Lattices and Topology

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Lecture 1: Basics of lattice theory

Introduction

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But it wasn’t until the 1930ies and 1940ies that lattice theory became an independent branch of mathematics with its own internal problematics, thanks to the work of such mathematicians as Garett Birkhoff (1911 – 1996), Marshall Stone (1903 - 1989) , Alfred Tarski (1902 - 1983), and Robert Dilworth (1914 - 1993).

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Further advances in lattice theory were obtained by Bjarni Jónsson, Bernhard Banaschewski, George Grätzer, and many many others..

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The logical significance of these theorems lies in the fact that they are essentially equivalent to results about relational and topological completeness of some well-known propositional calculi.

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Lecture 1: Basics of lattice theory

- Partial orders and lattices
- Lattices as algebras
- Distributive laws, Birkhoff's characterization of distributive lattices
- Boolean lattices and Heyting lattices

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Lecture 2: Representation of distributive lattices

- Join-prime and meet-prime elements
- Birkhoff's duality between finite distributive lattices and finite posets
- Prime filters and prime ideals
- Representation of distributive lattices

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- Topological spaces
- Closure and interior
- Separation axioms
- Compactness
- Compact Hausdorff spaces
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Lecture 5: Spectral duality and applications to logic

- Spectral duality
- Distributive lattices in logic
- Relational completeness of IPC and CPC
- Topological completeness of IPC and CPC

Posets

A pair $(P, \leqslant)$ is called a **poset** (shorthand for **p***artially* **o***rdered* **s***et*) if P is a nonempty set and $\leqslant$ is a partial order on P; that is $\leqslant$ is a binary relation on P which is reflexive, antisymmetric, and transitive.

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- Transitive: If $p \leqslant q$ and $q \leqslant r$, then $p \leqslant r$ for all $p, q, r \in P$.

Hasse diagrams

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Hasse diagrams

Example: Let $P = \{a, b, c, d, e\}$ with

$$\begin{array}{ccccc}
a \leqslant a & a \leqslant b & a \leqslant c & a \leqslant d & a \leqslant e \\
& b \leqslant b & & b \leqslant d & b \leqslant e \\
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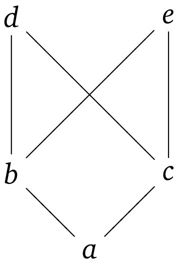
Then the corresponding Hasse diagram looks like this:



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Hasse diagrams of linear orders look like this:

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The latter requirement is necessary since there exist 1-1 and onto order-preserving maps whose inverses aren't order-preserving.

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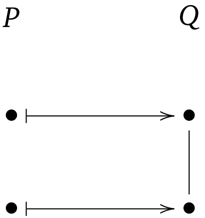

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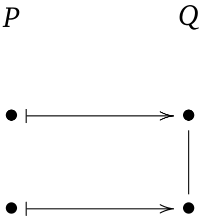

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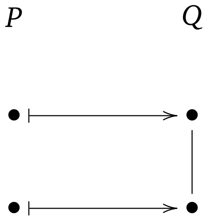

It is clearly 1-1 onto order-preserving. However it's inverse is not order-preserving.

Suprema and infima

Let $(P, \leqslant)$ be a poset. Whenever there exists $p \in P$ such that $q \leqslant p$ for each $q \in P$, we call p the largest or top element of P and denote it by 1.

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Similarly, whenever there exists $p \in P$ such that $p \leqslant q$ for each $q \in P$, we call p the least or bottom element of P and denote it by 0.

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Let $(P, \leqslant)$ be a poset and let $S \subseteq P$. We call $u \in P$ an upper bound of S if $s \leqslant u$ for all $s \in S$. We denote the set of upper bounds of S by S^u.

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We say that $S \subseteq P$ possesses a least upper bound (shortly lub), or supremum, or join, if there exists a least element in S^u.

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If S has glb, then we denote it by $\mathrm{Inf}(S)$ or $\bigwedge S$.

Lattices

We call a poset $(P, \leqslant)$ a lattice if

$$p \vee q = \mathrm{Sup}\{p, q\}$$

and

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exist for all $p, q \in P$.

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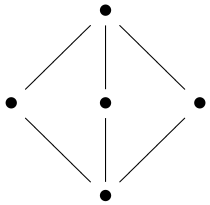
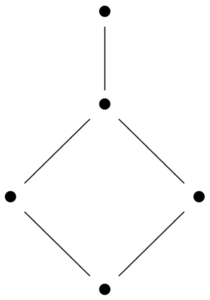
Examples:

(1) Here are Hasse diagrams of a couple of finite lattices:





Lattices

(2) Any linearly ordered set is a lattice, where

$$a \vee b = \max(a, b) = \begin{cases} b & \text{if } a \leqslant b, \\ a & \text{if } a \geqslant b \end{cases}$$

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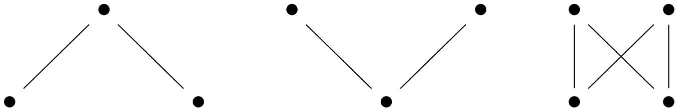
(3) The following posets, however, are not lattices:



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Fact: Let L be a lattice. Then all nonempty finite subsets of L possess suprema and infima.

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Proof (Sketch): Let $a_1, a_2, ..., a_n \in L$. Then an easy induction gives:

$$\bigvee \{a_1, a_2, ..., a_n\} = (...(a_1 \vee a_2) \vee ...) \vee a_n$$

and

$$\bigwedge \{a_1, a_2, ..., a_n\} = (...(a_1 \wedge a_2) \wedge ...) \wedge a_n.$$

Therefore, $\bigvee \{a_1, a_2, ..., a_n\}$ and $\bigwedge \{a_1, a_2, ..., a_n\}$ exist in L.

Complete lattices

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(1) Let $\mathbb{Z}$ be the lattice of all integers with the usual (linear) ordering. Then the set of positive integers has no supremum and the set of negative integers has no infimum in $\mathbb{Z}$.

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(2) Let $\mathbb{N}$ denote the set of non-negative integers. Then the set $\mathscr{P}_{\text{fin}}\mathbb{N}$ of finite subsets of $\mathbb{N}$ is a lattice with set-theoretic union and intersection as lattice operations.

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(2) Let $\mathbb{N}$ denote the set of non-negative integers. Then the set $\mathscr{P}_{\text{fin}}\mathbb{N}$ of finite subsets of $\mathbb{N}$ is a lattice with set-theoretic union and intersection as lattice operations. However, the set of all finite subsets of $\mathscr{P}_{\text{fin}}\mathbb{N}$ has no supremum.

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(2) The powerset $\mathscr{P}X$ of a set X is a complete lattice with respect to the order $\leqslant = \subseteq$. In fact, for each $S \subseteq \mathscr{P}X$ we have $\bigvee S = \bigcup S$ and $\bigwedge S = \bigcap S$.

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Example: Let $\mathbb{Q}$ be the set of rational numbers, and let $L = [0, 1] \cap \mathbb{Q}$. Then L is bounded, but it is not complete.

Lattices as algebras

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3. $a \vee a = a = a \wedge a$ (idempotency).

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3. $a \vee a = a = a \wedge a$ (idempotency).
4. $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ (absorption).

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3. $a \vee a = a = a \wedge a$ (idempotency).
4. $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ (absorption).

Moreover, $a \leqslant b$ iff $a \wedge b = a$ iff $a \vee b = b$.

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Fact: We have that $\leqslant$ is a partial order on L, that $\mathrm{Sup}\{a, b\} = a \vee b$, and that $\mathrm{Inf}\{a, b\} = a \wedge b$ for each $a, b \in L$.

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Then we can define $\leqslant$ on L as follows:

$$a \leqslant b \text{ iff } a \wedge b = a \text{ iff } a \vee b = b.$$

Fact: We have that $\leqslant$ is a partial order on L, that $\mathrm{Sup}\{a, b\} = a \vee b$, and that $\mathrm{Inf}\{a, b\} = a \wedge b$ for each $a, b \in L$.

Thus, we can think of lattices as algebras $(L, \vee, \wedge)$, where $\vee, \wedge : L^2 \to L$ are two binary operations on L satisfying the commutativity, associativity, idempotency, and absorption laws.

Lattice homomorphisms and isomorphisms

A map $f : L \longrightarrow K$ between two lattices L and K is called a lattice homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

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A lattice isomorphism is a 1-1 and onto lattice homomorphism.

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(3) Let $(P, \leqslant)$ be a poset. We call $A \subseteq P$ an upset of P if $x \in A$ and $x \leqslant y$ imply $y \in A$. Let $\mathscr{U}(P)$ denote the set of upsets of P. Then $(\mathscr{U}(P), \cup, \cap)$ is a distributive lattice.

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Dually, A is called a downset of P if $x \in A$ and $y \leqslant x$ imply $y \in A$. Let $\mathscr{D}(P)$ denote the set of downsets of P. Then $(\mathscr{D}(P), \cup, \cap)$ is a distributive lattice.

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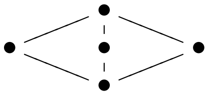
Examples:

(1) The lattice depicted below, and called the diamond, is not distributive.



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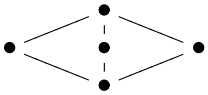
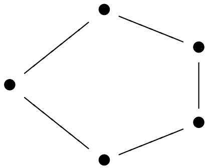


(2) Another non-distributive lattice, called the pentagon, is shown below.



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We say that a lattice K is isomorphic to a (bounded) sublattice S of L if there exists a (bounded) lattice isomorphism from K onto S.

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Proof (Idea): Clearly if either the diamond or the pentagon can be embedded into L, then L is non-distributive.

The converse is more difficult to prove. The rough idea is to show that if L is not distributive, then we can build either the diamond or the pentagon inside L. We skip the details.

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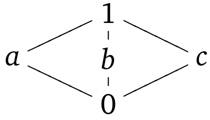
(1) In the diamond, the elements a, b, and c are complements of each other.



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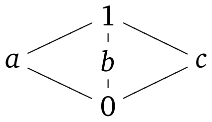
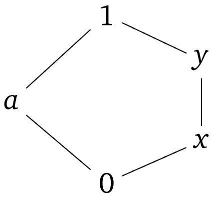


(2) In the pentagon, both x and y are complements of a.



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We denote the complement of a by $\neg a$.

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Thus, in a Heyting lattice L we have:

$$a \wedge x \leqslant b \text{ iff } x \leqslant a \rightarrow b$$

for all $a, b, x \in L$.

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On the other hand, not every bounded distributive lattice is a Heyting lattice.

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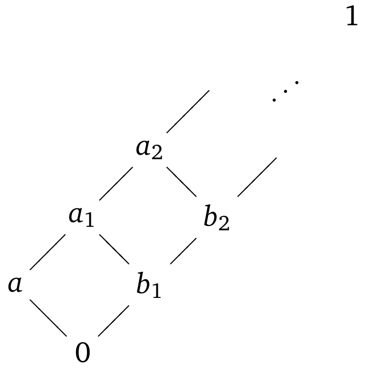
Let L be the lattice shown below:



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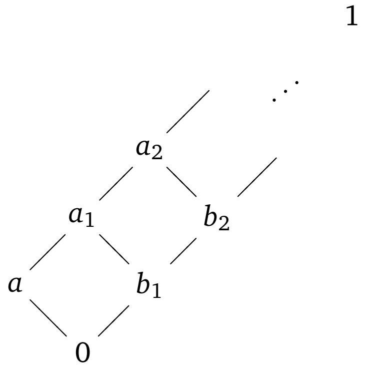

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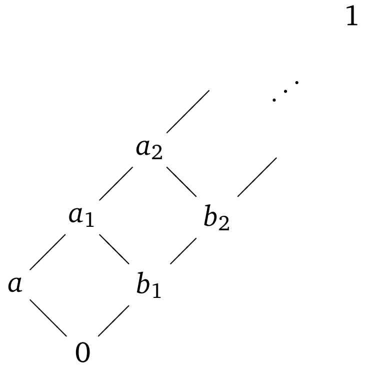

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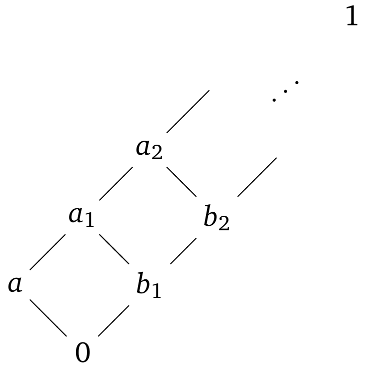

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does not have a largest element. Thus L is not a Heyting lattice.

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This is exactly the reason that L is not a Heyting lattice because a complete distributive lattice is a Heyting lattice iff the $(\wedge, \bigvee)$-distributivity holds in it.