots in the matrix.

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$$\begin{pmatrix}
* & * & * & * & \circledast & 0 & 0 & 0 \\
\circledast & * & * & \circledast & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & \circledast & * & 0 & 0 & 0 \\
0 & 0 & \circledast & * & * & * & * & \circledast \\
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0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 \\
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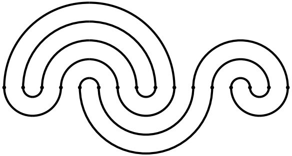
\end{pmatrix}$$

So the needed form φ just picks the matrix entry values at these spots.

Switching to meanders

Instead of angular lines, one can also work with *meanders* of special kind:



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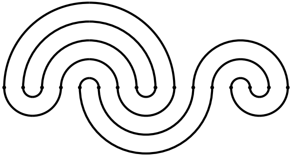
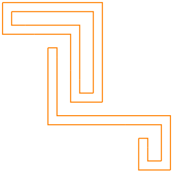


This contains the same information as the previous picture



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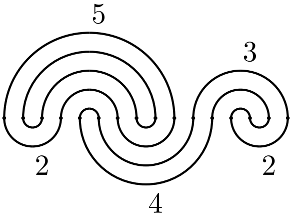

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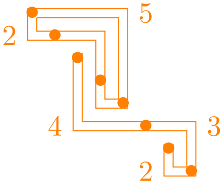

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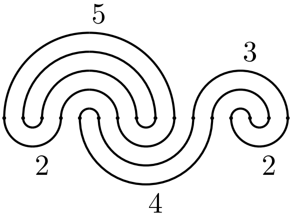

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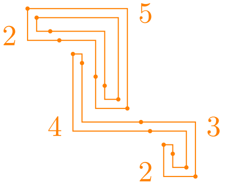
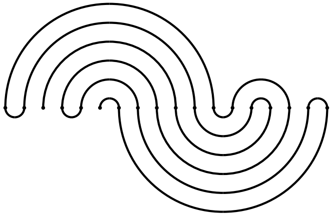
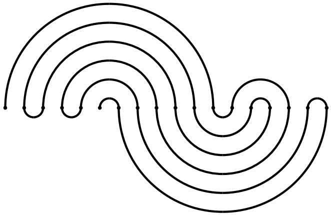


Which of these two graphs is connected?





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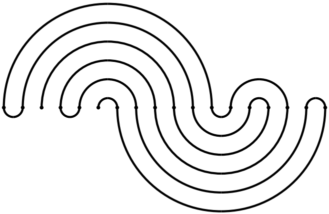
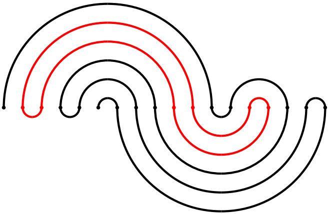
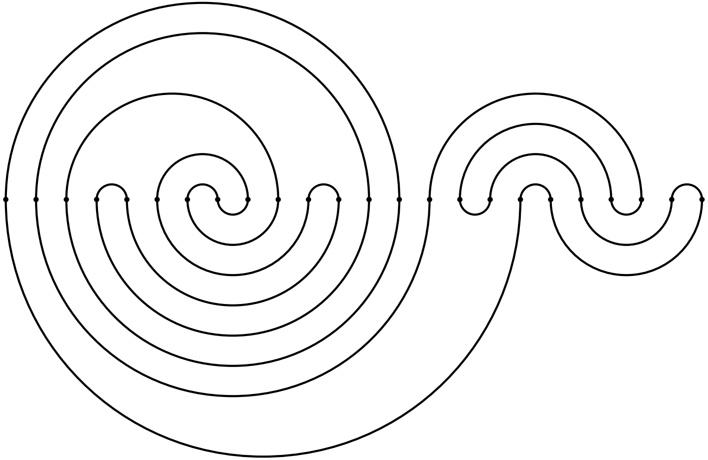




Meanders



SUR UN THÉORÈME DE GÉOMÉTRIE.

Par M. **H. Poincaré** (Paris).

Adunanza del 10 marzo 1912.

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INTRODUCTION.

Je n'ai jamais présenté au public un travail aussi inachevé; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre. J'ai démontré, il y a longtemps déjà,

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Rendiconti del Circolo matematico di Palermo, 33, 1912, 375–407

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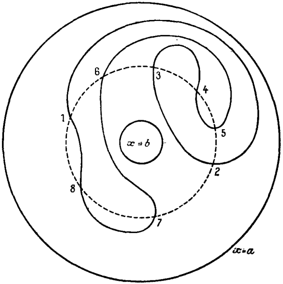
nous rencontrons successivement en parcourant l'horizontale $x = c$ depuis $y = -\infty$ jusqu'à $y = +\infty$.

Si dans cette suite prolongée nous considérons les numéros qui correspondent aux extrémités des arcs primaires extérieurs, ces numéros se suivront dans l'ordre numérique croissant.

5° La différence des numéros des deux extrémités d'un arc primaire (et a fortiori d'un arc quelconque) est au plus égale à $m - 1$.

Le *rang* d'un point d'intersection sera par définition la place qu'occupe son numéro dans la suite (1). Ce qui caractérise un arc ultime, c'est que les rangs de ses deux extrémités sont consécutifs, ainsi que leurs numéros. Ce qui caractérise la distribution normale, c'est que les rangs se suivent dans le même ordre que les numéros, et que l'on peut toujours s'arranger pour que le rang d'un point soit égal à son numéro.



(Fig. 1).

On se rendra mieux compte de ce qui précède en se reportant à la figure 1; sur

A BRANCHED COVERING OF $\mathbf{CP}^2 \to S^4$, HYPERBOLICITY AND PROJECTIVITY TOPOLOGY

V. I. Arnol'd

UDC 514.755

In a 1971 paper on the topology of real algebraic curves [1], I used as a known and obvious fact that the quotient space of the complex projective plane CP^2 with respect to the involution of complex conjugation is diffeomorphic to the four-dimensional sphere S^4.

From Kreck's report at the topology conference in Baku (1987), I learned that this fact, used in the contemporary differential topology of four-dimensional manifolds, is now known as the Kuiper–Massey theorem, since the proof mentioned by Kuiper in [2] establishes only a homeomorphism and piecewise-linear equivalence of the quotient to the four-dimensional sphere, whereas the proof of a diffeomorphism "is considerably more complex and relies on the general theory of smoothing of four dimensional manifolds."

Since my original proof is completely elementary and establishes at once a diffeomorphism of the quotient to the four-dimensional sphere, I decided to publish it here. The proof is based on a quite unexpected connection between the involution of complex conjugation and the geometry of hyperbolic polynomials, i.e., the principal symbols of hyperbolic equations with partial derivatives.

And, specifically, the sphericity of the quotient is a particular case of a more general fact: the components of the intersection of the characteristic cone of a hyperbolic equation with a sphere centered at the origin which correspond to successive waves, first, second, etc., are homeomorphic to spheres and possess specific characteristics of convexity. In particular, the first wave corresponds to the convex component, and this simple general fact

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В. И. Арнольд, “Разветвленное накрытие $\mathbf{CP}^2 \to \mathbf{S}^4$, гиперболичность и проективная топология”, Сиб. мат. ж., 29:5 (1988), 36–47

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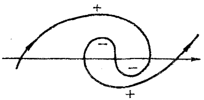

Fig. 1. Signs of the arcs of a meander in the plane.

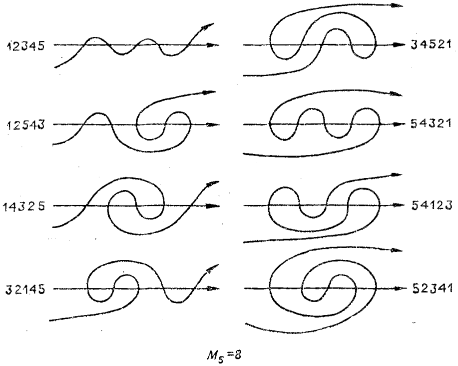

Fig. 2. Meandering permutations on five elements.

Now consider a meander in the projective plane RP^2 with a fixed oriented line RP^1. To define the sign of an arc, we pass to a covering sphere. The line is doubly covered by an

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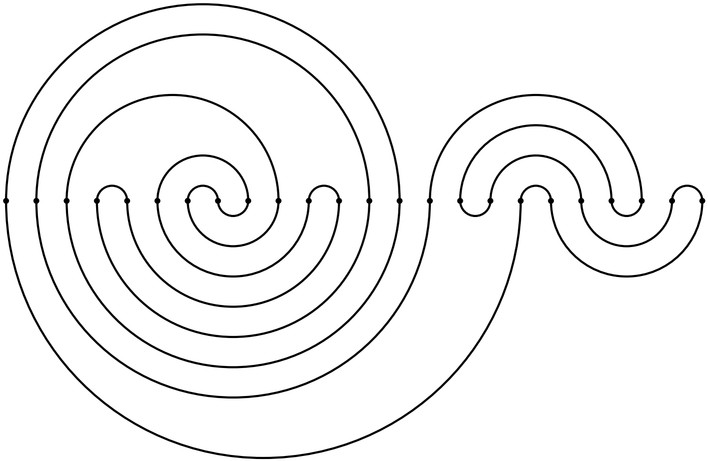

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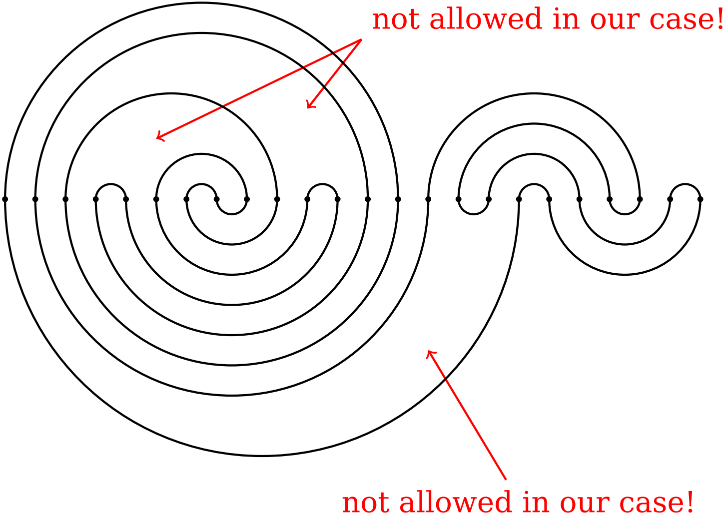

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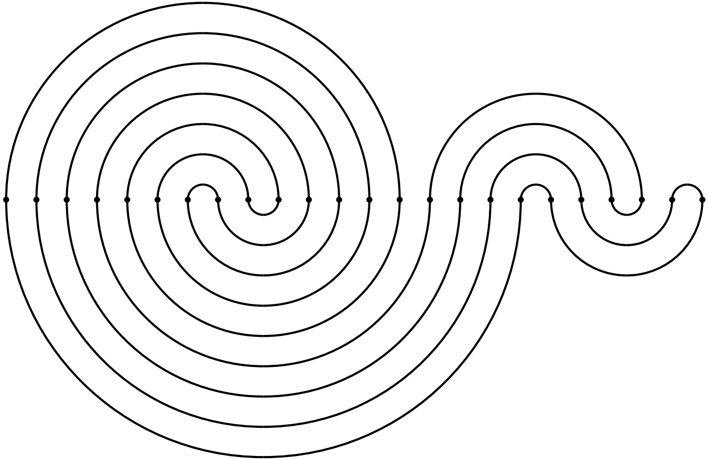

Enumeration, asymptotics

n	M_n
1	1
2	2
3	8
4	42
5	262
6	1828
7	1380
8	110954
9	933458
10	8152860
11	73424650
12	678390116
13	6405031050
14	61606881612

Enumeration, asymptotics

n	M_n	L_n
1	1	1
2	2	2
3	8	6
4	42	14
5	262	34
6	1828	68
7	1380	150
8	110954	296
9	933458	586
10	8152860	1140
11	73424650	2182
12	678390116	4130
13	6405031050	7678
14	61606881612	14368

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9	933458	586
10	8152860	1140
11	73424650	2182
12	678390116	4130
13	6405031050	7678
14	61606881612	14368

$M_n \sim (12.26...)^n \qquad L_n \sim (1.748648...)^n$

Enumeration, asymptotics

$n = 4$:

Enumeration, asymptotics

$n = 4$:

$n = 5$:

Enumeration, asymptotics

$n = 4$:

$n = 5$:

n	L_n
…	
105	248742274995715373879042070
106	434962771573005719770576034
107	760597063369550445571334010
108	1330016842349701088401439208
109	2325732108141510145312701272
110	4066887817970878716400628884
111	7111557640719424745330990326

Enumeration — Panyushev's algorithm

Number of connected components for $(a_1, ..., a_k; b_1, ..., b_l)$ is the same as for $(d - r, r, a_2, ..., a_k; b_2, ..., b_l)$, where $d = a_1 - b_1$ and r is the residue of b_1 modulo d

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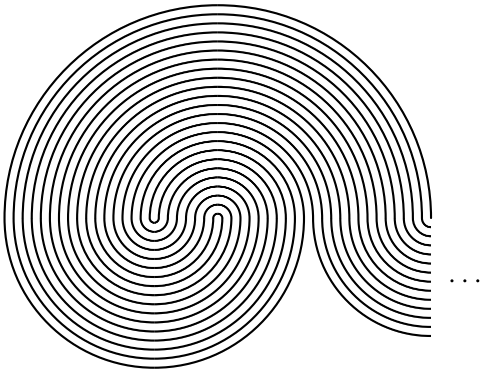

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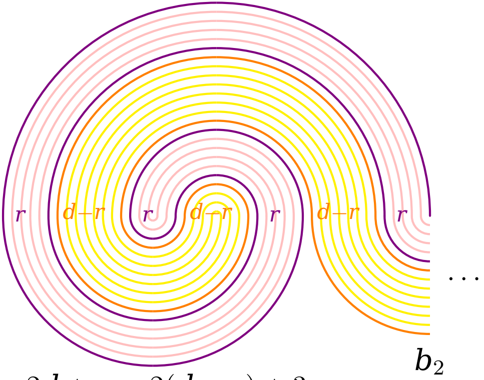
$a_1 = 3d + r = 3(d-r) + 4r$



$b_1 = 2d + r = 2(d-r) + 3r$

Step zero – indecomposability

Number of all pairs $(a_1, ..., a_k; b_1, ..., b_\ell)$ with $a_1 + ... + a_k = b_1 + ... + b_\ell = n$ is $A(n) = 2^{n-1} \times 2^{n-1} = 4^{n-1}$, so

$$\sum_{n \geqslant 1} A(n) q^n = \frac{q}{1 - 4q};$$

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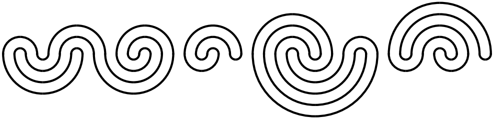

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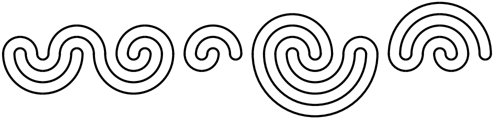

A pair $(a_1, ..., a_k; b_1, ..., b_\ell)$ is 0-indecomposable iff $a_1 + ... + a_i = b_1 + ... + b_j$ implies either $i = j = 0$ or $i = k$, $j = \ell$.

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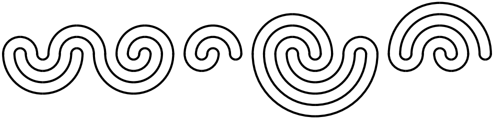

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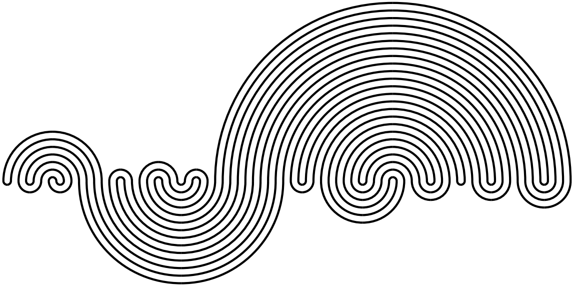
Number of 0-indecomposable pairs $I_0(n) = 3^{n-1}$, so

$$\sum_n I_0(n) q^n = \frac{q}{1-3q}.$$

Step one – irreducibility



Step one – irreducibility

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A pair $(a_1, ..., a_k; b_1, ..., b_\ell)$ is irreducible iff

$$
\begin{aligned}
& b_1 + ... + b_j - (a_1 + ... + a_{i-1}) \\
& \quad = b_{j+j'+1} + b_{j+j'+2} + ... + b_l - (a_{i+1} + a_{i+2} + ... + a_k) \geqslant 0 \\
& \qquad \Rightarrow j' = 0
\end{aligned}
$$

Let $I_1(n)$ be the number of irreducible pairs with sum n.

Step one — irreducibility

Selecta Mathematica Sovietica
Vol. 11, No. 2 (1992)

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Meanders*

S. K. Lando and A. K. Zvonkin

Five miles meandering with a mazy motion.
S. T. Coleridge, *Kubla Khan*

By slow Meander's margent green
And in the violet-embroidered vale.
John Milton, *Comus*

1. Introduction. Formulation of the problem

1.1. Definition. Let us fix a straight line on the plane and $2n$ points on it. Consider a simple connected not self-intersecting closed curve intersecting the line in exactly those points. The equivalence class of such curves with respect to isotopies of the plane leaving the line fixed is called a (closed) *meander* of order n.

For the generating function of the analog of irreducibility for general meanders, Lando and Zvonkin have a simpler functional equation but no explicit identification of the solution

Step one — irreducibility

5.7. Functional equation for generating functions

Let $B(x)$ be the generating function for the number of systems of meanders, i.e. for the squares of Catalan numbers:

$$B(x) = \sum_{n=0}^{\infty} (\mathrm{Cat}_n)^2 x^n.$$

Theorem. *The functions $B(x)$ and $N(x)$ satisfy the functional equation*

$$B(x) = N(xB^2(x)). \tag{5.3}$$

Proof. Replace each of the letters a, b, c, d in equality (5.2) by the same formal variable t. We obtain the relation

$$B(t^2) = 1 + t^2B^2(t^2) + 2t^4B^4(t^2) + 8t^6B^6(t^2) + 46t^8B^8(t^2)$$
$$+ \cdots + N_n t^{2n} B^{2n}(t^2) + \cdots = N(t^2B^2(t^2)).$$

Replacing t^2 by x we obtain (5.3).

Step one — irreducibility

Let $I_1(b, n)$ be the number of those irreducible $(a_1, ..., a_k; b_1, ..., b_l)$ with $a_1 > b_1 = b$;

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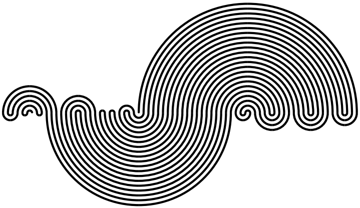

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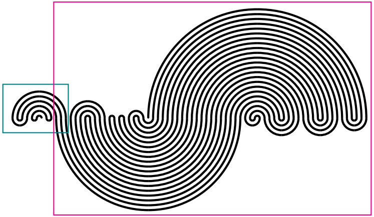

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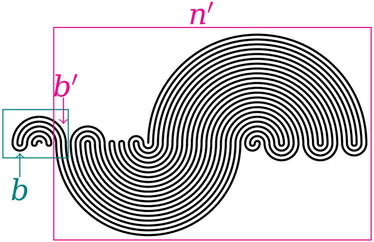

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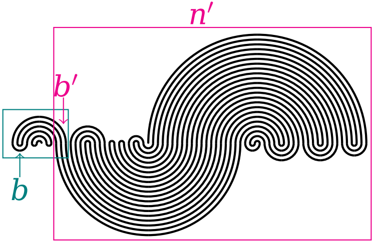

Then

$$I_1(b, n) = \sum_{b'} P(|b - b'|, n - b - b') + \sum_{b', n'} P(|b - b'|, n - b - n') I_1(b', n'),$$

where $P(d, m)$ is the number of irreducible pairs of the form $(b_1 + m + b_l; b_1, ..., b_l)$ with $|b_1 - b_l| = d$.

Step one — irreducibility

This gives the following functional equation for $f(z,q) = \sum I_1(b,n) z^b q^n$:

$$f(z,q) = R_0 + R_1 f(1,q) - R_2 f(qz,q),$$

where

$$R_0 = \frac{q^2(1 - q - 4q^2 + 2q^3 - q^2 z + q^3 z + 4q^4 z) z (1 - z)}{(1 - q)(1 - 3q^2)(2 - z)(1 - qz)(1 - 2q^2 z)},$$

$$R_1 = \frac{(1 + q - 2qz) z}{(2 - z)(1 - qz)},$$

$$R_2 = \frac{q(1 + 2q)(1 - z)(1 - q^2 z)}{(1 - 3q^2)(2 - z)(1 - 2q^2 z)}.$$

Step one — irreducibility

We obtain the following expression for $F(q) = f(1,q) = \sum I_1(n)q^n$:

$$F(q) = \frac{1-2q}{1+q-4q\frac{{}_2\phi_1\left({2,2q \atop 4q^2};q,qz\right)}{{}_2\phi_1\left({2,2q \atop 4q^2};q,z\right)}} - \frac{1}{1-q},$$

where $z = -\frac{q^2(1+2q)}{2(1-3q^2)}$ and

$${}_2\phi_1\left({a,b \atop c};q,z\right) := \sum_{n=0}^{\infty} \frac{(1-a)(1-b)(1-aq)(1-bq)\cdots(1-aq^{n-1})(1-bq^{n-1})}{(1-c)(1-cq)\cdots(1-cq^{n-1})(1-q)\cdots(1-q^n)} z^n$$

is the *basic hypergeometric series*.

Step one — irreducibility

More recently Don Zagier derived a formula for our two-variable generating function too:

$$f(z,q) = A(q)\frac{qz}{1-qz} + B(q)\left(R_2(q)\frac{2q^2z}{1-2q^2z} + R_3(q)\frac{2q^3z}{1-2q^3z} + \dots\right)$$

where $A(q)$, $B(q)$ are explicit rational functions of q and of $F(q)$ from the previous slide, while $R_2(q)$, $R_3(q)$, ... are explicitly given rational functions of q.

Step one — irreducibility

One would hope for a connection of $f(z,q)$ with some kind of Jacobi forms; unfortunately $f(z,q)$ itself is not one. As a function of z with fixed q it is something ugly.

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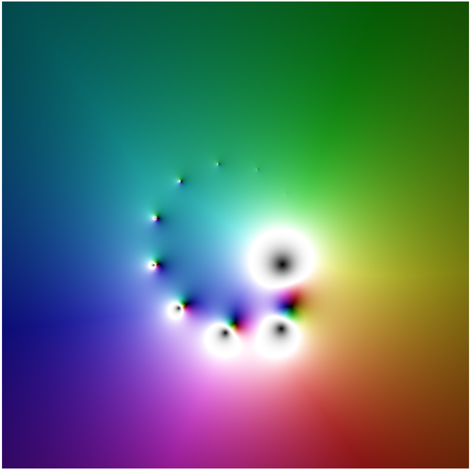

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Your typical Jacobi form looks way much nicer:

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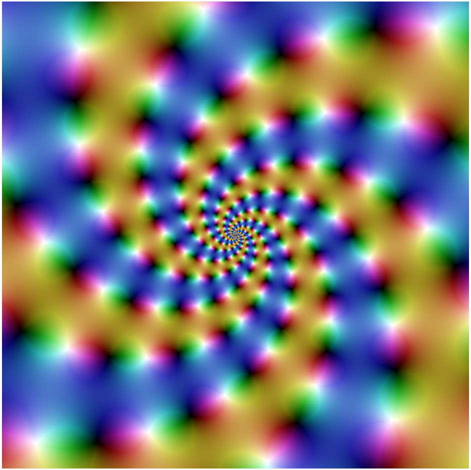
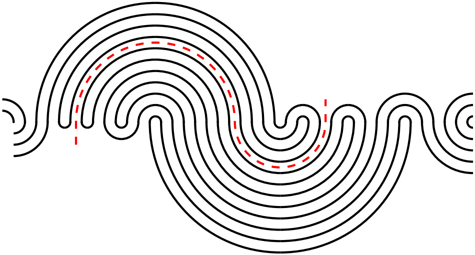


Further steps – higher irreducibilities



Other classical types

We just described meander graphs corresponding to classical Lie algebras of type A. For other classical types the corresponding graphs have been introduced by Panyushev and Yakimova.

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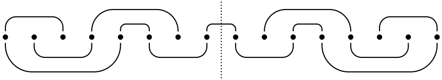

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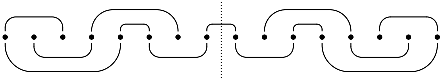
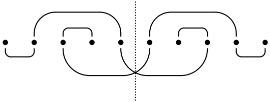
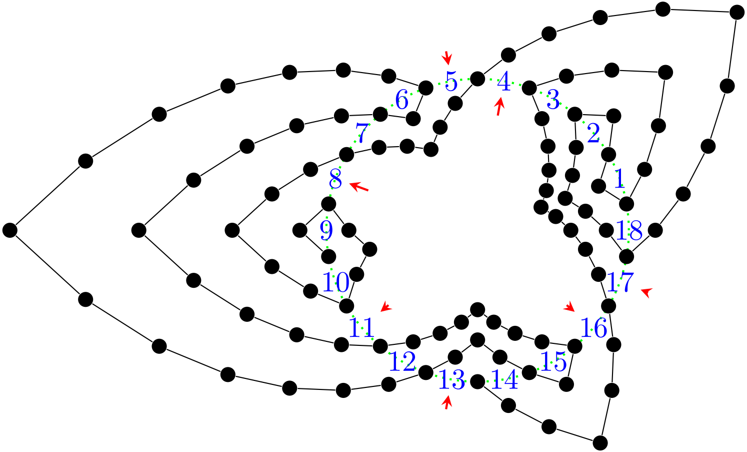


Meander graphs of type D look like

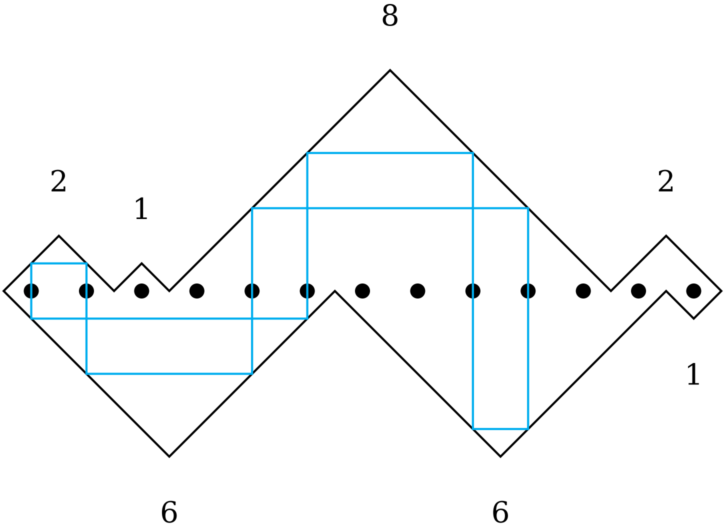


Lieanders – the Kac-Moody case



Lieanders – billiards

Each pair $(a_1, \ldots, a_k; b_1, \ldots, b_\ell)$ determines a *billiard*.



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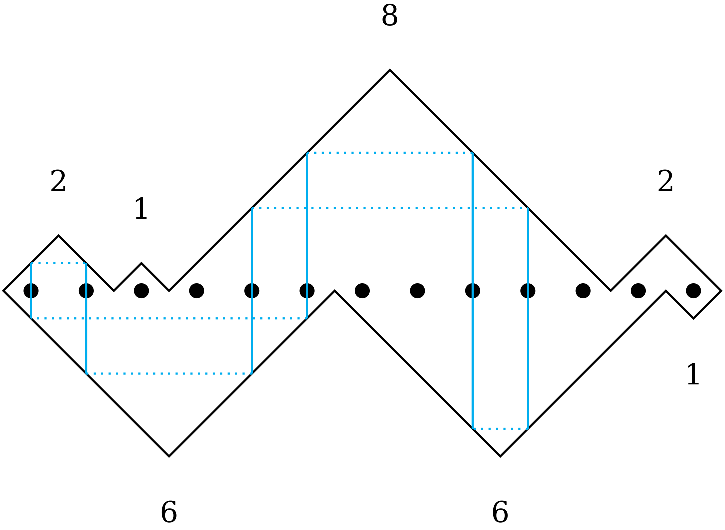

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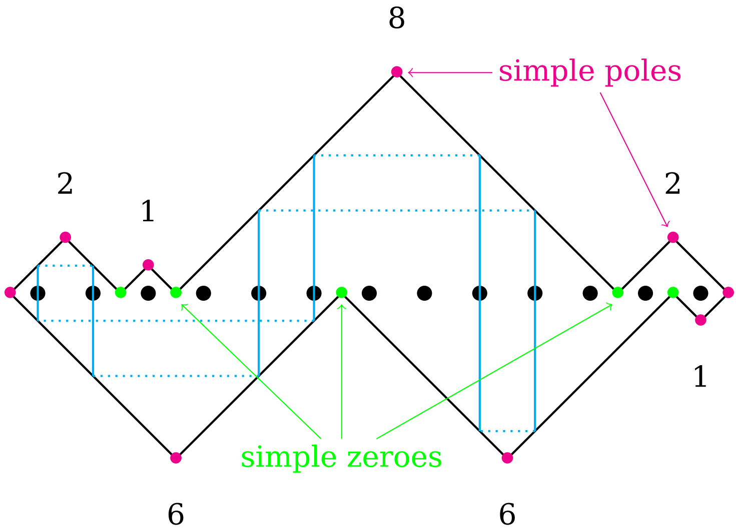

Lieanders – billiards

Pillowcases and quasimodular forms

Alex Eskin* and Andrei Okounkov†

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To Vladimir Drinfeld on his 50th birthday.

Summary. We prove that natural generating functions for enumeration of branched coverings of the pillowcase orbifold are level 2 quasimodular forms. This gives a way to compute the volumes of the strata of the moduli space of quadratic differentials.

Lieanders – billiards

In "Algebraic geometry and number theory", pp. 1-25. Birkhäuser Boston, 2006.

Pillowcases and quasimodular forms

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QUASIMODULARITY AND LARGE GENUS LIMITS OF SIEGEL-VEECH CONSTANTS

DAWEI CHEN, MARTIN MÖLLER, AND DON ZAGIER

Contents

Liouville quantum gravity

https://youtu.be/HHnJVkPIaMY
“On the geometry of uniform meandric systems” —
Ewain Gwynne with collaborators

Liouville quantum gravity

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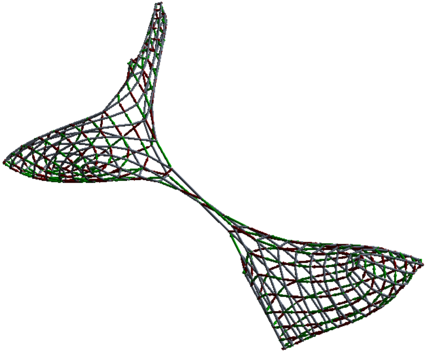
"On the geometry of uniform meandric systems" — Ewain Gwynne with collaborators

Meander graphs can be used to encode on a surface of genus zero, a metric in which all edges have approximately equal arclengths.

Liouville quantum gravity

It seems that doing that to *Lie*ander graphs gives some special kinds of metrics.



Thank you for having listened!