Lattices and Topology

Guram Bezhanishvili and Mamuka Jibladze

ESSLLI'08
11-15.VIII.2008

Lecture 3: Topology

Review of lecture 2

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.
- With each finite distributive lattice L we associated its dual poset L_* of join-prime elements.

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.
- With each finite distributive lattice L we associated its dual poset L_* of join-prime elements.
- Conversely, with each finite poset P we associated its dual lattice $P^* = \mathscr{U}(P)$ of upsets.

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.
- With each finite distributive lattice L we associated its dual poset L_* of join-prime elements.
- Conversely, with each finite poset P we associated its dual lattice $P^* = \mathscr{U}(P)$ of upsets.
- We showed that these constructions are mutually inverse in the sense that ${L_*}^*$ is isomorphic to L and that ${P^*}_*$ is order-isomorphic to P,

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.
- With each finite distributive lattice L we associated its dual poset L_* of join-prime elements.
- Conversely, with each finite poset P we associated its dual lattice $P^* = \mathscr{U}(P)$ of upsets.
- We showed that these constructions are mutually inverse in the sense that ${L_*}^*$ is isomorphic to L and that ${P^*}_*$ is order-isomorphic to P, thus obtaining the Birkhoff duality between finite distributive lattices and finite posets.

Review of lecture 2

- We have discussed the Birkhoff duality between finite distributive lattices and finite posets.
- Our main tools were join-prime and meet-prime elements.
- With each finite distributive lattice L we associated its dual poset L_* of join-prime elements.
- Conversely, with each finite poset P we associated its dual lattice $P^* = \mathscr{U}(P)$ of upsets.
- We showed that these constructions are mutually inverse in the sense that ${L_*}^*$ is isomorphic to L and that ${P^*}_*$ is order-isomorphic to P, thus obtaining the Birkhoff duality between finite distributive lattices and finite posets.
- We saw that the Birkhoff duality provides a representation of finite distributive lattices as lattices of sets with set-theoretic union and intersection.

Review of lecture 2

- We next turned to infinite distributive lattices.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.
- This forced us to generalize the concept of join-prime element to that of prime filter.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.
- This forced us to generalize the concept of join-prime element to that of prime filter. Similarly, the concept of meet-prime element generalizes to that of prime ideal.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.
- This forced us to generalize the concept of join-prime element to that of prime filter. Similarly, the concept of meet-prime element generalizes to that of prime ideal.
- We next described a representation of a (possibly infinite) distributive lattice L as a sublattice of the lattice of all upsets of the poset $\mathscr{X}(L)$ of prime filters of L.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.
- This forced us to generalize the concept of join-prime element to that of prime filter. Similarly, the concept of meet-prime element generalizes to that of prime ideal.
- We next described a representation of a (possibly infinite) distributive lattice L as a sublattice of the lattice of all upsets of the poset $\mathscr{X}(L)$ of prime filters of L. This representation required the Stone lemma.

Review of lecture 2

- We next turned to infinite distributive lattices.
- Unfortunately the Birkhoff duality does not extend straightforwardly to the infinite case because there may be lack of join-prime elements.
- This forced us to generalize the concept of join-prime element to that of prime filter. Similarly, the concept of meet-prime element generalizes to that of prime ideal.
- We next described a representation of a (possibly infinite) distributive lattice L as a sublattice of the lattice of all upsets of the poset $\mathscr{X}(L)$ of prime filters of L. This representation required the Stone lemma.
- A precise description of this sublattice can be done by topological means.

Short outline of lecture 3

Lecture 3: Topology

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces
- Closure and interior

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces
- Closure and interior
- Separation axioms

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces
- Closure and interior
- Separation axioms
- Compactness

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces
- Closure and interior
- Separation axioms
- Compactness
- Compact Hausdorff spaces

Short outline of lecture 3

Lecture 3: Topology

- Topological spaces
- Closure and interior
- Separation axioms
- Compactness
- Compact Hausdorff spaces
- Stone spaces

Topology

The origins of topology go back to Georg Cantor (1845 – 1918) and his fundamental work concerning the nature of infinity.

Topology

The origins of topology go back to Georg Cantor (1845 – 1918) and his fundamental work concerning the nature of infinity.

The definition of a topological space that we use today was first given by Felix Hausdorff (1868 - 1942) in 1914.

Topology

The origins of topology go back to Georg Cantor (1845 – 1918) and his fundamental work concerning the nature of infinity.

The definition of a topological space that we use today was first given by Felix Hausdorff (1868 - 1942) in 1914.

Yet another definition which we will encounter in this lecture was given by Kazimierz Kuratowski (1896 – 1980) in 1922.

Topology

The origins of topology go back to Georg Cantor (1845 – 1918) and his fundamental work concerning the nature of infinity.

The definition of a topological space that we use today was first given by Felix Hausdorff (1868 - 1942) in 1914.

Yet another definition which we will encounter in this lecture was given by Kazimierz Kuratowski (1896 – 1980) in 1922.

As shown by McKinsey and Tarski in 1944, it opens the door to connect topology with modal logic.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset$, X and closed under finite intersections and arbitrary unions.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

As an immediate consequence of the definition we see that topologies provide a new set of examples of distributive lattices.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

As an immediate consequence of the definition we see that topologies provide a new set of examples of distributive lattices. In fact τ is a sublattice of $\mathscr{P}(X)$.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

As an immediate consequence of the definition we see that topologies provide a new set of examples of distributive lattices. In fact τ is a sublattice of $\mathscr{P}(X)$.

Let X be a topological space. The complement of an open set of X is called a closed set.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

As an immediate consequence of the definition we see that topologies provide a new set of examples of distributive lattices. In fact τ is a sublattice of $\mathscr{P}(X)$.

Let X be a topological space. The complement of an open set of X is called a closed set. Let δ denote the collection of closed sets of X.

Topology

Hausdorff's Definition: A topological space is a pair (X, τ), where X is a set and τ is a collection of subsets of X—called open sets—containing $\emptyset, X$ and closed under finite intersections and arbitrary unions.

More precisely, τ is a topology on X if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$.

As an immediate consequence of the definition we see that topologies provide a new set of examples of distributive lattices. In fact τ is a sublattice of $\mathscr{P}(X)$.

Let X be a topological space. The complement of an open set of X is called a closed set. Let δ denote the collection of closed sets of X. Thus, we have $\delta = \{X - U : U \in \tau\}$.

Topology

We immediately obtain:

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:

(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

(2) Now set $\tau = \{\emptyset, X\}$.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

(2) Now set $\tau = \{\emptyset, X\}$. Then τ is a topology on X, called the trivial topology.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

(2) Now set $\tau = \{\emptyset, X\}$. Then τ is a topology on X, called the trivial topology.

(3) Let $(P, \leqslant)$ be a poset. We set $\tau_{\leqslant} = \mathscr{U}(P)$.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

(2) Now set $\tau = \{\emptyset, X\}$. Then τ is a topology on X, called the trivial topology.

(3) Let $(P, \leqslant)$ be a poset. We set $\tau_{\leqslant} = \mathscr{U}(P)$. Then $(P, \tau_{\leqslant})$ is a topological space in which the intersection of any collection of opens is again open.

Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \Rightarrow F \cup G \in \delta$.
3. $S \subseteq \delta \Rightarrow \bigcap S \in \delta$.

Similarly to τ, we have that δ is also a sublattice of $\mathscr{P}(X)$.

Examples:
(1) Let X be a nonempty set. We set $\tau = \mathscr{P}(X)$. Then (X, τ) is a topological space in which every set is open. Such spaces are called discrete spaces.

(2) Now set $\tau = \{\emptyset, X\}$. Then τ is a topology on X, called the trivial topology.

(3) Let $(P, \leqslant)$ be a poset. We set $\tau_{\leqslant} = \mathscr{U}(P)$. Then $(P, \tau_{\leqslant})$ is a topological space in which the intersection of any collection of opens is again open. Spaces with this property are called Alexandroff spaces.

Topology

(4) In particular, the “smallest” example of a non-discrete nontrivial topology is given by the Sierpiński space

Topology

(4) In particular, the “smallest” example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

Topology

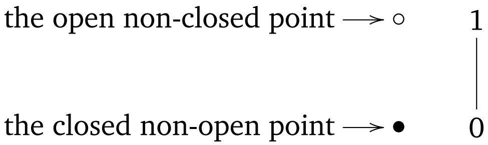
(4) In particular, the "smallest" example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

This space is important enough that it is worth drawing a picture of it

Topology

(4) In particular, the "smallest" example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

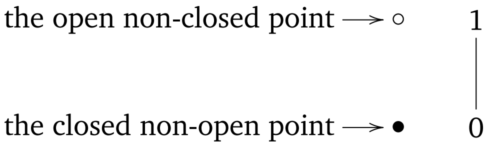
This space is important enough that it is worth drawing a picture of it.



Topology

(4) In particular, the "smallest" example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

This space is important enough that it is worth drawing a picture of it.

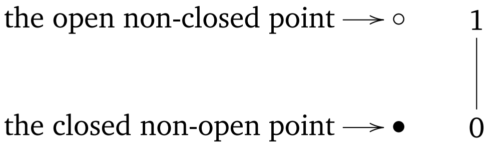


(5) Consider the real line $\mathbb{R}$.

Topology

(4) In particular, the "smallest" example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

This space is important enough that it is worth drawing a picture of it.

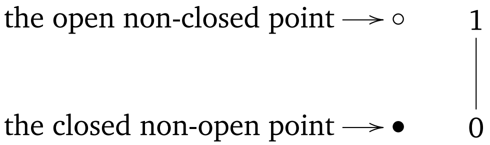


(5) Consider the real line $\mathbb{R}$. We call $U \subseteq \mathbb{R}$ open if for each $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Topology

(4) In particular, the "smallest" example of a non-discrete nontrivial topology is given by the Sierpiński space. This is the Alexandroff space corresponding to the two-element poset $\{0 \leqslant 1\}$. Thus $\{0\}$ is a closed set which is not open and $\{1\}$ is an open set which is not closed.

This space is important enough that it is worth drawing a picture of it.



(5) Consider the real line $\mathbb{R}$. We call $U \subseteq \mathbb{R}$ open if for each $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$. Then $\mathbb{R}$ becomes a topological space.

Interior and closure

Let X be a topological space and let $x \in X$.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\text{int}(A)$ denote the interior of A.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\text{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\operatorname{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\operatorname{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

It is easy to verify that the interior and closure satisfy the following conditions:

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\mathrm{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

It is easy to verify that the interior and closure satisfy the following conditions:

- $\mathrm{int}(X) = X$

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\operatorname{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

It is easy to verify that the interior and closure satisfy the following conditions:

- $\operatorname{int}(X) = X$,
- $\operatorname{int}(A) \subseteq A$

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\mathrm{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

It is easy to verify that the interior and closure satisfy the following conditions:

- $\mathrm{int}(X) = X$,
- $\mathrm{int}(A) \subseteq A$,
- $\mathrm{int}(A) \subseteq \mathrm{int}(\mathrm{int}(A))$

Interior and closure

Let X be a topological space and let $x \in X$. We call open sets containing x open neighborhoods of x.

For $A \subseteq X$, we say that x belongs to the interior of A if there exists an open neighborhood U of x contained in A. Let $\mathrm{int}(A)$ denote the interior of A.

We also say that x belongs to the closure of A if each open neighborhood U of x has nonempty intersection with A. Let $\overline{A}$ denote the closure of A.

It is easy to verify that the interior and closure satisfy the following conditions:

- $\mathrm{int}(X) = X$,
- $\mathrm{int}(A) \subseteq A$,
- $\mathrm{int}(A) \subseteq \mathrm{int}(\mathrm{int}(A))$,
- $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.

Interior and closure

- $\overline{\varnothing} = \varnothing$

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other:

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other:
$\mathrm{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \mathrm{int}(X - A)$.

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other: $\mathrm{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \mathrm{int}(X - A)$.

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions.

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other: $\text{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \text{int}(X - A)$.

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions.

Indeed, let $\overline{} : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function satisfying the above four conditions.

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other: $\operatorname{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \operatorname{int}(X - A)$.

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions.

Indeed, let $\overline{} : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function satisfying the above four conditions. Call $A \subseteq X$ a closed subset of X if $A = \overline{A}$.

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other:

$\mathrm{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \mathrm{int}(X - A)$.

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions.

Indeed, let $\overline{} : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function satisfying the above four conditions. Call $A \subseteq X$ a closed subset of X if $A = \overline{A}$. Then $\tau = \{X - A : A \text{ is closed}\}$ is a topology on X, and every topology on X arises this way!

Interior and closure

- $\overline{\varnothing} = \varnothing$,
- $A \subseteq \overline{A}$,
- $\overline{\overline{A}} \subseteq \overline{A}$,
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The closure and interior operators are dual to each other: $\mathrm{int}(A) = X - \overline{X - A}$ and $\overline{A} = X - \mathrm{int}(X - A)$.

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions.

Indeed, let $\overline{} : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function satisfying the above four conditions. Call $A \subseteq X$ a closed subset of X if $A = \overline{A}$. Then $\tau = \{X - A : A \text{ is closed}\}$ is a topology on X, and every topology on X arises this way! The same, of course, is true if we work with $\mathrm{int} : \mathscr{P}(X) \to \mathscr{P}(X)$ instead.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

There is a natural way to equip Y with a topology "coming from X", called the subspace topology.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

There is a natural way to equip Y with a topology "coming from X", called the subspace topology.

Open sets of the subspace topology on Y are of the form $Y \cap U$ for $U \in \tau$.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

There is a natural way to equip Y with a topology "coming from X", called the subspace topology.

Open sets of the subspace topology on Y are of the form $Y \cap U$ for $U \in \tau$.

Examples:

(1) The subspace topology of any finite subset of the real line $\mathbb{R}$ is discrete.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

There is a natural way to equip Y with a topology "coming from X", called the subspace topology.

Open sets of the subspace topology on Y are of the form $Y \cap U$ for $U \in \tau$.

Examples:

(1) The subspace topology of any finite subset of the real line $\mathbb{R}$ is discrete.

(2) On the other hand, the subspace topology of $\mathbb{Q} \subset \mathbb{R}$ is by no means discrete.

Subspaces

Let (X, τ) be a topological space and Y be a subset of X.

There is a natural way to equip Y with a topology "coming from X", called the subspace topology.

Open sets of the subspace topology on Y are of the form $Y \cap U$ for $U \in \tau$.

Examples:

(1) The subspace topology of any finite subset of the real line $\mathbb{R}$ is discrete.

(2) On the other hand, the subspace topology of $\mathbb{Q} \subset \mathbb{R}$ is by no means discrete.

(3) Neither is the subspace topology of the interval $(0, 1) \subset \mathbb{R}$.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

(4) The following map is not continuous.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

(4) The following map is not continuous. Let X be any set containing more than one element.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

(4) The following map is not continuous. Let X be any set containing more than one element. Let τ be the discrete topology and η be the trivial topology on X.

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

(4) The following map is not continuous. Let X be any set containing more than one element. Let τ be the discrete topology and η be the trivial topology on X. Then the identity map from (X, η) to (X, τ) is not continuous

Continuous maps

Given two topological spaces (X, τ) and (X', τ'), a map $f : X \to X'$ is called continuous if the inverse image $f^{-1}(U)$ of any open subset U of X' is open in X.

Examples:

(1) If X is a discrete space and Y is any space, then any map $f : X \to Y$ is continuous.

(2) Each polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a continuous map from the real line $\mathbb{R}$ to itself.

(3) For partial orders $(P, \leqslant)$ and $(P', \leqslant')$, a map $f : P \to P'$ is continuous between the corresponding Alexandroff spaces iff f is order-preserving.

(4) The following map is not continuous. Let X be any set containing more than one element. Let τ be the discrete topology and η be the trivial topology on X. Then the identity map from (X, η) to (X, τ) is not continuous because the inverse image of any singleton subset of (X, τ) is not open in (X, η).

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Two spaces are called homeomorphic if there exists a homeomorphism between them.

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Two spaces are called homeomorphic if there exists a homeomorphism between them.

Examples:
(1) The identity map from a space (X, τ) to itself is a homeomorphism.

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Two spaces are called homeomorphic if there exists a homeomorphism between them.

Examples:
(1) The identity map from a space (X, τ) to itself is a homeomorphism.

(2) A map between Alexandroff spaces is a homeomorphism iff it is an order-isomorphism between the corresponding partial orders.

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Two spaces are called homeomorphic if there exists a homeomorphism between them.

Examples:
(1) The identity map from a space (X, τ) to itself is a homeomorphism.

(2) A map between Alexandroff spaces is a homeomorphism iff it is an order-isomorphism between the corresponding partial orders.

(3) The real line $\mathbb{R}$ is homeomorphic to its subspace $(-1, 1) \subseteq \mathbb{R}$.

Homeomorphism

A homeomorphism is a continuous bijection whose inverse is also continuous.

Two spaces are called homeomorphic if there exists a homeomorphism between them.

Examples:
(1) The identity map from a space (X, τ) to itself is a homeomorphism.

(2) A map between Alexandroff spaces is a homeomorphism iff it is an order-isomorphism between the corresponding partial orders.

(3) The real line $\mathbb{R}$ is homeomorphic to its subspace $(-1, 1) \subseteq \mathbb{R}$. One possible homeomorphism $f : (-1, 1) \to \mathbb{R}$ is given by

$$f(t) = \frac{t}{1 - t^2}.$$

Separation properties

Let X be a topological space.

Separation properties

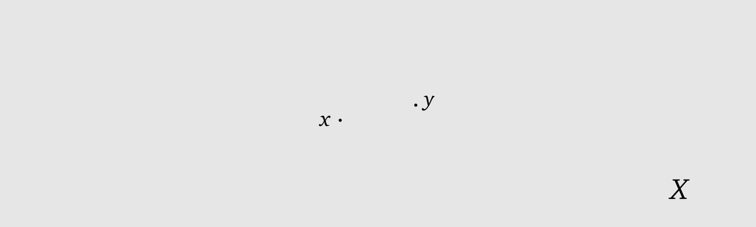
Let X be a topological space.

We call X T_0 if for each pair x, y of distinct points of X, there exists an open set U of X containing exactly one of x, y.

Separation properties

Let X be a topological space.

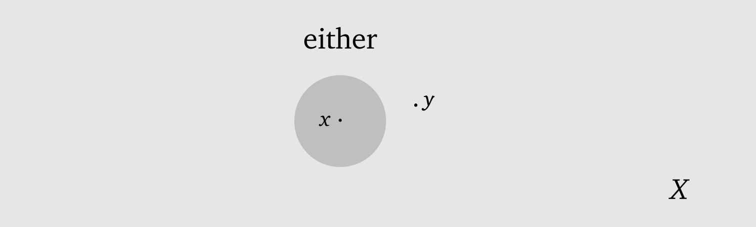
We call X T_0 if for each pair x, y of distinct points of X, there exists an open set U of X containing exactly one of x, y.



Separation properties

Let X be a topological space.

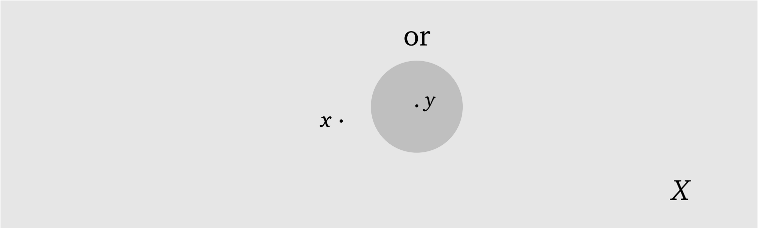
We call X T_0 if for each pair x, y of distinct points of X, there exists an open set U of X containing exactly one of x, y.



Separation properties

Let X be a topological space.

We call X T_0 if for each pair x, y of distinct points of X, there exists an open set U of X containing exactly one of x, y.

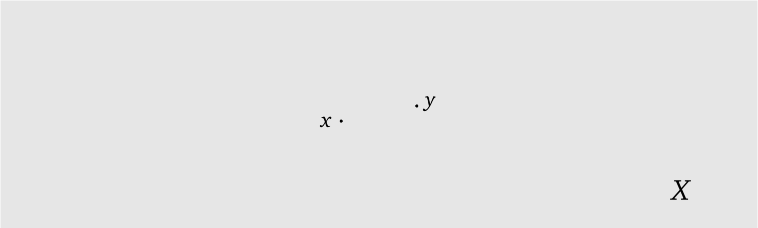


Separation properties

We call X T_1 if for each pair x, y of distinct points of X, there exists an open set U of X such that $x \in U$ and $y \notin U$.

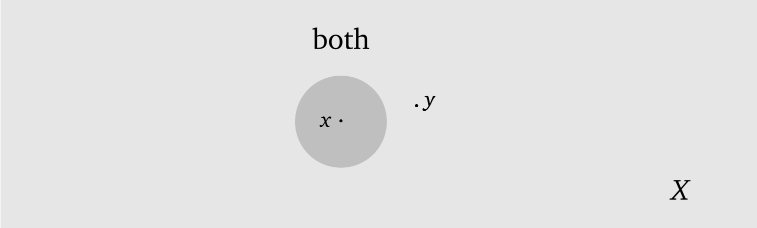
Separation properties

We call X T_1 if for each pair x, y of distinct points of X, there exists an open set U of X such that $x \in U$ and $y \notin U$.



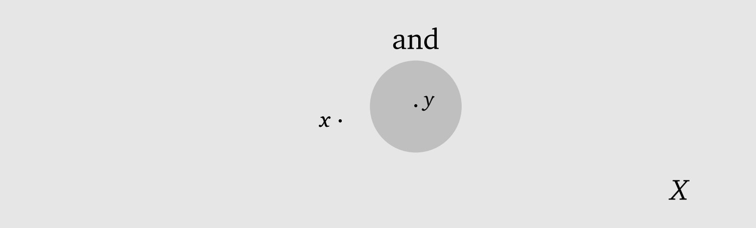
Separation properties

We call X T_1 if for each pair x, y of distinct points of X, there exists an open set U of X such that $x \in U$ and $y \notin U$.



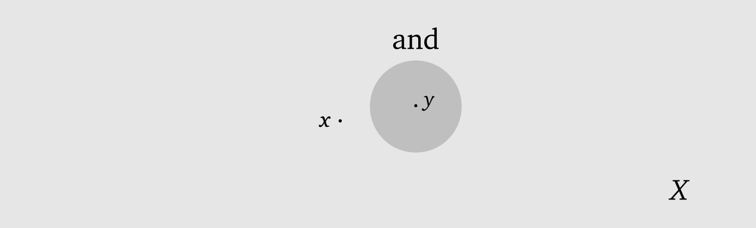
Separation properties

We call X T_1 if for each pair x, y of distinct points of X, there exists an open set U of X such that $x \in U$ and $y \notin U$.



Separation properties

We call X T_1 if for each pair x, y of distinct points of X, there exists an open set U of X such that $x \in U$ and $y \notin U$.



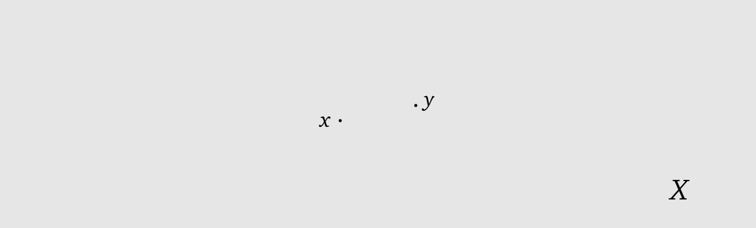
Equivalently, X is T_1 iff each singleton $\{x\}$ is closed.

Separation properties

We call X T_2 or Hausdorff if for each pair x, y of distinct points of X, there exist disjoint open sets U, V of X such that $x \in U$ and $y \in V$.

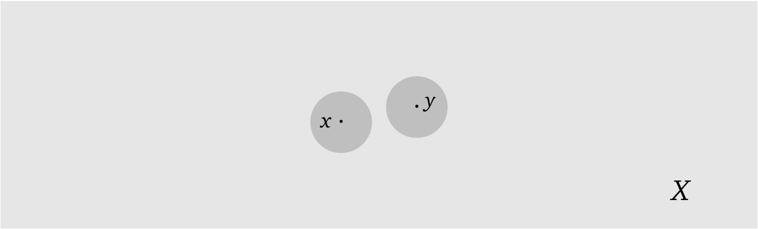
Separation properties

We call X T_2 or Hausdorff if for each pair x, y of distinct points of X, there exist disjoint open sets U, V of X such that $x \in U$ and $y \in V$.



Separation properties

We call X T_2 or Hausdorff if for each pair x, y of distinct points of X, there exist disjoint open sets U, V of X such that $x \in U$ and $y \in V$.



Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology. The set S with the cofinite topology is an example of a T_1-space which is not T_2.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology. The set S with the cofinite topology is an example of a T_1-space which is not T_2.

If a set S has more than one element, then the trivial topology on S is not T_0.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology. The set S with the cofinite topology is an example of a T_1-space which is not T_2.

If a set S has more than one element, then the trivial topology on S is not T_0. All the other spaces we have considered so far are T_0.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology. The set S with the cofinite topology is an example of a T_1-space which is not T_2.

If a set S has more than one element, then the trivial topology on S is not T_0. All the other spaces we have considered so far are T_0.

Any finite Hausdorff space (in fact, any finite T_1-space) is discrete.

Separation properties

Clearly each T_2-space is T_1, and each T_1-space is T_0.

The Sierpiński space is an example of a T_0-space which is not T_1.

Let S be an infinite set. Then the cofinite sets of S together with the empty set form a topology on S we call the cofinite topology. The set S with the cofinite topology is an example of a T_1-space which is not T_2.

If a set S has more than one element, then the trivial topology on S is not T_0. All the other spaces we have considered so far are T_0.

Any finite Hausdorff space (in fact, any finite T_1-space) is discrete.

An example of a non-discrete Hausdorff space is the real line $\mathbb{R}$.

Sober spaces

Let X be a topological space.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

The uniqueness requirement means that sober spaces are T_0.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

The uniqueness requirement means that sober spaces are T_0. On the other hand, not all T_0 spaces are sober.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

The uniqueness requirement means that sober spaces are T_0. On the other hand, not all T_0 spaces are sober. For example, the cofinite topology on an infinite set is T_1 but not sober.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

The uniqueness requirement means that sober spaces are T_0. On the other hand, not all T_0 spaces are sober. For example, the cofinite topology on an infinite set is T_1 but not sober. Nevertheless, one can show that each Hausdorff space is sober.

Sober spaces

Let X be a topological space. Clearly the closures of singletons $\overline{\{x\}}$ are join-prime elements of the lattice of closed subsets of X.

We call X sober if each join-prime element of the lattice of closed sets has the form $\overline{\{x\}}$ for a unique point x.

The uniqueness requirement means that sober spaces are T_0. On the other hand, not all T_0 spaces are sober. For example, the cofinite topology on an infinite set is T_1 but not sober. Nevertheless, one can show that each Hausdorff space is sober.

An important property of sober spaces is that one can recover points of such a space from knowing only its lattice of closed sets.

Specialization order

Let X be a T_0-space. We define the specialization order $\leqslant_\tau$ of X by

$$x \leqslant_\tau y \text{ iff } x \in \overline{\{y\}}$$

Specialization order

Let X be a T_0-space. We define the specialization order $\leqslant_\tau$ of X by

$$x \leqslant_\tau y \text{ iff } x \in \overline{\{y\}}$$

Equivalently $x \leqslant_\tau y$ iff $x \in U$ implies $y \in U$ for each open set U.

Specialization order

Let X be a T_0-space. We define the specialization order $\leqslant_\tau$ of X by

$$x \leqslant_\tau y \text{ iff } x \in \overline{\{y\}}$$

Equivalently $x \leqslant_\tau y$ iff $x \in U$ implies $y \in U$ for each open set U.

It is easy to see that $\leqslant_\tau$ is reflexive and transitive.

Specialization order

Let X be a T_0-space. We define the specialization order $\leqslant_\tau$ of X by

$$x \leqslant_\tau y \text{ iff } x \in \overline{\{y\}}$$

Equivalently $x \leqslant_\tau y$ iff $x \in U$ implies $y \in U$ for each open set U.

It is easy to see that $\leqslant_\tau$ is reflexive and transitive. Moreover $\leqslant_\tau$ is antisymmetric because X is T_0.

Specialization order

Let X be a T_0-space. We define the specialization order $\leqslant_\tau$ of X by

$$x \leqslant_\tau y \text{ iff } x \in \overline{\{y\}}$$

Equivalently $x \leqslant_\tau y$ iff $x \in U$ implies $y \in U$ for each open set U.

It is easy to see that $\leqslant_\tau$ is reflexive and transitive. Moreover $\leqslant_\tau$ is antisymmetric because X is T_0.

Note that if X is T_1, then the specialization order is trivial.

Specialization order

Thus each T_0-topology τ on a set X gives rise to the partial order $\leqslant_\tau$ on X.

Specialization order

Thus each T_0-topology τ on a set X gives rise to the partial order $\leqslant_\tau$ on X. Conversely, as we already saw, each partial order $\leqslant$ on X gives rise to the Alexandroff topology $\tau_\leqslant$ on X.

Specialization order

Thus each T_0-topology τ on a set X gives rise to the partial order $\leqslant_\tau$ on X. Conversely, as we already saw, each partial order $\leqslant$ on X gives rise to the Alexandroff topology $\tau_\leqslant$ on X.

Therefore we obtain the following correspondences:

$$\tau \;\mapsto\; \leqslant_\tau \;\mapsto\; \tau_{\leqslant_\tau}$$

and

$$\leqslant \;\mapsto\; \tau_\leqslant \;\mapsto\; \leqslant_{\tau_\leqslant} .$$

Specialization order

Thus each T_0-topology τ on a set X gives rise to the partial order $\leqslant_\tau$ on X. Conversely, as we already saw, each partial order $\leqslant$ on X gives rise to the Alexandroff topology $\tau_\leqslant$ on X.

Therefore we obtain the following correspondences:

$$\tau \quad \mapsto \quad \leqslant_\tau \quad \mapsto \quad \tau_{\leqslant_\tau}$$

and

$$\leqslant \quad \mapsto \quad \tau_\leqslant \quad \mapsto \quad \leqslant_{\tau_\leqslant} \ .$$

It turns out that for any partial order $\leqslant$ we have $\leqslant_{\tau_\leqslant} = \leqslant$.

Specialization order

Thus each T_0-topology τ on a set X gives rise to the partial order $\leqslant_\tau$ on X. Conversely, as we already saw, each partial order $\leqslant$ on X gives rise to the Alexandroff topology $\tau_\leqslant$ on X.

Therefore we obtain the following correspondences:

$$\tau \quad \mapsto \quad \leqslant_\tau \quad \mapsto \quad \tau_{\leqslant_\tau}$$

and

$$\leqslant \quad \mapsto \quad \tau_\leqslant \quad \mapsto \quad \leqslant_{\tau_\leqslant} \quad .$$

It turns out that for any partial order $\leqslant$ we have $\leqslant_{\tau_\leqslant} = \leqslant$.

On the other hand $\tau_{\leqslant_\tau} = \tau$ iff τ is an Alexandroff topology.

Specialization order

This establishes complete balance between partial orders and Alexandroff T_0-spaces.

Specialization order

This establishes complete balance between partial orders and Alexandroff T_0-spaces.

In particular, since every finite space is automatically Alexandroff, we obtain complete balance between finite T_0-spaces and finite posets.

Specialization order

This establishes complete balance between partial orders and Alexandroff T_0-spaces.

In particular, since every finite space is automatically Alexandroff, we obtain complete balance between finite T_0-spaces and finite posets.

On the other hand, there are plenty of infinite non-Alexandroff spaces.

Specialization order

This establishes complete balance between partial orders and Alexandroff T_0-spaces.

In particular, since every finite space is automatically Alexandroff, we obtain complete balance between finite T_0-spaces and finite posets.

On the other hand, there are plenty of infinite non-Alexandroff spaces.

In particular, $\mathbb{R}$ and $\mathbb{Q}$ are such examples.

Specialization order

This establishes complete balance between partial orders and Alexandroff T_0-spaces.

In particular, since every finite space is automatically Alexandroff, we obtain complete balance between finite T_0-spaces and finite posets.

On the other hand, there are plenty of infinite non-Alexandroff spaces.

In particular, $\mathbb{R}$ and $\mathbb{Q}$ are such examples. More generally, each non-discrete (infinite) T_1-space is not Alexandroff.

Compactness

We will need one more fundamental concept in topology, that of compactness.

Compactness

We will need one more fundamental concept in topology, that of compactness.

A topological space X is called compact if for each family $\mathscr{U}$ of open subsets of X with $X = \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $X = \bigcup \mathscr{U}_0$.

Compactness

We will need one more fundamental concept in topology, that of compactness.

A topological space X is called compact if for each family $\mathscr{U}$ of open subsets of X with $X = \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $X = \bigcup \mathscr{U}_0$. In other words, X is compact if any open cover of X has a finite subcover.

Compactness

We will need one more fundamental concept in topology, that of compactness.

A topological space X is called compact if for each family $\mathscr{U}$ of open subsets of X with $X = \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $X = \bigcup \mathscr{U}_0$. In other words, X is compact if any open cover of X has a finite subcover.

Similarly, we call a subset C of X compact if for each family $\mathscr{U}$ of open subsets of X with $C \subseteq \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $C \subseteq \bigcup \mathscr{U}_0$.

Compactness

We will need one more fundamental concept in topology, that of compactness.

A topological space X is called compact if for each family $\mathscr{U}$ of open subsets of X with $X = \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $X = \bigcup \mathscr{U}_0$. In other words, X is compact if any open cover of X has a finite subcover.

Similarly, we call a subset C of X compact if for each family $\mathscr{U}$ of open subsets of X with $C \subseteq \bigcup \mathscr{U}$, there exists a finite subfamily $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $C \subseteq \bigcup \mathscr{U}_0$. In other words, C is compact if any open cover of C has a finite subcover.

Compactness

Examples:

(1) Each finite space is compact.

Compactness

Examples:
(1) Each finite space is compact.

(2) The cofinite topology is compact.

Compactness

Examples:

(1) Each finite space is compact.

(2) The cofinite topology is compact.

(3) $[0, 1]$ is compact in $\mathbb{R}$.

Compactness

Examples:

(1) Each finite space is compact.

(2) The cofinite topology is compact.

(3) $[0, 1]$ is compact in $\mathbb{R}$.

(4) On the other hand, $\mathbb{R}$ is not compact.

Compactness

Examples:

(1) Each finite space is compact.

(2) The cofinite topology is compact.

(3) $[0, 1]$ is compact in $\mathbb{R}$.

(4) On the other hand, $\mathbb{R}$ is not compact.

(5) Any infinite discrete space is not compact.

Compactness

Examples:

(1) Each finite space is compact.

(2) The cofinite topology is compact.

(3) $[0, 1]$ is compact in $\mathbb{R}$.

(4) On the other hand, $\mathbb{R}$ is not compact.

(5) Any infinite discrete space is not compact.

(6) $\mathbb{Q}$ is not compact.

Compactness

Examples:

(1) Each finite space is compact.

(2) The cofinite topology is compact.

(3) $[0, 1]$ is compact in $\mathbb{R}$.

(4) On the other hand, $\mathbb{R}$ is not compact.

(5) Any infinite discrete space is not compact.

(6) $\mathbb{Q}$ is not compact.

(7) In fact, a subset of $\mathbb{Q}$ is compact iff it is finite.

Compactness

Lemma: A closed subset of a compact space is compact.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Proof: Let $\{U_i : i \in I\}$ be an open cover of $f(X)$.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Proof: Let $\{U_i : i \in I\}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Proof: Let $\{U_i : i \in I\}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. As X is compact, there is a finite cover $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ of X.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Proof: Let $\{U_i : i \in I\}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. As X is compact, there is a finite cover $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ of X. But then $U_1, \ldots, U_n$ is a finite cover of $f(X)$.

Compactness

Lemma: A closed subset of a compact space is compact.

Proof: Let C be closed and $\{U_i : i \in I\}$ be an open cover of C. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since X is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of X. Then $U_1, \ldots, U_n$ is a finite cover of C. Therefore C is compact.

Lemma: If X is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in Y.

Proof: Let $\{U_i : i \in I\}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. As X is compact, there is a finite cover $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ of X. But then $U_1, \ldots, U_n$ is a finite cover of $f(X)$. Therefore $f(X)$ is compact.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:
(1) Each finite discrete space is a compact Hausdorff space.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:
(1) Each finite discrete space is a compact Hausdorff space.

(2) The interval $[0, 1]$ is a compact Hausdorff space.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:
(1) Each finite discrete space is a compact Hausdorff space.

(2) The interval $[0, 1]$ is a compact Hausdorff space.

(3) The real line $\mathbb{R}$ is Hausdorff, but it is not compact.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:

(1) Each finite discrete space is a compact Hausdorff space.

(2) The interval $[0, 1]$ is a compact Hausdorff space.

(3) The real line $\mathbb{R}$ is Hausdorff, but it is not compact. Therefore $\mathbb{R}$ is not compact Hausdorff.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:
(1) Each finite discrete space is a compact Hausdorff space.

(2) The interval $[0, 1]$ is a compact Hausdorff space.

(3) The real line $\mathbb{R}$ is Hausdorff, but it is not compact. Therefore $\mathbb{R}$ is not compact Hausdorff.

Compact Hausdorff spaces have many pleasant properties.

Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

Examples:
(1) Each finite discrete space is a compact Hausdorff space.

(2) The interval $[0, 1]$ is a compact Hausdorff space.

(3) The real line $\mathbb{R}$ is Hausdorff, but it is not compact. Therefore $\mathbb{R}$ is not compact Hausdorff.

Compact Hausdorff spaces have many pleasant properties. We only mention two because they will be useful for our considerations.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$. Since X is Hausdorff, for each $c \in C$ there exist disjoint open sets U_c and V_c such that $x \in U_c$ and $c \in V_c$.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$. Since X is Hausdorff, for each $c \in C$ there exist disjoint open sets U_c and V_c such that $x \in U_c$ and $c \in V_c$. The sets V_c cover C.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$. Since X is Hausdorff, for each $c \in C$ there exist disjoint open sets U_c and V_c such that $x \in U_c$ and $c \in V_c$. The sets V_c cover C. Since C is compact, there exists a finite cover $V_{c_1}, \ldots, V_{c_n}$.

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$. Since X is Hausdorff, for each $c \in C$ there exist disjoint open sets U_c and V_c such that $x \in U_c$ and $c \in V_c$. The sets V_c cover C. Since C is compact, there exists a finite cover $V_{c_1}, \ldots, V_{c_n}$. Therefore $x \in U_{c_1} \cap \cdots \cap U_{c_n}$ is an open set disjoint from C

Compact Hausdorff spaces

We saw that any closed subset of a compact space is compact. It turns out that the converse—which is not true in general—is true for compact Hausdorff spaces.

Theorem: A subset of a compact Hausdorff space is compact iff it is closed.

Proof: It is sufficient to show that if $C \subseteq X$ is compact, then $X - C$ is open. Let $x \in X - C$. Since X is Hausdorff, for each $c \in C$ there exist disjoint open sets U_c and V_c such that $x \in U_c$ and $c \in V_c$. The sets V_c cover C. Since C is compact, there exists a finite cover $V_{c_1}, \ldots, V_{c_n}$. Therefore $x \in U_{c_1} \cap \cdots \cap U_{c_n}$ is an open set disjoint from C. Thus $X - C$ is open.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact. Therefore $f(F)$ is compact in Y.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact. Therefore $f(F)$ is compact in Y. But Y is compact Hausdorff.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact. Therefore $f(F)$ is compact in Y. But Y is compact Hausdorff. Thus $f(F)$ is closed in Y.

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact. Therefore $f(F)$ is compact in Y. But Y is compact Hausdorff. Thus $f(F)$ is closed in Y. Consequently g is continuous,

Compact Hausdorff spaces

Theorem: A continuous 1-1 onto map between compact Hausdorff spaces is a homeomorphism.

Proof: Let X, Y be compact Hausdorff and $f : X \to Y$ be 1-1 onto continuous. The map f has an inverse $g : Y \to X$; let us show that g is continuous. For this it is sufficient to show that for a closed subset F of X, $g^{-1}(F) = f(F)$ is closed. Since X is compact, F is compact. Therefore $f(F)$ is compact in Y. But Y is compact Hausdorff. Thus $f(F)$ is closed in Y. Consequently g is continuous, and so f is a homeomorphism.

Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of Stone spaces which will play a prominent role in our considerations.

Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of Stone spaces which will play a prominent role in our considerations.

To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is clopen if it is both closed and open.

Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of Stone spaces which will play a prominent role in our considerations.

To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is clopen if it is both closed and open.

Let X be a topological space.

Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of Stone spaces which will play a prominent role in our considerations.

To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is clopen if it is both closed and open.

Let X be a topological space. If the topology on X is generated by clopen subsets of X, then X is called zero-dimensional.

Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of Stone spaces which will play a prominent role in our considerations.

To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is clopen if it is both closed and open.

Let X be a topological space. If the topology on X is generated by clopen subsets of X, then X is called zero-dimensional.

We call a compact Hausdorff space a Stone space if it is zero-dimensional.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\}$$

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$. Moreover a subset of X containing 0 is clopen iff it is cofinite in X.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$.
Moreover a subset of X containing 0 is clopen iff it is cofinite in X.
Hence any open subset of X is a union of clopen subsets of X,

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$. Moreover a subset of X containing 0 is clopen iff it is cofinite in X.

Hence any open subset of X is a union of clopen subsets of X, so X is a Stone space.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$.
Moreover a subset of X containing 0 is clopen iff it is cofinite in X.
Hence any open subset of X is a union of clopen subsets of X, so X is a Stone space.

X is a countably infinite Stone space.

Stone spaces — examples

Examples:

(1) Each finite discrete space is a Stone space.

(2) Consider the following subspace of $\mathbb{R}$:

$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\}$$

It is easy to see that in X all singletons are clopen except $\{0\}$. Moreover a subset of X containing 0 is clopen iff it is cofinite in X.
Hence any open subset of X is a union of clopen subsets of X, so X is a Stone space.

X is a countably infinite Stone space. Now we give an example of an uncountable Stone space.

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0,1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0,1]$ defined as the complement of a certain union of open intervals:

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Formally, C is

$$[0, 1] - \left((\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \right.$$
$$\left. \cup (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27}) \cup \cdots \right),$$

Stone spaces

(3) The Cantor set C is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Formally, C is

$$[0,1] - \Big((\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27}) \cup \cdots \Big),$$

or, more precisely, $C = [0, 1] - \bigcup_{n=1}^{\infty} U_n$,

where $U_1 = (\frac{1}{3}, \frac{2}{3})$ and $U_{n+1} = \frac{1}{3}U_n \cup (1 - \frac{1}{3}U_n)$.

Stone spaces

There are many other examples of infinite Stone spaces.

Stone spaces

There are many other examples of infinite Stone spaces. As we will see in Lecture 4, there is a complete balance between Stone spaces and Boolean lattices.

Stone spaces

There are many other examples of infinite Stone spaces. As we will see in Lecture 4, there is a complete balance between Stone spaces and Boolean lattices.

(4) The interval $[0, 1]$ is a typical example of a compact Hausdorff space which is not a Stone space.