A variety of bi-Heyting algebras not generated by complete algebras

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Background

While there are several Kripke incomplete modal logics known, all of those above **S4** essentially reduce to the one discovered by Kit Fine in 1972 (“An incomplete logic containing S4”).

Background

An incomplete logic containing S4

by

KIT FINE

(University of Edinburgh)

This paper uses the standard terminology of modal logic. It should suffice to say that: all *logics* contain the minimal logic K and are closed under necessitation, substitution and modus ponens; *frames* consist of a relation defined on a non-empty set of points; *models* consist of a frame with a valuation; and truth-at-a-point is defined and notated in an obvious way; with the formula $\Box A$ true at a point iff A is true at all accessible points. The formula A is *true in* (*satisfied by*) a model if it is true in all (some) points of the model; A is *strongly verified* in a model if all substitution-instances of A are true in the model; and A is *valid* in a frame if A is true in all models based upon the frame. A set of formulas is true, strongly verified, or valid if all of its members are. Unless otherwise stated, all logics contain *S4* and all models and frames possess reflexive and transitive relations.

A logic is *complete* if any formula valid in all frames that validate the logic is in the logic. This paper exhibits a logic L containing *S4* that is not complete.[1]

To define L we need the formulas below. The point of the construction will become apparent in the course of the proof. For distinct sentence letters $p_0, p_1, q_0, q_1, r_0, r_1, t$ and s, and for $m \geq 0$:

$$B_0 = q_0, \quad C_0 = r_0,$$
$$B_1 = q_1, \quad C_1 = r_1,$$
$$B_{m+2} = \Diamond B_{m+1} \wedge \Diamond C_m \wedge -\Diamond C_{m+1},$$
$$C_{m+2} = \Diamond C_{m+1} \wedge \Diamond B_m \wedge -\Diamond B_{m+2},$$
$$A_m = \Diamond B_{m+1} \wedge \Diamond C_{m+1} \wedge -\Diamond B_{m+2},$$

[1] Thomason in [3] has independently constructed an incomplete logic. It contains T, but not *S4*.

His terrifying axioms for the Kripke incomplete modal logic **F** above **S4** involve eight variables.

Background

$$D = (p_0 \vee p_1) \wedge \Box(p_0 \to -p_1 \wedge \Diamond p_1) \wedge \Box(-(p_0 \vee p_1) \to \Box(-(p_0 \vee p_1))),$$
$$E = D \wedge \Diamond A_0 \wedge \Box(B_1 \to \Diamond B_0 \wedge -\Diamond C_0) \wedge \Box(C_1 \to \Diamond C_0 \wedge -\Diamond B_0) \wedge \Box(B_0 \to -\Diamond B_1) \wedge \Box(C_0 \to -\Diamond C_1),$$
$$F = \Diamond((p_0 \vee p_1) \wedge -\Diamond A_0 \wedge \Diamond A_1),$$
$$G = E \to F,$$
$$H = -(s \wedge \Box(s \to \Diamond(-s \wedge t \wedge \Diamond((-s \wedge -t) \wedge \Diamond s)))).$$

The logic L is the smallest logic to contain $S4$, G and H. The strategy of the proof is to show that $-E$ is not in L but that it *should* be, i.e., that it is valid in all frames that validate L. To show that $-E$ is not in L we construct a model $\mathfrak{A}$ that strongly verifies L but satisfies E.

Lemma 1. *Any frame that validates L also validates $-E$.*

Proof. The proof requires three preliminary results. For any $m \geq 0$ and any formula A, let A^m be the result of substituting B_{m+1} for $B_i = q_i$ and C_{m+i} for $C_i = r_i$, $i = 0$ or 1.

(1) $B_n{}^m = B_{m+n}$, $C_n{}^m = C_{m+n}$ *and* $A_n{}^m = A_{m+n}$ *for* m, $n \geq 0$.

Proof. Prove for the B's and C's by induction on n. (i) $n = 0$ or 1. $B_n{}^m = B_{m+n}$ and $C_n{}^m = C_{m+n}$ by definition. (ii) $n > 1$. $B_n{}^m = (\Diamond B_{n-1} \wedge \Diamond C_{n-2} \wedge -\Diamond C_{n-1})^m = \Diamond B_{n-1}{}^m \wedge \Diamond C_{n-2}{}^m \wedge -\Diamond C_{n-1}{}^m = \Diamond B_{m+n-1} \wedge \Diamond C_{m+n-2} \wedge -\Diamond C_{m+n-1}$ (by IH) $= B_{m+n}$. Similarly for $C_n{}^m$.

Now for any m, $n \geq 0$, $A_n{}^m = (\Diamond B_{n+1} \wedge \Diamond C_{n+1} \wedge -\Diamond B_{n+2})^m = \Diamond B_{n+1}{}^m \wedge \Diamond C_{n+1}{}^m \wedge -\Diamond B_{n+2}{}^n = \Diamond B_{m+n+1} \wedge \Diamond C_{m+n+1} \wedge -\Diamond B_{m+n+2}$ (by IH) $= A_{m+n}$.

Let K be the conjunction of the last four conjuncts of E.

(2) *If* $(\mathfrak{A}, w) \models E$ *then* $(\mathfrak{A}, w) \models K^m$ *for any model* $\mathfrak{A}$, *point* w *in* $\mathfrak{A}$, *and* $m \geq 0$.

Proof. $K^m = \Box(B_{m+1} \to \Diamond B_m \wedge -\Diamond C_m) \wedge \Box(C_{m+1} \to \Diamond C_m \wedge -\Diamond B_m) \wedge \Box(B_m \to -\Diamond B_{m+1}) \wedge \Box(C_m \to -\Diamond C_{m+1})$. We distinguish two cases. (i) $m = 0$. Then $K^0 = K$. (ii) $m > 0$. The first two conjuncts are theorems of $S4$ and so true at $(\mathfrak{A}, w)$. In particular, $w \models \Box(B_m \to$

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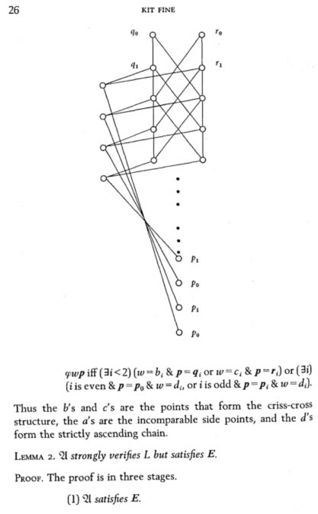
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φwp iff $(\exists i<2)\,(w=b_i\ \&\ p=q_i$ or $w=c_i\ \&\ p=r_i)$ or $(\exists i)$ (i is even & $p=p_0$ & $w=d_i$, or i is odd & $p=p_i$ & $w=d_i$).

Thus the b's and c's are the points that form the criss-cross structure, the a's are the incomparable side points, and the d's form the strictly ascending chain.

Lemma 2. *$\mathfrak{A}$ strongly verifies L but satisfies E.*

Proof. The proof is in three stages.

(1) *$\mathfrak{A}$ satisfies E.*

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He managed to achieve incompleteness using a clever combination of a certain two-variable axiom (which we will call **Ш**) with the three-variable formula $\mathbf{bb}_2$ by Dov Gabbay and Dick De Jongh (1974, ▸ “A sequence of decidable finitely axiomatizable intermediate logics with the disjunction property”) expressing bounded branching in Kripke structures

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Litak's paper also contains a modification/simplification of Shehtman's arguments (that part actually required a correction, as suggested by Guillaume Massas; the corrected version is currently at ▸ arXiv:1808.06284).

Background – generalizations

Kripke semantics can be viewed as a particular case of a number of other semantics.

For topological semantics of both modal logics above **S4** and of superintuitionistic logics, Kripke frames correspond to topological spaces with the property that *all points possess smallest neighborhoods*.

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For topological semantics of both modal logics above **S4** and of superintuitionistic logics, Kripke frames correspond to topological spaces with the property that *all points possess smallest neighborhoods*.

On the algebraic side, these correspond to **S4**-algebras of the form $(\mathscr{P}X, \Box)$ where $\mathscr{P}X$ is a full powerset and $\Box$ is *totally multiplicative*, i. e. distributes over arbitrary intersections.

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In 2004 (▸ "Modal incompleteness revisited") Litak showed that even if you replace $\mathscr{P}X$ with any complete Boolean algebra and drop total multiplicativity requirement on $\Box$, the Fine logic **F** will remain incomplete. His terminology for that, suggested by Dick De Jongh: **F** is *completely incomplete*.

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While I cannot provide an answer, I can provide a very well known *reply* to this question: this is related to the Kuznetsov problem!

Background – generalizations

(c) K_{tfs}—class of twice finitely sliced SL, i.e., such that they are not included either in LC or in LC′;

(d) K_{lt}—class of locally tabular SL, i.e., $l \in \mathscr{L}$ such that in the variety Ml all the finitely generated algebras are finite (compare [2b], [6d]);

(e) K_{fa}—class of f.a. logics;

(f) K_{hfa}—class of hereditarily f.a. logics, i.e., $l \in \mathscr{L}$ for which all $l' \in \mathscr{L}, l \subseteq l'$, are f.a.;

(g) K_{top}—class of topologizable SL, i.e., $l \in \mathscr{L}$ such that l is the logic of some topological space, i.e., the logic of the pseudoboolean algebra of all its open sets [12];

(h) K_m—class of modelable SL (in connection with Kripke's models), i.e., $l \in \mathscr{L}$ such that $l = L^*\mathfrak{M}$ for some partially ordered set $\mathfrak{M}$ (see [20b]), where $L^*\mathfrak{M}$ is the logic of the topological space obtained from $\mathfrak{M}$, when open sets are defined as subsets closed under increasing;

(i) K_1—class of SL which are approximable by the algebras of the form $\mathfrak{A}_1 + \mathfrak{A}_2 + \cdots + \mathfrak{A}_m$, where all terms are finite or isomorphic to Z_∞;

(j) K_2—class of SL which are approximable by algebras for each of which there exists a natural n such that it has no n pairwise incomparable elements;

(k) K_3—class of SL which are approximable by the algebras with the descending chain condition.

It may be proved that

$$K_t \subset K_{tfs} \subset K_{fs} \subset K_{lt} \subset K_{hfa} \subset K_{fa} \subset K_1 \subset K_2 \subset K_3 \subseteq K_m \subseteq K_{top} \subseteq \mathscr{L}.$$

There remain open the questions of coincidence of the last four classes of this chain. For the first nine classes of it the examples of their difference are respectively the logics LC″, LC′, LC, LZ_∞ (see [1], [7]), LI, $L(Z_\infty + Z_7 + Z_2)$ (see [7]), $L((Z_\infty \times Z_2) + Z_2)$ (× denotes the Cartesian product) and $L(Z_\infty^2 + Z_7 + Z_2)$.

During his plenary lecture at the International Congress of Mathematicians in Vancouver (1974) (delivered by Ershov, actually) Kuznetsov asked (among other things) whether every superintuitionistic logic is topologically complete.

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During his plenary lecture at the International Congress of Mathematicians in Vancouver (1974) (delivered by Ershov, actually) Kuznetsov asked (among other things) whether every superintuitionistic logic is topologically complete. The question remains unanswered to this day.

Background – generalizations

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In this language, Kuznetsov's problem translates into the question whether every variety of Heyting algebras is generated by Heyting algebras of all open sets of topological spaces.

Main result

Our aim is to show that there exists a variety of bi-Heyting algebras that is not generated by complete bi-Heyting algebras.

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Thus we provide a negative solution of Kuznetsov's problem in the setting of Heyting-Brouwer logics.

Esakia duality

Our approach (see ▸ arXiv:2104.05961) is based on Esakia duality, mentioned in the yesterday's talk ▸ "Toward choice-free Esakia duality" by Wesley Holliday.

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Clearly then the dual distributive lattice of $(X, \leqslant)$ is a bi-Heyting algebra iff for any clopen $C \subseteq X$, both $\downarrow C$ and $\uparrow C$ are clopen. We might call such $(X, \leqslant)$ bi-Esakia spaces.

Esakia duality – completeness

We require one more fact from duality theory: characterization of those $(X, \leqslant)$ whose dual (bi-)Heyting algebras are complete.

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This has been addressed by Guram Bezhanishvili and Nick Bezhanishvili in ▸ "Profinite Heyting algebras" (2008): the algebra of clopen upper sets of $(X, \leqslant)$ is complete iff $(X, \leqslant)$ is extremally order disconnected, which means that closure of each open upper set is (cl)open.

Esakia duality – completeness

What is crucial for us is that if moreover $(X, \leqslant)$ is a bi-Esakia space, then it follows from the results of Guram Bezhanishvili and John Harding in ▸ “MacNeille completions of Heyting algebras” (2004) that we may calculate arbitrary joins $\bigvee_i D_i$ in the lattice of clopen *lower* sets of $(X, \leqslant)$ by the formula

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(if $(X, \leqslant)$ were only an extremally order-disconnected Esakia space rather than bi-Esakia space, we would additionally need to take $\downarrow$ of that closure).

The Fine frame, algebraically

Let us return to the Fine frame $\mathscr{F}$, and let $\mathscr{F}^+$ be the Heyting algebra of all its upper sets.

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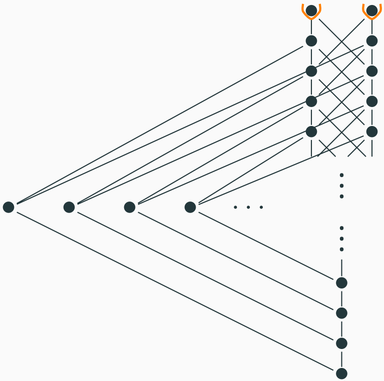
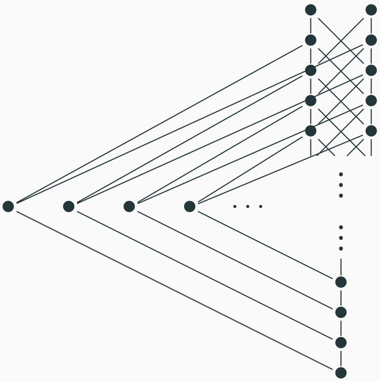
Let us return to the Fine frame $\mathscr{F}$, and let $\mathscr{F}^+$ be the Heyting algebra of all its upper sets.

Consider the Fine algebra $\mathbb{A}$ – the Heyting subalgebra of $\mathscr{F}^+$ generated by its two maximal singleton subsets:



The Fine frame, algebraically

As it turns out, the dual Esakia space of the Fine algebra $\mathbb{A}$ is obtained by adding a single limit point to the Fine frame $\mathbb{F}$.



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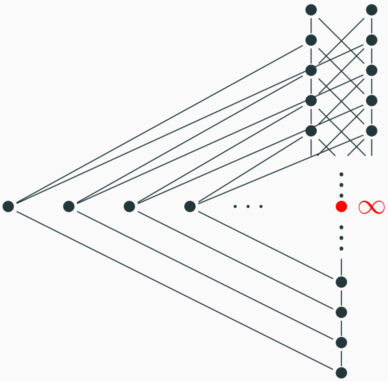

The Fine frame, algebraically

This fact allows us to prove that the Fine algebra $\mathbb{A}$ is a bi-Heyting algebra, by observing that its dual Esakia space is a bi-Esakia space.

The axioms

Now to tell you what we actually proved and how, we must take a look at the actual axioms.

Let me begin with the Gabbay-De Jongh formula $\mathbf{bb}_2$. It is the following formula in three propositional variables x, y, z:

$$
\begin{aligned}
&[(x \to (y \vee z)) \to (y \vee z)] \\
\wedge&[(y \to (x \vee z)) \to (x \vee z)] \\
\wedge&[(z \to (x \vee y)) \to (x \vee y)] \\
&\to (x \vee y \vee z)
\end{aligned}
$$

$\mathbf{bb}_2$

The meaning of $\mathbf{bb}_2$ is easiest to understand in the topological semantics: a topological space validates $\mathbf{bb}_2$ if and only if for any three closed sets C_1, C_2, C_3 with common intersection $C = C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3$, if C is nowhere dense in all of the C_1, C_2, C_3 then $C = \varnothing$.

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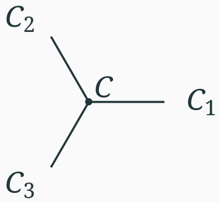
In other words, $\mathbf{bb}_2$ forbids situations like this:



In the language of (bi-)Esakia spaces, closed sets get replaced with clopen lower sets, while “nowhere dense” becomes “nowhere cofinal”, in the following sense:

Definition

Given lower sets D, E in a poset, with $D \subset E$, we say that D is *nowhere cofinal in* E, if $\downarrow(E - D) = E$.

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For clopen downsets of an Esakia space, this is equivalent to $D \cap \max(E) = \varnothing$.

The axiom **III**

Let us now turn to the Shehtman axiom **III**.

To digest more easily the things that follow, let me also introduce an alternative notation for implication: I will denote

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Let us fix two propositional variables p, q and denote $m := p \wedge q$, $u := p \vee q$.

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$$ {}_{\overline{a}\urcorner}b := b \to a. $$

Let us fix two propositional variables p, q and denote $m := p \wedge q$, $u := p \vee q$.

We will need a uniform substitution σ determined by

$$ \sigma(p) = p \vee {}_{\overline{m}\urcorner}p, \qquad \sigma(q) = q \vee {}_{\overline{m}\urcorner}q. $$

The axiom III

Let $d := {}_{\overline{m}\rceil}u \vee {}_{\overline{m}\rceil}{}_{\overline{m}\rceil}u$. Then, the axiom **III** is equivalent to

$$(\sigma^2(d) \to \sigma^3(d)) \to \sigma^2(d).$$

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We will try to explicate some of the semantical meaning of this below. Here let me only mention that under a valuation $\mathcal{V}$ in a topological space, a formula like $p \vee {}_{\overline{m}\urcorner}p$ can be interpreted as a certain *boundary*.

Namely,

$$-\mathcal{V}(p \vee {}_{\overline{m}\urcorner}p) = \partial_{-\mathcal{V}(m)}(-\mathcal{V}(p));$$

that is, $-\mathcal{V}(p \vee {}_{\overline{m}\urcorner}p)$ is the boundary of $-\mathcal{V}(p)$ as a subspace of $-\mathcal{V}(m)$.

The axiom III

Let $d := \overline{m}\rceil u \vee \overline{m}\rceil\overline{m}\rceil u$. Then, the axiom **III** is equivalent to

$$(\sigma^2(d) \to \sigma^3(d)) \to \sigma^2(d).$$

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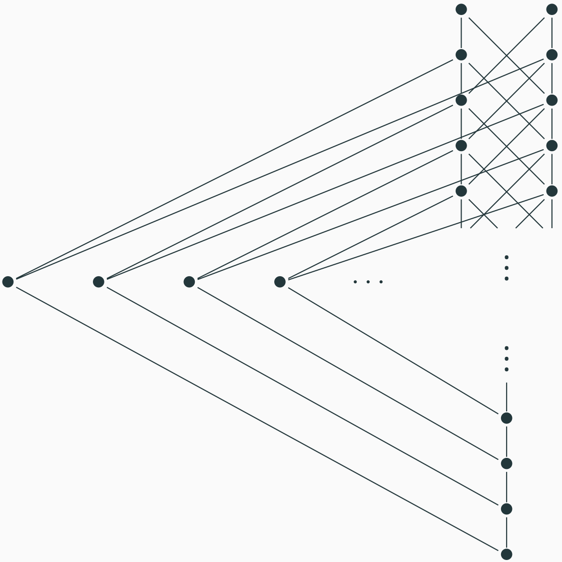
Probably the best way to explain the meaning of **III** is to see what it does in a particular valuation on the Fine frame.

The valuation



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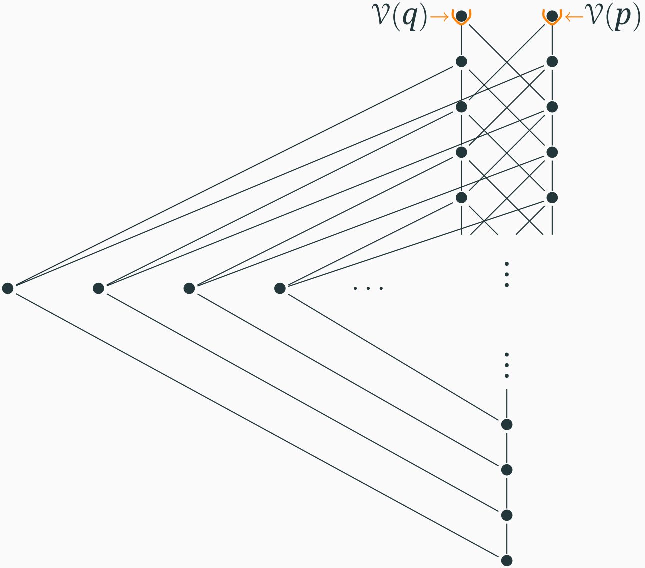

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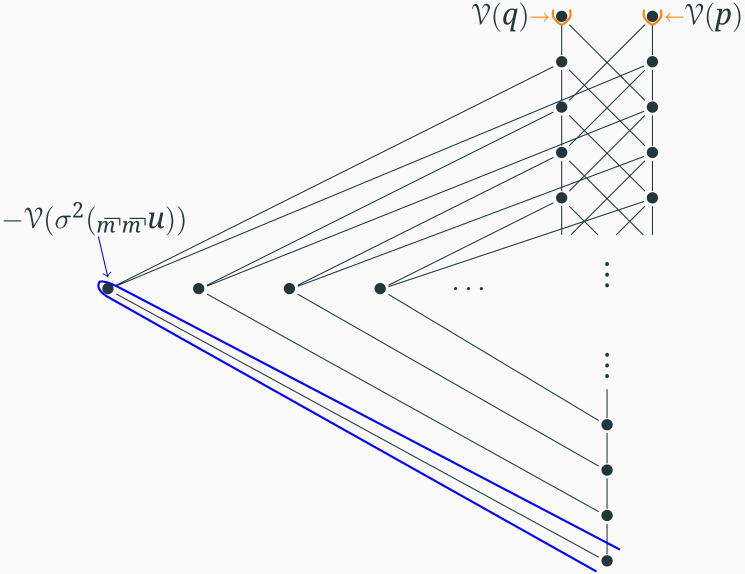

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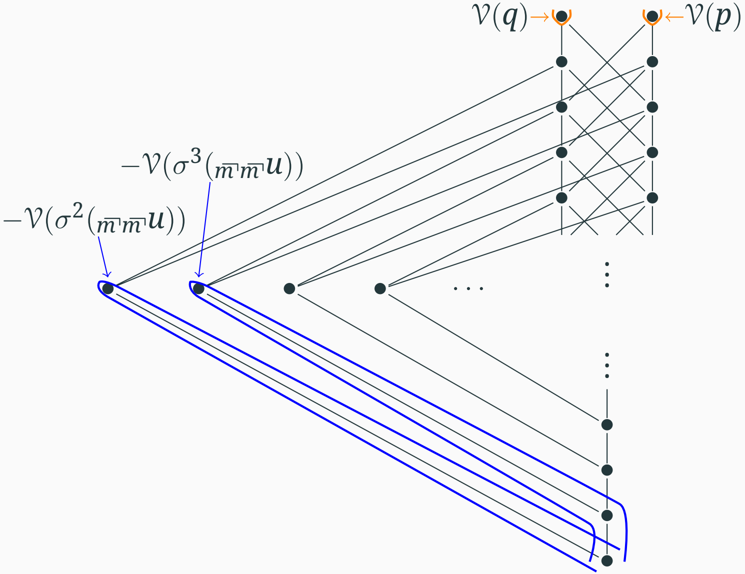

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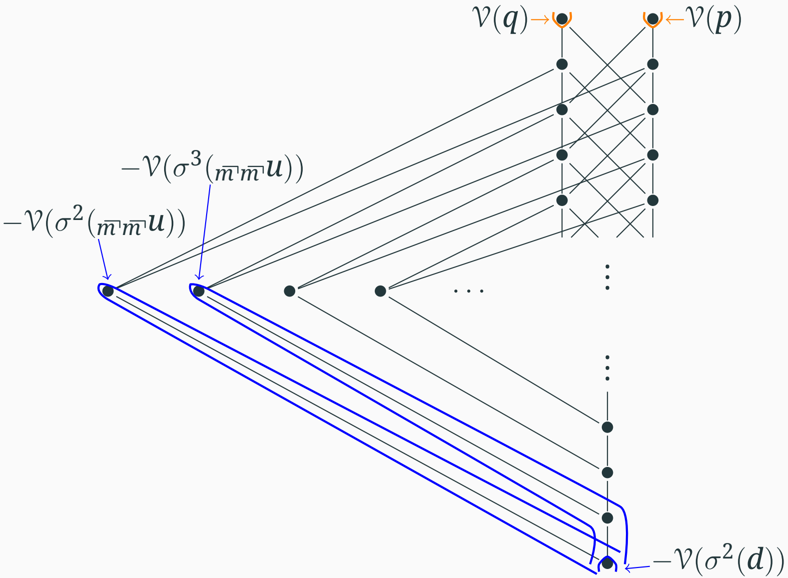

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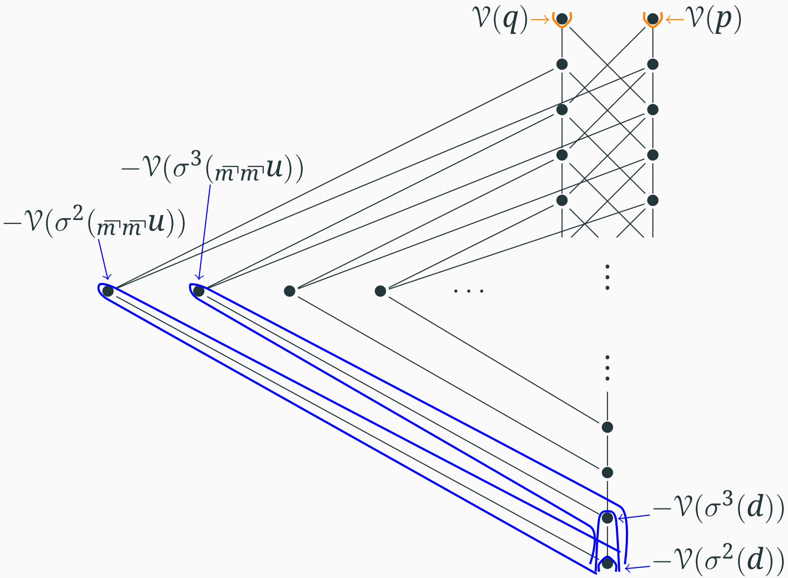

$\mathbf{bb}_2 + \mathbf{III}$

In particular, the axiom **III** effectively says that $-\mathcal{V}(\sigma^2(d))$ is nowhere cofinal in $-\mathcal{V}(\sigma^3(d))$ under any valuation $\mathcal{V}$.

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This implies that whenever $-\mathcal{V}(\sigma^2(\boldsymbol{d})) \neq \varnothing$, we in fact have a strictly growing infinite chain

$$-\mathcal{V}(\sigma^2(\boldsymbol{d})) \subsetneq -\mathcal{V}(\sigma^3(\boldsymbol{d})) \subsetneq -\mathcal{V}(\sigma^4(\boldsymbol{d})) \subsetneq \cdots$$

Theorem

Here is, finally, what we managed to prove:

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Suppose $(X, \leqslant)$ *is an extremally order-disconnected bi-Esakia space validating* $\mathbf{III}$ *such that* $(X, \leqslant) \not\models \sigma^2(d)$*. Then* $(X, \leqslant) \not\models \mathbf{bb}_2$*.*

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Dually, any complete bi-Heyting algebra which validates $\mathbf{Ш} + \mathbf{bb}_2$ also validates $\sigma^2(d)$.

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Thus $\mathbf{III} + \mathbf{bb}_2$ is incomplete with respect to complete bi-Heyting algebras.

How do we prove it

Here is the idea of the proof:

We consider some valuation $\mathcal{V}$ on an extremally order-disconnected bi-Esakia space $(X, \leqslant)$ with $-\mathcal{V}(\sigma^2(d)) \neq \varnothing$; by assumption $\mathcal{V}$ (any valuation, in fact) validates **III**.

It then turns out that **III** allows us to use the Fine frame as a kind of blueprint to construct C_1, C_2, C_3 required to refute $\mathbf{bb}_2$, namely

$$C_1 = -\mathcal{V}\sigma^4(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^7(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^{10}(\overline{m}\neg\overline{m}\neg u) \vee \cdots$$
$$C_2 = -\mathcal{V}\sigma^2(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^5(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^8(\overline{m}\neg\overline{m}\neg u) \vee \cdots$$
$$C_3 = -\mathcal{V}\sigma^3(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^6(\overline{m}\neg\overline{m}\neg u) \vee -\mathcal{V}\sigma^9(\overline{m}\neg\overline{m}\neg u) \vee \cdots$$

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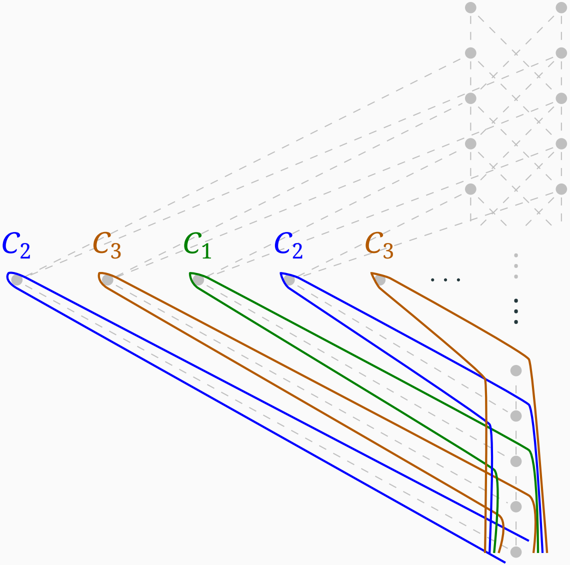

Completely incomplete Heyting-Brouwer logics

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Adapting this technique allows us to prove that there are continuum many completely incomplete Heyting-Brouwer logics.

Consequences for varieties of Heyting algebras

Corollary

There exist continuum many varieties of Heyting algebras not generated by complete bi-Heyting algebras.

This implies that there are continuum many superintuitionistic logics incomplete with respect to complete bi-Heyting algebras.

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This result was also obtained by Guillaume Massas using the techniques related to the semantics of the Propositional Lax Logic (the one that made appearance in the talk by Sebastian Melzer ▸ “Canonical formulas for IK4” on Thursday).

Obstacles towards Kuznetsov's problem

Needless to say, we cannot get rid of that "bi-". In showing $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3$ we essentially use that infinite (well, countable, but...) joins of clopen lower sets distribute over their finite meets, which requires certain amount of co-Heytingness.

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Moreover in proving that C is nowhere cofinal in C_1, C_2, C_3 we also use the fact that to compute infinite joins of clopen lower sets one only needs to take closure of their union, which is already a lower set, so that further generating lower set from it is not needed.

Thank you for your patience!