Proceedings of A.Razmadze
Mathematical Institute,
Vol. **118**, 1998

LOWER BAGDOMAIN AS A GLUEING

M. JIBLADZE

Abstract. Recent generalization of Diaconesu's theorem by Moerdijk is further extended to a certain class of internal categories in the category of toposes and geometric morphisms. The generalization is used to construct the Lower Bagdomain topos of Vickers and Johnstone as a glueing of a natural diagram of toposes indexed by the category of finite cardinals.

One of the important tools for working with the classifying toposes is Diaconescu's theorem, describing geometric morphisms to the topos of presheaves on a small category $\mathbb{C}$ in terms of flat functors on $\mathbb{C}$. In [4], this theorem is generalized from discrete categories to topological categories, with the additional requirement that the map assigning to an arrow its source is étale (a local homeomorphism). This generalization enables the author of [4] to interpret geometric morphisms to many new interesting toposes, e. g. those of sheaves on a simplicial space, in the sense of [1].

To deal with arbitrary Grothendieck toposes, we present a further generalization of Diaconescu's theorem, now from topological categories to "topical categories", i. e. internal categories in the category of toposes and geometric morphisms. After extending suitably the notion of a principal bundle to this case, we will show that for certain such categories $\mathcal{C}$, called here domain-étale, geometric morphisms to the classifying topos of $\mathcal{C}$ correspond to principal $\mathcal{C}$-bundles. As in [4], this in particular gives an interpretation of geometric morphisms to a generalized glueing of a small diagram of toposes $D : \mathbb{C} \to \mathrm{TOP}$ in terms of D-augmented principal $\mathbb{C}$-bundles. It

1991 *Mathematics Subject Classification.* 18B25.
Key words and phrases. Topos, principal C-bandle, lower bagdomain.

is hoped that this interpretation can be used to construct representing objects for many interesting functors on toposes. In this note we present just one example of such an application of this generalized Diaconescu theorem. It concerns a particular description of the lower bagdomain topos from [3]: for a topos $\mathbf{E}$, its lower bagdomain $\mathcal{B}_L(\mathbf{E})$ will be expressed as glueing of a certain diagram naturally associated with $\mathbf{E}$.

Everywhere in the sequel, TOP is the category of Grothendieck toposes, although most probably some of the toposes involved might be assumed unbounded. Morphisms of toposes mean geometric morphisms. When possible, we avoid mentioning 2-categorical aspects of our constructions; e. g. pullback of toposes really means bi-pullback, TOP-valued functors are really pseudofunctors, i. e. functors up to coherent canonical 2-isomorphisms, etc.

This research has been completed during author's visit to the University of Utrecht, supported by the Netherlands Organization for Scientific Research (NWO). The author is especially grateful to Ieke Moerdijk for much help in various ways, for interest, and for patience.

1. Domain-Étale Topical Categories

We will systematically consider internal categories in TOP; usually they are called *topical categories*. Structure items of such a category $\mathcal{C}$ will be denoted using the standard simplicial notations for its nerve. Thus, $\mathcal{C}$ looks like

$$\mathcal{C}_2 \xrightarrow{d_1} \mathcal{C}_1 \begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow[d_1]{} \end{smallmatrix} \mathcal{C}_0,$$

where $\mathcal{C}_2 = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ is moreover projected onto $\mathcal{C}_1$ via d_0, d_2.

Definition 1.1. One says that a topical category $\mathcal{C}$ *has étale domain*, or, is *domain-étale*, if the geometric morphism $d_0 : \mathcal{C}_1 \to \mathcal{C}_0$ is étale, i. e. a local homeomorphism, or a slice, that is, $d_0 \simeq (X^* \dashv \prod_X) : \mathcal{C}_0/X \to \mathcal{C}_0$, for some object $X \in \mathcal{C}_0$.

Example 1.1. For any internal category $\mathbb{C}$ in a topos $\mathbf{X}$, its *externalization* $\mathbf{X}/\mathbb{C} = (\mathbf{X}/C_2 \to \mathbf{X}/C_1 \begin{smallmatrix} \to \\ \leftarrow \\ \to \end{smallmatrix} \mathbf{X}/C_0)$ is evidently a domain-étale topical category. For any domain-étale $\mathcal{C}$ and any $\mathcal{C}$*-diagram*, i. e. an internal functor $q : \mathcal{D} \to \mathcal{C}$ in TOP which is a *discrete opfibration*, that is, the square

$$\begin{tikzcd}
\mathcal{D}_1 \arrow[r, "d_0"] \arrow[d, "q_1"'] & \mathcal{D}_0 \arrow[d, "q_0"] \\
\mathcal{C}_1 \arrow[r, "d_0"] & \mathcal{C}_0
\end{tikzcd}$$

is pullback, the category $\mathcal{D}$ is evidently also domain-étale. This is no longer true for *internal presheaves* on $\mathcal{C}$, i. e. for those internal functors $p : \mathcal{P} \to \mathcal{C}$ for which the square

$$\begin{array}{ccc} \mathcal{P}_1 & \xrightarrow{d_1} & \mathcal{P}_0 \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_0}\big\downarrow \\ \mathcal{C}_1 & \xrightarrow{d_1} & \mathcal{C}_0 \end{array}$$

is pullback. However if we additionally require that p_0 is étale, then also p_1 will be, hence also $d_0p_1 = p_0d_0$, and then also $d_0 : \mathcal{P}_1 \longrightarrow \mathcal{P}_0$. So if $p : \mathcal{P} \to \mathcal{C}$ is an étale presheaf in the sense of the definition to follow, then $\mathcal{P}$ is also domain-étale.

Definition 1.2. An *étale presheaf* on a topical category $\mathcal{C}$ is a $\mathcal{C}$-presheaf, i. e., a discrete fibration $p : \mathcal{P} \to \mathcal{C}$, such that $p_0 : \mathcal{P}_0 \to \mathcal{C}_0$ is étale. With the evident notion of morphism, the étale $\mathcal{C}$-presheaves form the category $|\mathcal{C}|$.

Proposition 1.1. *For domain-étale $\mathcal{C}$, the category $|\mathcal{C}|$ is a topos.*

Proof. It is clear that to give an étale presheaf on $\mathcal{C}$ is the same as to give an object P of $\mathcal{C}_0$ with an action $a : d_1^*P \to d_0^*P$, satisfying certain conditions. Since d_0 is étale, the pullback square

$$\begin{array}{ccc} \mathcal{C}_2 & \xrightarrow{d_2} & \mathcal{C}_1 \\ {\scriptstyle d_0}\big\downarrow & & {\scriptstyle d_0}\big\downarrow \\ \mathcal{C}_1 & \xrightarrow{d_1} & \mathcal{C}_0 \end{array}$$

satisfies the Beck-Chevalley condition $d_0^*d_{1*} \cong d_{2*}d_0^*$. This enables one to define a comonad structure on $d_{1*}d_0^*$ in such a way that an action a as above becomes equivalent to a coalgebra structure over this comonad. So $|\mathcal{C}| \simeq (\mathcal{C}_0)_{d_{1*}d_0^*}$; since the comonad is evidently left exact, the proposition follows. ■

Remark 1.1. Actually, $|\mathcal{C}|$ is a topos, equipped with a surjection $\mathcal{C}_0 \to |\mathcal{C}|$, for any topical category $\mathcal{C}$ – this is a very particular case of Theorem 2.5 from [5]. We only included the above sketchy argument to make explicit the involved comonad.

One sees easily that for an internal functor $f : \mathcal{C}' \to \mathcal{C}$ in TOP, the induced functor $f^* : |\mathcal{C}| \to |\mathcal{C}'|$ admits application of the standard adjoint lifting argument to acquire a right adjoint, hence a geometric morphism $|f| : |\mathcal{C}'| \to |\mathcal{C}|$.

Example 1.2. Note that for any étale presheaf $p : \mathcal{P} \to \mathcal{C}$ one has $|\mathcal{C}|/(\mathcal{P}, p) \simeq |\mathcal{P}|$. For a $\mathcal{C}$-diagram $q : \mathcal{D} \to \mathcal{C}$, relationship between $|\mathcal{D}|$ and $|\mathcal{C}|$ is more complicated. We will describe the particular case we need: let $\mathcal{C}$ be externalization of a sma ll category $\mathbb{C}$, i. e. of an internal category in the topos **S**

of sets. Then one easily observes $|\mathbf{S}/\mathbb{C}| \simeq \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ (in fact this is equally true for any topos $\mathbf{X}$ in place of $\mathbf{S}$, i. e. for any internal category $\mathbb{C}$ in $\mathbf{X}$ one has $|\mathbf{X}/\mathbb{C}| \simeq \mathbf{X}^{\mathbb{C}^{\mathrm{op}}}$). Now, $\mathbf{S}/\mathbb{C}$-diagrams as above are in one-to-one correspondence with TOP/$\mathbf{S}$-valued (pseudo)functors on $\mathbb{C}$ – this is a straightforward generalization of the correspondence between set-valued functors on a small category and discrete opfibrations over it. The $\mathbf{S}/\mathbb{C}$-diagram corresponding to $D : \mathbb{C} \to \mathrm{TOP}/\mathbf{S}$ will be denoted $\tilde{D}$. Thus, its topos of objects $\tilde{D}_0$ is the coproduct in TOP of the $D(c)$, for all $c \in \mathbb{C}_0$, equipped with the evident projection q_0 to $\mathbf{S}/\mathbb{C}_0$. We recall the glueing construction $\mathrm{Gl}(D)$ for a functor D as above: objects of $\mathrm{Gl}(D)$ are families $((X_c)_{c\in C_0}, (x_\gamma : \gamma^* X_c \to X_{c'})_{(c' \xrightarrow{\gamma} c)\in C_1}) \in \prod_{c\in C_0} D(c)_0 \times \prod_{(c'\xrightarrow{\gamma} c)\in C_1} D(c')_1$ satisfying $x_{s_0(c)} : s_0(c)^* X_c \xrightarrow{\cong} X_c$, and

$$\begin{array}{ccc} \gamma_2^*\gamma_1^* X_c & \xrightarrow{\cong} & (\gamma_1\gamma_2)^* X_c \\ {\scriptstyle \gamma_2^* x_{\gamma_1}}\big\downarrow & & {\scriptstyle x_{\gamma_1\gamma_2}}\big\downarrow \\ \gamma_2^* X_{c'} & \xrightarrow{x_{\gamma_2}} & X_{c''}, \end{array}$$

where the $\cong$'s denote the canonical isomorphisms of the pseudofunctor D. Then, Proposition 5.1 of [4] can be generalized in a straightforward way, giving an equivalence $|\tilde{D}| \simeq \mathrm{Gl}(D)$, which moreover relates $|\tilde{D}| \xrightarrow{|q|} |\mathbf{S}/\mathbb{C}| \simeq \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ to $\mathrm{Gl}(D) \to \mathrm{Gl}(1) \simeq \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$, where 1 is the constant functor with value $\mathbf{S}$.

2. Generalized Diaconescu's Theorem

Definition 2.1. For a topical category $\mathcal{C}$, a *principal $\mathcal{C}$-bundle* over a topos $\mathbf{X}$ is a discrete opfibration $e : \mathbf{X}/\mathbb{E} \to \mathcal{C}$, where $\mathbb{E}$ is a **cofiltered** internal category in $\mathbf{X}$. For any geometric morphism $f : \mathbf{Y} \to \mathbf{X}$, the composite $\mathbf{Y}/f^*\mathbb{E} \to \mathbf{X}/\mathbb{E} \to \mathcal{C}$ is also a principal $\mathcal{C}$-bundle, denoted $f^*(\mathbb{E}, e)$. A morphism from $e' : \mathbf{X}/\mathbb{E}' \to \mathcal{C})$ to $e : \mathbf{X}/\mathbb{E} \to \mathcal{C})$ is an internal functor $g : \mathbb{E}' \to \mathbb{E}$ together with a transformation

$$\begin{array}{ccc} \mathbf{X}/E'_0 & \xrightarrow{\mathbf{X}/g_0} & \mathbf{X}/E_0 \\ & {\scriptstyle e'_0}\searrow \quad \Rightarrow \quad \swarrow{\scriptstyle e_0} & \\ & \mathcal{C}_0 & \end{array}$$

compatible with actions of the respective discrete opfibrations. With this notion of morphism, principal $\mathcal{C}$-bundles over $\mathbf{X}$ form a category which we will denote $\mathrm{Prin}(\mathbf{X},\mathcal{C})$.

Remark 2.1. One might show that specifying a morphism of principal bundles as above is equivalent to specifying a principal $\mathcal{C}$-bundle $f : \mathbf{X^2}/\mathbb{F} \to \mathcal{C}$ over the Sierpinski topos $\mathbf{X^2}$ of $\mathbf{X}$, such that $0^*(\mathbb{F}, f) = (\mathbb{E}, e)$ and $1^*(\mathbb{F}, f) = (\mathbb{E}', e')$, for the points $0, 1 : \mathbf{X} \to \mathbf{X^2}$.

Example 2.1. The generic example of a principal $\mathcal{C}$-bundle lives in $|\mathcal{C}|$. Since $\mathcal{C}$ has étale domain, the diagram

$$
\begin{tikzcd}
* \downarrow \mathcal{C} = \big(\mathcal{C}_3 \xrightarrow{d_1} \mathcal{C}_2 \arrow[rr, shift left=2, "d_0"] \arrow[rr, shift right=2, "d_1"'] \arrow[dr] & & \mathcal{C}_1\big) \arrow[ll, "s_0"'] \arrow[dl, "d_0"] \\
& \mathcal{C}_0 &
\end{tikzcd}
$$

represents externalization of an internal category in $\mathcal{C}_0$. Moreover each of the $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ have compatible $\mathcal{C}$-actions given by $\mathcal{C}_i \times_{\mathcal{C}_0} \mathcal{C}_1 = \mathcal{C}_{i+1} \xrightarrow{d_i} \mathcal{C}_i$, hence the above diagram determines an internal category $\mathbb{E}_{\mathcal{C}}$ in $|\mathcal{C}|$; this internal category is cofiltered – in fact, after forgetting the $d_{1*}d_0^*$-coalgebra structure it even has an initial object $s_0 : \mathcal{C}_0 \to \mathcal{C}_1$. There is a discrete opfibration $|\mathcal{C}|/\mathbb{E}_{\mathcal{C}} \to \mathcal{C}$ given by the geometric morphism

$$
e_{\mathcal{C}} : |\mathcal{C}|/(\mathcal{C}_1 \xrightarrow{d_0} \mathcal{C}_0) \simeq \left| \begin{tikzcd} \mathcal{C}_3 \arrow[r, "d_2"] & \mathcal{C}_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & \mathcal{C}_1 \arrow[l, "s_1"'] \end{tikzcd} \right| \to \mathcal{C}_0
$$

induced by $\mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0$. This is indeed a discrete opfibration as $\mathcal{C}_1 \times_{\mathcal{C}_0} (|\mathcal{C}|/\mathbb{E}_{\mathcal{C}})_0 \simeq \mathcal{C}_1 \times_{\mathcal{C}_0} (|\mathcal{C}|/(\mathcal{C}_1 \xrightarrow{d_0} \mathcal{C}_0)) \simeq (|\mathcal{C}|/(\mathbb{E}_{\mathcal{C}})_0)/e_{\mathcal{C}}^*(* \downarrow \mathcal{C})_0 \simeq (|\mathcal{C}|/\mathbb{E}_{\mathcal{C}})_1$.

The following is a slight further extension of the generalization of Diaconescu's theorem given in [4] (Chapter II, Theorem 4.1):

Theorem 2.1. *The assignment $f \mapsto f^*(\mathbb{E}_{\mathcal{C}}, e_{\mathcal{C}})$ establishes an equivalence of categories*

$$\mathrm{TOP}(\mathbf{X}, |\mathcal{C}|) \xrightarrow{f^*} \mathrm{Prin}(\mathbf{X}, \mathcal{C}),$$

for any domain-étale topical category $\mathcal{C}$ and any topos $\mathbf{X}$.

Proof. One constructs a functor in the opposite direction by assigning to a principal $\mathcal{C}$-bundle $(\mathbb{E}, e)$ the composite $\mathbf{X} \xrightarrow{l_{\mathbb{E}}} \mathbf{X}^{\mathbb{E}^{\mathrm{op}}} \simeq |\mathbf{X}/\mathbb{E}| \xrightarrow{|e|} |\mathcal{C}|$, where $l_{\mathbb{E}} = (\varinjlim_{\mathbb{E}^{\mathrm{op}}} \dashv \mathrm{const})$ is the geometric morphism existing by cofilteredness of $\mathbb{E}$. To define it on morphisms of bundles, one just replaces the $\mathbf{X}$ above by $\mathbf{X}^{\mathbf{2}}$, according to Remark 2.1.

Starting from a principal bundle $(\mathbb{E}, e)$, one sees directly that the internal category in $\mathbf{X}^{\mathbb{E}^{\mathrm{op}}}$ corresponding to $|e|^*(\mathbb{E}_{\mathcal{C}}, e_{\mathcal{C}})$ has the presheaf of objects isomorphic to

$$\begin{array}{ccc} E_2 & \begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow[d_1]{}\end{array} & E_1 \\ \downarrow d_0 & & \downarrow d_0 \\ E_1 & \begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow[d_1]{}\end{array} & E_0 \end{array}$$

and the presheaf of morphisms isomorphic to

$$\begin{array}{ccc} E_3 & \begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow[d_1]{}\end{array} & E_2 \\ \downarrow d_0 d_0 & & \downarrow d_0 d_0 \\ E_1 & \begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow[d_1]{}\end{array} & E_0. \end{array}$$

The upper horizontal pairs of these diagrams have split coequalizers equal to E_0 and E_1 respectively and the corresponding morphisms between them constitute precisely $\mathbb{E}$; moreover one easily checks that the induced discrete opfibration over $\mathcal{C}$ is just e. This shows that the composition

$$\mathrm{Prin}(\mathbf{X},\mathcal{C}) \to \mathrm{TOP}(\mathbf{X},|\mathcal{C}|) \to \mathrm{Prin}(\mathbf{X},\mathcal{C})$$

is isomorphic to the identity. For the second composition, take any geometric morphism $f : \mathbf{X} \to |\mathcal{C}|$ and consider the diagram

$$\begin{array}{ccccc} \mathbf{X}^{f^*\mathbb{E}_{\mathcal{C}}^{\mathrm{op}}} & \xrightarrow{q} & |\mathcal{C}|^{\mathbb{E}_{\mathcal{C}}^{\mathrm{op}}} & \simeq ||\mathcal{C}|/\mathbb{E}_{\mathcal{C}}| \xrightarrow{b} |\mathcal{C}| \\ {}_l\uparrow\downarrow_p & & \downarrow_g & \\ \mathbf{X} & \xrightarrow{f} & |\mathcal{C}| & \end{array}$$

where the square is pullback and $b = |e_{\mathcal{C}}|$ is induced by the generic principal bundle. Note that the Beck-Chevalley condition is satisfied for the square, and the morphisms l and p form an adjoint pair, i. e. $l_* \cong p^*$. We have to check that the composite bql is isomorphic to f. But $(bql)_* = b_*q_*l_* \cong b_*q_*p^* \cong b_*g^*f_*$, so it remains to check that b_*g^* is isomorphic to the identity, or, equivalently, that $g_!b^*$ is (here $g_! \dashv g^*$). Now as before, one sees directly that, given an étale presheaf on $\mathcal{C}$, i. e. $P \in \mathcal{C}_0$ with a $d_{1*}d_0^*$-coalgebra structure, the functor $b^* = |e_{\mathcal{C}}|^*$ carries it to one on $|\mathcal{C}|/\mathbb{E}_{\mathcal{C}}$ which corresponds to an $\mathbb{E}_{\mathcal{C}}$-presheaf in $|\mathcal{C}|$ represented by $P \times_{\mathcal{C}_0} \mathcal{C}_1 \leftleftarrows P \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$, with the compatible $d_{1*}d_0^*$-coalgebra structures. Taking $g_!$ of it means passing to the coequalizer; and that pair has a split coequalizer equal to P, with it's original action. ∎

Remark 2.2. One should probably say more about defining f^* above on morphisms. We refer to [2] for the detailed analysis of the much more general situation. Our case falls within the scope of Lemma 7.6 there.

For a small category $\mathbb{C}$, a principal $\mathbf{S}/\mathbb{C}$-bundle over $\mathbf{X}$ is the same as a flat functor $\mathbb{C} \to \mathbf{X}$, and one recovers the original form of Diaconescu's theorem. Now suppose given a functor $D : \mathbb{C} \to \mathrm{TOP}$ as in Example 1.2 above. As we saw there, it gives rise to a discrete opfibration $\tilde{D} \to \mathbf{S}/\mathbb{C}$ in TOP with $|\tilde{D}| \simeq \mathrm{Gl}(D)$. As in [4], one can describe principal $\tilde{D}$-bundles over $\mathbf{X}$ in terms of D-augmented principal $\mathbf{S}/\mathbb{C}$-bundles, in the following sense:

Definition 2.2. A flat functor $F : \mathbb{C} \to \mathbf{X}$ augmented to $D : \mathbb{C} \to \mathrm{TOP}$ is one equipped with a natural transformation $\mathbf{X}/F \to D$, where $\mathbf{X}/F$ is F followed by the canonical functor $\mathbf{X} \to \mathrm{TOP}$, $I \mapsto \mathbf{X}/I$. A morphism of augmented flat functors is a pair (f, φ) as in the diagram

$$\begin{array}{ccccc} \mathbf{X}/F & & \xrightarrow{\mathbf{X}/f} & & \mathbf{X}/F' \\ & \searrow & \overset{\varphi}{\Rightarrow} & \swarrow & \\ & & D & & \end{array} .$$

With the obvious identities and composition the D-augmented flat functors form a category $\mathrm{Flat}(\mathbb{C}, \mathbf{X}) \Downarrow D$.

Thus an augmentation of F to D is a system of geometric morphisms $\mathrm{aug}_c : \mathbf{X}/F(c) \to D(c)$, natural in $(\gamma : c \to c') \in \mathbb{C}$.

Proposition 2.1. *For any diagram $D : \mathbb{C} \to \mathrm{TOP}$ with small $\mathbb{C}$, and any topos $\mathbf{X}$, there is an equivalence of categories*

$$\mathrm{TOP}(\mathbf{X}, \mathrm{Gl}(D)) \simeq \mathrm{Flat}(\mathbb{C}, \mathbf{X}) \Downarrow D.$$

Proof. By the theorem above, the lhs is equivalent to $\mathrm{Prin}(\mathbf{X}, \tilde{D})$, so let us construct an equivalence from the latter category to D-augmented flat functors $\mathbb{C} \to \mathbf{X}$. In one direction, given a principal $\tilde{D}$-bundle $\mathbf{X}/\mathbb{E} \to \tilde{D}$, the composite $\mathbf{X}/\mathbb{E} \to \tilde{D} \to \mathbf{S}/\mathbb{C}$ is clearly a discrete opfibration, hence yields a principal $\mathbf{S}/\mathbb{C}$-bundle, i. e., a flat functor; moreover $\mathbf{X}/\mathbb{E} \to \tilde{D}$ yields an augmentation of this flat f unctor. It is straightforward also to assign to a morphism of principal $\tilde{D}$-bundles a morphism of corresponding augmented flat functors. Conversely, given $F : \mathbb{C} \to \mathbf{X}$ with an augmentation $(\mathrm{aug}_c : \mathbf{X}/F(c) \to D(c))_c$, taking coproduct of all the aug_c in TOP yields $\mathbf{X}/E_0 \to \sum_c D(c) = \tilde{D}_0$, whereas action of $\mathbb{C}$ on D and on $\mathbf{X}/F$ yields an obvious $\tilde{D}$-action on $\mathbf{X}/\mathbb{E}$. ∎

3. Gluing Together the Bagdomain

We refer to [3] for the definition of the lower bagdomain topos $\mathcal{B}_L(\mathbf{E})$ of a topos $\mathbf{E}$. One might equivalently determine it as follows: first, for a topos $\mathbf{X}$, define the category $\mathrm{Fam}(\mathbf{X}, \mathbf{E})$ of $\mathbf{X}$-valued families of points of $\mathbf{E}$ as the category whose objects are geometric morphisms $\mathbf{X}/I \to \mathbf{E}$, for various $I \in \mathbf{X}$, and morphisms looking like

$$
\begin{tikzcd}
\mathbf{X}/I \arrow[rr, "\mathbf{X}/f"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\overset{\varphi}{\Rightarrow}"] & & \mathbf{X}/J \arrow[dl] \\
& \mathbf{E} &
\end{tikzcd}
.
$$

This construction is obviously contravariantly functorial in $\mathbf{X}$; moreover any object $p : \mathbf{X}/I \to \mathbf{E}$ of $\mathrm{Fam}(\mathbf{X}, \mathbf{E})$ determines a functor $(-)^*p$: $\mathrm{TOP}(\mathbf{Y}, \mathbf{X}) \to \mathrm{Fam}(\mathbf{Y}, \mathbf{E})$, assigning to $f : \mathbf{Y} \to \mathbf{X}$ the object $\mathbf{Y}/f^*I \to \mathbf{X}/I \to \mathbf{E}$. Then, $\mathcal{B}_L(\mathbf{E})$ is defined as the topos equipped with an universal family of points of $\mathbf{E}$, i. e., there is an object $I_\mathbf{E}$ in $\mathcal{B}_L(\mathbf{E})$ and a geometric morphism $p_\mathbf{E} : \mathcal{B}_L(\mathbf{E})/I_\mathbf{E} \to \mathbf{E}$ such that for any $\mathbf{X}$, the functor $(-)^*p_\mathbf{E} : \mathrm{TOP}(\mathbf{X}, \mathcal{B}_L(\mathbf{E})) \to \mathrm{Fam}(\mathbf{X}, \mathbf{E})$ is an equivalence.

In fact, rigorous definition of the functor $(-)^*p$ on morphisms requires some more words. This is another particular case of the situation mentioned in the Remark 2.2 above.

Example 3.1. Let $\mathbb{C} = \mathbb{S}_{\mathrm{fin}}^{\mathrm{op}}$ be the opposite of the "standard" category of finite sets (more precisely, it is the full subcategory on objects $n = \{0, \ldots, n-1\}$, $n \in \mathbb{N}$); so $|\mathbf{S}/\mathbb{C}| = \mathbf{S}^{\mathbb{S}_{\mathrm{fin}}}$ is the ce lebrated Object Classifier, and all flat functors $F : \mathbb{C} \to \mathbf{X}$ for any topos $\mathbf{X}$ are of the kind $n \mapsto I^n$ for an object $I \in \mathbf{X}$ unique up to isomorphism. In other words, $\mathbf{S}^{\mathbb{S}_{\mathrm{fin}}}$ contains the generic object U such that the functor $(-)^*U : \mathrm{TOP}(\mathbf{X}, \mathbf{S}^{\mathbb{S}_{\mathrm{fin}}}) \to \mathbf{X}$ is an equivalence for any topos $\mathbf{X}$. Since $\mathrm{Fam}(\mathbf{X}, \mathbf{S}) \simeq \mathbf{X}$, this shows clearly that $\mathbf{S}^{\mathbb{S}_{\mathrm{fin}}}$ can serve as $\mathcal{B}_L(\mathbf{S})$.

Now any topos $\mathbf{E}$ determines a functor $\mathbf{E}^\bullet : \mathbb{S}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{TOP}$ sending n to $\mathbf{E}^n$ (n-th power of $\mathbf{E}$ in TOP) and $\varphi : n \to m$ to $\mathbf{E}^\varphi : \mathbf{E}^m \to \mathbf{E}^n$. By Proposition 2.1, we know that $\mathrm{Gl}(\mathbf{E}^\bullet)$ classifies $\mathbf{E}^\bullet$-augmented flat functors on $\mathbb{S}_{\mathrm{fin}}^{\mathrm{op}}$.

Given $I \in \mathbf{X}$, an $\mathbf{E}^\bullet$-augmentation of the corresponding flat functor is a natural family $(\mathrm{aug}_n : \mathbf{X}/I^n \to \mathbf{E}^n)_{n \in \mathbb{N}}$; naturality requirement for the map $\iota_i : 1 \to n$ that picks some element i forces the square

$$
\begin{tikzcd}
\mathbf{X}/I^n \arrow[r, "\mathrm{aug}_n"] \arrow[d, "\mathbf{X}/\pi_i"'] & \mathbf{E}^n \arrow[d, "\pi_i"] \\
\mathbf{X}/I \arrow[r, "\mathrm{aug}_1"] & \mathbf{E}
\end{tikzcd}
$$

to commute, π_i denoting the corresponding product projections. Hence the whole family is uniquely determined by $\mathrm{aug}_1 : \mathbf{X}/I \to \mathbf{E}$ which might be arbitrary. Moreover a morphism of augmented flat functors is determined by a diagram like

$$
\begin{tikzcd}
\mathbf{X}/I \arrow[rr, "\mathbf{X}/f"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\Rightarrow"] & & \mathbf{X}/J \arrow[dl] \\
& \mathbf{E} &
\end{tikzcd}
.
$$

This means that $\mathrm{Gl}(\mathbf{E}^\bullet)$ has the same universal property as the lower bagdomain $\mathcal{B}_L(\mathbf{E})$ of $\mathbf{E}$. Hence one obtains

Proposition 3.1. *There is an equivalence*

$$\mathcal{B}_L(\mathbf{E}) \simeq \mathrm{Gl}(\mathbf{E}^\bullet)$$

for any topos $\mathbf{E}$.

REFERENCES

1. P. Deligne, Théorie de Hodge, III. *Inst. Hautes Études Sci. Publ. Math.* **44**(1975), 5–77.

2. P. T. Johnstone, Fibrations and partial products in a 2-category. *Appl. Categ. Structures* **1**(1993), 141–179.

3. P. T. Johnstone, Partial products, bagdomains and hyperlocal toposes. in *Applications of Categories in Computer Science*, 315–339, *London Math. Soc. Lecture Notes Ser.* 177, 1992.

4. I. Moerdijk, Classifying Spaces and Classifying Topos. *Lecture Notes in Math.* 1616, *Springer, New York* 1995.

5. I. Moerdijk, The classifying topos of a continuous groupoid, I. *Trans. Amer. Math. Soc.* **310**(1988), 629–668.

(Received 20.10.1997)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia