Lattices and Topology

Exercises for Lecture 2

1. Let $\mathscr{U}(P)$ be the lattice of upsets of a poset P.
1a. Show that for each $p \in P$ the upset $\uparrow p$ is a join-prime element of $\mathscr{U}(P)$.
1b. Show that if P is finite, then the converse also holds. That is, each join-prime element of $\mathscr{U}(P)$ has the form $\uparrow p$ for some $p \in P$.

2. Let L be a finite distributive lattice and $\phi : L \to {L_*}^*$ be given by

$$\phi(a) = \{j \in \mathfrak{J}(L) : j \leqslant a\}.$$

2a. Show that ϕ is a well-defined lattice homomorphism.
2b. Show that ϕ is onto.
2c. Show that ϕ is 1-1.

3. Show that a nonempty subset F of a lattice L is a filter iff for each $a, b \in L$ we have

$$a, b \in F \text{ iff } a \wedge b \in F.$$

By a dual argument, show that a nonempty subset I of L is an ideal iff for each $a, b \in L$ we have

$$a, b \in I \text{ iff } a \vee b \in I.$$

4. Prove that in a finite lattice each filter and each ideal are principal.

5. Show that a filter F of a lattice L is prime iff $L - F$ is an ideal, which is then a prime ideal. By a dual argument, show that an ideal I of L is prime iff $L - I$ is a filter, which is then a prime filter.

6. Prove that in a linearly ordered set, each upset is a prime filter and each downset is a prime ideal.

7. Show that an element a of a lattice L is join-prime iff the filter $\uparrow a$ is prime. By a dual argument, show that a is a meet-prime element iff $\downarrow a$ is a prime ideal.

8. Prove that in a finite lattice the map $a \mapsto \uparrow a$ establishes order-isomorphism between the posets $(\mathfrak{J}(L), \geqslant)$ and $(\mathscr{X}(L), \subseteq)$. By a dual argument, show that the map $a \mapsto \downarrow a$ establishes order-isomorphism between the posets $(\mathfrak{M}(L), \leqslant)$ and $(\mathscr{Y}(L), \subseteq)$. Conclude that the posets $(\mathfrak{J}(L), \leqslant)$ and $(\mathfrak{M}(L), \leqslant)$ are order-isomorphic.

9. Give a concrete example of a distributive lattice L which is not isomorphic to $\mathscr{U}(\mathscr{X}(L))$.