Lattices and Topology

Exercises for Lecture 3

1. Let X be a topological space and $A, B \subseteq X$.

1a. Show that $\text{int}(A) \subseteq \text{int}(\text{int}(A))$ and $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. By a dual argument, show that $\overline{\overline{A}} \subseteq \overline{A}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

1b. Prove that $\text{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \text{int}(X - A)$.

2. Let X be a set, and $\mathrm{i} : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function satisfying

- $\mathrm{i}(X) = X$,
- $\mathrm{i}(A) \subseteq A$,
- $\mathrm{i}(A) \subseteq \mathrm{i}(\mathrm{i}(A))$,
- $\mathrm{i}(A \cap B) = \mathrm{i}(A) \cap \mathrm{i}(B)$.

2a. Show that $\tau = \{A \subseteq X : \mathrm{i}(A) = A\}$ is a topology on X.

2b. Prove that every topology on X is obtained this way.

3. Let (X, τ) be a topological space and $Y \subseteq X$. Set $\tau_Y = \{U \cap Y : U \in \tau\}$. Show that τ_Y is a topology on Y.

4. Prove that the subspace topology of any finite subset of the real line $\mathbb{R}$ is discrete.

5. Let $(P, \leqslant)$ and $(P', \leqslant')$ be posets and $\tau_{\leqslant}$ and $\tau_{\leqslant'}$ be the corresponding Alexandroff topologies.

5a. Show that a map $f : (P, \tau_{\leqslant}) \to (P', \tau_{\leqslant'})$ is continuous iff $f : (P, \leqslant) \to (P', \leqslant')$ is order-preserving.

5b. Show that $f : (P, \tau_{\leqslant}) \to (P', \tau_{\leqslant'})$ is a homeomorphism iff $f : (P, \leqslant) \to (P', \leqslant')$ is an order-isomorphism.

6. Prove that a space X is T_1 iff each singleton subset of X is closed. Deduce that each finite T_1-space is discrete.

7. Let X be a topological space and $x \in X$. Show that $\overline{\{x\}}$ is a join-prime element of the lattice of closed subsets of X.

8. Let X be a topological space.

8a. Show that for each $x, y \in X$ we have $x \in \overline{\{y\}}$ iff $\overline{\{x\}} \subseteq \overline{\{y\}}$.

8b. Prove that X is T_0 iff for each $x, y \in X$, from $\overline{\{x\}} = \overline{\{y\}}$ it follows that $x = y$.

8c. Deduce that each sober space is T_0.

9. Show that the cofinite topology on an infinite set is not sober.

10*. Prove that each Hausdorff space is sober.

11. Let X be a topological space. Show that the specialization order of X is reflexive and transitive, and that it is antisymmetric iff X is T_0.

12. Let $(P, \leqslant)$ be a poset. Prove that $\leqslant_{\tau_{\leqslant}} = \leqslant$.

13. Let (X, τ) be a topological space.

13a. Show that $\tau \subseteq \tau_{\leqslant_\tau}$.

13b. Prove that $\tau = \tau_{\leqslant_\tau}$ iff τ is an Alexandroff topology.

14. Show that each cofinite topology is compact.

15*. Prove that a subset of $\mathbb{Q}$ is compact iff it is finite.

16. Let X be the following subset of $[0, 1]$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., 0\}.$$

Equip X with the subspace topology.

16a. Show that X is compact.

16b. Prove that a subset of X is clopen iff either it is a finite subset of X not containing 0 or it is a cofinite subset of X containing 0.

16c. Deduce that X is a Stone space.