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“EXPLAINING” THE ARDESHIR-RUITENBURG OPERATOR

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Abstract. We provide semantical description of a peculiar operator on Visser algebras introduced by Ardeshir and Ruitenburg, in the semantics of the Basic Propositional Calculus proposed by Ma and Sano using the dual of the Cantor-Bendixson derivative, as well as in the semantics provided by wK4-algebras.

1. Introduction

In [6] Albert Visser introduced the Basic Propositional Logic BPL. This is a rather peculiar subintuitionistic logic: adding to it **true** → **false** does not lead to contradiction. Instead it lands in a consistent extension of an even more interesting logic FPL, which is to the Provability Logic of the Peano Arithmetic as the Intuitionistic Propositional Logic IPL is to the modal logic S4 (via the celebrated Gödel translation).

We will work with BPL through its algebraic semantics; it should be clear from it what BPL actually is. Algebras that arise from BPL are usually called *Visser algebras* in the literature.

1.1. **Definition.** A *Visser algebra* [1] $(A, \dashrightarrow)$ is a bounded distributive lattice A together with a binary operation $\dashrightarrow\, : A \times A \to A$ satisfying

$$\begin{aligned} x &\leqslant 1 \dashrightarrow x,\\ (x \dashrightarrow x) &= 1,\\ (x \dashrightarrow y) \wedge (y \dashrightarrow z) &\leqslant x \dashrightarrow z,\\ x \dashrightarrow (y \wedge z) &= (x \dashrightarrow y) \wedge (x \dashrightarrow z),\\ (x \vee y) \dashrightarrow z &= (x \dashrightarrow z) \wedge (y \dashrightarrow z) \end{aligned}$$

for any $x, y, z \in A$.

1.1.1. *Remarks.* Let us point out that any Heyting algebra satisfies these conditions with the Heyting implication $\to$ taken for $\dashrightarrow$. In fact, a Visser algebra A satisfies $1 \dashrightarrow x = x$ for any $x \in A$ if and only if it is a Heyting algebra with $\dashrightarrow$ as the Heyting implication. Still better way to see where Visser algebras are situated with respect to Heyting algebras is to note the easy consequence of the above definition: for any x, y, z in a Visser algebra $(A, \dashrightarrow)$,

$$(x \wedge y) \dashrightarrow z \leqslant x \dashrightarrow (y \dashrightarrow z).$$

And, $\dashrightarrow$ can be used for the implication that turns A into a Heyting algebra if and only if the reverse inequality holds in A too.

In [1, 2], an operator ξ is defined on Visser algebras. It is given by

$$\xi a = ((1 \dashrightarrow a) \dashrightarrow a) \dashrightarrow (1 \dashrightarrow a)$$

and has the following properties:

$1 \dashrightarrow \xi a = \xi a$;
if $1 \dashrightarrow a = a$ then $\xi a = a$.

These imply that ξ is an idempotent operator whose image coincides with the set of all *Heyting elements* (those a with $1 \dashrightarrow a = a$).

In [4], a semantics for the BPL has been proposed, which provides certain class of Visser algebras: one takes the lattice $\mathrm{Op}(X)$ of all open sets of any topological space X with the binary operation

$$U \dashrightarrow V := \tau((X - U) \cup V),$$

where τ is the *coderivative* operator:

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1.2. **Definition.** For a subset S of a topological space X, the coderivative of S is

$$\tau S := \{x \in X \mid \exists\, x \in U \in \mathrm{Op}(X)\ \ U - S \subseteq \{x\}\ \}.$$

Note that when applied to an open set, the operator τ adds to it all isolated points of its complement[1].

Since in this interpretation we have $1 \dashrightarrow U = \tau U$, it follows that the above operator ξ in this case retracts all open sets onto all *perfect* open sets (the fixed points of τ). This seems quite surprising since in general finding the perfect hull of an open set requires either transfinite iteration of τ or (in general infinite) meet of all perfect opens containing the given open set, whereas ξ is a finitary construct.

We will describe the operator ξ in this situation; in particular it will be clear that, although it indeed captures all perfect open sets, ξU does not coincide with the perfect hull of U — it may be strictly larger.

2. Operators on Heyting algebras

We will use the following fact, cf. [7, 3.9.22]

2.1. **Proposition.** *Let* $\mathrm{T} : H \to H$ *be a multiplicative inflationary operator on a Heyting algebra* H*. Then the operator* $\mathrm{F_T} : H \to H$ *given by*

$$\mathrm{F_T}a := (\mathrm{T}a \to a) \to a$$

has the following properties:

- $\mathrm{F_T}a$ *is a fixed point of* T *for every* $a \in H$*;*
- $\mathrm{F_T}$ *and* T *have the same fixed points.*

Hence $\mathrm{F_T}$ *is an idempotent operator whose image coincides with the set of all fixed points of* T*. Moreover if* $x \leqslant \mathrm{F_T}a$ *then* $\mathrm{T}x \leqslant \mathrm{F_T}a$*, in particular, one has*

$$a \leqslant \mathrm{T}a \leqslant \mathrm{TT}a \leqslant \cdots \leqslant \mathrm{F_T}a$$

for all $a \in H$*.*

Proof. The first assertion follows from

$$\begin{aligned}
\mathrm{T}((\mathrm{T}a \to a) \to a) \wedge (\mathrm{T}a \to a) &= \mathrm{T}((\mathrm{T}a \to a) \to a) \wedge \mathrm{T}(\mathrm{T}a \to a) \wedge (\mathrm{T}a \to a)\\
&= \mathrm{T}(((\mathrm{T}a \to a) \to a) \wedge (\mathrm{T}a \to a)) \wedge (\mathrm{T}a \to a)\\
&= \mathrm{T}(a \wedge (\mathrm{T}a \to a)) \wedge (\mathrm{T}a \to a)\\
&= \mathrm{T}a \wedge (\mathrm{T}a \to a)\\
&= \mathrm{T}a \wedge a\\
&= a,
\end{aligned}$$

as by adjunction this gives

$$\mathrm{T}((\mathrm{T}a \to a) \to a) \leqslant (\mathrm{T}a \to a) \to a.$$

The second assertion follows since

$$\mathrm{F_T}a \to a = ((\mathrm{T}a \to a) \to a) \to a = \mathrm{T}a \to a.$$

These two then easily imply the rest. □

2.1.1. *Example.* If H is boolean, then $\mathrm{F_T} = \mathrm{T}$. In fact any inflationary multiplicative operator T on a boolean algebra has form $\mathrm{T}a = a \vee \mathrm{T}0$. Indeed, $a \vee \mathrm{T}0 \leqslant \mathrm{T}a$ is clear while $\mathrm{T}a - a \leqslant \mathrm{T}a \wedge \mathrm{T}{-a} = \mathrm{T}(a \wedge -a) = \mathrm{T}0$.

Note that $\mathrm{F_T}$ need not preserve order, and $\mathrm{F_T}\,\mathrm{T} \neq \mathrm{F_T}$ in general.

2.1.2. *Example.* Suppose H is linear. Then

$$\mathrm{F_T}a = \begin{cases} a & \text{if } \mathrm{T}a = a,\\ 1 & \text{if } \mathrm{T}a > a.\end{cases}$$

3. Ma-Sano semantics for wK4-algebras

Let us now consider, according to [4], the Visser algebra obtained from the lattice of open sets of a topological space equipped with the binary operation $\dashrightarrow$ defined as above by $U \dashrightarrow V := \tau(-U \cup V)$.

It will turn out that when T is the coderivative operator on open sets, then $\mathrm{F_T}$ coincides with the Ardeshir-Ruitenburg operator on the corresponding Visser algebra. We can as well work in the greater generality of any *wK4-algebra* [3].

[1]Since there might be non-unique interpretation of this term: for us, a point x of a topological space X is isolated iff the singleton set $\{x\}$ is open

3.1. **Definition.** A wK4-algebra is a modal algebra (B, τ) where B is a Boolean algebra and τ is a multiplicative operator such that, defining $\Box b := b \wedge \tau b$, the algebra $(B, \Box)$ is an S4-algebra. This holds if and only if

$$\tau 1 = 1$$

and

$$b \wedge \tau b \leqslant \tau\tau b$$

for all $b \in B$.

3.1.1. *Example.* The leading example is the algebra $B = \mathscr{P}(X)$ of all subsets of a topological space X, equipped with the coderivative operator τ as in 1.2 above. This justifies the terminology that follows.

An element $u \in B$ will be called *open* if $u = \Box u$ (equivalently, $u \in \Box(B)$, equivalently, $u \leqslant \tau u$). An open element u is *perfect* if $\tau u = u$. Dually, $\delta b = -\tau{-b}$ and $\Diamond b = b \vee \delta b$; an element $c \in B$ is *closed* if $c = \Diamond c$ (equivalently, $c \in \Diamond(B)$, equivalently, $\delta c \leqslant c$). A closed element c is *perfect* if $\delta c = c$. An open (resp. closed) element u (resp. c) is *regular* if $\Box\Diamond u = u$ (resp. $\Diamond\Box c = c$). An element $b \in B$ will be called *locally closed* if $b = c \wedge u$ for some closed c and open u.

Given a wK4-algebra (B, τ), let $H_\tau \subseteq B$ be the set of its open elements, $H_\tau = \Box(B)$. Since $(B, \Box)$ is an S4-algebra, H_τ is a sublattice of B and is a Heyting algebra when equipped with the implication

$$u \to v := \Box(-u \vee v).$$

To extend the semantics of [4] from topological spaces to arbitrary wK4-algebras, we are going to equip H_τ with the operation

$$u \underset{\tau}{\dashrightarrow} v := \tau(-u \vee v).$$

For this to work, we need to know that $\tau(-u \vee v)$ is open for any open u, v. It will be convenient for us to pass to the complements and prove the equivalent statement, that $\delta(c - d)$ is closed for any closed elements c, d, i. e. that $\delta(s)$ is closed for any locally closed s.

3.2. **Proposition.** *Let s be an element of a wK4-algebra (B, τ). If s is locally closed then δs is closed, i. e. $\delta\delta s \leqslant \delta s$ holds.*

Proof. Clearly s is locally closed iff there are closed elements c_1, c_2 with $c_2 \leqslant c_1$ such that $s = c_1 - c_2$. Thus $\delta c_1 \leqslant c_1$ and $\delta c_2 \leqslant c_2$. Then,

$$\delta c_1 = \delta(c_2 \vee (c_1 - c_2)) = \delta c_2 \vee \delta(c_1 - c_2) \leqslant c_2 \vee \delta(c_1 - c_2),$$

or equivalently

$$(\delta c_1) - c_2 \leqslant \delta(c_1 - c_2).$$

Since δ is order preserving, $\delta\delta(c_1 - c_2) \leqslant \delta\delta c_1 \leqslant \delta c_1$; also $\delta\delta(c_1 - c_2) \leqslant (c_1 - c_2) \vee \delta(c_1 - c_2)$ since we are in wK4. It follows that

$$\begin{aligned}\delta\delta(c_1 - c_2) \leqslant \delta c_1 \wedge ((c_1 - c_2) \vee \delta(c_1 - c_2)) = (\delta c_1 \wedge (c_1 - c_2)) \vee (\delta c_1 \wedge \delta(c_1 - c_2))\\ = ((\delta c_1) - c_2) \vee \delta(c_1 - c_2) = \delta(c_1 - c_2),\end{aligned}$$

which by definition means that $\delta(c_1 - c_2)$ is closed. □

3.2.1. *Remark.* It is interesting to compare 3.2 with the following fact, observed in [3]. For a topological space X, the wK4-algebra $(\mathscr{P}(X), \tau)$ corresponding to it (using τ as in 1.2) is a K4-algebra (i. e. $\delta\delta S \subseteq \delta S$ holds for all $S \subseteq X$), iff X is a T_d*-space* (i. e. every singleton subset of X is locally closed).

With 3.2 at hand, we have

3.3. **Corollary.** *For any wK4-algebra (B, τ), let $H_\tau \subseteq B$ be the set of its τ-open elements, i. e. $H_\tau = \{u \in B \mid u \leqslant \tau u\}$. For any $u, v \in H_\tau$ let*

$$u \underset{\tau}{\dashrightarrow} v := \tau(-u \vee v).$$

Then, $u \underset{\tau}{\dashrightarrow} v \in H_\tau$.

Proof. This is straightforward from 3.2, given that an element is open iff its complement is closed, and that

$$-(u \underset{\tau}{\dashrightarrow} v) = \delta((-v) - (-u)).$$

□

3.4. **Theorem.** *If (B, τ) is a wK4-algebra, then $(H_\tau, \underset{\tau}{\dashrightarrow})$ is a Visser algebra.*

Proof. It is straightforward to verify all of the identities in 1.1 for $\underset{\tau}{\dashrightarrow}$, using properties of τ. □

4. The Ardeshir-Ruitenburg operator

Given a wK4-algebra $(\boldsymbol{B},\tau)$ as above, since $(\boldsymbol{H}_\tau,\underset{\tau}{\dashrightarrow})$ is a Visser algebra, we can, after [1], consider the operator $\xi:\boldsymbol{H}_\tau\to\boldsymbol{H}_\tau$ defined as follows:

$$\xi u := ((1\underset{\tau}{\dashrightarrow}u)\underset{\tau}{\dashrightarrow}u)\underset{\tau}{\dashrightarrow}(1\underset{\tau}{\dashrightarrow}u) = (\tau u\underset{\tau}{\dashrightarrow}u)\underset{\tau}{\dashrightarrow}\tau u.$$

On the other hand, since $(\boldsymbol{B},\square)$ is an S4-algebra, $\boldsymbol{H}_\tau$ is a Heyting algebra, with the implication

$$u\to v := \square(-u\vee v),$$

so, as in 2.1, we have the operator F_τ on $\boldsymbol{H}_\tau$ given by

$$\mathrm{F}_\tau(u) = (\tau u\to u)\to u.$$

We then have

4.1. Theorem. *For any wK4-algebra $(\boldsymbol{B},\tau)$, the operators ξ and F_τ above coincide.*

Proof. Let us begin by showing that $\xi(0)=\mathrm{F}_\tau(0)$. We have

$$\xi(0) = \tau(-\tau(-\tau 0)\vee\tau 0) = \tau(\delta\tau 0\vee\tau 0) = \tau\Diamond\tau 0$$

and

$$\mathrm{F}_\tau(0) = \neg\neg\tau 0 = \square\Diamond\tau 0 = \Diamond\tau 0\wedge\tau\Diamond\tau 0,$$

so definitely $\mathrm{F}_\tau(0)\leqslant\xi(0)$. For the reverse inequality we have to show $\tau\Diamond\tau 0\leqslant\Diamond\tau 0$. Indeed

$$\begin{aligned}\tau\Diamond\tau 0-\Diamond\tau 0 = \tau\Diamond\tau 0-\Diamond\Diamond\tau 0 = \tau\Diamond\tau 0\wedge\square(-\Diamond\tau 0) = \tau\Diamond\tau 0\wedge\tau(-\Diamond\tau 0)\wedge(-\Diamond\tau 0)\\ = \tau(\Diamond\tau 0\wedge(-\Diamond\tau 0))\wedge(-\Diamond\tau 0) = \tau 0\wedge(-\Diamond\tau 0)\leqslant\tau 0\wedge(-\tau 0) = 0,\end{aligned}$$

so that indeed $\tau\Diamond\tau 0\leqslant\Diamond\tau 0$.

For general $u\in\boldsymbol{H}_\tau$, let us relativize the whole thing to the algebra $(\boldsymbol{B}_u,\tau_u)$, where $\boldsymbol{B}_u=[u,1]$ and $\tau_u=\tau$. It is straightforward to show that this is a wK4-algebra, so that we get operators F_{τ_u} and ξ_u on $\boldsymbol{H}_{\tau_u}$. Moreover the bottom element 0_u of $\boldsymbol{B}_u$ is u, and $\boldsymbol{H}_{\tau_u}=\boldsymbol{H}_\tau\cap[u,1]$, so that

$$\mathrm{F}_{\tau_u}(0_u) = \neg_u\neg_u\tau u = (\tau(u)\to u)\to u = \mathrm{F}_\tau(u),$$

while for $u\leqslant b$,

$$\delta_u(b) = -_u\tau_u(-_u b) = u\vee-\tau(u\vee-b) = u\vee\delta(b-u),$$

so that

$$\Diamond_u(b) = b\vee\delta_u(b) = b\vee\delta(b-u),$$

hence

$$\xi_u(0_u) = \tau_u\Diamond_u\tau_u(0_u) = \tau(\tau u\vee\delta(\tau u-u)) = \xi(u).$$

As we have already shown $\mathrm{F}_{\tau_u}(0_u)=\xi_u(0_u)$, the equality $\mathrm{F}_\tau(u)=\xi(u)$ follows. □

References

[1] Mohammad Ardeshir and Wim Ruitenburg, *Basic propositional calculus I*, Mathematical Logic Quarterly **44** (1998), 317–343, DOI 10.1002/malq.19980440304.

[2] Majid Alizadeh, Mohammad Ardeshir, and Wim Ruitenburg, *Boolean Algebras in Visser Algebras*, Notre Dame Journal of Formal Logic **57** (2016), no. 1, 141–150, DOI 10.1215/00294527-3339473.

[3] Leo Esakia, *Weak transitivity – a restitution*, Logical Investigations **8** (2001), 244–255 (Russian). "Nauka", Moscow.

[4] Minghui Ma and Katsuhiko Sano, *Alternative Semantics for Visser's Basic Propositional Logic*, 10th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2013 (Gudauri, Georgia, September 23–27, 2013) (Martin Aher, Daniel Hole, Emil Jeřábek, and Clemens Kupke, eds.), Lecture Notes in Computer Science, vol. 8984, Springer-Verlag, pp. 257–275, DOI 10.1007/978-3-662-46906-4_15.

[5] Yasuhito Suzuki, Frank Wolter, and Michael Zakharyaschev, *Speaking about transitive frames in propositional languages*, Journal of Logic, Language and Information **7** (1998), 317–339, DOI 10.1023/A:1008237600846.

[6] Albert Visser, *A propositional logic with explicit fixed points*, Studia Logica **40** (1981), no. 2, 155–175, DOI 10.1007/bf01874706.

[7] J. Todd Wilson, *The assembly tower and some categorical and algebraic aspects of frame theory*, Carnegie-Mellon Univ. Pittsburgh PA School Of Computer Science, 1994, https://www.researchgate.net/publication/35295449. Ph. D. thesis CMU-CS-94-186.