Lattices and Topology

Exercises for Lecture 1

1. Give a detailed proof that all nonempty finite subsets of a lattice possess suprema and infima.

2. Show that each complete lattice is bounded.

3. Prove that in a lattice the operations $a \vee b = \mathrm{Sup}\{a, b\}$ and $a \wedge b = \mathrm{Inf}\{a, b\}$ satisfy commutativity, associativity, and absorption laws.

4. Let $\wedge, \vee : L \times L \to L$ be binary operations on a set L satisfying commutativity, associativity, idempotency, and absorption laws. Define the binary relation $\leq$ on L by

$$a \leq b \text{ iff } a \wedge b = a.$$

4a. Prove that $a \leq b$ iff $a \vee b = a$.
4b. Prove that $\leq$ is a partial order.
4c. Prove that $a \wedge b = \mathrm{Inf}\{a, b\}$ and $a \vee b = \mathrm{Sup}\{a, b\}$.

5. Prove that the inverse of a lattice isomorphism is a lattice isomorphism.

6. Let L be a lattice. Prove that for each $a, b, c \in L$ we have:

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c) \quad \text{and} \quad a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c).$$

7. Prove that each linearly ordered set is a distributive lattice.

8. Show that in a bounded linearly ordered set the only elements possessing a complement are 0 and 1.

9. Prove that every finite distributive lattice is a Heyting lattice.

10*. Prove that a complete lattice L is a Heyting lattice iff the following *infinite distributive law*

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds in L for all $a \in L$ and $S \subseteq L$.