Survey of mathematical work of Dito Pataraia

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First publications of Dito:

- Duality of K-theory with K-homology for C*-algebras, 1983
- Topological K-homology for C*-algebras, 1984
- Homology properties of algebraic K-theory, 1987
- On Quillen's $+$ construction of perfect groups, 1990

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The fourth, in English, in the volume 1437 of the Springer Lecture Notes in Mathematics, dedicated to the work during 1987-1988 of the members of the regular seminar of the Department of Algebra run by Hvedri Inassaridze at the Razmadze Mathematical Institute.

The Pataraia fundamental group

Internal categories in the left exact cosimplicial category. *Georgian Math. J.* **4** (1997), 533–556.

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Equivalence of these definitions is a deep fact that, among other things, relates Algebraic Topology to other areas of mathematics, for example to Galois Theory and Algebraic Geometry.

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Dito had his own approach, which is less known but, in my opinion, conceptually very significant and ბევრის მთქმელი.

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A typical groupoid is the collection of all isomorphisms between several objects, with *many* identities, and multiplication only defined for *composable* isomorphisms.

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So for $g_1, g_2 \in G$ their composite $g_1 g_2$ is only defined if, say, g_1 acts as an isomorphism from some $x_1 \in X$ to some $y_1 \in X$, g_2 from x_2 to y_2, *and* $x_1 = y_2$, and then the composite will act from x_2 to y_1.

The Pataraia fundamental group

The simplest examples of groupoids are the most important ones.

For each collection X of objects, there is the discrete groupoid $\mathrm{disc}(X)$ consisting of identity isomorphisms only.

It has $G = X$ and all of the $e, s, t, {}^{-1}$ the identity maps of X. Then also $G \times_X G$ can be identified with X, and the composition is this identification.

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It has $G = X \times X$, with $e : X \to X \times X$ the diagonal, $s, t : X \times X \to X$ the projections, composition $(y, z)(x, y) = (x, z)$ and inversion $(x, y)^{-1} = (y, x)$.

The Pataraia fundamental group

For a fixed space X, Dito considers all possible maps from groupoids $(G, Y, e, s, t, ^{-1})$ to the indiscrete groupoid $\mathrm{ind}(X)$, such that $Y \to X$ and $G \to X \times X$ are local homeomorphisms.

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For any given $f : Y \to X$ it is seemingly easy to construct such a map, taking $(f, f) : Y \to X \times X$.

But if one starts with a local homeomorphism f, the map (f, f) is almost never a local homeomorphism again!

The Pataraia fundamental group

Then for any given local homeomorphism $f : Y \to X$ Dito manages to construct a groupoid $\mathrm{disc}(f) = (G = \tilde{Y}, Y, s, t, e, ^{-1})$ which is “as discrete as possible”, i. e. has as few as possible non-identity isomorphisms, and a local homeomorphism $\tilde{Y} \to X \times X$ defining the needed map $\mathrm{disc}(f) \to \mathrm{ind}(X)$.

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It has not been studied how what he obtains compares to various versions of the fundamental group for “bad” spaces.

The main point: the very nature of Dito’s construction is such that it admits much more general reformulation (replacing local homeomorphisms with étale maps, discrete fibrations, etc.) and recovers most of the known relationships between versions of the fundamental group in Algebraic Geometry, Galois Theory and Topos Theory.

The Pataraia fundamental group

Let me also remark that relationship between Dito's construction and the very general category-theoretic Galois theory of G. Janelidze has never been studied too. I believe it can be very interesting to compare them in the cases of spaces X whose spaces of connected components $\pi_0(X)$ are badly behaved.

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More generally, (for specialists) in the cases when the inverse image functors do not possess left adjoints.

The Pataraia fixed point theorem

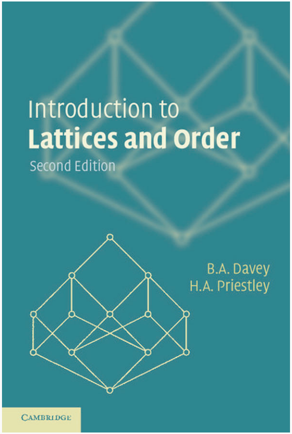
The next result of Dito I want to tell about is usually cited like this:

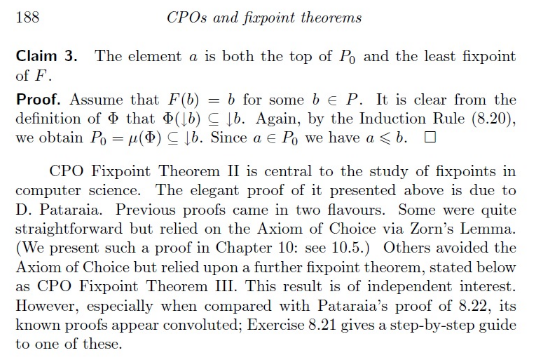
Pataraia, D. (1997), *A constructive proof of the fixed-point theorem for DCPOs*, unpublished talk, given at the “65th Meeting of the Peripatetic Seminar on Sheaves and Logic (PSSL65), Aarhus, Denmark.”

It has been reproduced with proof and used for important applications (mostly for computer science) in several publications.

The Pataraia fixed point theorem

In fact it is by now an established “textbook theorem”.





188 *CPOs and fixpoint theorems*

Claim 3. The element a is both the top of P_0 and the least fixpoint of F.

Proof. Assume that $F(b) = b$ for some $b \in P$. It is clear from the definition of Φ that $\Phi(\downarrow b) \subseteq \downarrow b$. Again, by the Induction Rule (8.20), we obtain $P_0 = \mu(\Phi) \subseteq \downarrow b$. Since $a \in P_0$ we have $a \leqslant b$. ☐

CPO Fixpoint Theorem II is central to the study of fixpoints in computer science. The elegant proof of it presented above is due to D. Pataraia. Previous proofs came in two flavours. Some were quite straightforward but relied on the Axiom of Choice via Zorn’s Lemma. (We present such a proof in Chapter 10: see 10.5.) Others avoided the Axiom of Choice but relied upon a further fixpoint theorem, stated below as CPO Fixpoint Theorem III. This result is of independent interest. However, especially when compared with Pataraia’s proof of 8.22, its known proofs appear convoluted; Exercise 8.21 gives a step-by-step guide to one of these.

The Pataraia fixed point theorem

The theorem states:

For any partially ordered set $(P, \leqslant)$ possessing suprema for arbitrary directed subsets, any increasing order preserving self-map $e : P \to P$ has a fixed point.

That is, there is an $x \in P$ with $e(x) = x$. Recall that e *increasing* means $x \leqslant e(x)$ for all $x \in P$.

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A non-constructive proof has been found by Bourbaki (1949) and Witt (1951), and only requires existence of suprema for arbitrary *chains*. Starting from any x one forms a transfinite chain

$$x \leqslant e(x) \leqslant e^2(x) \leqslant \cdots \leqslant \sup_n e^n(x) \leqslant e(\sup_n e^n(x)) \leqslant \cdots \leqslant e^\alpha(x) \leqslant \cdots$$

where $e^{\alpha+1}(x) = e(e^\alpha(x))$ and $e^\alpha(x) = \sup_{\beta<\alpha} e^\beta(x)$ for any limit ordinal α.

The Pataraia fixed point theorem

But as it turns out, in constructive mathematics, where you do not have ordinals and transfinite induction available, chain-completeness is not in fact sufficient, one needs full directed-completeness: there are counterexamples!

The Pataraia fixed point theorem

Dito’s proof is both astonishingly simple and astonishingly clever. Let

$$E(P) = \{e : P \to P \mid \ e \text{ is order preserving and increasing}\}$$

and consider it as a partially ordered set under pointwise order, $e \leqslant e'$ iff $e(x) \leqslant e'(x)$ for all $x \in P$.

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Then, whenever P has suprema of arbitrary directed subsets, so does $E(P)$ – these suprema can be computed pointwise, $(\sup_{e \in D} e)(x) = \sup_{e \in D} e(x)$.

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Then, whenever P has suprema of arbitrary directed subsets, so does $E(P)$ – these suprema can be computed pointwise, $(\sup_{e\in D} e)(x) = \sup_{e\in D} e(x)$.

But $E(P)$ is itself directed! Indeed it is closed under composition, and for any $e, e' \in E(P)$ we have $e \leqslant e \circ e'$ and $e' \leqslant e \circ e'$.

The Pataraia fixed point theorem

It follows that $\top := \sup_{e \in E(P)} e$ exists in $E(P)$, i. e. $E(P)$ has largest element $\top$. Then, for any $e \in E(P)$ we have $e \circ \top \leqslant \top$. But e is also increasing, so $\top \leqslant e \circ \top$. Thus $e \circ \top = \top$, i. e. $e(\top(x)) = \top(x)$ for any $x \in P$.

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Note that for a computer scientist, what Dito achieves is even much more valuable than a mere fixed point theorem: he constructs a *closure* operator $\top : P \to P$ which to every $x \in P$ assigns a $\top(x) \geqslant x$ which is a *simultaneous* fixed point for *all* $e \in E(P)$.

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Clearly this $\top : P \to P$ is an important map telling much about the directed-complete partial order $(P, \leqslant)$. I don’t know whether it has been studied by anybody.

The Pitts problem

Probably the most important contribution of Dito to mathematics is unfortunately still unpublished.

He only has a paper about his preliminary work on a particular case,

Description of all functions definable by formulæ of the 2nd order intuitionistic propositional calculus on some linear Heyting algebras, *Journal of Applied Non-Classical Logics* **16**(2006), 457–483.

I will try to briefly describe what it is about.

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Classically, subsets of any set X form a Boolean algebra $\wp(X)$.

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$\wp(X)$ is only a Heyting algebra: you have all the usual operations $\cap$, $\cup$, including the *implication* $\to$,

$$S \to T := \bigcup\{S' \mid S \cap S' \subseteq T\},$$

you have $S \cap (S \to T) = S \cap T$ and $T \subseteq S \to T$, but $X \setminus S$ is $S \to \emptyset$, and in general $S \cup (S \to \emptyset) = X$ cannot be proved.

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His proof consists of two parts: first, he shows how to build an elementary topos from any hocHa;

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His proof consists of two parts: first, he shows how to build an elementary topos from any hocHa;

second, he constructs a *cylindrification* of any Heyting algebra.

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The second part is much more involved and requires subtle step-by-step reduction of higher order intuitionistic formulæ using second order quantifiers constructed by Pitts himself in his 1992 paper.

Joint publications of Dito

Combinatorics of necklaces and Hermite reciprocity (with A. Elashvili and M. Jibladze). *J. Algebr. Comb.* **10**(1999), 173–188.

Scattered toposes (with L. Esakia and M. Jibladze). *Ann. Pure Appl. Logic* **103**(2000), 97–107.

What I miss most