Invent. math. 159, 245–316 (2005)
DOI: 10.1007/s00222-004-0389-9

Inventiones mathematicae

On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation

L. Desvillettes[1], **C. Villani**[2]

[1] CMLA, ENS Cachan, 61 avenue du Président Wilson, 94235 Cachan, France (e-mail: desville@cmla.ens-cachan.fr)
[2] UMPA, ENS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France (e-mail: cvillani@umpa.ens-lyon.fr)

Oblatum 13-I-2003 & 17-V-2004
Published online: 2 September 2004 –

Abstract. As part of our study of convergence to equilibrium for spatially inhomogeneous kinetic equations, started in [21], we derive estimates on the rate of convergence to equilibrium for solutions of the Boltzmann equation, like $O(t^{-\infty})$. Our results hold conditionally to some strong but natural estimates of smoothness, decay at large velocities and strict positivity, which at the moment have only been established in certain particular cases. Among the most important steps in our proof are 1) quantitative variants of Boltzmann's H-theorem, as proven in [52,60], based on symmetry features, hypercontractivity and information-theoretical tools; 2) a new, quantitative version of the instability of the hydrodynamic description for non-small Knudsen number; 3) some functional inequalities with geometrical content, in particular the Korn-type inequality which we established in [22]; and 4) the study of a system of coupled differential inequalities of second order, by a treatment inspired from [21]. We also briefly point out the particular role of conformal velocity fields, when they are allowed by the geometry of the problem.

Contents

I. Introduction and main results

This work is the sequel of our program started in [21] about the trend to thermodynamical equilibrium for spatially inhomogeneous kinetic equations. In the present paper, we shall derive estimates on the rate of convergence to equilibrium for smooth solutions of the Boltzmann equation, providing a first quantitative basis for the maximum entropy principle in this context.

There are several reasons for giving the Boltzmann equation a central role in this program. First, the problem of convergence to equilibrium for this equation is famous for historical reasons, since (together with the *H* theorem, that we shall recall below) it was one of the main elements of the controversy between Boltzmann and his peers, and one of the most spectacular predictions of Boltzmann's approach. At the level of partial differential equations, the problems which one has to overcome when studying the Boltzmann equation are typical of those associated with the combination of transport phenomena and collisions – and for very few models do these problems arise with such intensity as in the case of the Boltzmann equation. Finally, the Boltzmann equation establishes a beautiful bridge between statistical mechanics and fluid mechanics, a property which will be central in our treatment.

In this introductory section, we shall first introduce briefly the model, then recall Boltzmann's famous *H* theorem, and finally state our main result, the proof of which will be the object of the rest of the paper.

I.1. The Boltzmann equation. Let Ω_x be a position space for particles in a gas obeying the laws of classical mechanics. For simplicity we shall assume that Ω_x is either a smooth (say C^1) bounded connected open subset of $\mathbb{R}^N$ ($N \geq 2$) or the N-dimensional torus $\mathbb{T}^N$. The latter case is not so relevant from the physical point of view, but it has the advantage to avoid the subtle problems caused by boundaries, and is therefore commonly used in theoretical and numerical studies. Without loss of generality we shall assume that Ω_x has unit Lebesgue measure:

$$|\Omega_x| = 1. \tag{1}$$

The unknown in Boltzmann's description is a time-dependent probability density $(f_t)_{t\geq 0}$ on the phase space $\Omega_x \times \mathbb{R}^N$ (to think of as a tangent bundle); it will be denoted either $f(t, x, v)$ or $f_t(x, v)$ and stands for the density of the gas in phase space. If one assumes that the gas is dilute, that particles interact via binary, elastic, microscopically reversible collisions, and that there are no correlations between particles which are just about to collide (Boltzmann's chaos assumption), then one can argue, and in some cases "prove", that it is reasonable to use Boltzmann's evolution equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f). \tag{2}$$

Here ∇_x stands for the gradient operator with respect to the position variable x, and, accordingly, $v \cdot \nabla_x$ is the classical transport operator, while Q is the quadratic Boltzmann collision operator,

$$Q(f, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} (f' f'_* - f f_*) B(v - v_*, \sigma) \, d\sigma \, dv_*. \tag{3}$$

The above formula gives the value of the function $Q(f, f)$ at (t, x, v), the parameters v_* and σ live in $\mathbb{R}^N$ and S^{N-1} respectively, we used the common shorthands $f = f(t, x, v)$, $f_* = f(t, x, v_*)$, $f'_* = f(t, x, v'_*)$, $f' = f(t, x, v')$, and (v', v'_*) stand for the pre-collisional velocities of two particles which interact and will have velocities (v, v_*) as a result of the interaction:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma. \tag{4}$$

The nonnegative function $B = B(v - v_*, \sigma)$, which we call Boltzmann's collision kernel, only depends on the modulus of the relative collision velocity, $|v - v_*|$, and on the cosine of the deviation angle θ, i.e.

$$\cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

It is linked to the cross-section Σ by the formula $B = |v - v_*|\Sigma$. It is not our purpose here to describe the collision kernel precisely; see [58, Chap. 1, Sect. 3] for some elements of classification in a mathematical perspective. Our only explicit restrictions on B will be that it is strictly positive, in the sense

$$B \geq K_B \min(|v - v_*|^{\gamma_-}, |v - v_*|^{-\beta_-}) \quad (K_B > 0,\ \gamma_- \geq 0,\ \beta_- \geq 0) \tag{5}$$

and that B is not too much singular, more precisely that Q is continuous in the sense of bilinear mappings, in a scale of weighted Sobolev spaces (see Condition (19) below). This is the case for instance if

$$B \leq C_B \max(|v - v_*|^{\gamma_+}, |v - v_*|^{-\beta_+}) \quad (C_B > 0,\ \gamma_+ \geq 0,\ 0 < \beta_+ < N) \tag{6}$$

(the critical case $\beta_+ = N$ can be handled at the price of some additional assumptions, see [1]) or if

$$\begin{gathered} B = b(\cos\theta)\, \Phi(|v - v_*|), \\ \Phi(|z|) + |z||\Phi'(|z|)| \leq C_B \max(|z|^{\gamma_+}, |z|^{-\beta_+}) \\ (C_B > 0,\ \gamma_+ \geq 0,\ 0 < \beta_+ < N), \\ b(\cos\theta) \simeq \theta^{-(1+\nu)} \quad \text{as} \quad \theta \to 0 \quad (\nu \in (0, 2)). \end{gathered} \tag{7}$$

The last set of conditions corresponds to what is called "non-cutoff cross-sections". Much more general conditions can be found in [1], and a survey of plausible assumptions in [58]. In particular, the model of hard spheres in dimension $N = 3$, $B = |v - v_*|$ (constant cross-section) satisfies Assumption (6), while collision kernels associated with inverse power law forces like $1/(\text{distance})^s$, $2 < s < \infty$ in dimension $N = 3$, satisfy Assumption (7).

Many models which are obtained as limits, or variants, of the Boltzmann equation, can also be included in our analysis. An example of interest is the Landau equation for Coulomb interaction: in this model, the Boltzmann collision operator is replaced by the Landau collision operator,

$$(8) \qquad Q_L(f, f) = \nabla_v \cdot \left(\int_{\mathbb{R}^n} a(v - v_*) \big[f_* (\nabla f) - f (\nabla f)_* \big] \, dv_* \right),$$

where the matrix-valued kernel a takes the form

$$a_{ij}(z) = \frac{L}{|z|} \left[\delta_{ij} - \frac{z_i z_j}{|z|^2} \right] \qquad (L > 0).$$

See [2, 31] and the references included for mathematical and physical background about this model, which is of great importance in plasma physics.

In the sequel, we shall sometimes need to use bilinear forms of the Boltzmann collision operator. We shall write

$$Q(g, f) = \int (f' g'_* - f g_*) \, B \, d\sigma \, dv_*,$$

and

$$Q^{\mathrm{sym}}(g, f) = \frac{1}{2} \big[Q(g, f) + Q(f, g) \big].$$

Equation (2) must be supplemented with boundary conditions. Realistic boundary conditions are quite complicated and sometimes controversial [15]; here we shall limit ourselves to some of the most common model cases. By convention, we shall say that we consider

- *periodic boundary conditions* if $\Omega_x = \mathbb{T}^N$, in which case of course there are no boundaries;
- *bounce-back boundary conditions* if $f_t(x, -v) = f_t(x, v)$ for all $t > 0$, $x \in \partial\Omega_x$, $v \in \mathbb{R}^N$;
- *specular reflection boundary conditions* if $f_t(x, R_x v) = f_t(x, v)$ for all $t > 0$, $x \in \partial\Omega_x$, $v \in \mathbb{R}^N$, where $R_x v \equiv v - 2\langle v, n \rangle n$, and $n = n(x)$ stands for the unit outwards normal to $\partial\Omega$ at x.

We shall treat all of these boundary conditions at the same time, pointing out the differences whenever needed. Specular reflection is probably the most natural mechanism, however in some modelling problems it does not lead to realistic conclusions [15]. Compared with the other two cases, it will lead to more interesting mathematical situations, and significant additional difficulties. Although we did not take into account accomodation conditions (often called Maxwell boundary conditions), this extension is certainly feasible in the case when the boundary has fixed, constant temperature.

Accomodation conditions with variable temperature are considerably more complicated and seem definitely out of reach at the moment.

Let us now introduce the **hydrodynamical fields** associated to a kinetic distribution $f(x, v)$ (not necessarily solution of Boltzmann equation). These are the $(N+2)$ scalar fields of *density* ρ (scalar), *mean velocity* u (vector-valued) and *temperature* T (scalar) defined by the formulas

$$\rho = \int_{\mathbb{R}^N} f \, dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^N} f v \, dv, \quad T = \frac{1}{N\rho} \int_{\mathbb{R}^N} f |v-u|^2 \, dv. \tag{9}$$

In the above, of course $f = f(x, v)$, $\rho = \rho(x)$, $u = u(x)$, $T = T(x)$. Whenever $f = f_t$ is a solution of the Boltzmann equation, we shall denote by ρ_t, u_t, T_t the associated time-dependent hydrodynamic fields. They will play a key role in the sequel.

The total mass, momentum and kinetic energy of the gas can be expressed in terms of these fields:

$$\int_{\Omega_x \times \mathbb{R}^N} f \, dv \, dx = \int_{\Omega_x} \rho \, dx; \qquad \int_{\Omega_x \times \mathbb{R}^N} f v \, dv \, dx = \int_{\Omega_x} \rho u \, dx,$$

$$\int_{\Omega_x \times \mathbb{R}^N} f \frac{|v|^2}{2} \, dv \, dx = \int_{\Omega_x} \left(\rho \frac{|u|^2}{2} + \frac{N}{2} \rho T \right) dx.$$

Whenever f_t is a (well-behaved) solution of the Boltzmann equation, one has the *global conservation laws* for mass and energy

$$\frac{d}{dt} \int f_t(x, v) \, dv \, dx = 0, \qquad \frac{d}{dt} \int f_t(x, v) \, \frac{|v|^2}{2} \, dv \, dx = 0.$$

In the case $\Omega_x = \mathbb{T}^N$, also

$$\frac{d}{dt} \int f_t(x, v) v \, dv \, dx = 0.$$

Therefore, without loss of generality we shall impose

$$\int f_t(x, v) \, dv \, dx = 1, \qquad \int f_t(x, v) \frac{|v|^2}{2} \, dv \, dx = \frac{N}{2}; \tag{10}$$

and, in the case of periodic boundary conditions, also

$$\int f_t(x, v) \, v \, dv \, dx = 0. \tag{11}$$

As we recall in the next subsection, these conservation laws are generally enough to uniquely determine the stationary states of the Boltzmann equation. There is however one important exception to this rule: the case of specular boundary conditions with some symmetry properties. In particular, in dimension $N = 2$ or $N = 3$, for *axisymmetric* domains Ω_x, there is an additional conservation law, namely angular momentum along the axis of symmetry,

$$\int f(x,v)\big[(v\wedge(x-x_0))\cdot\omega\big]\,dv\,dx,$$

where the point x_0 and the unit vector ω determine an axis around which Ω_x is axisymmetric (in dimension $N=2$, the formula above remains valid if Ω is a disk of center x_0, and ω is taken to be a unit vector orthogonal to a plane containing Ω_x). In dimension 3, this provides either one (if Ω_x has just one axis of symmetry) or three (if there are at least two axes, in which case Ω_x has spherical symmetry) conservation laws, and these additional laws have to be taken into account in the study of convergence to equilibrium. In this paper, for simplicity we shall rule out these cases by assuming that, if specular reflection is enforced, then the dimension is either 2 or 3 and *Ω_x is not an axisymmetric domain*. More explicitly, Ω_x is preserved by no nontrivial one-parameter continuous family of isometries, a condition which can be stated in any dimension.

Boundary conditions for f result in boundary conditions for the mean velocity u: the bounce-back condition implies

$$u=0 \qquad \text{on } \partial\Omega_x,$$

while specular reflection implies

$$u\cdot n=0 \qquad \text{on } \partial\Omega_x,$$

where n stands for the outer unit normal vector field on $\partial\Omega_x$.

After these preparations, we state Boltzmann's H theorem.

I.2. Boltzmann's *H* theorem. The proposition below comprises both the H theorem (points (i) and (ii)) and an important consequence, the uniqueness of the stationary state (point (iii)). We do not wish to be too precise here about the meaning of a "smooth" solution; this just means that all the integrability and differentiability properties which are needed in the proof of the theorem are satisfied. For instance, a rapidly decreasing function f, satisfying $|\log f|\le C(1+|v|^{q_0})$, will do.

Proposition 1. *(i) Let $(f_t)_{t\ge0}$ be a smooth solution of the Boltzmann equation* (2). *Then the H functional (negative of the entropy)*

$$H(f)=\int_{\Omega_x\times\mathbb{R}^N} f\log f$$

is nonincreasing as a function of t. Moreover, one can define a nonnegative functional D on the set $L^1_+(\mathbb{R}^N_v)$ of nonnegative densities, thereafter called "H-dissipation" or "entropy production", such that

$$\frac{d}{dt}H(f_t)=-\int_{\Omega_x} D(f_t(x,\cdot))\,dx.$$

(ii) Assume that the collision kernel B satisfies $B > 0$ almost everywhere, and let f be any nonnegative density on $\mathbb{R}^N_v$ with finite second moment. Then, $D(f) = 0$ if and only if f has the special form

$$f(v) = \frac{\rho\, e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{N/2}} \equiv M_{\rho\, u\, T}(v). \tag{12}$$

In particular, if $f = f(x, v)$ is any probability distribution on $\Omega_x \times \mathbb{R}^N$, then the total entropy production $\int_{\Omega_x} D(f(x,\cdot))\,dx$ vanishes if and only if there exist functions $\rho(x)$, $u(x)$ and $T(x)$ such that

$$f(x, v) = M_{\rho\, u\, T}(v) \equiv \frac{\rho(x)\, e^{-\frac{|v-u(x)|^2}{2T(x)}}}{(2\pi T(x))^{N/2}}. \tag{13}$$

(iii) Let again (f_t) be a smooth solution of the Boltzmann equation for which the value of H does not change as time goes by. Further assume that the boundary conditions are either periodic, or bounce-back, or specular, and in the latter case assume that Ω_x lies in dimension 2 or 3 and is not axisymmetric. Then, f takes the particular form

$$f(t, x, v) = \frac{\rho\, e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{N/2}}, \tag{14}$$

where ρ, u and T are constants, and moreover $u \equiv 0$ in the case of bounce-back or specular boundary conditions.

We shall use the following terminology: a velocity distribution of the form (12) will be called a *Maxwellian* distribution; a distribution of the form (13) will be called a *local Maxwellian* (in the sense that the constants ρ, u and T appearing there depend on the position x); and a distribution of the form appearing in the right-hand side of (14) will be called a *global Maxwellian*.

If we impose the normalizations (1), (10), (11), we can identify uniquely the global Maxwellian of point (iii) as

$$M(x, v) = M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{N/2}}. \tag{15}$$

In the sequel, we shall make this assumption without further comment.

The proof of Proposition 1 is well-known, but it is good to sketch it briefly, since the main tools underlying this paper are quantitative variants of statements (ii) and (iii). Although the assumption of smoothness can be considerably relaxed (see in particular [18], or the references in [58, Chap. 1, Sect. 2.5]), we shall assume f to be very well-behaved.

To prove point (i), it suffices to establish, by means of well-chosen changes of variables, the explicit formula

$$(16) \qquad D(f) = \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}} (f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} \, B \, d\sigma \, dv \, dv_*.$$

Since the function $(X - Y)\log(X/Y)$ is always nonnegative, the functional D has to be nonnegative too.

To prove point (ii), one should show that solutions of the functional equation

$$(17) \qquad f' f'_* = f f_* \qquad (\forall v, v_*, \sigma)$$

are Maxwellian distributions. For smooth distributions, a beautiful proof was given by Boltzmann himself: one averages equation (17) over σ, takes logarithms of both sides and then applies the operator $(v - v_*) \wedge (\nabla_v - \nabla_{v_*})$, to conclude that $\nabla(\log f)$ is proportional to v, up to an additive (vector) constant. See [58] for more details, references and comments. Note that this part of the theorem is a statement about the functional D, not about the equation.

Finally, point (iii) amounts to proving that any local Maxwellian solving the Boltzmann equation is necessarily a global Maxwellian. For this we first note that if f is a local Maxwellian, then $Q(f, f) = 0$, hence if it solves the Boltzmann equation it also solves the free transport equation $(\partial_t + v \cdot \nabla_x) f = 0$, in particular $(\partial_t + v \cdot \nabla_x)(\log f) = 0$. This can be rewritten as an explicit equation of the form $P_x(v) = 0$, where P_x is a polynomial in the N velocity variables $v_1, \dots, v_N$, with coefficients in the space of rational fractions generated by ρ, u, T and their derivatives with respect to t or x (we shall perform the explicit computation in Sect. III.2, since this will be needed there). If we use the linear independence of the functions 1, $v_i - u_i$ $(1 \le i \le N)$, $(v_i - u_i)(v_j - u_j)$ $(1 \le i \le j \le N)$, $|v - u|^2 (v_j - u_j)$ $(1 \le j \le N)$ over $\mathbb{R}$, and express the fact that their respective coefficients are identically vanishing, we find the following set of equations:

$$(18) \qquad \begin{cases} \dfrac{\partial_t \rho}{\rho} - \dfrac{N}{2} \dfrac{\partial_t T}{T} + u \cdot \left(\dfrac{\nabla_x \rho}{\rho} - \dfrac{N}{2} \dfrac{\nabla_x T}{T} \right) = 0, \\[2ex] \dfrac{\partial_t u}{T} + \dfrac{(u \cdot \nabla_x) u}{T} + \dfrac{\nabla_x \rho}{\rho} - \dfrac{N}{2} \dfrac{\nabla_x T}{T} = 0, \\[2ex] \forall i \neq j \qquad \dfrac{\partial_{x_i} u_j + \partial_{x_j} u_i}{T} = 0, \\[2ex] \forall i \qquad \dfrac{\partial_{x_i} u_i}{T} + \dfrac{\partial_t T + u \cdot \nabla_x T}{2T^2} = 0, \\[2ex] \dfrac{\nabla_x T}{2T} = 0. \end{cases}$$

The solutions to this system consist in a finite-dimensional vector space which can be made explicit. By taking into account the boundary conditions, one can then prove that ρ, u and T do not depend on x neither on t. A complete proof can be found in, e.g., Desvillettes [18].

I.3. Statement of the problem and main result. On the basis of Proposition 1, one can argue that f_t converges to the equilibrium distribution M as $t \to \infty$. This equilibrium distribution solves a variational principle: it achieves the minimum of the H functional under the constraints (10) (complemented with (11) in the case of periodic boundary conditions). In other words, one expects that *as time becomes large, the particle distribution approaches the state which has maximum entropy under the constraints imposed by the conservation laws.* This is one of the most basic instances of the "maximum entropy principle" popularized by Boltzmann.

Our goal here is precisely to investigate the long-time behavior of a solution f. This topic is closely related to another famous issue, the so-called *hydrodynamic limit* (also known as limit of small Knudsen number[1]); in particular, the H theorem plays a crucial role in both problems. However, we insist that there are important differences between them. For instance, the limit $t \to \infty$ is essentially a global problem, in the sense that the geometry of Ω_x and the boundary conditions play an important part, while the hydrodynamic limit is mainly a local problem[2]. In particular, as we shall see, the roles of local Maxwellians in both problems are completely different.

If f is any reasonable solution of the Boltzmann equation, satisfying certain a priori bounds of compactness (in particular, ensuring that no kinetic energy is allowed to "leak" at large velocities), then it is quite easy to prove by a soft compactness argument, based on Proposition 1, that f_t does indeed converge to M as $t \to \infty$. Of course, these a priori bounds may be quite tricky; as a matter of fact, they have been established only in the spatially homogeneous situation (which means that the distribution function does not depend on the position variable; see the survey in [58]) or in a close-to-equilibrium setting (see in particular [55,32,31] for the three-dimensional torus, and [48] for a convex bounded open set), and still constitute a famous open problem for spatially inhomogeneous initial data far from equilibrium. But once these bounds are settled, the convergence to equilibrium is readily proven.

Our goal in this paper is considerably more ambitious: we are interested in the study of *rates of convergence* for the Boltzmann equation, and wish to derive *constructive* bounds for this convergence. The previously mentioned

---

[1] The Knudsen number Kn is essentially the ratio between the typical mean free path and the typical length scale of the problem. The Boltzmann equation (2) should be written with a coefficient $1/\mathrm{Kn}$ in front of the quadratic collision operator. To simplify, one may define the hydrodynamic limit problem as the study of the asymptotic behavior of solutions to the Boltzmann equation in the limit $\mathrm{Kn} \to 0$.

[2] This rule of course admits some exceptions, for instance the study of sound waves in the incompressible limit, see [6,40,29] or the survey in [57].

argument, based on compactness, is definitely nonconstructive and fails to give any bound on the rate of convergence, neither of course does it provide any hint of how the various parameters in this problem may affect these rates.

There are several reasons why one may be interested in explicit bounds on the rate of convergence. In particular, one may look for qualitative properties of the solution, or one may just prefer constructive arguments to nonconstructive ones. More importantly, one may wish to make sure that the conclusion is physically relevant at least in some range of physical parameters, since one usually expects a given model to be valid only in a certain physical regime. In the case of the Boltzmann equation, this point is central, because it was one of the key issues in the controversy between Boltzmann and Zermelo: the latter argued that Boltzmann's conclusions were irrelevant because they would contradict Poincaré's recurrence theorem, stating that the gas should return (with probability 1) arbitrarily close to its initial state. Poincaré himself used the same argument to question the validity of the foundations of kinetic theory. The solution to this apparent paradox is by now well-understood: the accuracy with which Boltzmann's model describes the gas is expected to break down on very large time scales, depending on physical parameters such as the actual number of particles in the gas. As a consequence, the physical relevance of Boltzmann's conclusion can be ensured only if one is able to control the time scale on which convergence to equilibrium holds true, and show that it is small with respect to the time scale on which the Boltzmann modelling is realistic.

The first thing that one might be tempted to do, in order to study rates of convergence, is to apply standard techniques of linearization around the equilibrium. This was used for instance by Ukai [55] as soon as in the seventies, to prove exponential convergence to equilibrium in the three-dimensional torus; the case of a convex domain with specular reflection was treated soon after by Shizuta and Asano [48]. However, we see three important reasons for not be content with that. A first reason is that spectral gaps for the Boltzmann operator do not always exist[3]: there is no spectral gap for so-called "soft potentials". A second reason is that the classical theory of the linearized Boltzmann equation has been developed under extremely stringent integrability assumptions (typically, $\int f^2/M \, dv \, dx < +\infty$), which seem completely out of reach in a nonlinear setting... except, precisely, when one is very close to equilibrium! This leads us to our third and most fundamental reason, which has to do with the nature of linearization: this technique is likely to provide excellent estimates of convergence only after the solution has entered a narrow neighborhood of the equilibrium state, narrow enough that only linear terms are prevailing in the Boltzmann equation. But by nature, it cannot say anything about the time needed to enter

[3] Even when they exist, their computation is tricky. When we started the present program, only the particular case of "Maxwellian molecules" was known. It is only very recently that Baranger and Mouhot [5], partly motivated by our problems, extended these bounds to much more general kernels, by a quite clever argument.

such a neighborhood; the latter has to be estimated by techniques which would be well-adapted to the fully nonlinear equation. This is where our contribution takes place, and this is why we shall definitely not rely on linearization techniques. Instead, we shall stick as close as possible to the physical mechanism of entropy production.

Let us now discuss the type of estimates which will be used in the present paper. Regularity in recent work on kinetic theory is usually measured in terms of estimates for

- moments in the velocity variable ($\int f|v|^s \, dv \, dx$), controlling the size of distribution tails;
- Sobolev norms for the distribution function in both the position and velocity variables;
- lower bounds of the form $f(t,x,v) \geq K_0 \exp(-A_0|v|^{q_0})$ for some $K_0, A_0 > 0$, $q_0 \geq 2$ (in some situations, one can even achieve $q_0 = 2$, which is optimal).

For more information see [58, Chap. 2]. For instance, as we shall discuss more precisely in Subsect. I.5, these three types of estimates can be derived for the close-to-equilibrium solutions constructed by Guo [32]. As we already mentioned, estimates of the form $\int f^2/M < +\infty$ are not known in the nonlinear setting, even for very simplified situations. One of our contributions in the present work is to show that one can dispend with them, and be content with the estimates of moments, Sobolev norms and lower bounds mentioned above. More precisely, *our main result reduces the problem of the long-time behavior to the problem of establishing a priori bounds on moments, Sobolev norms and lower bounds for the distribution f, uniformly in time*.

We shall combine moment and Sobolev estimates by the use of weighted Sobolev norms: with standard symbolic notation, whenever $f = f(x,v)$ is a function of position and velocity, we define

$$\|f\|^2_{H^k_s(\Omega_x \times \mathbb{R}^N)} \equiv \sum_{|\alpha| \leq k} \int_{\Omega_x \times \mathbb{R}^N} (D^\alpha f)^2 (1+|v|^2)^s \, dv \, dx \qquad (k \in \mathbb{N}, \ s > 0).$$

By interpolation, the definition can be extended to arbitrary positive values of k. Note that D^α is a differential operator with respect to both x and v variables. It is standard that $\|f\|_{H^k_s}$ can be controlled by $\int f|v|^{s'} \, dv \, dx$ and $\|f\|_{H^{k'}_0}$ for s', k' large enough.

Our main result here is

Theorem 2. *Let B be a collision kernel satisfying the positivity condition* (5)*, and such that the associated collision operator satisfies*

$$\|Q^{\mathrm{sym}}(g,h)\|_{L^2(\mathbb{R}^N_v)} \leq C_B \|g\|_{H^{k_0}_{s_0}(\mathbb{R}^N_v)} \|h\|_{H^{k_0}_{s_0}(\mathbb{R}^N_v)} \tag{19}$$

for some $k_0, s_0 \geq 0$. Let $(f_t)_{t\geq 0}$ be a smooth solution of the Boltzmann equation (2)*, such that for all $k, s > 0$,*

$$\sup_{t\geq 0} \|f_t\|_{H^k_s(\Omega_x\times\mathbb{R}^N)} \leq C_{k,s} < +\infty, \tag{20}$$

and

(21)
$$\forall t \geq 0,\ x\in\Omega_x,\ v\in\mathbb{R}^N, \quad f_t(x,v) \geq K_0 e^{-A_0|v|^{q_0}} \quad (A_0, K_0 > 0;\ q_0 \geq 2).$$

Assume that either $\Omega_x = \mathbb{T}^N$ or Ω_x is a C^1, bounded, connected open subset of $\mathbb{R}^N$. Assume either periodic, or bounce-back, or specular reflection boundary conditions, and in the last case assume that $N = 2$ or 3 and that Ω_x is not axisymmetric. Assume the normalization conditions (1)*,* (10) *and* (11) *in case of periodic boundary conditions. Then,*

$$\|f_t - M\| = O(t^{-\infty}), \tag{22}$$

and the constants involved can be computed in terms of $C_B, k_0, s_0, K_0, A_0, q_0, K_B, \gamma_-, \beta_-$ and $C_{k,s}$.

More explicitly, for all $\varepsilon > 0$ and for all $k, s > 0$ there exists a constant C_ε, only depending on $C_B, k_0, s_0, K_0, A_0, q_0, K_B, \gamma_-, \beta_-$ and on $C_{k',s'}$ for some k', s' large enough (explicitly computable in terms of the abovementioned constants and on k, s), such that

$$\forall t \geq 0, \qquad \|f_t - M\|_{H^k_s} \leq C_\varepsilon\, t^{-1/\varepsilon}.$$

Remarks.

1. Whenever f and g are two probability densities on $\Omega_x \times \mathbb{R}^N$, we define the **Kullback relative information** by

$$H(f|g) = \int_{\Omega_x\times\mathbb{R}^N} f \log\frac{f}{g}.$$

It is easy to check that if f satisfies (10), then

$$H(f|M) = H(f) - H(M).$$

Moreover, the Csiszár-Kullback-Pinsker inequality asserts

$$H(f|M) \geq \frac{1}{2}\|f - M\|^2_{L^1}.$$

Combining this with standard interpolation in Sobolev spaces, and with the assumed bounds on (f_t) in Theorem 2, we see that (22) is an easy consequence of the seemingly simpler result

$$H(f_t) - H(M) = O(t^{-\infty}) \qquad \text{with explicit constants}. \tag{23}$$

In words, controlling the speed of convergence of the entropy to its equilibrium value is enough to control the speed of convergence of the solution to equilibrium, in very strong sense.

2. Our assumptions in this theorem are a priori and seem quite restrictive, shedding doubt on the relevance of the result itself. In Subsect. I.5, we shall comment on these assumptions in more detail, and discuss precisely the range of application of the results.

3. For various reasons, our method of proof cannot yield exponential decay, even in situations where there is a spectral gap for the linearized problem. However, in such situations we do hope to get exponential decay in future work by combining the present result with a new spectral study of the Boltzmann equation *in a non-self-adjoint setting*. This point is crucial: as we discussed earlier, the traditional framework of an L^2 space with inverse Maxwellian weight would not be large enough; while the present results allow one to get into a regime in which it is possible to linearize the Boltzmann equation in weighted L^2 space with *polynomial* weights. Related problems have been solved, in a much simpler but already delicate setting, by Gallay and Wayne [28] in the context of mathematical fluid mechanics. Hypoelliptic-type phenomena will also inevitably arise in this study. In spite of all these difficulties, this result of exponential convergence does not seem out of reach.

I.4. Ingredients. The proof of Theorem 2 is certainly at least as interesting as the result itself. It is quite intricate, but rests on a few well-identified principles, which apply with a lot of generality to many variants of the Boltzmann equation. As we mentioned, it does not make any use of linearization. It is definitely not self-contained, and builds upon several preceding results of the authors:

- the quantitative versions of Boltzmann's H theorem proven in [60];
- some estimates about systems of second-order differential inequalities, from [21], playing the role of a Gronwall lemma in this context;
- finally, the Korn-like inequality established in [22].

These works were in turn inspired by the ideas of many authors; some details and references are provided in the bibliographical notes at the end of the paper.

For pedagogical reasons, before entering the details of the proof we shall first give a lengthy – but, we believe, necessary – summary of the strategy. This will be the object of the whole next section. But before that, we discuss precisely the situations in which our results apply.

I.5. Range of application. What exactly is the range of application of Theorem 2?

This discussion is related to the huge work in progress which is the building up of a rigorous mathematical theory for the Boltzmann equation.

We recall that the study of the Cauchy problem for this equation, started by Carleman [10] in the thirties, is still far from completion, in spite of spectacular advances like the DiPerna-Lions theory [24] or its recent extension [1]. For general initial data, regularity and positivity estimates like the ones used in Theorem 2 seem definitely out of reach at the moment. In spite of that, our assumptions are (in our opinion) not so unsatisfactory, as we shall explain.

First of all, at least in the case of periodic boundary conditions, the positivity assumption (21) can be shown to be a consequence of the regularity assumption (20): this is a particular case of recent results by Mouhot [43]. He proves that for a large class of collision kernels, including hard spheres and inverse power law interactions, any solution of the Boltzmann equation in $\mathbb{T}^N \times \mathbb{R}^N_v$, satisfying

$$\inf_{t,x} \rho_t \geq \rho_0,$$

$$\sup_{t,x} \left(\int f_t(x,v)(1+|v|^2+|\log f_t(x,v)|)\, dv + \|f_t(x,\cdot)\|_{L^\infty} + \left\| D_v^2 f_t(x,\cdot) \right\|_{L^\infty} \right) \leq C_0$$

automatically satisfies a lower bound of the form (20) for $t \geq t_0 > 0$, and the constants only depend on ρ_0, C_0, t_0. For cutoff interactions, the estimate on $\|D_v^2 f_t(x,\cdot)\|_{L^\infty}$ may be dropped and the exponent $q_0 = 2$ is admissible in (21). If the interaction is not only cutoff but also of hard potential type (e.g. for hard spheres), the estimate on $\|f_t\|_{L^\infty}$ may also be dropped.

By the way, the lower bound (21), which is only used to apply the entropy production estimates proven in [60], can be dispended with for the Landau equation. Indeed, in that case, a pointwise lower bound on $\rho_t(x)$, together with the smoothness estimates (20), are enough to apply the results in [53].

Secondly, our regularity assumptions (20), although quite strong, can all be proven in certain situations. This is the case in a close-to-equilibrium setting, as shown in a beautiful recent series of works by Guo [31–33], treating both hard and soft potentials with cutoff on one hand, and the Landau equation (with soft potentials including the Coulomb case) on the other hand. It should be emphasized that in both settings, *uniform (in time) a priori estimates were established prior to any knowledge of convergence to equilibrium*. This feature, which may look somewhat surprising, is particularly striking for the Landau equation with Coulomb interaction [31]: spectral estimates for the linearized operator were so weak, that at the time when this paper was written, Guo was unable to get any rate of convergence, even starting very close to equilibrium [34]. On the other hand, his global in time estimates in (any) Sobolev norm are sufficient for our analysis to apply, and prove decay to equilibrium like $O(t^{-\infty})$ if the initial datum is smooth enough. Here is a precise statement[4]:

[4] At the moment, the constants in the following theorem are not explicit, since Guo's estimates are partly based on compactness arguments.

Theorem 3 (rates of convergence for soft potentials). *Consider the kinetic equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \qquad x \in \mathbb{T}^3, \tag{24}$$

where Q is

- *either the Boltzmann collision operator with a collision kernel $B = |v - v_*|^\gamma\, b(\cos\theta)$, where $-3 < \gamma \le -1$ and $b \in L^1(\sin\theta\, d\theta)$ (cutoff very soft potentials);*
- *or the Landau collision operator* (8) *(Coulomb interaction).*

Then, given any $\varepsilon > 0$ there exist $K \in \mathbb{N}$ and $\delta > 0$ such that, for any initial datum f_0 satisfying the normalizations (1), (10), (11), *and the assumption*

$$\left\| \frac{f_0 - M}{\sqrt{M}} \right\|_{H^K(\mathbb{T}^3)} \le \delta,$$

there exists a unique smooth solution $(f_t)_{t\ge 0}$ of (24) *starting from f_0, and*

$$\|f_t - M\|_{L^2} = O(t^{-1/\varepsilon}).$$

This theorem is readily proven by putting together the main results in [32,31,43] and in the present paper (taking into account the fact that the assumption of pointwise lower bound is not necessary for the Landau equation).

Even if it holds in a close-to-equilibrium setting, Theorem 3 on its own may be considered as a first justification for our work: before that, there was no estimate of any rate of convergence for such degenerate situations[5].

There are other, less degenerate situations in which our results can also be applied to existing solutions, but yield worse results than linearization techniques. This is true in particular for the solutions constructed in a close-to-equilibrium context for cutoff hard potentials by Ukai [55] in the torus; in that case linearization is able to yield exponential decay to equilibrium. This is also true of the solutions built by Caflisch [9], still in a close-to-equilibrium context but now for cutoff soft potentials with $\gamma \ge -1$; in that case rates like $O(e^{-t^\alpha})$ can be achieved by linearization.

On the other hand, our main result will of course apply to smooth solutions of the Boltzmann equation if they are ever constructed and estimated uniformly in time. This topic probably remains one of the outstanding unsolved problems in partial differential equations, although it seems a safe bet that present-day techniques should soon be able to solve this problem conditionally to global in time a priori estimates on the hydrodynamic fields

[5] Since then, motivated by the present work, Yan Guo announced an alternative proof of these decay estimates, in collaboration with Bob Strain (May 2004). Among other tools, their proof uses a clever interpolation method.

ρ, u and T. One can also note that our results hold for cutoff as well as non-cutoff collision kernels, although the Cauchy problem for the non-cutoff case is notoriously more delicate, and to this day has not been solved even in a close-to-equilibrium context for the full Boltzmann equation.

I.6. Non-smooth initial data. To conclude this section, we shall discuss very briefly what can be hoped if the initial datum is not smooth. We consider two very distinct cases.

- For so-called "non-cutoff" collision kernels, in which grazing collisions play an important role, it can be proven that the Boltzmann equation has a regularizing effect [1,59] and it is expected that the solution becomes immediately very smooth for positive times, even if the initial datum is not smooth. Then the conclusion of the theorem would still apply.
- For so-called "cutoff" collision kernels, there is no regularization; however, the conclusion of the theorem will still hold true if one can establish three ingredients:
  (a) a result of exponential decay of singularities, meaning that f_t can be decomposed into the sum of a part which is as smooth as required (in terms of Sobolev regularity), and a part which has exactly the same smoothness as the initial datum, but whose amplitude (say in L^1 norm) decays exponentially fast;
  (b) a result of propagation of smoothness, meaning that the solution is very smooth (with weighted Sobolev estimates) if the initial datum is very smooth;
  (c) a stability theorem, meaning that two solutions of the Boltzmann equation, starting with initial data that are close enough, depart from each other at most exponentially fast as time goes by (say in L^1 norm).

In the particular case of spatially homogeneous distributions, the third result has been known for long [35], while the first two are part of the regularity study in [44]. Current research is going on for the adaptation of these results to the spatially inhomogeneous case, under a priori conditions of uniform integrability.

II. The strategy

In this section, we describe the plan of the proof of Theorem 2, and write down the system of differential inequalities upon which our estimates of convergence are based.

Apart from its systematic nature, in our opinion one of the most attractive features of our proof is the fact that it follows physical intuition quite closely, from various points of view. Also it treats both the approach to local Maxwellians and the relaxation to the global Maxwellian simultaneously.

As in the whole theory of the Boltzmann equation, we shall have to overcome two crucial difficulties. First, the complexity of the collision operator Q makes it very hard to extract fine estimates on the solution. Secondly, the fact that this operator is *localized* in the t and x variables (and only acts upon the velocity dependence) entails a degeneracy in the x direction, which makes it very hard to estimate the speed of spatial homogeneization. Here a third important difficulty (related to the second one) will appear: the existence of the huge family of local Maxwellians, which make the entropy production vanish.

The starting point of our study is the H theorem, in the form of point (i) in Proposition 1. We wish to give a precise formulation to the soft principle that *if at some time t the distribution function f_t is still far from equilibrium, then in the next few instants of time a lot of entropy will be produced.* Then our proof is nothing but a quantitative version of points (ii) and (iii) in Proposition 1. Both points are really subtle, and of completely different natures. As we already noted, point (ii) is a property of the collision kernel, while point (iii) involves the whole Boltzmann equation. The quantitative version of point (ii) will be related to information theory, while the quantitative version of point (iii) will belong to the world of fluid mechanics. The geometry of the domain Ω_x will also enter into play. We shall make use, as much as possible, of functional inequalities, as a robust way to encode information about the smoothness of the solution, the geometry of the domain, etc. In the sequel, we try to focus on what is most important for the reader to understand the principle of the proof; much more comments will be made in Sect. VII.

II.1. Quantitative H theorem. Our first tool is a quantitative version of Boltzmann's H Theorem. The bound that we use is taken from a recent work by the second author [60]. See the bibliographical comments at the end of the paper, or the introduction of [60], for further references and related work on the subject.

Theorem 4 (quantitative H Theorem from [60]). *Let B be a collision kernel satisfying* (5)*, and let D be the associated entropy production functional, as defined in* (16)*. Let f be a nonnegative smooth function such that*

$$(25)\qquad \forall v \in \mathbb{R}^N, \quad f(v) \geq K_0 e^{-A_0|v|^{q_0}} \quad (K_0, A_0 > 0;\ q_0 \geq 2),$$

and

$$(26)\qquad \forall k > 0,\ s > 0, \qquad \|f\|_{H^k_s(\mathbb{R}^N_v)} \leq C_{k,s} < +\infty.$$

Let $\rho = \int f\,dv$, $u = (\int fv\,dv)/\rho$, $T = (\int f|v-u|^2\,dv)/(N\rho)$, and let

$$(27)\qquad M^f_{\rho\,u\,T} = \frac{\rho e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{N/2}}.$$

Then, for all $\varepsilon > 0$ *there exists an explicit constant* $K_\varepsilon > 0$*, only depending on* A_0*,* K_0*,* q_0 *and on* $C_{k,s}$ *for some* k, s *large enough, such that*

$$D(f) \geq K_\varepsilon \left(\int_{\mathbb{R}^N} f \log \frac{f}{M^f_{\rho\, u\, T}}\, dv \right)^{1+\varepsilon}. \tag{28}$$

The following easy consequence shows that the entropy production controls how far f_t is from the associated local Maxwellian. Whenever $f = f(x, v)$ is a probability distribution on $\Omega_x \times \mathbb{R}^N$, we define $M^f_{\rho\, u\, T}$ as the local Maxwellian (13) with the same fields of density, mean velocity and temperature as f. Then we have:

Corollary 5. *Let* f *be a solution of the Boltzmann equation, satisfying the assumptions of Theorem 2. Then, for all* $\varepsilon > 0$ *there is a positive constant* K'_ε*, only depending on the bounds appearing in the assumptions of Theorem 2, such that*

$$-\frac{d}{dt} H(f_t|M) \geq K'_\varepsilon \, H\big(f_t \big| M^{f_t}_{\rho_t\, u_t\, T_t}\big)^{1+\varepsilon}. \tag{29}$$

Proof. Here as well as often in the sequel, we shall omit the explicit dependence of f, ρ, u, T upon t. To arrive at (29) it suffices to note that (with obvious notation)

$$\begin{aligned} -\frac{d}{dt} H(f|M) &= \int_{\Omega_x} D(f(x,\cdot))\, dx \\ &\geq \inf_{x \in \Omega_x,\, t \geq 0} K_\varepsilon(f_t(x,\cdot)) \int_{\Omega_x} \left(\int_{\mathbb{R}^N} f \log \frac{f}{M^f}\, dv \right)^{1+\varepsilon} dx \\ &\geq \inf_{x \in \Omega_x,\, t \geq 0} K_\varepsilon(f_t(x,\cdot)) \left(\int_{\Omega_x \times \mathbb{R}^N} f \log \frac{f}{M^f}\, dv\, dx \right)^{1+\varepsilon}, \end{aligned}$$

where, in view of (1), the last inequality is a consequence of Jensen's inequality. □

The fact that the exponent in (29) can be chosen arbitrarily close to 1 will be crucial to obtain the $O(t^{-\infty})$ rate of convergence in the end.

If we were dealing with a spatially homogeneous situation, then (29) would be enough to conclude: it could be rewritten as

$$-\frac{d}{dt} H(f_t|M) \geq K'_\varepsilon \, H(f_t|M)^{1+\varepsilon},$$

and then the desired estimates would follow by Gronwall's lemma. But the crucial point now is that it is really $H(f_t|M^{f_t})$ which appears in the right-hand side of (29), so this inequality can only be used to control how close f is to a *local Maxwellian*, and gives no information whatsoever about how close it is to a global Maxwellian. This is of course not a drawback of

Theorem 4, it is intrinsic to the H theorem since $f = M^f_{\rho u T}$ does satisfy $D(f) = 0$.

All this analysis can be recast for the Landau collision operator (8), only in simpler terms, since positivity estimates are not even necessary [20, 53].

II.2. Instability of the hydrodynamic regime. Despite the last remark, the reader may feel that equation (29) alone is sufficient to assert at least part of the trend to equilibrium result, namely that f approaches $M^f_{\rho u T}$ as $t \to \infty$. In some sense this is true, since it implies, with the abbreviation $M^{f_t} = M^{f_t}_{\rho_t u_t T_t}$,

$$\text{(30)} \qquad \int_0^{+\infty} H\big(f_t|M^{f_t}\big)^{1+\varepsilon}\,dt \le K_\varepsilon^{-1}\int_0^{+\infty}\int_{\Omega_x} D(f_t(x,\cdot))\,dx\,dt = K_\varepsilon^{-1}[H(f_0) - H(M)] < +\infty.$$

Then one could show that $H(f_t|M^{f_t})$ is the sum of a monotonically decreasing function of t and a gently varying function (we shall establish variants of this in the sequel), and it would follow that $H(f_t|M^{f_t})$ does converge to 0 as $t \to \infty$ – hopefully, with a rate like $O(t^{-1/2+0})$, which would not be so good, but still something. In particular, f_t should resemble a local Maxwellian as $t \to \infty$. Up to extraction of a subsequence of times (t_n) if necessary, this implies the convergence of $f(t_n + \cdot, \cdot, \cdot)$ to a solution of the Boltzmann equation which is a local Maxwellian; hence it has to be the global Maxwellian. Thus there is a unique cluster point for f as $t \to \infty$, and convergence follows.

This way of arguing, essentially the only one which was ever invoked in a nonlinear context, looks natural, but *cannot* lead to constructive estimates, because of the impossibility to a priori estimate the rate of convergence of $H(f_t|M^{f_t})$. Even if this is a very smooth function, nothing prevents it from being very small over very long periods of time (so f would be very close to local Maxwellian), then going up only for very short periods. Thus, we have to use more information coming from f being a solution of the Boltzmann equation, and prove that f does not get stuck too close to the family of local Maxwellians.

This difficulty was already noticed by Truesdell [54] and Grad [30], who analyzed the problem and suggested a strategy to overcome it. Grad's argument is however hardly more constructive, and would only work for a linearized situation. The method which we introduced in [21] (and which since then has been used again in several works [8, 25]) suggests to search for a *second-order differential inequality* on the quantity $H(f|M^f_{\rho u T})$, which could be used to quantify the idea that the "manifold" of local Maxwellians is not "stable", and that the eventuality that f spends much of its time close to a local and non global Maxwellian can be ruled out. Here we shall also use this idea, with an important modification: the replacement of $H(f|M^f_{\rho u T})$ by the squared L^2 norm $\|f - M^f_{\rho u T}\|^2_{L^2(\Omega_x\times\mathbb{R}^N)}$. Of course this will result in additional complications when it comes to couple this information with (28); but the point is that the control of $H(f|M^f_{\rho u T})$ would require much more stringent assumptions on the tail behavior of f (something like $f/M^f_{\rho u T}$ bounded from above and below) than we wish to use. To sum up, we will look for a *second-order (in time) differential inequality on* $\|f - M^f_{\rho u T}\|^2_{L^2}$, *which will put into quantitative form the idea that* f *cannot stay too close to* $M^f_{\rho u T}$ *for too long.* We call this mechanism **instability of the hydrodynamic description**, since the hydrodynamic regime is precisely a regime such that f can be replaced by $M^f_{\rho u T}$ up to a very small error. This instability is by no means in contradiction with classical physics: our discussion takes place at a scale where the Knudsen number is not vanishingly small, but of order unity. In fact, the typical time scale for this instability would become very large as the Knudsen number would become very small.

There seems to be some paradoxical element in this strategy: in order to eventually prove that the solution of the Boltzmann equation will look like a Maxwellian for long time, we shall use inequalities proving that the Boltzmann equation cannot stay too close to local Maxwellians. The contradiction is however not so shocking if one recalls that the set of local Maxwellians is not stable under the flow associated with the Boltzmann equation.

Why look for a second-order equation? There are two main reasons:

- First, let us imagine for a second that f will coincide with $M^f_{\rho u T}$ at some time t_0; then $\|f - M^f_{\rho u T}\|^2_{L^2}$ will vanish smoothly for $t \to t_0$, and it is likely to vanish at order 2 (i.e. like $(t - t_0)^2$; we shall see that this is true generically, but sometimes false). To get information about how fast f will, shortly after t_0, depart from local Maxwellian state, it is natural to try to find an estimate on the second-order time derivative of $\|f - M^f_{\rho u T}\|^2_{L^2}$.
- Secondly, the action of taking *twice* the time-derivative will have the effect to make the transport operator $v \cdot \nabla_x$ in the Boltzmann equation enter *twice* the computations; and by applying twice a first-order differential operator, one has a chance to obtain something which acts like a *diffusion* operator[6]. Thus it is no wonder that, in the regime when $f \simeq M^f_{\rho u T}$, the dominant terms in our differential inequalities will be formally the same as those which would have been obtained by the action of some diffusion operator in the x variable: typically, Dirichlet forms like $\int |\nabla_x F|^2$, for various functions F. In this way, we will be rid of the degeneracy in the x variable, which is as usual one of the major problems in the study of the Boltzmann equation, and recover some kind of ellipticity in x.

---

[6] This is in fact what happens in the Chapman-Enskog derivation of (say) the compressible Navier-Stokes equation from the Boltzmann equation.

In implementing this approach, we shall encounter many difficulties, starting with the complexity of computations. In view of our results from [21] on a simpler model, a natural but naive guess would be that

$$\text{(31)}\quad \frac{d^2}{dt^2}\left\| f - M^f_{\rho\, u\, T}\right\|^2_{L^2} \geq K \int_{\Omega_x} \left(|\nabla_x \rho|^2 + |\nabla_x u|^2 + |\nabla_x T|^2\right) dx - C\left(\left\| f - M^f_{\rho\, u\, T}\right\|^2_{L^2}\right)^{1-\varepsilon}.$$

Assume for a moment that inequality (31) holds true. This implies that f cannot stay for too long too close to some local equilibrium for which gradients of the hydrodynamic fields ρ, u and T would not have been small enough. Indeed, the right-hand side would be strictly positive in such a situation, so the function $\| f - M^f_{\rho\, u\, T}\|^2_{L^2}$ would be a strictly convex function of time, which of course cannot stay very small for too long. Nothing could be said when ρ, u and T have very small gradients; but, because of the conservation laws, in such a situation $M^f_{\rho\, u\, T}$ should be very close to the global Maxwellian M.

The guess (31) is however false. What we shall be able to establish, at the end of a very intricate computation, is that for ε small enough,

$$\text{(32)}\quad \frac{d^2}{dt^2}\left\| f - M^f_{\rho\, u\, T}\right\|^2_{L^2} \geq K \left(\int_{\Omega_x} |\nabla_x T|^2\, dx + \int_{\Omega_x} |\{\nabla_x u\}|^2\, dx\right) - C\left\| f - M^f_{\rho\, u\, T}\right\|^{1-\varepsilon}_{L^2} \| f - M\|^{1-\varepsilon}_{L^2},$$

where K and C are explicit constants, depending on the assumed bounds on f (as in Theorem 2), and on ε as well. Here we use the following conventions: $\nabla_x u$ is the matrix function defined by

$$(\nabla_x u)_{ij} = \frac{\partial u_j}{\partial x_i},$$

its symmetric part (or deformation tensor) will be denoted by

$$\nabla_x^{\mathrm{sym}} u = \frac{\nabla_x u + (\nabla_x u)^T}{2},$$

its divergence by

$$\nabla_x \cdot u = \mathrm{div}_x\, u = \mathrm{tr}\,(\nabla_x u) = \sum_{i=1}^N \frac{\partial u_i}{\partial x_i},$$

and finally $\{\nabla_x u\}$ stands for the traceless part of the symmetric part of ∇u, or traceless Reynolds tensor, or deviator,

$$\text{(33)}\quad \{\nabla_x u\} = \nabla_x^{\mathrm{sym}} u - \frac{(\nabla_x \cdot u)}{N} I_N,$$

where I_N is the identity $N \times N$ matrix. When A is a matrix, we write $|A|$ for the Hilbert-Schmidt norm of the matrix A. Moreover, in the sequel, we shall often omit the subscript x and write ∇ for ∇_x.

Here we insist on an important point: the appearance of the parameter ε in (32). In the way we perform the estimates, it seems to be unavoidable, because the computation leads to some L^2 norms of derivatives of f (with respect to both x and v), which we control by L^2 norms of f via an *interpolation* procedure. At this point a lot of smoothness will be required from the solution.

As a consequence of (32), one sees that if f is close enough to $M^f_{\rho\, u\, T}$, then

$$\frac{d^2}{dt^2}\big\|f - M^f_{\rho\, u\, T}\big\|^2_{L^2} \geq K \int_{\Omega_x} |\nabla T|^2,$$

which shows that inhomogeneities in the temperature result in the hydrodynamic instability, as we wished. On the other hand, it may come as a surprise that the right-hand side of (32) does not contain any information about the density ρ. However, this should not look so surprising if one replaces this in the context of hydrodynamics, and recalls that compressible Navier-Stokes equations, for instance, have diffusion at the level of u and T, but never at the level of ρ, or only indirectly (via the diffusion of u, which drives ρ). Similarly, the deviator field $\{\nabla u\}$ is known to play an important role in compressible fluid dynamics; in an incompressible context it could be replaced by $\nabla^{\mathrm{sym}} u$, which would greatly simplify the analysis.

In any case, (32) cannot be improved in the sense that the left-hand side can be shown to vanish for very smooth and very integrable distributions satisfying $f = M^f_{\rho\, u\, T}$, $\nabla T = 0$ and $\{\nabla u\} = 0$. We therefore conclude to the existence of “quasi-equilibria”, which are particular local Maxwellian states enjoying the property that whenever a solution of the Boltzmann equation passes through them (or approaches them very closely), then it will stay “quite close” to the manifold of local Maxwellians. Here “quite close” means that the distance between f and $M^f_{\rho\, u\, T}$ will vanish in time up to order (almost) 2 instead of 1. We shall see in the Appendix that the classification of quasi-equilibria depends on the dimension and the boundary conditions.

Coming back to our problem, we need to find additional information to avoid the trap caused by these quasi-equilibria, which make the right-hand side of (32) vanish. To state things informally, on the whole what we shall quantify is that

a) gradients of temperature imply departure of f from the set of local Maxwellian states (as we discussed above);
b) symmetric gradients of velocity imply departure of f from the set of local Maxwellians with constant temperature;
c) gradients of density imply departure of f from the set of local Maxwellians with constant temperature and homogeneous velocity field.

The preceding statements can be reformulated in a slightly more precise and more intuitive way, in terms of transversality. For this see the first comment in Subsect. VII.1.

To couple points (a), (b) and (c) together, and conclude that f does depart from local Maxwellian anyway, we need to show that inhomogeneous velocity fields have nonzero symmetric gradients. This fact holds true as a consequence of our boundary conditions (in particular Ω_x being not axisymmetric for specular reflection) and will be quantified by some variant of **Korn inequalities**, recalled in Subsect. IV.1. Upon use of this inequality, one can restate point b) as

b') gradients of velocity imply departure of f from the set of local Maxwellians with constant temperature.

To measure how far f is from being a local Maxwellian with constant temperature, we shall define the average temperature

$$\langle T\rangle_\rho = \int_{\Omega_x} \rho T\, dx,$$

introduce the particular local Maxwellian

$$M^f_{\rho\, u\, \langle T\rangle} = \frac{\rho\, e^{-\frac{|v-u|^2}{2\langle T\rangle_\rho}}}{(2\pi\langle T\rangle_\rho)^{N/2}} \tag{34}$$

where $\rho = \rho(x)$, $u = u(x)$, and consider the squared L^2 norm

$$\big\| f - M^f_{\rho\, u\, \langle T\rangle}\big\|^2_{L^2}.$$

Similarly, to measure how far f is from being a local Maxwellian with constant temperature and uniform velocity field, we introduce the particular local Maxwellian

$$M^f_{\rho\, 0\, 1} = \rho M = \frac{\rho\, e^{-\frac{|v|^2}{2}}}{(2\pi)^{N/2}}, \tag{35}$$

and the squared L^2 norm

$$\big\| f - M^f_{\rho\, 0\, 1}\big\|^2_{L^2}.$$

The choice of the parameters in $M^f_{\rho\, 0\, 1}$ comes from the fact that if the temperature is constant and the velocity field uniform, then the conservation laws, together with our normalization conventions, impose $u \equiv 0$, $T \equiv 1$.

For $\| f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}$ and $\| f - M^f_{\rho\, 0\, 1}\|^2_{L^2}$ we shall derive again a differential inequality of second order, similar to that in (32), except that the squared gradient terms on the right-hand side will be replaced by $|\nabla^{\mathrm{sym}} u|^2$ and $|\nabla\rho|^2$ respectively; after a little bit of transformation they will lead to formulas (44) and (45) below. In particular, in the first case the traceless part of the symmetrized gradient of u has been replaced by the full symmetrized gradient, and we can apply our Korn inequality, in the form

$$\|\nabla^{\mathrm{sym}} u\|_{L^2} \geq K \|\nabla u\|_{L^2}.$$

II.3. Putting both features together. Let us now see how to couple our quantitative H theorem with our quantitative "hydrodynamic instability" statement. For this we shall make use of five ingredients.

• The first one is the formula of **additivity of the entropy**: the entropy can be decomposed into the sum of a purely hydrodynamic part, and (by contrast) of a purely kinetic part. In terms of H functional: one can write

$$H(f|M) = \mathcal{H}(\rho, u, T) + H\big(f\big|M^f_{\rho u T}\big), \quad \mathcal{H}(\rho, u, T) = \int_{\Omega_x} \rho \log \frac{\rho}{T^{N/2}}\, dx.$$

This simple form of the hydrodynamical entropy is however not very convenient for the use that we wish to make. Taking into account the conservation laws, we note that

$$\int_{\Omega_x} \rho \log \rho = \int_{\Omega_x} (\rho \log \rho - \rho + 1),$$
$$-\frac{N}{2}\int_{\Omega_x} \rho \log T = \frac{N}{2}\int_{\Omega_x} \rho\,(T - \log T - 1) + \int_{\Omega_x} \rho \frac{|u|^2}{2}.$$

Thus $\mathcal{H}(\rho, u, T)$ can be written as the sum of three nonnegative terms: $\int(\rho \log \rho - \rho + 1)$, $\int \rho |u|^2/2$ and $(N/2)\int \rho(T - \log T - 1)$, which vanish if and only if $\rho \equiv 1$, $u \equiv 0$, $T \equiv 1$ respectively. With somewhat sloppy notation, we shall write

$$\text{(36)} \quad \mathcal{H}(\rho|1) = \int_{\Omega_x} (\rho \log \rho - \rho + 1)\, dx, \quad \mathcal{H}(u|0) = \int_{\Omega_x} \rho \frac{|u|^2}{2}\, dx,$$
$$\mathcal{H}(T|1) = \frac{N}{2}\int_{\Omega_x} \rho(T - \log T - 1)\, dx.$$

It should be noted that $\mathcal{H}(T|1)$, for instance, does not depend only on T but also on ρ.

From this we see that the hydrodynamic part of the H functional controls how close ρ, u, T are to 1, 0, 1 respectively. We shall need a slightly more precise decomposition: write $\Phi(X) = (N/2)(X - \log X - 1)$, and note that $\mathcal{H}(T|1) = \langle \Phi(T)\rangle_\rho$; so one can define

$$\text{(37)} \quad \begin{cases} \mathcal{H}(T|1) = \mathcal{H}(T|\langle T\rangle_\rho) + \mathcal{H}(\langle T\rangle_\rho|1); \\ \mathcal{H}(T|\langle T\rangle_\rho) = \langle \Phi(T)\rangle_\rho - \Phi(\langle T\rangle_\rho), \qquad H(\langle T\rangle_\rho|1) = \Phi(\langle T\rangle_\rho), \end{cases}$$

and note that both terms are nonnegative, the first one because of Jensen's inequality and the convexity of Φ. The interest of this decomposition is that now one part of our entropy controls how close T is from being constant, independently of the value of this constant.

With these definitions one can check the following additivity rules, which are formally appealing if one recalls $M = M_{101}$:

$$
(38) \qquad \begin{cases} H\big(f|M^f_{\rho u T}\big) + \mathcal{H}(T|\langle T\rangle_\rho) & = H\big(f|M^f_{\rho u \langle T\rangle}\big) \\ H\big(f|M^f_{\rho u \langle T\rangle}\big) + \mathcal{H}(\langle T\rangle_\rho|1) + \mathcal{H}(u|0) & = H\big(f|M^f_{\rho 0 1}\big) \\ H\big(f|M^f_{\rho 0 1}\big) + \mathcal{H}(\rho|1) & = H(f|M). \end{cases}
$$

• The second ingredient is the use of various functional inequalities which relate the *gradients* appearing in the right-hand sides of (43)–(45) to the hydrodynamical $\mathcal{H}$ functionals defined above. We already mentioned that a Korn inequality would be used to replace the symmetrized gradient of (44) with a complete gradient. Here we will also use **Poincaré inequalities**, combined with upper and lower bounds for ρ and T, to arrive at

$$
(39) \qquad \begin{cases} \int_{\Omega_x} |\nabla T|^2 \geq K\mathcal{H}(T|\langle T\rangle_\rho), \\ \int_{\Omega_x} |\nabla u|^2 \geq K\mathcal{H}(u|0), \\ \int_{\Omega_x} |\nabla \rho|^2 \geq K\mathcal{H}(\rho|1). \end{cases}
$$

These inequalities will be proven in Subsect. IV.2, right after we recall the necessary Korn inequalities in Subsect. IV.1. The missing term $\mathcal{H}(\langle T\rangle_\rho|1)$ will be controlled by means of the global conservation laws (10).

• Next, we would like to reconcile the relative information functionals appearing in the right-hand side of (28) or (36)–(37), with the squared L^2 norms appearing in (43)–(45). It is well-known that Kullback's relative information behaves in several respects like a squared norm, and that it controls the square of L^1 norm. To take advantage of this fact, we would like to estimate the L^2 norms by L^1 norms, losing as little as possible with respect to the exponents. For this we shall use interpolation with a high-order Sobolev norm, to arrive at

$$
(40) \qquad H(f|g) \geq K\|f-g\|^{2(1+\varepsilon)}_{L^2},
$$

where g is either $M^f_{\rho u T}$, or $M^f_{\rho u \langle T\rangle}$, or $M^f_{\rho 0 1}$. Here as before, ε is arbitrarily small, and the constant K depends on smoothness estimates for f at large order (depending on ε). A proof of estimate (40) will be included in the derivation of (43)–(45).

• Our next tool is a result about second-order differential inequalities. It states that whenever $h(t)$ is a *nonnegative* solution of a differential inequality like

$$
h''(t) + Ch(t)^{1-\varepsilon} \geq \alpha > 0
$$

on some time interval I, then either I is very short, or the *average value* of h over I is not too small: see Lemma 12 in Sect. VI for a precise statement. This lemma plays in our context the role that Gronwall's lemma plays for first-order differential inequalities; we shall systematically apply it in various situations for which the solution f is either close to $M^f_{\rho\, u\, T}$, or close to $M^f_{\rho\, u\, \langle T\rangle}$, or close to $M^f_{\rho\, 0\, 1}$, and use it to recover estimates about the average entropy production.

A noticeable point about this lemma is that, in contrast with the case of first-order differential inequalities, the restriction on the length of the time interval here cannot be removed.

• With the preceding remark comes a tricky point: since we do not say anything about intervals with too short length, nothing could a priori prevent the solution f to oscillate rapidly between the various regimes in which it is close to $M^f_{\rho\, u\, T}$, or $M^f_{\rho\, u\, \langle T\rangle}$, or $M^f_{\rho\, 0\, 1}$, and then nothing could be said about average entropy production. To rule out this scenario, we shall show that the values of the hydrodynamic entropies cannot oscillate too much, by establishing bounds on their variations, in terms of distance to global equilibrium:

$$\left|\frac{d}{dt}\mathcal{H}(\rho|1), \quad \frac{d}{dt}\mathcal{H}(u|0), \quad \frac{d}{dt}\mathcal{H}(T|1)\right| \leq CH(f|M)^{1-\varepsilon}.$$

We call this phenomenon **damping of hydrodynamical oscillations**, and it will be established in Sect. V.

We will not need such bounds for the kinetic part of the H functional, and this is good news, because the moment bounds that we wish to use seem to be a priori too weak to imply sharp enough control on the oscillations for this kinetic part (although one may possibly use the additivity of the entropy and the fact that the H functional is always decreasing to circumvent this difficulty).

II.4. The system. All the abovementioned bounds lead to a large system of coupled differential inequalities and functional inequalities. For convenience, let us recast it explicitly here. Recall the definitions (27), (33), (34), (35), (36), (37).

Proposition 6. *Let B be a collision kernel, and let f be a solution of the Boltzmann equation, both of them satisfying the same assumptions as in Theorem 2. Assume $\int f_0\, dv\, dx = 1$, $\int f_0 |v|^2\, dv\, dx = N$, and, in case of periodic boundary conditions, $\int f_0 v\, dv\, dx = 0$. Let ε be small enough, say $\varepsilon \leq 1/10$. Then, f and the associated hydrodynamic fields ρ, u, T satisfy a system of inequalities made of:*

a) Some conservation laws and boundary conditions:

$$\int_{\Omega_x} \rho\, dx = 1, \qquad \int_{\Omega_x} (\rho|u|^2 + N\rho T)\, dx = N, \tag{41}$$

$$\int_{\Omega_x} \rho u\, dx = 0 \qquad \textit{in the case of periodic boundary conditions},$$

$$u = 0 \qquad \textit{on } \partial\Omega_x \textit{ for bounce-back conditions},$$

$$u \cdot n = 0 \qquad \textit{on } \partial\Omega_x \textit{ for specular reflection};$$

b) a quantitative version of Boltzmann's H theorem:

$$-\frac{d}{dt} H(f|M) \geq K_H H\big(f|M^f_{\rho\, u\, T}\big)^{1+\varepsilon}, \tag{42}$$

where $K_H > 0$ *only depends on* ε*,* B *and on estimates of smoothness, moments and positivity for* f*;*

c) three differential inequalities of second order, expressing the instability of the hydrodynamic description for f*: if* $\delta_1, \delta_2, \delta_3$ *are small enough, say in* $(0, 1/10)$*, then*

$$\frac{d^2}{dt^2}\big\|f - M^f_{\rho\, u\, T}\big\|^2_{L^2} \geq K_1 \left[\int_{\Omega_x} |\nabla T|^2\, dx + \int_{\Omega_x} |\{\nabla u\}|^2\, dx\right] - \frac{C_1}{{\delta_1}^{1-\varepsilon}} \left(\big\|f - M^f_{\rho\, u\, T}\big\|^2_{L^2}\right)^{1-\varepsilon} - \delta_1 H(f|M); \tag{43}$$

$$\frac{d^2}{dt^2}\big\|f - M^f_{\rho\, u\, \langle T\rangle}\big\|^2_{L^2} \geq K_2 \int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\, dx - \frac{C_2}{{\delta_2}^{1-\varepsilon}} \left(\big\|f - M^f_{\rho\, u\, \langle T\rangle}\big\|^2_{L^2}\right)^{1-\varepsilon} - \delta_2 H(f|M); \tag{44}$$

$$\frac{d^2}{dt^2}\big\|f - M^f_{\rho\, 0\, 1}\big\|^2_{L^2} \geq K_3 \int_{\Omega_x} |\nabla \rho|^2\, dx - \frac{C_3}{{\delta_3}^{1-\varepsilon}} \left(\big\|f - M^f_{\rho\, 0\, 1}\big\|^2_{L^2}\right)^{1-\varepsilon} - \delta_3 H(f|M). \tag{45}$$

Here the positive constants K_i *and the negative constants* C_i *only depend on* ε*,* B *and on smoothness and moments bounds for* f*, together with positivity estimates for* ρ *and* T *(which are deduced from positivity estimates on* f*);*

d) Three rules of additivity of the entropy, and three interpolation inequalities, which couple the three equations in c):

$$\begin{cases} H\big(f|M^f_{\rho\, u\, T}\big) + \mathcal{H}(T|\langle T\rangle_\rho) = H\big(f|M^f_{\rho\, u\, \langle T\rangle}\big) \\ H\big(f|M^f_{\rho\, u\, \langle T\rangle}\big) + \mathcal{H}(\langle T\rangle_\rho|1) + \mathcal{H}(u|0) = H\big(f|M^f_{\rho\, 0\, 1}\big) \\ H\big(f|M^f_{\rho\, 0\, 1}\big) + \mathcal{H}(\rho|1) = H(f|M) \end{cases} \tag{46}$$

and

$$\text{(47)}\qquad \begin{cases} H\big(f|M^f_{\rho u T}\big) \ge K_I \big\| f - M^f_{\rho u T}\big\|_{L^2}^{2(1+\varepsilon)}, \\ H\big(f|M^f_{\rho u \langle T\rangle}\big) \ge K_I \big\| f - M^f_{\rho u \langle T\rangle}\big\|_{L^2}^{2(1+\varepsilon)}, \\ H\big(f|M^f_{\rho 0 1}\big) \ge K_I \big\| f - M^f_{\rho 0 1}\big\|_{L^2}^{2(1+\varepsilon)}, \end{cases}$$

where $K_I > 0$ only depends on ε, B, smoothness bounds on f and lower bounds on ρ, T;

e) Geometrical functional inequalities of Korn-type,

$$\text{(48)}\qquad \int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\,dx \ge K_K \int_{\Omega_x} |\nabla u|^2\,dx,$$

where K_K only depends on geometrical information about Ω_x and on the type of boundary conditions. In particular, in the case of specular reflection ($u \cdot n = 0$), K_K depends on how much Ω_x departs from being axisymmetric; and of Poincaré-type,

$$\text{(49)}\qquad \int_{\Omega_x} |\nabla T|^2\,dx \ge K_T\,\mathcal{H}(T|\langle T\rangle_\rho),$$

$$\text{(50)}\qquad \int_{\Omega_x} |\nabla u|^2\,dx \ge K_u\,\mathcal{H}(u|0),$$

$$\text{(51)}\qquad \int_{\Omega_x} |\nabla \rho|^2\,dx \ge K_\rho\,\mathcal{H}(\rho|1),$$

where the constants K_T, K_u, K_ρ only depend on geometrical information about Ω_x, on the type of boundary conditions, and on some bounds on ρ, T;

f) Four first-order inequalities expressing the damping of hydrodynamical oscillations,

(52)
$$\left|\frac{d}{dt}\mathcal{H}(\rho|1),\ \frac{d}{dt}\mathcal{H}(u|0),\ \frac{d}{dt}\mathcal{H}(\langle T\rangle_\rho|1),\ \frac{d}{dt}\mathcal{H}(T|\langle T\rangle_\rho)\right| \le C_S H(f|M)^{1-\varepsilon},$$

where C_S only depends on f via bounds of smoothness, moments and positivity.

In Sect. VI, we shall prove that this system implies a decay of $H(f|M)$ like $O(t^{-1/k\varepsilon})$ if ε is small enough, where k is an explicit constant independent of ε. This will conclude the proof of Theorem 2.

III. Instability of the hydrodynamic description

We now start to implement the program outlined in the previous section. The final goal of this section is to establish the inequalities (43) to (45) when f is a smooth solution of the Boltzmann equation. Again, we shall never display explicitly the time-dependence.

To arrive at (43)–(45), we shall use the identity

$$\frac{d^2}{dt^2}\|f-g\|_{L^2}^2 = 2\int \left[\frac{\partial}{\partial t}(f-g)\right]^2 dv\,dx + 2\int (f-g)\,\frac{\partial^2}{\partial t^2}(f-g)\,dv\,dx. \tag{53}$$

In this inequality we shall replace f by a smooth solution of the Boltzmann equation, and g by one of the three local Maxwellian distributions $M^f_{\rho\,u\,T}$, $M^f_{\rho\,u\,\langle T\rangle}$, $M^f_{\rho\,0\,1}$. Isolating the "dominant" terms will be a little bit tricky.

We shall split the derivation of (43)–(45) into several stages. In a first step, we write down the "well-known" equations satisfied by ρ, u and T, and deduce some equations satisfied by the three local Maxwellians above. In a second step (Subsect. III.2), for pedagogical reasons and because this will enable to identify dominant terms, we establish the desired differential inequalities at a particular time t_0 such that $f_{t_0} = g_{t_0}$. In this particular case, only the first term on the right-hand side of (53) survives, and can be bounded below in a rather simple way. Finally, in Subsect. III.3 we show that the error terms with respect to the situation of Subsect. III.2 are of order $O(\|f-g\|^2)$ and we arrive at (43)–(45).

Before beginning to fulfill this plan, we record an easy proposition about the regularity of hydrodynamic fields.

Proposition 7. *Let f satisfy the estimates* (25) *and* (26)*, and let ρ, u and T be the associated hydrodynamic fields. Then, ρ and T are bounded from above and below, uniformly in t, by explicit constants only depending on A_0, K_0 and q_0. Moreover, for all $k' > 0$, the Sobolev norms $\|\rho\|_{H^{k'}(\Omega_x)}$, $\|u\|_{H^{k'}(\Omega_x)}$, $\|T\|_{H^{k'}(\Omega_x)}$ are bounded by explicitable constants only depending on A_0, K_0, q_0 and $C_{k,s}$ for $k = k(q_0, k')$ and $s = s_0(q_0, k')$ large enough.*

The proof of this proposition is almost obvious and we skip it. As a consequence, ρ, u and T will be assumed to be very smooth, and ρ, T to be uniformly bounded below. An easy consequence is as follows:

Corollary 8. *Let f satisfy the same smoothness and positivity assumptions as above, and let g be either $M^f_{\rho\,u\,T}$, $M^f_{\rho\,u\,\langle T\rangle}$ or $M^f_{\rho\,0\,1}$ the associated local Maxwellians, as defined in* (27), (34) *and* (35). *Then, for all $k', s' > 0$, the Sobolev norm $\|g\|_{H^{k'}_{s'}}$ is bounded from above by an explicit constant, independently of t.*

III.1. Equations for hydrodynamical fields. To any probability distribution $f(x, v)$ we associate, besides the fields ρ, u and T, two additional fields D (matrix-valued) and R (vector-valued). The field D is the "deviator", or traceless part of the symmetric deformation tensor, defined by

$$D_{ij}(x) = \int_{\mathbb{R}^N} f(x, v)\left[(v-u)_i(v-u)_j - \frac{|v-u|^2}{N}\delta_{ij}\right] dv, \tag{54}$$

where δ_{ij} stands for the Kronecker symbol, and R is the heat transfer flux,

$$R(x) = \int_{\mathbb{R}^N} f(x, v)\frac{|v-u|^2}{2}(v-u)\, dv. \tag{55}$$

Of course,

$$\int_{\mathbb{R}^N} f v_i v_j\, dv = \rho u_i u_j + \rho T \delta_{ij} + D_{ij}. \tag{56}$$

• Let now $f = f_t(x, v)$ be a smooth solution of the Boltzmann equation, and let ρ, u, T, D, R be the associated fields (we do not write explicitly the time dependence). By repeated use of (56) and the identity

$$\int Q(f, f)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0, \tag{57}$$

it is not difficult to establish the equations

(58)
$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u) = 0 \\ \dfrac{\partial}{\partial t}(\rho u) + \nabla_x \cdot \big(\rho u \otimes u + \rho T I_N + D\big) = 0 \\ \dfrac{\partial}{\partial t}\big(\rho|u|^2 + N\rho T\big) + \nabla_x \cdot \big(\rho|u|^2 u + (N+2)\rho u T + 2Du + 2R\big) = 0. \end{cases}$$

Here we used the notation $I_N =$ identity matrix of order N, $(a \otimes b)_{ij} = a_i b_j$. This (non closed) system of conservation laws can be recast in the non-divergence form

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\rho + \rho\nabla\cdot u = 0 \\ \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)u + \nabla T + \dfrac{T\nabla\rho}{\rho} + \dfrac{\nabla\cdot D}{\rho} = 0 \\ \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)T + \dfrac{2T}{N}\nabla\cdot u + \dfrac{2}{\rho N}(\nabla u : D + \nabla\cdot R) = 0, \end{cases} \tag{59}$$

where the second equation should be understood as a system of N scalar equations, and we used the standard notation $A : B = \sum_{ij} A_{ij} B_{ij}$.

From (59) we also deduce, recalling that $\langle T \rangle_\rho = \int_{\Omega_x} \rho T$,

$$
\begin{aligned}
(60)\quad (\partial_t + u \cdot \nabla_x)\langle T \rangle_\rho = \partial_t \langle T \rangle_\rho &= \int (\partial_t \rho) T + \int \rho(\partial_t T) \\
&= -\int \nabla \cdot (\rho u) T - \int \rho u \cdot \nabla T - \frac{2}{N} \int \rho T \, \nabla \cdot u \\
&\quad - \frac{2}{N} \int \nabla u : D - \frac{2}{N} \int \nabla \cdot R.
\end{aligned}
$$

Recall the boundary conditions $u \cdot n = 0$ (for specular reflection) or $u = 0$ (for bounce-back), or no boundary, to find that the integration by parts formula

$$
-\int \nabla \cdot (\rho u) T = \int \rho u \cdot \nabla T
$$

is valid, and so the first two terms in the last line of (60) cancel out.

We shall show that the last term also vanishes:

$$
(61)\qquad \int_{\Omega_x} \nabla \cdot R = \int_{\partial \Omega_x} R \cdot n = 0.
$$

Let us prove (61) in the case of specular reflection (for bounce-back, the proof is slightly simpler). We have, for any $x \in \partial \Omega_x$,

$$
R \cdot n = \int f_t(x, v) \frac{|v - u|^2}{2} (v - u) \cdot n \, dv.
$$

Since $u \cdot n = 0$, this can be rewritten

$$
(62)\qquad R \cdot n = \int f_t(x, v) \frac{|v - u|^2}{2} v \cdot n \, dv
$$

$$
(63)\qquad = -\int f_t(x, v) \frac{|R_x v - u|^2}{2} v \cdot n \, dv,
$$

where $R_x v = v - 2\langle v, n \rangle n$ and we used $f_t(x, v) = f_t(x, R_x v)$ on $\partial \Omega_x$. Summing up (62) and (63), and using $|R_x v|^2 = |v|^2$, we find

$$
\begin{aligned}
R \cdot n &= -\frac{1}{2} \int_{\mathbb{R}} f_t(x, v) \big[v \cdot u - (R_x v) \cdot u \big] \, dv \\
&= -\int_{\mathbb{R}} f_t(x, v)(v \cdot n)(n \cdot u) \, dv = \rho (u \cdot n)^2 = 0.
\end{aligned}
$$

So in the end,

$$
(64)\quad (\partial_t + u \cdot \nabla_x)\langle T \rangle_\rho = -\frac{2}{N} \int_{\Omega_x} \rho T (\nabla \cdot u) dx - \frac{2}{N} \int_{\Omega_x} \nabla u : D \, dx.
$$

• Next, let $M_{\rho u T}$ be an *arbitrary* smooth local Maxwellian with parameters ρ, u and T (not necessarily related to any solution of the Boltzmann equation). By evaluating $(\partial_t + v\cdot\nabla_x)\log M_{\rho u T}$ and expanding this in powers of $v-u$, we obtain

$$
\begin{aligned}
(65)\quad &(\partial_t + v\cdot\nabla_x) M_{\rho u T}\\
&= M_{\rho u T}\Bigg\{\left[\frac{\partial_t\rho + u\cdot\nabla_x\rho}{\rho} - \frac{N}{2}\frac{(\partial_t + u\cdot\nabla_x)T}{T}\right]\\
&\quad + \frac{v-u}{\sqrt{T}}\cdot\left[\sqrt{T}\,\frac{\nabla_x\rho}{\rho} - \frac{N}{2}\frac{\nabla_x T}{\sqrt{T}} + \frac{(\partial_t + u\cdot\nabla_x)\,u}{\sqrt{T}}\right]\\
&\quad + \sum_{1\le i<j\le N}\left(\frac{v-u}{\sqrt{T}}\right)_i\left(\frac{v-u}{\sqrt{T}}\right)_j\left[\partial_{x_j}u_i + \partial_{x_i}u_j\right]\\
&\quad + \sum_{1\le i\le N}\left(\frac{v_i-u_i}{\sqrt{T}}\right)^2\left[\partial_{x_i}u_i + \frac{1}{2}\left(\frac{(\partial_t + u\cdot\nabla_x)T}{T}\right)\right]\\
&\quad + \left|\frac{v-u}{\sqrt{T}}\right|^2\left(\frac{v-u}{\sqrt{T}}\right)\cdot\frac{\nabla_x T}{2\sqrt{T}}\Bigg\}.
\end{aligned}
$$

Combining (59) and (65), with ρ, u, T replaced (with obvious notation) by either (ρ^f, u^f, T^f), or $(\rho^f, u^f, \langle T\rangle_\rho^f)$, or $(\rho^f, 0, 1)$, we arrive at a set of equations for $M^f_{\rho\, u\, T}$, $M^f_{\rho\, u\,\langle T\rangle}$, $M^f_{\rho\, 0\, 1}$. First,

$$
\begin{aligned}
(66)\quad &(\partial_t + v\cdot\nabla_x) M^f_{\rho\, u\, T}\\
&= M^f_{\rho\, u\, T}\Bigg\{\left[\frac{\nabla u : D}{\rho T} + \frac{\nabla\cdot R}{\rho T}\right]\\
&\quad + \frac{v-u}{\sqrt{T}}\cdot\left[-\left(\frac{N}{2}+1\right)\frac{\nabla T}{\sqrt{T}} - \frac{\nabla\cdot D}{\rho\sqrt{T}}\right]\\
&\quad + \sum_{1\le i<j\le N}\left(\frac{v-u}{\sqrt{T}}\right)_i\left(\frac{v-u}{\sqrt{T}}\right)_j\left[\partial_{x_j}u_i + \partial_{x_i}u_j\right]\\
&\quad + \sum_{1\le i\le N}\left(\frac{v_i-u_i}{\sqrt{T}}\right)^2\left[\partial_{x_i}u_i - \frac{\nabla\cdot u}{N} - \frac{\nabla u : D}{N\rho T} - \frac{\nabla\cdot R}{N\rho T}\right]\\
&\quad + \left|\frac{v-u}{\sqrt{T}}\right|^2\left(\frac{v-u}{\sqrt{T}}\right)\cdot\frac{\nabla T}{2\sqrt{T}}\Bigg\}\\
&\equiv M^f_{\rho\, u\, T}\,\mathcal{P}_x\left(\frac{v-u}{\sqrt{T}}\right),
\end{aligned}
$$

where $\mathcal{P}_x$ is a polynomial of degree 3, in N variables, with x-dependent coefficients, given by the formula above.

Next,

(67) $$(\partial_t + v\cdot\nabla_x) M^f_{\rho\,u\,\langle T\rangle}$$

$$= M^f_{\rho\,u\,\langle T\rangle}\Bigg\{\left[-\nabla\cdot u + \frac{\int \nabla u : D}{\langle T\rangle_\rho} + \frac{\int \rho T \nabla\cdot u}{\langle T\rangle_\rho}\right] + \frac{v-u}{\sqrt{\langle T\rangle_\rho}}\cdot\left[-\frac{\nabla T}{\sqrt{\langle T\rangle_\rho}} - \left(\frac{T}{\sqrt{\langle T\rangle_\rho}} - \sqrt{\langle T\rangle_\rho}\right)\frac{\nabla\rho}{\rho} - \frac{\nabla\cdot D}{\rho\sqrt{\langle T\rangle_\rho}}\right] + \sum_{1\le i<j\le N}\left(\frac{v-u}{\sqrt{\langle T\rangle_\rho}}\right)_i\left(\frac{v-u}{\sqrt{\langle T\rangle_\rho}}\right)_j\left[\partial_{x_j}u_i + \partial_{x_i}u_j\right] + \sum_{1\le i\le N}\left(\frac{v_i-u_i}{\sqrt{\langle T\rangle_\rho}}\right)^2\left[\partial_{x_i}u_i - \frac{1}{N\langle T\rangle_\rho}\int \nabla u : (D+\rho T I_N)\right]\Bigg\}$$

$$\equiv M^f_{\rho\,u\,\langle T\rangle}\,\mathcal{Q}_x\left(\frac{v-u}{\sqrt{\langle T\rangle_\rho}}\right),$$

where $\mathcal{Q}_x$ is another explicit polynomial with x-dependent coefficients.

Finally,

(68) $$(\partial_t + v\cdot\nabla_x) M^f_{\rho\,0\,1} = M^f_{\rho\,0\,1}\left\{-\frac{\nabla\cdot(\rho u)}{\rho} + v\cdot\frac{\nabla\rho}{\rho}\right\}. \equiv M^f_{\rho\,0\,1}\,\mathcal{R}_x(v).$$

III.2. Estimates at local equilibrium. Assume that for some time $t = t_0$, the functions f and g in (53) coincide. We shall establish (43)–(45) for such a time t_0. Explicitly, this means, depending on the cases,

(69)
$$\left.\frac{d^2}{dt^2}\right|_{t=t_0} \|f - M^f_{\rho\,u\,T}\|^2_{L^2} \ge K_1\left(\int_{\Omega_x} |\nabla T|^2\,dx + \int_{\Omega_x} |\{\nabla u\}|^2\,dx\right),$$

(70) $$\left.\frac{d^2}{dt^2}\right|_{t=t_0} \|f - M^f_{\rho\,u\,\langle T\rangle}\|^2_{L^2} \ge K_2\int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\,dx,$$

(71) $$\left.\frac{d^2}{dt^2}\right|_{t=t_0} \|f - M^f_{\rho\,0\,1}\|^2_{L^2} \ge K_3\int_{\Omega_x} |\nabla\rho|^2\,dx$$

(of course t_0 in (69) is defined as a time at which $f = M^f_{\rho\,u\,T}$, in (70) as a time at which $f = M^f_{\rho\,u\,\langle T\rangle}$, etc.)

For this we first note that, since f solves the Boltzmann equation (2), and since $Q(f,f)|_{t=t_0} = Q(g,g)|_{t=t_0} = 0$,

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0}(f-g) = -v\cdot\nabla_x f_{t_0} - \left.\frac{\partial}{\partial t}\right|_{t=t_0} g = -\,(\partial_t + v\cdot\nabla_x)|_{t=t_0}\, g.$$

By plugging in formulas (66), (67) and (68), and using (53), we find

$$\frac{d^2}{dt^2}\bigg|_{t=t_0} \|f - M^f_{\rho\, u\, T}\|^2_{L^2} = 2\int \frac{\rho^2 e^{-\frac{|v-u|^2}{T}}}{(2\pi T)^N}\mathcal{P}_x\left(\frac{v-u}{\sqrt{T}}\right)^2 dv\, dx$$
$$= 2\int \frac{\rho^2 e^{-|w|^2}}{(2\pi)^N}\mathcal{P}_x(w)^2\, dw\, dx$$
$$\frac{d^2}{dt^2}\bigg|_{t=t_0} \|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2} = 2\int \frac{\rho^2 e^{-\frac{|v-u|^2}{\langle T\rangle_\rho}}}{(2\pi\langle T\rangle_\rho)^N}\mathcal{Q}_x\left(\frac{v-u}{\sqrt{\langle T\rangle_\rho}}\right)^2 dv\, dx$$
$$= 2\int \frac{\rho^2 e^{-|w|^2}}{(2\pi)^N}\mathcal{Q}_x(w)^2\, dw\, dx$$
$$\frac{d^2}{dt^2}\bigg|_{t=t_0} \|f - M^f_{\rho\, 0\, 1}\|^2_{L^2} = 2\int \frac{\rho^2 e^{-|v|^2}}{(2\pi)^N}\mathcal{R}_x(v)^2\, dv\, dx.$$

Since the functions 1, $w_i (1 \le i \le N)$, $w_i w_j (1 \le i < j \le N)$, $w_i^2 (1 \le i \le N)$, $|w|^2 w_i (1 \le i \le N)$ are linearly independent and all belong to $L^2(e^{-|w|^2}\, dw)$, there exists a numerical constant $\kappa > 0$ (not too difficult to estimate explicitly) such that for all $a \in \mathbb{R}$, $\mathrm{b} \in \mathbb{R}^N$, $c_{ij} \in \mathbb{R} (i < j)$, $d_i \in \mathbb{R} (1 \le i \le N)$, $\mathrm{e} \in \mathbb{R}^N$,

$$\text{(72)}\quad \int_{\mathbb{R}^N} \frac{e^{-|w|^2}}{(2\pi)^N}\Big|a + \mathrm{b}\cdot w + \sum_{i<j} c_{ij} w_i w_j + \sum_i d_i w_i^2 + (\mathrm{e}\cdot w)|w|^2\Big|^2 dw$$
$$\ge \kappa\Big(a^2 + |\mathrm{b}|^2 + \sum_{i<j} c_{ij}^2 + \sum_i d_i^2 + |\mathrm{e}|^2\Big).$$

We use this to estimate the various second derivatives above. First,

$$\text{(73)}\quad \frac{d^2}{dt^2}\bigg|_{t=t_0} \|f - M^f_{\rho\, u\, T}\|^2_{L^2}$$
$$\ge 2\kappa \int \rho^2 \bigg\{\Big[\frac{\nabla u : D}{\rho T} + \frac{\nabla\cdot R}{\rho T}\Big]^2$$
$$+ \Big[-\Big(\frac{N+2}{2}\Big)\frac{\nabla T}{\sqrt{T}} - \frac{\nabla\cdot D}{\rho\sqrt{T}}\Big]^2$$
$$+ \sum_{1\le i<j\le N}\Big[\partial_{x_j} u_i + \partial_{x_i} u_j\Big]^2$$
$$+ \sum_{1\le i\le N}\Big[\partial_{x_i} u_i - \frac{\nabla\cdot u}{N} - \frac{\nabla u : D}{N\rho T} - \frac{\nabla\cdot R}{N\rho T}\Big]^2$$
$$+ \Big|\frac{\nabla T}{2\sqrt{T}}\Big|^2\bigg\}\, dx.$$

Recall that ρ is uniformly bounded below and T uniformly bounded from above. From the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ we deduce that

$$\left[\frac{\nabla u : D}{\rho T} + \frac{\nabla \cdot R}{\rho T}\right]^2 + \sum_{1\le i\le N}\left[\partial_{x_i} u_i - \frac{\nabla \cdot u}{N} - \frac{\nabla u : D}{N\rho T} - \frac{\nabla \cdot R}{N\rho T}\right]^2 \ge \frac{1}{2}\sum_i \left(\partial_{x_i} u_i - \frac{\nabla \cdot u}{N}\right).$$

Moreover,

$$\frac{1}{2}\sum_{i<j}(\partial_{x_i} u_i + \partial_{x_j} u_j)^2 + \sum_i \left(\partial_{x_i} u_i - \frac{\nabla \cdot u}{N}\right)^2 = |\{\nabla u\}|^2.$$

Thus we arrive at (69).

Similarly,

$$(74)\quad \frac{d^2}{dt^2}\bigg|_{t=t_0} \left\| f - M^f_{\rho\, u\, \langle T\rangle} \right\|^2_{L^2} \ge 2\kappa \int \rho^2 \Bigg\{\left[-\nabla\cdot u + \frac{1}{\langle T\rangle_\rho}\int \nabla u : (D + \rho T I_N)\right]^2 + \left[\frac{\nabla T}{\sqrt{\langle T\rangle_\rho}} + \left(\frac{T}{\sqrt{\langle T\rangle_\rho}} - \sqrt{\langle T\rangle_\rho}\right)\frac{\nabla\rho}{\rho} + \frac{\nabla \cdot D}{\rho\sqrt{\langle T\rangle_\rho}}\right]^2 + \sum_{1\le i<j\le N}\left[\partial_{x_j} u_i + \partial_{x_i} u_j\right]^2 + \sum_{1\le i\le N}\left[\partial_{x_i} u_i - \frac{1}{N\langle T\rangle_\rho}\int \nabla u : (D+\rho T I_N)\right]^2\Bigg\}\, dx.$$

In particular, this is greater than a constant multiple of

$$\inf_{a\in\mathbb{R}} \int_{\Omega_x} [-\nabla\cdot u + a]^2\, dx = \int_{\Omega_x}\left(\nabla\cdot u - \int_{\Omega_x}\nabla\cdot u\right)^2 dx = \int_{\Omega_x}(\nabla\cdot u)^2,$$

since any of our three boundary conditions implies $\int_{\Omega_x} \nabla\cdot u = \int_{\partial\Omega_x} u\cdot n = 0$. Moreover, denoting $b = \int \nabla u : (D+\rho T I_N)/\langle T\rangle_\rho$, we have

$$(-\nabla\cdot u + b)^2 + \sum_i (\partial_{x_i} u_i - b/N)^2 \ge \frac{1}{2}\sum_i(\partial_{x_i} u_i - \nabla\cdot u/N)^2.$$

Also we note that

$$\frac{1}{N}(\nabla \cdot u)^2 + \sum_i \left(\partial_{x_i} u_i - \frac{\nabla \cdot u}{N}\right)^2 + \frac{1}{2}\sum_{i<j}(\partial_{x_j} u_i + \partial_{x_i} u_j)^2 = |\nabla^{\mathrm{sym}} u|^2.$$

This leads to (70).

Finally, the derivation of (71) is almost trivial:

$$(75) \quad \left.\frac{d^2}{dt^2}\right|_{t=t_0} \|f - M^f_{\rho\, 0\, 1}\|^2_{L^2} \geq 2\kappa \int \rho^2 \left\{ \left(\frac{\nabla \cdot (\rho\, u)}{\rho}\right)^2 + \left|\frac{\nabla \rho}{\rho}\right|^2 \right\} dx$$
$$\geq 2\kappa \int |\nabla \rho|^2\, dx.$$

Remark. The three inequalities which we just established are *essentially local*: they are very close to hold pointwise in x. This is in contrast with the inequalities to come in Sect. IV, which are global.

III.3. The general case. In this subsection we establish (43)–(45) for any time. Let g stand for either $M^f_{\rho\, u\, T}$, or $M^f_{\rho\, u\, \langle T\rangle}$, or $M^f_{\rho\, 0\, 1}$.

a) Estimate of the first-order term: Using $Q(g, g) = 0$, we write

$$\int_{\Omega_x \times \mathbb{R}^N} [\partial_t (f - g)]^2\, dv\, dx = \int_{\Omega_x \times \mathbb{R}^N} [-v \cdot \nabla_x f + Q(f, f) - \partial_t g]^2\, dv\, dx$$
$$= \int [-v \cdot \nabla_x (f - g) + Q(f, f) - Q(g, g) - (\partial_t + v \cdot \nabla_x) g]^2\, dv\, dx$$
$$(76) \geq \frac{1}{3} \int \Big[(\partial_t + v \cdot \nabla_x) g\Big]^2 dv\, dx - \int \Big[Q(f, f) - Q(g, g)\Big]^2 dv\, dx$$
$$- \int \Big[v \cdot \nabla_x (f - g)\Big]^2 dv\, dx.$$

The first term in the right-hand side of (76) has been estimated from below in the previous subsection; we keep this lower bound. Now we shall show

$$(77) \qquad \int \Big[v \cdot \nabla_x (f - g)\Big]^2 dv\, dx \leq C \|f - g\|^{2(1-\varepsilon)}_{L^2}$$

$$(78) \qquad \int \Big[Q(f, f) - Q(g, g)\Big]^2 dv\, dx \leq C \|f - g\|^{2(1-\varepsilon)}_{L^2}.$$

We first prove (77). This inequality follows immediately from our smoothness assumptions on f, together with Corollary 8 and the following simple interpolation lemma, which will be used again later. We use the natural notation $v^\alpha = v_1^{\alpha_1} \cdots v_N^{\alpha_N}$.

Lemma 9. *Let h be a smooth function of x, v. Then, for all multi-indices α, β, and for all* $\eta < 1$,

$$\int \left(v^\alpha \partial^\beta_{x,v} h\right)^2 dv\, dx \leq \|h\|^{2\eta}_{H^{|\beta|}_{|\alpha|/\eta}} \|h\|^{2\eta(1-\eta)}_{H^{|\beta|/\eta}} \|h\|^{2(1-\eta)^2}_{L^2}.$$

Proof of Lemma 9. First, by Hölder's inequality,

$$\begin{aligned}\int \left(v^\alpha \partial^\beta_{x,v} h\right)^2 dv\, dx &\leq \left(\int |v|^{2|\alpha|/\eta} \left|\partial^\beta_{x,v} h\right|^2 dv\, dx\right)^\eta \left(\int \left|\partial^\beta_{x,v} h\right|^2 dv\, dx\right)^{1-\eta} \\ &\leq \|h\|^{2\eta}_{H^{|\beta|}_{|\alpha|/\eta}} \|h\|^{2(1-\eta)}_{H^{|\beta|}}.\end{aligned}$$

Next, by interpolation,

$$\|h\|_{H^{|\beta|}} \leq \|h\|^{\eta}_{H^{|\beta|/\eta}} \|h\|^{1-\eta}_{L^2}.$$

This concludes the proof. □

Next, we establish (78). On one hand,

$$\int [Q(f,f) - Q(g,g)]^2\, dv\, dx = \|Q^{\text{sym}}(f+g, f-g)\|^2_{L^2(\Omega_x \times \mathbb{R}^N)};$$

on the other hand, our continuity assumption (19) implies that

$$\|Q^{\text{sym}}(f+g, f-g)\|^2_{L^2(\mathbb{R}^N)} \leq C_B \|f+g\|^2_{H^{k_0}_{s_0}} \|f-g\|^2_{H^{k_0}_{s_0}},$$

for k_0 and s_0 large enough (depending on B). In particular,

$$\begin{aligned}&\|Q^{\text{sym}}(f+g, f-g)\|^2_{L^2(\Omega_x \times \mathbb{R}^N)} \\ &\qquad \leq C_B \|f+g\|^2_{L^\infty(\Omega_x; H^{k_0}_{s_0}(\mathbb{R}^N))} \|f-g\|^2_{L^2(\Omega_x; H^{k_0}_{s_0}(\mathbb{R}^N))}.\end{aligned}$$

From this we conclude the proof of (78) as before, after controlling the L^∞ norm by some Sobolev norm.

b) Estimate of the second-order term

In this subsection we establish, for $0 < \delta < 1/10$, with the same notation as before,

$$\left| \int_{\Omega_x \times \mathbb{R}^N} (f-g) \frac{\partial^2}{\partial t^2}(f-g)\, dv\, dx \right| \leq \frac{C}{\delta^{1-\varepsilon}} \|f-g\|^{2(1-\varepsilon)}_{L^2} + \delta H(f|M) \tag{79}$$

and this will conclude the proof of (43)–(45).

We start with

(80)
$$\left|\int_{\Omega_x\times\mathbb{R}^N}(f-g)\,\frac{\partial^2}{\partial t^2}(f-g)\,dv\,dx\right| \le \|f-g\|_{L^2}\left(\left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2}+\left\|\frac{\partial^2 g}{\partial t^2}\right\|_{L^2}\right).$$

If we manage to show

$$\text{(81)}\qquad \forall\eta>0,\qquad \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2}+\left\|\frac{\partial^2 g}{\partial t^2}\right\|_{L^2}\le C_\eta\|f-M\|_{L^1}^{1-\eta},$$

where $M = M_{1\,0\,1}$ is the equilibrium state, then, in view of the Csiszár-Kullback-Pinsker inequality, the right-hand side of (80) will be bounded by

$$\begin{aligned} C_\eta\|f-g\|_{L^2}\|f-M\|_{L^1}^{1-\eta} &\le C_\eta\|f-g\|_{L^2}H(f|M)^{\frac{1-\eta}{2}}\\ &\le C_\eta\left(\frac{\|f-g\|_{L^2}}{\delta^{(1-\eta)/2}}\right)^{\frac{2}{1+\eta}}+\frac{(1-\eta)\delta}{2}H(f|M). \end{aligned}$$

Then we obtain (80) by choosing $1+\eta=(1-\varepsilon)^{-1}$ and noting that $\delta^{\frac{1-\eta}{1+\eta}}=\delta^{(1-\varepsilon)(1-\eta)}\ge\delta^{1-\varepsilon}$. So we just have to prove (81).

To prove (81), we first establish the estimate on $\partial^2 f/\partial t^2$. From the Boltzmann equation (2), we deduce

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} &= -v\cdot\nabla_x\cdot\left(\frac{\partial f}{\partial t}\right)+2Q^{\text{sym}}\left(f,\frac{\partial f}{\partial t}\right)\\ &= v\otimes v:\nabla_x^2 f - v\cdot\nabla_x Q^{\text{sym}}(f,f)-2Q^{\text{sym}}(f,v\cdot\nabla_x f)\\ &\qquad\qquad +2Q^{\text{sym}}(f,Q(f,f))\\ &= v\otimes v:\nabla_x^2 f-2v\cdot Q^{\text{sym}}(\nabla_x f,f)-2Q^{\text{sym}}(f,v\cdot\nabla_x f)\\ &\qquad\qquad +2Q^{\text{sym}}(f,Q(f,f))\\ &= v\otimes v:\nabla_x^2(f-M)-2v\cdot Q^{\text{sym}}(\nabla_x(f-M),f)\\ &\quad -2Q^{\text{sym}}(f,v\cdot\nabla_x(f-M))+2Q^{\text{sym}}(f,Q(f,f)-Q(M,M)). \end{aligned}$$

Combining this with our continuity assumptions on Q and the interpolation lemma 9, we arrive at

$$\forall\eta>0,\qquad \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2}\le C_\eta\|f-M\|_{L^2}^{1-\eta},$$

where C_η depends on bounds on M in weighted Sobolev norms of order high enough. The desired bound will then be a consequence of another interpolation lemma:

Lemma 10. *Let h be a smooth function on $\Omega_x \times \mathbb{R}^N$. Then, for all $\alpha < 1$, there exists $\ell > 0$, and a numeric constant $C_\alpha > 0$ such that*

$$\|h\|_{L^2} \le C_\alpha \|h\|_{L^1}^{1-\alpha} \|h\|_{H^\ell}^{\alpha}.$$

This kind of interpolation lemma is classical. Let us however give a short proof for completeness.

Proof of Lemma 10. From our assumptions it follows that $\varphi(x) \equiv \|h(x,\cdot)\|_{L^2(\mathbb{R}^N_v)}$ is a smooth function of $x \in \Omega_x$. It is not difficult to check that whenever $k \in \mathbb{N}$, then $\|\varphi\|_{H^k(\Omega_x)} \le C_k \|h\|_{H^k(\Omega_x \times \mathbb{R}^N)}$ for some numeric constant C_k; by interpolation this also holds true for all $k > 0$. Let $H^{-k}(\Omega_x)$ be defined as the dual space of $H^k(\Omega_x)$:

$$\|u\|_{H^{-k}(\Omega_x)} = \sup_{\|\psi\|_{H^k(\Omega_x)} \le 1} \left| \int_{\Omega_x} u\psi \, dx \right|.$$

By definition,

$$\|\varphi\|^2_{L^2(\Omega_x)} \le \|\varphi\|_{H^{-k}(\Omega_x)} \|\varphi\|_{H^k(\Omega_x)}.$$

On the other hand, if $\beta \in (0,1)$, by interpolation,

$$\|\varphi\|_{H^k(\Omega_x)} \le \|\varphi\|_{L^2(\Omega_x)}^{1-\beta/2} \|\varphi\|_{H^{k'}(\Omega_x)}^{\beta/2},$$

if we choose $k' = k(1-\beta)/\beta$. Combining the two inequalities above, we obtain

$$\|\varphi\|_{L^2(\Omega_x)}^{1+\beta/2} \le \|\varphi\|_{H^{-k}(\Omega_x)} \|\varphi\|_{H^{k'}(\Omega_x)}^{\beta/2}.$$

If we choose $1 + \beta/2 = (1-\alpha)^{-1}$, this turns into

$$\|\varphi\|_{L^2(\Omega_x)} \le \|\varphi\|_{H^{-k}(\Omega_x)}^{1-\alpha} \|\varphi\|_{H^{k'}(\Omega_x)}^{\alpha}.$$

Now, by duality the Sobolev embedding implies

$$L^1(\Omega_x) \subset H^{-k}(\Omega_x),$$

for any $k > N/2$; let us set $k = N/2 + 1$ for instance. We conclude that

$$\text{(82)} \qquad \begin{aligned} \|h\|_{L^2(\Omega_x \times \mathbb{R}^N)} = \|\varphi\|_{L^2(\Omega_x)} &\le C \|\varphi\|_{L^1(\Omega_x)}^{1-\alpha} \|\varphi\|_{H^{k'}(\Omega_x)}^{\alpha} \\ &\le C \|\varphi\|_{L^1(\Omega_x)}^{1-\alpha} \|h\|_{H^{k'}(\Omega_x \times \mathbb{R}^N)}^{\alpha}. \end{aligned}$$

Next, from Lemma 4.4 in [60] we know that

$$\varphi(x) \le C_\alpha \|h(x,\cdot)\|_{L^1(\mathbb{R}^N_v)}^{1-\alpha} \|h(x,\cdot)\|_{H^m(\mathbb{R}^N_v)}^{\alpha},$$

if $m = (N+1)/(2\alpha)$. This implies

$$\|\varphi\|_{L^1(\Omega_x)} \le C_\alpha \Big\| \|h(x,\cdot)\|^{1-\alpha}_{L^1(\mathbb{R}^N_v)} \Big\|_{L^1(\Omega_x)} \sup_{x\in\Omega_x} \|h(x,\cdot)\|^{\alpha}_{H^m(\mathbb{R}^N_v)}$$
$$\le C_\alpha \|h\|^{1-\alpha}_{L^1(\Omega_x\times\mathbb{R}^N)} \|h\|^{\alpha}_{H^{m'}(\Omega_x\times\mathbb{R}^N)},$$

where m' is large enough, and we used Jensen's inequality (recall that $|\Omega_x| = 1$) and the Sobolev embedding of $L^\infty(\Omega_x)$ into $H^k(\Omega_x)$ for $k > N/2$. Combining this with (82), we conclude the proof of the lemma. □

It now remains to prove the second part of (81). We shall do this only for $g = M^f_{\rho u T}$, the other two cases being handled similarly (in a slightly simpler way). First we compute

$$(83)\quad \frac{\partial^2}{\partial t^2} M^f_{\rho u T} = M^f_{\rho u T} \left\{ \frac{\partial^2_{tt}\rho}{\rho} + \frac{\partial^2_{tt} u\cdot(v-u)}{T} + \left(\frac{|v-u|^2}{2T} - \frac{N}{2}\right)\frac{\partial^2_{tt}T}{T} \right.$$
$$- \frac{(\partial_t u)^2}{T} - \frac{|v-u|^2}{2T}\left(\frac{\partial_t T}{T}\right)^2 + \left(\frac{|v-u|^2}{2T} - \frac{N+4}{2}\right)\frac{\partial_t u\cdot(v-u)}{T}\,\frac{\partial_t T}{T}$$
$$\left. + \left(\frac{\partial_t u\cdot(v-u)}{T}\right)^2 + 2\,\frac{\partial_t\rho}{\rho}\,\frac{\partial_t u\cdot(v-u)}{T} + 2\,\frac{\partial_t\rho}{\rho}\left(\frac{|v-u|^2}{2T} - \frac{N}{2}\right)\frac{\partial_t T}{T} \right\}.$$

Then we see that each of the terms $\partial_t\rho$, $\partial_t u$, $\partial_t T$, $\partial^2_{tt}\rho$, $\partial^2_{tt}u$, $\partial^2_{tt}T$ can be written in terms of $f - M$ and $Q(f,f) = Q(f,f) - Q(M,M)$. Let us give the explicit formula for the terms in ρ and u (we skip those for T because they are extremely long):

$$\partial_t\rho = -\nabla_x\cdot \int_{\mathbb{R}^N} (f-M)\, v\, dv,$$
$$\partial_t u = \frac{1}{\rho}\left[u\,\nabla_x\cdot\int_{\mathbb{R}^N}(f-M)\,v\,dv - \nabla_x\cdot\int_{\mathbb{R}^N}(f-M)\,v\otimes v\,dv\right],$$
$$\partial^2_{tt}\rho = \nabla_x\nabla_x : \int_{\mathbb{R}^N}(f-M)\,v\otimes v\,dv,$$
$$\partial^2_{tt}u = \frac{1}{\rho}\nabla_x\nabla_x : \int_{\mathbb{R}^N}(f-M)\,v\otimes v\otimes v\,dv$$
$$- \frac{1}{\rho}\nabla_x\cdot\int_{\mathbb{R}^N}\Big(Q(f,f) - Q(M,M)\Big)\,v\otimes v\,dv$$
$$- \frac{u}{\rho}\,\nabla_x\nabla_x : \int_{\mathbb{R}^N}(f-M)\,v\otimes v\,dv$$
$$- 2\,\frac{\nabla_x\cdot(\rho u)}{\rho^2}\,\nabla_x\cdot\int_{\mathbb{R}^N}(f-M)\,v\otimes v\,dv$$
$$+ 2u\,\frac{\nabla_x\cdot(\rho u)}{\rho^2}\,\nabla_x\cdot\int_{\mathbb{R}^N}(f-M)\,v\,dv.$$

As a consequence,

$$\frac{\partial^2}{\partial t^2} M^f_{\rho u T} = \sum_{\mathcal{A}_x} M^f_{\rho u T} \mathcal{A}_x(v),$$

where the $\mathcal{A}_x$ are a family of polynomials in v with x-dependent coefficients, more precisely of the form

$$\frac{\partial_x^\alpha (\rho, u, T)}{\rho^a \, T^b} \partial_x^\beta \left(\int_{\mathbb{R}^N} (f - M) \mathcal{B}(v) \, dv \right)$$

or

$$\frac{\partial_x^\alpha (\rho, u, T)}{\rho^a \, T^b} \partial_x^\beta \left(\int_{\mathbb{R}^N} [Q(f, f) - Q(M, M)] \mathcal{B}(v) \, dv \right),$$

where $\mathcal{B}$ stands for polynomials in v. Using our bounds on f and interpolation of $H^k(\Omega_x)$ between $L^2(\Omega_x)$ and $H^\ell(\Omega_x)$ for ℓ large enough, we can conclude that

$$\left\| \frac{\partial^2}{\partial t^2} M^f_{\rho u T} \right\|_{L^2} \leq C_\alpha \| f - M \|_{L^2}^{1-\alpha}.$$

The end of the proof is as before.

IV. Some geometrical inequalities

In this section we compile the geometrical inequalities entering the system presented in Subsect. II.4. They are of two kinds: variants of Korn's inequality (inequality (48)), and variants of Poincaré's inequality (inequalities (49) to (51)). Exactly which versions we need depends on the conservation laws and boundary conditions. Throughout this section, Ω_x is either the torus $\mathbb{T}^N$, or a smooth, bounded, connected open subset of $\mathbb{R}^N$.

IV.1. Korn-type inequalities

• In the case of specular boundary conditions in a domain Ω_x of $\mathbb{R}^2$ or $\mathbb{R}^3$ which is not axisymmetric, the Korn inequality that we need is the one that was established in [22]: if $\Omega_x \subset \mathbb{R}^3$ is not axisymmetric, then

$$u \cdot n = 0 \quad \text{on } \partial\Omega_x \quad \Longrightarrow \int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2 \, dx \geq K_K(\Omega_x) \int_{\Omega_x} |\nabla u|^2$$

for some constant $K_K(\Omega_x) > 0$, only depending on Ω_x.

As shown in [22], one can quantify the degree of non-axisymmetry of Ω_x by comparing the (normalized) Lebesgue measure $\mathcal{L}$ on Ω_x with the "radially symmetrized" version $\mathcal{L}_\sigma$ of $\mathcal{L}$ around some axis $\sigma \in S^2$, going through the center of mass of Ω_x. For instance, the constant $K_K(\Omega_x)$ is of order const.η if Ω_x is convex and $\inf_{\sigma\in S^2} W_2(\mathcal{L},\mathcal{L}_\sigma) \geq$ const.η. See [22] for details. In the same reference we also derive the variants which should be used to treat the case of an axisymmetric domain; we will not discuss them here.

• In the case of bounce-back boundary conditions ($u = 0$ on $\partial\Omega_x$), we should use the "standard" Korn inequality,

$$u = 0 \quad \text{on } \partial\Omega_x \quad \Longrightarrow \int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\,dx \geq K_K(\Omega_x)\int_{\Omega_x} |\nabla u|^2\,dx.$$

• Finally, in the case of a periodic box ($\Omega_x = \mathbb{T}^N$), we can use the variant

$$\int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\,dx \geq K_K(\Omega_x)\int_{\Omega_x} |\nabla u|^2\,dx.$$

In fact, an even stronger inequality holds, in which the symmetrized gradient is replaced by its traceless part:

$$\int_{\Omega_x} |\{\nabla u\}|^2\,dx \geq K_K(\Omega_x)\int_{\Omega_x} |\nabla u|^2\,dx. \tag{84}$$

The interest of (84) is to provide a simpler proof of convergence to equilibrium in the case of periodic boundary conditions. Let us give a more precise statement.

Proposition 11. *Let $u \in H^1(\mathbb{T}^N;\mathbb{R}^N)$. Then*

$$\big\|\{\nabla u\}\big\|_{L^2}^2 \geq \frac{1}{2N}\big\|\nabla u\big\|_{L^2}^2.$$

Proof. We write $u = (u_1,\dots,u_N)$ and

$$u_i(x) = \sum_{k\in\mathbb{Z}^N} \widehat{u_i}(k)e^{2i\pi k\cdot x}$$

the Fourier series for each component of u. By Parseval's identity,

$$\begin{aligned}
&\big\|\{\nabla u\}\big\|_{L^2}^2 \\
&= \sum_{k\in\mathbb{Z}^N}\sum_{1\leq i\leq N}\sum_{1\leq j\leq N}\left(\frac{1}{2}k_i\widehat{u_j}(k) + \frac{1}{2}k_j\widehat{u_i}(k) - \frac{1}{N}\sum_{1\leq \ell\leq N} k_\ell\widehat{u_\ell}(k)\delta_{ij}\right)^2 \\
&= \sum_{k\in\mathbb{Z}^N}\left[\frac{1}{4}\sum_{i\neq j}\Big(k_i\widehat{u_j}(k) + k_j\widehat{u_i}(k)\Big)^2 + \sum_i\left(k_i\widehat{u_i}(k) - \frac{1}{N}\sum_{1\leq\ell\leq N} k_\ell\widehat{u_\ell}(k)\right)^2\right].
\end{aligned}$$

Now we apply the well-known “discrete Poincaré inequality”

$$\sum_{1\le i,j\le N}(a_i-a_j)^2 \le 4N \sum_{1\le i\le N}\left(a_i-\frac{1}{N}\sum_{1\le \ell\le N}a_\ell\right)^2,$$

to find

$$\begin{aligned}\left\|\{\nabla u\}\right\|_{L^2}^2 \ge \sum_{k\in\mathbb{Z}^N}\Bigg[\frac{1}{4}\sum_{i\neq j}\Big(k_i\widehat{u_j}(k)+k_j\widehat{u_i}(k)\Big)^2 \\ +\frac{1}{4N}\sum_{ij}\Big(k_i\widehat{u_j}(k)-k_j\widehat{u_i}(k)\Big)^2\Bigg] \\ \ge \frac{1}{4N}\sum_{k\in\mathbb{Z}^N}\sum_{i\neq j}\left(k_i^2+k_j^2\right)\left(|\widehat{u_i}(k)|^2+|\widehat{u_j}(k)|^2\right) \\ \ge \frac{1}{2N}\sum_{1\le i\le N}\sum_{k\in\mathbb{Z}^N}|k|^2|\widehat{u_i}(k)|^2 = \frac{1}{2N}\|\nabla u\|_{L^2}^2.\end{aligned}$$

IV.2. Poincaré-type inequalities

• We start with an inequality for the density ρ. Since $\int\rho = 1$ and $|\Omega_x| = 1$, the standard Poincaré(-Wirtinger) inequality yields

$$\int_{\Omega_x}|\nabla\rho|^2\,dx \ge K(\Omega_x)\int\left(\rho-\int\rho\right)^2 dx = K(\Omega_x)\int(\rho-1)^2.$$

As an immediate consequence, there is a constant $K'(\Omega_x) > 0$ such that

$$\int|\nabla\rho|^2\,dx \ge K'(\Omega_x)\int_{\Omega_x}(\rho\log\rho-\rho+1)\,dx = K'(\Omega_x)\int_{\Omega_x}\rho\log\rho\,dx.$$

This is inequality (51).

• We now turn to an analogous inequality for the temperature. The difference is that now the relevant average is not with respect to Lebesgue measure, but with respect to the density ρ; accordingly, we set $\langle T\rangle_\rho = \int\rho T$. To prove (49), we write

$$\int_{\Omega_x}|\nabla T|^2\,dx \ge K\int_{\Omega_x}(T-\langle T\rangle)^2 = \frac{K}{2}\int\int[T(x)-T(y)]^2\,dx\,dy,$$

where the average $\langle T\rangle$ is here with respect to Lebesgue measure, and then bound this quantity below by

$$\frac{K}{2\|\rho\|_{L^\infty}^2}\int\int\rho(x)\rho(y)[T(x)-T(y)]^2\,dx\,dy = \frac{K}{\|\rho\|_{L^\infty}^2}\int_{\Omega_x}(T-\langle T\rangle_\rho)^2\,dx.$$

To establish (49), it remains to bound this expression below in terms of $\mathcal{H}(T|\langle T\rangle_\rho)$. This is done as follows ($\Phi(X)$ stands for $X - \log X - 1$ as before):

$$\begin{aligned}\mathcal{H}(T|\langle T\rangle_\rho) &= \langle \Phi(T)\rangle_\rho - \Phi(\langle T\rangle_\rho) = \langle \Phi(T) - \Phi(\langle T\rangle_\rho)\rangle_\rho \\ &= \Big\langle \Phi'(\langle T\rangle_\rho)\cdot(T - \langle T\rangle_\rho) \\ &\qquad + \int_0^1 (1-s)\Phi''\big(\langle T\rangle_\rho + s(T-\langle T\rangle_\rho)\big)\,ds\,(T-\langle T\rangle_\rho)^2\Big\rangle_\rho \\ &\le \Big[\sup_{\theta\in[T,\langle T\rangle_\rho]} \Phi''(\theta)\Big]\big\langle (T-\langle T\rangle_\rho)^2\big\rangle_\rho\end{aligned}$$

(with the convention that $[T, \langle T\rangle_\rho] = [\langle T\rangle_\rho, T]$ if $T > \langle T\rangle_\rho$). Since Φ'' is bounded on any interval $[T_0, +\infty)$ and since we assume a lower bound on T, therefore on $\langle T\rangle_\rho$, we conclude that $\int_{\Omega_x}(T-\langle T\rangle_\rho)^2\,dx$ dominates $\mathcal{H}(T|\langle T\rangle_\rho)$, and thus we have established (49).

Remark. In the case of periodic and bounce-back conditions, it is possible to use also the inequality

$$\int |\nabla T|^2 \ge K\mathcal{H}(T|1) - C\mathcal{H}(u|0),$$

which is an easy consequence of the above and the equality $\int \rho T + (\int \rho|u|^2)/N = 1$; this simplifies the overall scheme of proof.

• We end up with some Poincaré inequalities for vector-valued functions. Two of our three cases are immediate:

– For periodic boundary conditions, we apply the inequality

$$\int |\nabla u_i|^2\,dx \ge K\big\langle (u_i - \langle u_i\rangle_\rho)^2\big\rangle_\rho,$$

which was proven above for T in place of u_i. Since $\langle u_i\rangle_\rho$ is normalized to 0, by summing up for $i = 1, \dots N$ we obtain

$$\int |\nabla u|^2 \ge K\int \rho|u|^2.$$

– For bounce-back boundary conditions, we know $u = 0$ on $\partial\Omega_x$, and we can use the "standard" Poincaré inequality

$$\int |\nabla u_i|^2 \ge K\int u_i^2 \ge K\int \rho u_i^2.$$

By summing up for $i = 1, \dots, N$, we obtain the same conclusion.

– The case of specular boundary conditions is (once again) slightly more delicate. We treat it as follows. We cover $\overline{\Omega_x}$ by a finite number of neighborhoods $U_1, \dots, U_\ell$ satisfying the following: there exists a neighborhood V_i of $\overline{U_i}$ such that either V_i is included in Ω_x, or C^1 local coordinates $(x_1, \dots, x_N)$ can be defined in V_i in such a way that
  (i) $A_i \equiv U_i \cap \Omega_x$ (resp. $V_i \cap \Omega_x$) is represented by the equations $(a_k < x_k < b_k)$ (resp. $a'_k < x_k < b'_k$);
  (ii) $\partial\Omega_x \cap U_i$ is described by the equation $(x_N = 0)$;
  (iii) the lines $(x_1 = x_1^0, \dots, x_{N-1} = x_{N-1}^0)$ arrive orthogonally to $\partial\Omega_x$.

See Fig. 1 for an illustration. To ensure point (iii), one might locally extend the normal vector field into a smooth, unit-norm vector field defined inside Ω_x, then choose the integral curves for this vector field as the lines of constant $(x_1, \dots, x_N)$ coordinates.

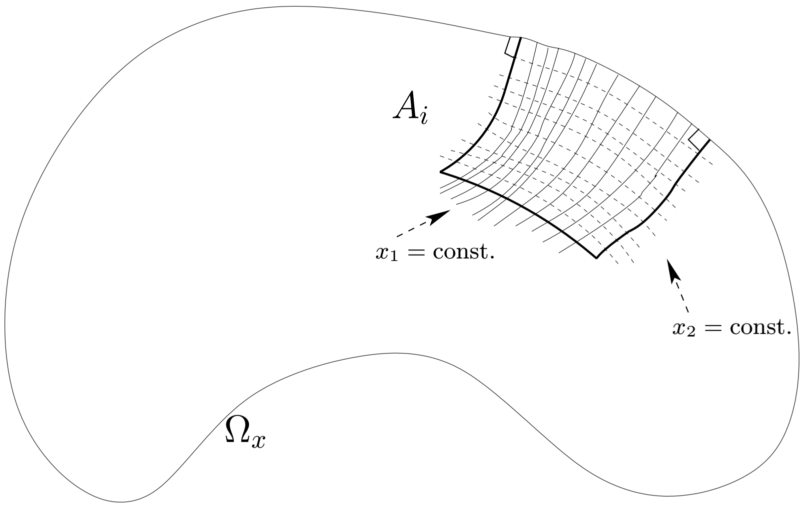

Fig. 1. Local coordinates for A_i

To prove the desired Poincaré inequality, it is sufficient to show

$$\int_{A_i} u_k^2 \le C \int_{\Omega_x} |\nabla u|^2. \tag{85}$$

We shall establish (85) only in the case when $\partial\Omega_x \cap A_i$ is not empty, the reasoning being almost exactly the same in the other case. We shall write the proof in a slightly informal way for simplicity.

We first extend all the lines $(x_{i_1}, \dots, x_{i_{N-1}}) = \text{const.}$ $(i_1, \dots, i_{N-1}$ distinct), i.e. the integral curves of the 1-form dx_k, for all k, into smooth lines contained in Ω_x, ending up on some part of $\partial\Omega_x$, *orthogonally*. For each k, the family of lines thus obtained can be parameterized by $x_k \in (\alpha_k, \beta_k)$.

Such extensions are portraited on the left of Fig. 2; their existence might seem not so obvious in general, so we should briefly justify it. Pick any point z in the boundary $\partial\Omega_x$, out of U_i, and a neighborhood W of z in $\mathbb{R}^N$ such that $W \cap U_i = \emptyset$, $W \cap \Omega$ is diffeomorphic to a C^1 subgraph, and a local extension of the unit normal vector field can be defined in W. Pick w in W and z in $\partial V_i \cap \Omega_x$. Since Ω_x is connected, there is a smooth path in Ω going from w to z, which can be enlarged in a tubular neighborhood T (locally homeomorphic to a cylinder). We choose a small ball B around z. Using local coordinates all along T, we can define a smooth family of lines joining $B \cap V_i$ to a portion of ∂W, which itself can be joined to $\partial\Omega$ by a family of lines arriving orthogonally. Since $w \in W$, we can join w to the boundary $\partial\Omega$ by a line arriving orthogonally to $\partial\Omega$. Since $(V_i \cap \Omega_x) \setminus U_i$ is homeomorphic to a ball, it can be deformed continuously into whichever of its points, in particular z; in that way we can continuously join z to all the lines $(x_{i_1}, \dots, x_{i_{N-1}}) = \text{const.}$, defined in U_i. We consider the extension defined by

– first continuing up to B by the contracting map (stop when B is reached);
– then going along the tubular neighborhood T up to ∂W;
– then going up to $\partial\Omega_x$ along the extended normal unit vector field.

Thus each line has been extended into a piecewise smooth line arriving orthogonally at $\partial\Omega_x$ (see the right of Fig. 2). To conclude the argument it suffices to smooth things a little bit.

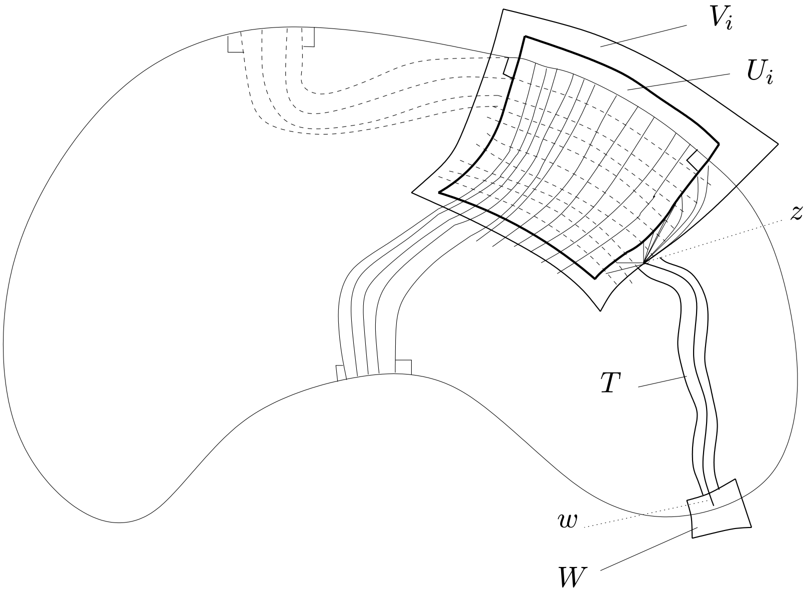

Fig. 2. Extending the network

Let k be any index in $\{1, \ldots, N\}$, we can define $u_k = \langle dx_k, u\rangle$ on A_i. From our definition and the boundary conditions, $u_k = 0$ on the line $(x_k = \alpha_k)$.

Then we write $A_i = \bigcup_{y\in U} \ell_y$, where ℓ_y is the line $x_k^{-1}((a_k, b_k))$ and y is some $(N-1)$-tuple of all coordinates distinct from k. Let $\overline{\ell}_y^{(k)}$ stand for the extended line, parameterized by (α_k, β_k), and let $\overline{A_i}^{(k)}$ be the union of all of these extended lines. We shall write $x = (y, x_k)$ and let y vary in $Y \equiv \prod_{j\neq k}(\alpha_j, \beta_j)$. Since u_k vanishes on $(x_k = \alpha_k)$, we can write

$$u_k(y, x_k) = \int_{\alpha_k}^{x_k} \frac{\partial u_k}{\partial x_k}(y, t_k)\, dt_k,$$

so

$$u_k(y, x_k)^2 \le C \int_{\alpha_k}^{x_k} \left[\frac{\partial u_k}{\partial x_k}(y, t_k)\right]^2 dt_k \le C \int_{\alpha_k}^{\beta_k} |\nabla u(y, t_k)|^2\, dt_k.$$

As a consequence,

$$\int_Y u_k(y, x_k)^2\, dy \le C \int_{Y\times(\alpha_k,\beta_k)} |\nabla u|^2\, dx.$$

In particular,

$$\int_{Y\times(a_k,b_k)} u_k(y, x_k)^2\, dy\, dt_k \le C \int_{Y\times(\alpha_k,\beta_k)} |\nabla u|^2\, dx.$$

Since everything is smooth and the change of variables from the Euclidean coordinates to the local system of coordinates constructed for V_i is nondegenerate, this implies

$$\int_{A_i} u_k^2\, dx \le C \int_{\overline{A_i}^{(k)}} |\nabla u|^2\, dx,$$

and this concludes the proof of (85). □

V. Damping of hydrodynamic oscillations

In this short section, we establish the bounds (52). We shall denote by C or C_ε various constants which depend only on the constants in the assumptions of Theorem 2. Throughout the section, f will be a smooth solution of the Boltzmann equation and ρ, u, T will be the associated hydrodynamic fields; we shall not write down explicitly the time dependence.

• First, since $\partial_t \rho = -\nabla \cdot (\rho u)$ and $\rho u = \int_{\mathbb{R}^N} f\, v\, dv = \int_{\mathbb{R}^N} (f - M)\, v\, dv$, we can write

$$\frac{d}{dt}\int_{\Omega_x} \rho \log \rho = -\int_{\Omega_x} \log \rho \nabla_x \cdot \left(\int_{\mathbb{R}^N} (f - M)\, v\, dv \right) dx$$
$$\leq \| \log \rho \|_{L^1(\Omega_x)} \left\| \nabla_x \cdot \int_{\mathbb{R}^N} (f - M)\, v\, dv \right\|_{L^\infty(\Omega_x)}.$$

Using easy interpolation inequalities in weighted Sobolev spaces and Hölder's inequality, it is not difficult to prove

$$\left\| \nabla_x \cdot \int_{\mathbb{R}^N} (f - M)\, v\, dv \right\|_{L^\infty(\Omega_x)} \leq C_\varepsilon \| f - M \|^{1-2\varepsilon}_{L^1(\Omega_x \times \mathbb{R}^N)}.$$

Combining this with the Csiszár-Kullback-Pinsker inequality, we conclude that

$$\left\| \nabla_x \cdot \int_{\mathbb{R}^N} (f - M)\, v\, dv \right\|_{L^\infty(\Omega_x)} \leq C_\varepsilon H(f|M)^{\frac{1}{2}-\varepsilon}. \tag{86}$$

Now, since ρ is bounded from above and below, we can write

$$\int_{\Omega_x} |\log \rho|\, dx \leq C \int_{\Omega_x} |\rho - 1|\, dx \leq C \left(\int (\rho - 1)^2\, dx \right)^{1/2} \tag{87}$$
$$\leq C \left[\int (\rho \log \rho - \rho + 1) \right]^{1/2} = C \mathcal{H}(\rho|1)^{1/2} \leq C H(f|M)^{1/2}.$$

Of course the combination of (86) and (87) implies

$$\left| \frac{d}{dt} \int \rho \log \rho \right| \leq C_\varepsilon H(f|M)^{1-\varepsilon}.$$

• Next, since

$$\partial_t(\rho u) = -\nabla_x \cdot \left(\int f\, v \otimes v\, dv \right) = -\nabla_x \cdot \left(\int (f - M)\, v \otimes v\, dv \right),$$

we can write

$$\left| \frac{d}{dt} \int_{\Omega_x} \rho |u|^2\, dx \right| = \left| \frac{d}{dt} \int_{\Omega_x} \frac{|\rho u|^2}{\rho}\, dx \right|$$
$$= \left| \int_{\Omega_x} \frac{|\rho u|^2}{\rho^2} \nabla_x \cdot (\rho u) - 2 \int_{\Omega_x} \left\langle \frac{\rho u}{\rho}, \nabla_x \cdot \int_{\mathbb{R}^N} (f - M)\, v \otimes v\, dv \right\rangle \right|.$$

Using the fact that ρ, u are smooth and ρ is bounded from above and below, we can bound the above expression by

$$C\int \rho|u|^2 + C\left(\int_{\Omega_x}\rho|u|^2\right)^{1/2}\left\|\nabla_x\cdot\int_{\mathbb{R}^N}(f-M)\,v\otimes v\,dv\right\|_{L^2(\Omega_x)};$$

and just as before we can bound this by

$$C_\varepsilon\left(\int_{\Omega_x}\rho|u|^2 + \|f-M\|^{2(1-\varepsilon)}_{L^1(\Omega_x\times\mathbb{R}^N)}\right)\le C_\varepsilon H(f|M)^{1-\varepsilon}.$$

- Next, by the conservation laws (10),

$$\frac{d}{dt}\int\rho T = -\frac{1}{N}\frac{d}{dt}\int\rho|u|^2$$

which easily leads to

$$\left|\frac{d}{dt}\langle T\rangle_\rho\right|\le C_\varepsilon H(f|M)^{1-\varepsilon}.$$

Since ρ, T are bounded from above and below, we deduce that

$$\left|\frac{d}{dt}\Phi(\langle T\rangle_\rho)\right|\le C_\varepsilon H(f|M)^{1-\varepsilon},$$

where $\Phi(X)=X-\log X-1$.

- Finally, to get an estimate on $(d/dt)\int\rho(T-\log T-1)$, it only remains to bound $(d/dt)\int\rho\log T$. From the hydrodynamical equations in Subsect. III.1,

$$\begin{aligned}\frac{d}{dt}\int\rho\log T &= \int\partial_t\rho\,\log T + \int\rho\frac{\partial_t T}{T}\\
&= \int\partial_t\rho\,\log T + \int\frac{1}{T}\left[\partial_t\left(\rho T+\frac{\rho|u|^2}{N}\right)\right]\\
&\qquad\qquad -\int\frac{1}{T}\,\partial_t\left(\frac{\rho|u|^2}{N}\right)-\int\partial_t\rho\\
&= \int\partial_t\rho\,\log T + \int\left(\frac{1}{T}-1\right)\partial_t\left(\rho T+\frac{\rho|u|^2}{N}\right)\\
&\qquad\qquad -\frac{1}{N}\int\left(\frac{1}{T}-1\right)\partial_t\left(\frac{|\rho u|^2}{\rho}\right)\\
&\qquad +\int\partial_t\left(\rho T+\frac{\rho|u|^2}{N}\right)-\frac{1}{N}\int\partial_t\left(\frac{|\rho u|^2}{\rho}\right)\end{aligned}$$

$$(88)\qquad = \int \partial_t \rho \, \log T + \int \left(\frac{1}{T} - 1 \right) \partial_t \left(\rho T + \frac{\rho |u|^2}{N} \right) - \frac{1}{N} \int \left(\frac{1}{T} - 1 \right) \partial_t \left(\frac{|\rho u|^2}{\rho} \right) - \frac{1}{N} \frac{d}{dt} \int \rho |u|^2.$$

The last term in (88) has already been treated. As for the first three, they can be handled by the formulas

$$| \log T| \leq C|T - 1|,$$

$$T - 1 = \frac{1}{N\rho} \int_{\mathbb{R}^N} (f - M)|v|^2 \, dv + \frac{1}{\rho} \left(\int_{\mathbb{R}^N} (f - M) \, dv \right) - \frac{1}{N\rho^2} \left(\int_{\mathbb{R}^N} (f - M) \, v \, dv \right)^2,$$

$$\partial_t \rho = -\nabla \cdot (\rho u) = -\nabla_x \cdot \int_{\mathbb{R}^N} (f - M) \, v \, dv$$

$$\partial_t \left(\rho T + \frac{\rho |u|^2}{N} \right) = -\nabla_x \cdot \int_{\mathbb{R}^N} (f - M)|v|^2 v \, dv,$$

$$\partial_t (\rho u) = -\nabla_x \cdot \int_{\mathbb{R}^N} (f - M) \, v \otimes v \, dv.$$

These identities ensure that all the integrands in (88) are formally of order at least 2 in $f - M$, and that these terms can be handled in exactly the same way as before. This concludes the proof of (52).

VI. Average entropy production bounds via differential inequalities

In this section, we analyze the system of inequalities presented in Sect. II.4 and derived in Sects. III to V. We shall demonstrate that this system implies convergence of $H(f_t|M)$ to 0 as $t \to \infty$, and more precisely

$$H(f_t|M) = O(t^{-\infty})$$

with explicit constants.

Our proof is horribly technical but the strategy is systematic and can be explained in a rather simple way. We shall first proceed to this informal explanation. We shall sometimes use the notation $h|_t = h(t)$, or even $h|_{t=t_0} = h(t_0)$, for functions h depending on time.

VI.1. Strategy. Let $t_0 > 0$ be arbitrary, and let

$$\alpha_0 = H(f|M)|_{t=t_0}.$$

We wish to find an upper bound on a duration T_0 such that

$$H(f|M)|_{t=t_0+T_0} = \lambda \alpha_0,$$

where $\lambda \in (0, 1)$ is fixed once for all, according to technical convenience, say

$$\lambda = \frac{4}{5}.$$

Of course the finiteness of T_0 is part of the estimate. We shall show that if ε is small enough, say $\varepsilon < 1/100$ for simplicity (this bound can be considerably relaxed if one is extremely courageous), then

$$T_0 \leq C_0\, \alpha^{-226\varepsilon}, \tag{89}$$

where the constant C_0 only depends on ε and on the various constants appearing in the system of Subsect. II.4. Once (89) is proven, a standard argument (as in [21, p. 36] for instance) leads to

$$H(f_t|M) = O(t^{-\frac{1}{227\varepsilon}}).$$

Since ε is as small as desired, this implies the conclusion.

We did not at all try to optimize the factor multiplying ε in the exponent of (89), and we shall not worry about the dependence of the constant C_0 with respect to ε; without any doubt there is room for improvement here. In all the sequel we shall consider ε as fixed and smaller than 0.01.

Let us proceed on our way towards (89). First of all, $H(f|M)$ is of the order α_0 throughout the whole time interval $[t_0, t_0 + T_0]$, in the sense that

$$\frac{4}{5}\alpha_0 \leq H(f_t|M) \leq \alpha_0. \tag{90}$$

Recall from (46) that

$$H(f|M) = \mathcal{H}(\rho|1) + \mathcal{H}(u|0) + \mathcal{H}(\langle T\rangle_\rho|1) + \mathcal{H}(T|\langle T\rangle_\rho) + H\big(f|M^f_{\rho u T}\big). \tag{91}$$

Only when the last of these terms, the "kinetic H functional" $H(f|M^f_{\rho u T})$, is large enough, can we use the quantitative H theorem of Subsect. II.1 to get an upper bound on the rate of entropy production. Here "large" would mean, for instance, "greater than a given fraction of α_0". As we already explained, since we are unable to prove that $H(f|M^f_{\rho u T})$ is always large (and since this may be false in certain situations), we shall only show that *on the average* $H(f|M^f_{\rho u T})$ is large enough, and this will lead to a bound on T_0. For this we shall divide and subdivide the time-interval $[t_0, t_0 + T_0]$ into various subintervals, according to the respective sizes of the various objects making up the right-hand side of (91). Here we just give an informal construction of this subdivision; in the next subsection we shall give precise definitions.

- We first divide $[t_0, t_0 + T_0]$ into a family of subintervals $I_1, \dots, I_\ell$ (a priori the family may be infinite; but it will result from the estimates to come that it is actually finite). These subintervals fall into two separate classes, denoted "G" or "B" (think of G as "good", B as "bad", in the sense of high or low kinetic entropy, respectively). By definition, $H(f|M^f_{\rho u T})$ is "large" on each interval of class G, and "small" on each interval of class B. The intervals are denoted symbolically by I^B or I^G, depending on their class. Any two consecutive intervals will be of different types.
- Then we subdivide again each interval I^B into a family of subintervals, denoted by I^{BG} and I^{BB}. Essentially, the intervals I^{BG} will be those on which ($H(f|M^f_{\rho u T})$ is small but) $\mathcal{H}(T|\langle T\rangle_\rho)$ is large; and the intervals I^{BB} will constitute the rest of I^B.
- Then we subdivide again each interval I^{BB} into a family of subintervals, denoted by I^{BBG} and I^{BBB}. Now the intervals I^{BBG} will be those on which $\mathcal{H}(u|0)$ is large; and the intervals I^{BBB} will constitute the rest.

Inequality (52) will be used at this point to make sure that we can define these various subdivisions in a convenient way.

To get information on the average entropy production on these various time intervals, we shall use repeatedly Lemma 6.1 from [21], which we state below in a slightly more explicit form than in [21]:

Lemma 12. *Let $h = h(t) \geq 0$ be a C^2 function of $t \in [t_1, t_2]$ such that*

$$\forall t \in (t_1, t_2), \qquad h''(t) + A\, h(t)^{1-\varepsilon} \geq \alpha,$$

where A, α are positive constants and $\varepsilon \in (0, 1/10)$. Then,

- *either $t_2 - t_1$ is small,*

$$t_2 - t_1 \leq 50\, \frac{\alpha^{\frac{\varepsilon}{2(1-\varepsilon)}}}{A^{\frac{1}{2(1-\varepsilon)}}};$$

- *or h is large on the average,*

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(t)\, dt \geq \frac{\alpha^{\frac{1}{1-\varepsilon}}}{100} \inf\left(\frac{1}{A}, \frac{1}{A^{\frac{3}{2}+3\varepsilon}}\right).$$

We use the following conventions: whenever $I = [t_1, t_2]$ we shall write

$$|I| = t_2 - t_1, \qquad \langle u\rangle_I = \frac{1}{|I|} \int_{t_1}^{t_2} u(t)\, dt;$$

in particular $\langle -\dot{H}\rangle_I$ will stand for the average entropy production over I. Our goal is to estimate $\langle -\dot{H}\rangle_{[t_0, t_0+T_0]}$ from below, and for this we shall proceed in the following order:

a) Type G: If $I = I^G$ is some interval of type G, then $H(f|M^f_{\rho u T})$ is large on I, and we shall obtain an estimate on $\langle -\dot{H}\rangle_I$ thanks to the H Theorem of Subsect. II.1.

b) Type BG: If $I = I^{BG}$ is some interval of type BG, then $\mathcal{H}(T|\langle T\rangle_\rho)$ is large over I, so we can use inequalities (43) and (49), in conjunction with Lemma 12, to find an estimate on $\langle H(f|M^f_{\rho u T})\rangle_I$, provided that $|I|$ is not too small. This will imply an estimate on $\langle -\dot{H}\rangle_I$ by means of the H Theorem – again, if $|I|$ is not too small, a case which we shall rule out if the length of the "ambient" interval I^B is itself not too small.

c) Type BBG: If $I = I^{BBG}$ is some interval of type BBG, then $\mathcal{H}(u|0)$ is large over I, so we can use inequalities (44) and (50), in conjunction with Lemma 12, to find an estimate on $\langle H(f|M^f_{\rho u T})\rangle_I$. Since $\mathcal{H}(T|\langle T\rangle_\rho)$ is small, we know that $H(f|M^f_{\rho u T}) \simeq H(f|M^f_{\rho u \langle T\rangle})$, and we obtain an estimate on $\langle H(f|M^f_{\rho u T})\rangle_I$, and therefore on $\langle -\dot{H}\rangle_I$ by means of the H Theorem again. This will work if the length of I^{BB} is not too small.

d) Type BBB: If $I = I^{BBG}$ is some interval of type BBB, then $H(f|M^f_{\rho u T})$, $\mathcal{H}(T|\langle T\rangle_\rho)$ and $\mathcal{H}(u|0)$ are quite small on I, and therefore also $\mathcal{H}(\langle T\rangle_\rho|1)$ because of the conservation laws (10). So from (90) and (46) we know that $\mathcal{H}(\rho|1)$ is large. Then we can use inequalities (45) and (51), in conjunction with Lemma 12, to find an estimate on $\langle H(f|M^f_{\rho 0 1})\rangle_I$. Since $\mathcal{H}(u|0)$ and $\mathcal{H}(T|1)$ are very small, we know that $\mathcal{H}(f|M^f_{\rho 0 1}) \simeq \mathcal{H}(f|M^f_{\rho u T})$, and then we conclude as in case c), provided that $|I|$ is not too small.

e) Type BB: Any interval $I = I^{BB}$ is a union of intervals I^{BBG} and I^{BBB}. If $|I|$ is not too small, then we have, from the studies in c) and d), lower bounds on $\langle -\dot{H}\rangle_I^{BBG}$ and $\langle -\dot{H}\rangle_I^{BBB}$, except for those intervals I^{BBB} with too short length. But we will also have lower bounds on the length of (most) intervals I^{BBG}, and thus the intervals I^{BBB} with very short length will not affect too much the average.

f) Type B: For those intervals we shall use the results of b), e), and a reasoning similar to that in e).

g) The whole interval $[t_0, t_0 + T_0]$: To find an estimate on $\langle -\dot{H}\rangle_{[t_0,t_0+T_0]}$ we shall combine the results of a), g) and again a reasoning similar to that in e).

The technical point in the implementation of this scheme will be to correctly quantify the respective sizes ("small", "large") of the various quantities involved.

VI.2. The subdivision. Let us now proceed to the precise definition of the multiple subdivision of $[t_0, t_0 + T_0]$. Along our way, we shall estimate the sizes of the various H functionals, and some lower bounds on the sizes of the intervals. All functions ρ, u, T will implicitly depend on t.

• As explained before, we split $[t_0, t_0 + T_0]$ into subintervals $I_1, I_2, \ldots$, each of them being of type B or G, and denoted I^B or I^G. An interval I^B is defined as a maximal interval satisfying

$$(92) \quad \forall t \in I^B, \quad H\big(f|M^f_{\rho u T}\big) \le \frac{\alpha_0}{2}; \qquad \exists t \in I^B, \quad H\big(f|M^f_{\rho u T}\big) < \frac{\alpha_0}{10}.$$

The intervals I^G are those which separate two consecutive intervals I^B; they have the property

$$(93) \qquad \forall t \in I^G, \quad H\big(f|M^f_{\rho u T}\big) \ge \frac{\alpha_0}{10}.$$

See Fig. 3 for an illustration.

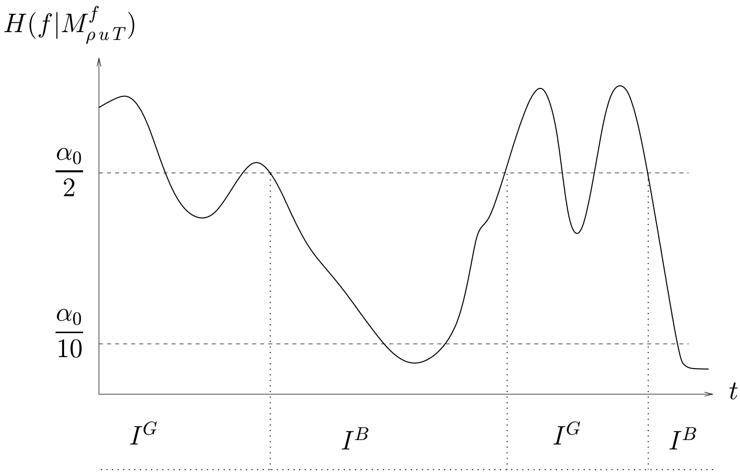

Fig. 3. Definition of I^B and I^G intervals

Due to our lack of control of the high-velocity contribution to rapid variations of $H(f|M^f_{\rho u T})$, we do not have a priori bounds on the size of these intervals.

• Next, we subdivide each interval $I^B = I_j$ again into subintervals I_{j1}, $I_{j2}, \ldots$, being successively of type BB and of type BG. By definition, the intervals I^{BB} are maximal subintervals of I^B such that

$$(94) \quad \forall s \in I^{BB}, \quad \mathrm{dist}\left(s, \left\{t \in I^B;\ \mathcal{H}(T|\langle T\rangle_\rho) \ge \eta_1 \times \frac{3\alpha_0}{10}\right\}\right) \ge \frac{\eta_1 \alpha_0^\varepsilon}{10 C_S},$$

where C_S is defined by (52) and η_1 is a small parameter, depending on α_0, that will be chosen later on, say in $(0, 1/10)$. Here we have used the natural notation $\mathrm{dist}(s, A) = \inf\{|s-t|;\ t \in A\}$, with the convention $\mathrm{dist}(s, \emptyset) = +\infty$. Finally, the subintervals I^{BG} are those which separate two consecutive subintervals I^{BB}. See Fig. 4 for an illustration.

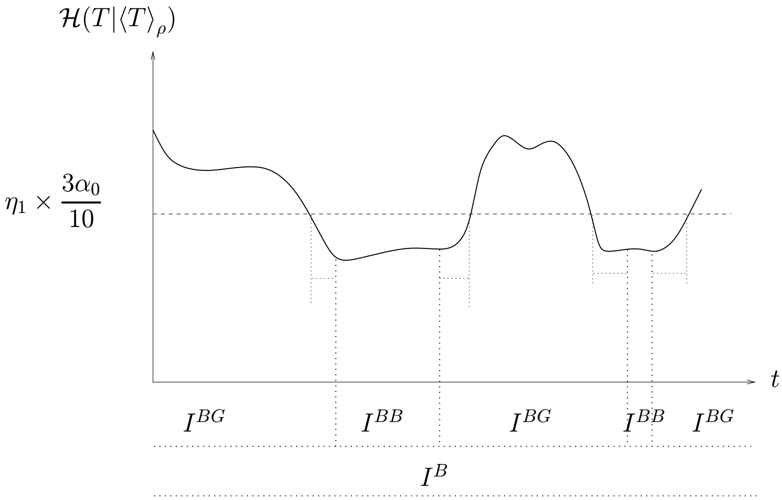

Fig. 4. Definition of I^{BB} and I^{BG} intervals

From the definition of I^{BB}, we have of course

(95) $$\forall t \in I^{BB}, \quad \mathcal{H}(T|\langle T\rangle_\rho) < \eta_1 \times \frac{3\alpha_0}{10}.$$

Moreover, from (52) and (91) we know that

(96)
$$\forall t \in I^{BG}, \quad \mathcal{H}(T|\langle T\rangle_\rho) \geq \eta_1 \times \frac{3\alpha_0}{10} - C_S \alpha_0^{1-\varepsilon} \times \eta_1 \frac{\alpha_0^\varepsilon}{10 C_S} = \eta_1 \times \frac{\alpha_0}{5}.$$

Finally, it follows from (94) that, for all interval I^{BG} *with the possible exception of at most two of them* (the "extreme" ones),

(97) $$|I^{BG}| \geq 2 \times \frac{\eta_1 \alpha_0^\varepsilon}{10 C_S} = \frac{\eta_1 \alpha_0^\varepsilon}{5 C_S}.$$

• Finally, we subdivide once again each interval $I^{BB} = I_{jk}$ into subintervals $I_{jk1}, I_{jk2}, \ldots,$ of type either BBG or BBB. By definition, the intervals I^{BBB} are the maximal intervals satisfying

(98) $$\forall s \in I^{BBB}, \quad \mathrm{dist}\left(s, \left\{t \in I^{BB};\ \mathcal{H}(u|0) \geq \eta_2 \times \frac{3\alpha_0}{10}\right\}\right) \geq \frac{\eta_2 \alpha_0^\varepsilon}{10 C_S},$$

where η_2 will be chosen later in $(0, 1/10)$, depending on α_0. The subintervals I^{BBG} will be those which separate two consecutive intervals of type BBB. Just as before, we have

$$\forall t \in I^{BBB}, \quad \mathcal{H}(u|0) < \eta_2 \times \frac{3\alpha_0}{10}, \tag{99}$$

$$\forall t \in I^{BBG}, \mathcal{H}(u|0) \geq \eta_2 \times \frac{\alpha_0}{5}, \tag{100}$$

and, for all I^{BBG} with at most two exceptions,

$$|I^{BBG}| \geq \eta_2 \times \frac{\alpha_0^{\varepsilon}}{5C_S}. \tag{101}$$

Remark. If we had more local conservation laws to handle, we could continue with definitions similar to (94) or (98). One reason why we used a different type of definition for the first subdivision is that we do not have the same kind of control on the variations of kinetic entropy, as we have on the variations of the hydrodynamic part of the entropy, as appearing in (52).

VI.3. Estimates of average entropy production. We now proceed to establish lower bounds on the average entropy production, according to the plan sketched in Subsect. VI.1. Throughout the proof, K_1', K_1'', K_2', $C_1^{(3)}$, etc. will stand for constants only depending on the constants appearing in the system of Subsect. II.4, and possibly on an upper bound H_0 of $H(f)$ at time 0 (which of course implies an upper bound on $H(f_0|M)$). All these constants do a priori depend on ε.

a) Type G: Let $I = I^G$ be some interval of type G. From (42) and (92) we know that

$$\forall t \in I \qquad -\frac{d}{dt}H(f|M) \geq K_H \left(\frac{\alpha_0}{10}\right)^{1+\varepsilon}.$$

And of course, the average entropy production over I satisfies

$$\langle -\dot{H} \rangle_I \geq \left(\frac{K_H}{10^{1+\varepsilon}}\right) \alpha_0^{1+\varepsilon}.$$

b) Type BG: Let $I = I^{BG}$ be some interval of type BG. From (43), (49) and (96) we have

$$\forall t \in I \quad \frac{d^2}{dt^2}\left\| f - M^f_{\rho u T}\right\|^2_{L^2} + \frac{C_1}{\delta_1^{1-\varepsilon}}\left(\left\| f - M^f_{\rho u T}\right\|^2_{L^2}\right)^{1-\varepsilon}$$
$$\geq \frac{K_1'\eta_1}{5}\alpha_0 - \delta_1 H(f|M) \geq \frac{K_1'\eta_1}{5}\alpha_0 - \frac{4}{5}\delta_1\alpha_0,$$

where we used (90) and we set $K_1' = K_1 \cdot K_T$, K_T being the Poincaré constant appearing in (49).

We now set $\delta_1 = \eta_1 K_1'/8$, so

$$\forall t \in I \qquad \frac{d^2}{dt^2}\|f - M^f_{\rho u T}\|^2_{L^2} + \frac{C_1'}{\eta_1^{1-\varepsilon}}\left(\|f - M^f_{\rho u T}\|^2_{L^2}\right)^{1-\varepsilon} \geq \frac{K_1'}{10}\eta_1\alpha_0,$$

where $C_1' = C_1(8/K_1')^{1-\varepsilon}$.

Applying Lemma 12 with $h(t) = \|f - M^f_{\rho u T}\|^2_{L^2}$, $\alpha = K_1'\eta_1\alpha_0/10$ and $A = A_1 C_1'/\eta_1^{1-\varepsilon}$, where $A_1 \geq \max(1, \eta_1^{1-\varepsilon}/C_1')$ will be chosen later, we get

(102)

$$\text{either} \quad |I| \leq 50\left(\frac{K_1'\eta_1\alpha_0}{10}\right)^{\frac{\varepsilon}{2(1-\varepsilon)}}\left(\frac{\eta_1^{1-\varepsilon}}{A_1 C_1'}\right)^{\frac{1}{2(1-\varepsilon)}} \leq C_1''\eta_1^{\frac{1}{2(1-\varepsilon)}} A_1^{-\frac{1}{2(1-\varepsilon)}}\alpha_0^{\frac{\varepsilon}{2(1-\varepsilon)}},$$

$$\text{or} \quad \langle\|f - M^f_{\rho u T}\|^2_{L^2}\rangle_I \geq \frac{1}{100}\left(\frac{K_1'\eta_1\alpha_0}{10}\right)^{\frac{1}{1-\varepsilon}} A_1^{-\left(\frac{3}{2}+3\varepsilon\right)}\left(\frac{\eta_1^{1-\varepsilon}}{C_1'}\right)^{\frac{3}{2}+3\varepsilon}$$

(103)
$$\geq K_1''\eta_1^{2.55} A_1^{-\left(\frac{3}{2}+3\varepsilon\right)}\alpha_0^{\frac{1}{1-\varepsilon}}.$$

Let us now assume that I satisfies (97), and let us choose A_1 large enough to make sure that the eventuality (102) does not happen: for this it suffices that

$$\frac{\eta_1\alpha_0^\varepsilon}{5C_S} > C_1''\eta_1^{\frac{1}{2(1-\varepsilon)}} A_1^{-\frac{1}{2(1-\varepsilon)}}\alpha_0^{\frac{\varepsilon}{2(1-\varepsilon)}},$$

so we choose

$$A_1 = \left(\frac{6C_S C_1''\eta_1^{\frac{1}{2(1-\varepsilon)}}\alpha_0^{\frac{\varepsilon}{2(1-\varepsilon)}}}{\eta_1\alpha_0^\varepsilon}\right)^{2(1-\varepsilon)} \equiv \frac{C_1^{(3)}\eta_1^{-1+2\varepsilon}}{\alpha_0^{2\varepsilon(\frac{1}{2}-\varepsilon)}}.$$

Plugging this back into (103), we obtain

$$\left\langle\|f - M^f_{\rho u T}\|^2_{L^2}\right\rangle_I \geq K_1^{(3)}\eta_1^{4.1}\alpha_0^{1+2.55\varepsilon}.$$

By using successively (41), the first equation in (47) and Jensen's inequality, we deduce

$$\begin{aligned}
\langle -\dot{H}\rangle_I &\geq K_H\left\langle H\left(f|M^f_{\rho u T}\right)^{1+\varepsilon}\right\rangle_I \\
&\geq K_H \cdot K_I\left\langle\left\|f - M^f_{\rho u T}\right\|^{2(1+\varepsilon)^2}_{L^2}\right\rangle_I \\
&\geq K_H \cdot K_I\left\langle\left\|f - M^f_{\rho u T}\right\|^2_{L^2}\right\rangle_I^{(1+\varepsilon)^2} \\
&\geq K_1^{(4)}\eta_1^{4.3}\alpha_0^{1+6\varepsilon}.
\end{aligned}$$

At this stage, we did not yet choose η_1.

c) Type BBG: Let $I = I^{BBG}$ be some interval of type BBG. From (44), (48), (50), (90), (95) and (100) we have

(104)
$$\frac{d^2}{dt^2}\|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2} + \frac{C_2}{\delta_2^{1-\varepsilon}}\big(\|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}\big)^{1-\varepsilon} \geq K_2'\eta_2\frac{\alpha_0}{5} - \delta_2 \times \frac{4\alpha_0}{5},$$

where $K_2' = K_2 \cdot K_K \times K_u$. We now impose

$$\delta_2 = \frac{K_2'\eta_2}{8}, \tag{105}$$

so that (104) becomes

$$\frac{d^2}{dt^2}\|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2} + \frac{C_2}{\eta^{1-\varepsilon}}\big(\|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}\big)^{1-\varepsilon} \geq K_2''\eta_2\alpha_0.$$

By a reasoning exactly similar to the one in b), we find that, if I satisfies (101), then

$$\big\langle \|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}\big\rangle_I \geq K_2^{(3)}\eta_2^{4.08}\alpha_0^{1+2.55\varepsilon}.$$

Hence, by Jensen's inequality and the second interpolation inequality in (47),

$$\begin{aligned}\big\langle H\big(f|M^f_{\rho\, u\, \langle T\rangle}\big)\big\rangle_I &\geq K_I\big\langle \|f - M^f_{\rho\, u\, \langle T\rangle}\|^{2(1+\varepsilon)}_{L^2}\big\rangle_I \\ &\geq K_I\big\langle \|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}\big\rangle_I^{1+\varepsilon} \geq K_2^{(4)}\eta_2^{4.13}\alpha_0^{1+4\varepsilon}.\end{aligned} \tag{106}$$

But, by (46) we know that

$$H\big(f|M^f_{\rho\, u\, T}\big) = H\big(f|M^f_{\rho\, u\, \langle T\rangle}\big) - \mathcal{H}(T|\langle T\rangle_\rho) \geq H\big(f|M^f_{\rho\, u\, \langle T\rangle}\big) - \frac{3\alpha_0\eta_1}{10}.$$

Taking mean values and using (106), we find

$$\big\langle H\big(f|M^f_{\rho\, u\, T}\big)\big\rangle_I \geq K_2^{(4)}\eta_2^{4.13}\alpha_0^{1+4\varepsilon} - \frac{3\alpha_0\eta_1}{10}.$$

We now impose

$$\eta_1 \leq \frac{5K_2^{(4)}\eta_2^{4.13}\min(\alpha_0^{4\varepsilon}, 1)}{3} \tag{107}$$

to obtain

$$\big\langle H\big(f|M^f_{\rho\, u\, T}\big)\big\rangle_I \geq \frac{K_2^{(4)}}{2}\eta_2^{4.13}\alpha_0^{1+4\varepsilon}.$$

Reasoning as in b), we conclude that

$$\langle -\dot{H}\rangle_I \geq K_2^{(5)}\eta_2^{4.26}\alpha_0^{1+7\varepsilon}.$$

d) Type BBB: Let $I = I^{BBB}$ be some interval of type BBB. From (92), (94) and (98), we have

$$H\big(f|M^f_{\rho u T}\big) \le \frac{\alpha_0}{2}, \quad \mathcal{H}(f|M) \ge \frac{4\alpha_0}{5},$$
$$\mathcal{H}(T|\langle T\rangle_\rho) \le \frac{3\eta_1\alpha_0}{10}, \quad \mathcal{H}(u|0) \le \frac{3\eta_2\alpha_0}{10}.$$

The conservation of energy in (10) implies

$$\langle T\rangle_\rho - 1 = -\frac{2}{N}\mathcal{H}(u|0) = -\frac{3\eta_2\alpha_0}{5N},$$

therefore, if η_2 is chosen small enough (depending only on the upper and lower bounds on T), we can find some small constant η' such that

$$\mathcal{H}(\langle T\rangle_\rho|1) \le \frac{3\eta'\alpha_0}{10}.$$

We assume that $\eta' < 1/10$, then the additivity implies

$$\mathcal{H}(\rho|1) \ge \alpha_0\left(\frac{4}{5} - \frac{1}{2} - \frac{3}{10}(\eta_1 + \eta_2 + \eta')\right) \ge \frac{\alpha_0}{5}.$$

From (45), (51) and (90) we obtain

$$\forall t \in I \quad \frac{d^2}{dt^2}\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2} + \frac{C_3}{\delta_3^{1-\varepsilon}}\big(\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2}\big)^{1-\varepsilon} \ge \frac{K_3'}{5}\alpha_0 - \frac{4}{5}\delta_3\,\alpha_0,$$

where $K_3' = K_3 \cdot K_\rho$. We now set $\delta_3 = K_3'/8$, and find

$$\forall t \in I \qquad \frac{d^2}{dt^2}\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2} + C_3'\big(\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2}\big)^{1-\varepsilon} \ge K_3''\alpha_0.$$

Reasoning as before, we see that for any $A_3 \ge \max(1, 1/C_3')$,

(108) $$\text{either} \quad |I| \le C_3'' A_3^{-\frac{1}{2(1-\varepsilon)}}\alpha_0^{\frac{\varepsilon}{2(1-\varepsilon)}},$$

(109) $$\text{or} \quad \big\langle\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2}\big\rangle_I \ge K_3'' A_3^{-\left(\frac{3}{2}+3\varepsilon\right)}\alpha_0^{\frac{1}{1-\varepsilon}}.$$

Since we have no lower bound on $|I|$, we cannot a priori exclude (108). However, we will make sure that if it ever occurs, then $|I| = O(\alpha_0^{2\varepsilon})$. For this we set

$$A_3 = \max\left\{\left[C_3''\alpha_0^{\varepsilon\left(\frac{1}{2(1-\varepsilon)}-2\right)}\right]^{2(1-\varepsilon)}, 1\right\}$$

and we conclude after a little bit of computation that either $|I| \le \alpha_0^{2\varepsilon}$, or

(110) $$\big\langle\big\|f - M^f_{\rho 0 1}\big\|^2_{L^2}\big\rangle_I \ge K_3^{(3)}\alpha_0^{1+9\varepsilon}.$$

Let us consider the case when (110) is satisfied. Then, by Jensen's inequality and the third line of (47),

$$\begin{aligned}\big\langle H\big(f|M^f_{\rho 0 1}\big)\big\rangle_I &\geq K_I\big\langle \big\| f - M^f_{\rho 0 1}\big\|^{2(1+\varepsilon)}_{L^2}\big\rangle_I \\ &\geq K_I\big\langle \big\| f - M^f_{\rho 0 1}\big\|^2_{L^2}\big\rangle_I^{1+\varepsilon} \geq K_3^{(4)}\alpha_0^{1+11\varepsilon}.\end{aligned}$$

Then,

$$\begin{aligned}H\big(f|M^f_{\rho u T}\big) &= H\big(f|M^f_{\rho 0 1}\big) - \mathcal{H}(T|\langle T\rangle_\rho) - \mathcal{H}(\langle T\rangle_\rho|1) - \mathcal{H}(u|0) \\ &\geq H\big(f|M^f_{\rho 0 1}\big) - (\eta_1+\eta_2+\eta')\frac{3\alpha_0}{10},\end{aligned}$$

so

$$\big\langle H\big(f|M^f_{\rho u T}\big)\big\rangle_I \geq K_3^{(4)}\alpha_0^{1+11\varepsilon} - (\eta_1+\eta_2+\eta')\frac{3\alpha_0}{10}.$$

We now require

$$\eta' \leq \eta_1 \leq \eta_2 \leq \frac{5}{6}K_3^{(4)}\alpha_0^{11\varepsilon}, \tag{111}$$

and conclude that

$$\big\langle H\big(f|M^f_{\rho u T}\big)\big\rangle_I \geq \frac{K_3^{(4)}}{2}\alpha_0^{1+11\varepsilon}.$$

Reasoning as before, we deduce

$$\langle -\dot{H}\rangle_I \geq K_3^{(5)}\alpha_0^{1+14\varepsilon}.$$

We can now choose the values of η_1 and η_2 according to (105), (107) and (111): we can ensure that these constraints are satisfied and at the same time take η_1 to be a constant multiple of $\alpha_0^{50\varepsilon}$ and η_2 a constant multiple of $\alpha_0^{11\varepsilon}$. This determines our lower bounds for cases a) and b).

Let us pause here to recapitulate: we have shown that whenever I is

- of type G, then $\langle -\dot{H}\rangle_I \geq K_0^{(6)}\alpha_0^{1+\varepsilon}$;
- of type BG, then $\langle -\dot{H}\rangle_I \geq K_1^{(6)}\alpha_0^{1+226\varepsilon}$ if $|I|$ satisfies (97);
- of type BBG, then $\langle -\dot{H}\rangle_I \geq K_2^{(6)}\alpha_0^{1+54\varepsilon}$ if $|I|$ satisfies (101);
- of type BBB, then $\langle -\dot{H}\rangle_I \geq K_3^{(6)}\alpha_0^{1+14\varepsilon}$, or $|I| \leq \alpha_0^{2\varepsilon}$.

We now resume our initial plan.

d) Type BB: Let $I = I^{BB}$ be some interval of type BB. It is made of intervals I^{BBG} and I^{BBB}. If we discard at most three of these intervals (located "at the edge" of I^{BB}), then we can pair each I^{BBB} with some I^{BBG} satisfying (101). Let be some pair (I^{BBB}, I^{BBG}), then

– either $\langle -\dot{H}\rangle_I^{BBB} \geq K_3^{(6)} \alpha_0^{1+14\varepsilon}$ and therefore

(112) $$\langle -\dot{H}\rangle_{I^{BBG}\cup I^{BBB}} \geq \inf\big[K_2^{(6)}, K_3^{(6)}\big] \inf\big(\alpha_0^{1+54\varepsilon}, \alpha_0^{1+14\varepsilon}\big),$$

– or $|I^{BBB}| \leq \alpha_0^{2\varepsilon}$ and $|I^{BBG}| \geq C^{(7)}\alpha_0^{12\varepsilon}$, (here we use (101)) so

$$\langle -\dot{H}\rangle_{I^{BBG}\cup I^{BBB}} \geq \langle -\dot{H}\rangle_I^{BBG} \frac{|I^{BBG}|}{|I^{BBB}| + |I^{BBG}|} \geq K_2^{(6)}\alpha_0^{1+54\varepsilon} \frac{1}{1 + \frac{C^{(7)}\alpha_0^{2\varepsilon}}{\alpha_0^{12\varepsilon}}}$$

$$\geq K_2^{(7)}\alpha_0^{1+64\varepsilon}.$$

Thus, on the complement of the (at most three) discarded intervals, we have $\langle -\dot{H}\rangle \geq K_2^{(8)}\alpha_0^{1+64\varepsilon}$. But the extreme intervals I' discarded all satisfy: either $|I'| \leq C^{(7)}\alpha_0^{12\varepsilon}$, or $|I'| \leq \alpha_0^{2\varepsilon}$, or

$$\langle -\dot{H}\rangle_{I'} \geq \inf\big(K_2^{(6)}, K_3^{(6)}\big) \inf\big(\alpha_0^{1+54\varepsilon}, \alpha_0^{1+14\varepsilon}\big).$$

We conclude in the end that

– either $|I| \leq C^{(9)}\alpha_0^{2\varepsilon}$,

– or $\langle -\dot{H}\rangle \geq K_2^{(9)}\alpha_0^{1+64\varepsilon}$.

e) Type B: Let $I = I^B$ be some interval of type B. We repeat the same reasoning as above (slightly simpler): on the complement of at most three intervals, we have

$$\langle -\dot{H}\rangle \geq \max\big(K_1^{(6)}, K_2^{(9)}\big) \max\big(\alpha_0^{1+64\varepsilon}, \alpha_0^{1+226\varepsilon}\big),$$

while the discarded intervals I' satisfy either similar estimates, or

$$|I'| \leq C^{(10)}\alpha_0^{2\varepsilon}.$$

Thus, again we obtain:

– either $|I| \leq C^{(10)}\alpha_0^{2\varepsilon}$,

– or $\langle -\dot{H}\rangle_I \geq K^{(10)}\alpha_0^{1+226\varepsilon}$.

f) Conclusion: We conclude by applying the same reasoning again, now to the whole interval $[t_0, t_0 + T_0]$, that

– either $T_0 \leq C^{(11)}\alpha_0^{2\varepsilon}$,

– or $\langle -\dot{H}\rangle_{[t_0,t_0+T_0]} \geq K^{(11)}\alpha_0^{1+226\varepsilon}$.

But, by definition of T_0, we know that

$$\langle -\dot{H}\rangle_{[t_0,t_0+T_0]} = \frac{\alpha_0/5}{T_0},$$

so the second case of the alternative can be rewritten as

$$T_0 \leq \frac{\alpha_0^{-226\varepsilon}}{5K^{(11)}}.$$

This concludes the whole argument, and the proof of Theorem 2.

VII. Further comments

We end this paper with a series of comments and references which, although not compulsory for the understanding of the proof, will hopefully enlighten some of the conclusions which can be drawn from our study, and some of its connections with various subjects.

VII.1. Qualitative behavior of the gas. Apart from the fact that the convergence to equilibrium occurs quite fast if the solution satisfies good a priori regularity bounds, what physical features may be derived from our study?

Let us first draw a nonrigorous picture of the Boltzmann dynamics. If we imagine the space of all probability distributions in x and v as an infinite-dimensional manifold, then we see that there is a distinguished infinite-dimensional sub-manifold $\mathcal{M}$, made of local Maxwellians, such that the collision operator Q vanishes on $\mathcal{M}$. Now, on this manifold the Boltzmann flow is in general *transverse*; it is only on a sub-manifold $\mathcal{M}'$ of $\mathcal{M}$ that the flow is tangent to $\mathcal{M}$. This sub-manifold $\mathcal{M}'$ will turn out to be made exactly of local Maxwellians with constant temperature, and conformal mean velocity field. But $\mathcal{M}'$ itself is transverse to the flow, except on a sub-sub-manifold $\mathcal{M}''$, made of local Maxwellians with constant temperature and vanishing velocity field. And finally, $\mathcal{M}''$ is transverse to the flow, except at one point, which is M.

To illustrate this point of view, let us imagine what goes on qualitatively if the initial datum is a local Maxwellian which is in equilibrium with respect to temperature and velocity (no temperature gradients, no mean velocity), but with inhomogeneous density (so f belongs to $\mathcal{M}''$ but does not coincide with M). One may have first guessed that f will stay close to $\mathcal{M}''$ in order to converge to M, but this is not the case: if one evolves from f_0, then after a very short time a nonzero velocity field should be induced, which in turn will induce temperature gradients. These in turn imply the breaking down of the hydrodynamical approximation, which means that f will depart from $\mathcal{M}$. This picture is purely heuristic, but can be justified under some very strong assumptions of regularity and decay for the density.

Another thing which is worth discussing is the influence of the **shape of the domain** on the speed of convergence. In the system of inequalities of Subsect. II.4, we have been careful to separate between information of geometric nature (shape of the domain) and boundary conditions on one hand, smoothness on the other hand. The geometry of the domain enters via the constants in Korn or Poincaré inequalities. This suggests that a rough idea of how fast/slow convergence to equilibrium is, depending on the domain, can be obtained by looking at these explicit values. For instance, in the case of specular reflection in dimension 3, the relaxation times become very large when Ω_x is very close to have a cylindrical shape, and this is reflected in our proof by the fact that the value of the constant K_K goes to 0 as Ω_x becomes cylindrical. Similarly, it would be natural to think that relaxation times become very large for domains which are very elongated along a certain direction; we do not know whether this guess can be justified, but it is backed by the fact that the Poincaré constants K_ρ and K_T become very small for such domains.

Finally, let us discuss **time-oscillations**. Our proof does not rule out the possibility that the entropy production undergo important oscillations in time, and actually most of the technical work is caused by this possibility. On the other hand, common sense may suggest that this is an artifact of the method of proof, rather than a physically relevant feature. We suspect that, on the contrary, these oscillations may occur in physical systems, and to back this claim we briefly comment on the fascinating numerical simulations performed on our suggestion by Filbet [27]. He simulates the full Boltzmann equation in a simplified geometry (one dimension of space, two dimensions of velocity, periodic boundary conditions, Knudsen number 0.25) with an accurate deterministic spectral code, and is able to observe spectacular oscillations in the entropy production and in the hydrodynamic entropy. More details on the numerical method are discussed in [27, 26, 16] (no need to say, tests have been performed with refined meshes to make sure that these oscillations are not due to numerical approximation). The strength of the oscillations depends a lot on the length L of the domain, which is consistent with the fact that such oscillations are never observed in the spatially homogeneous case ($L = 0$). On Fig. 5 are reproduced two of the most striking figures obtained by Filbet. The superimposed curves yield the time-evolution of the total H functional and of its kinetic part, respectively. In both cases, a local Maxwellian is chosen for initial datum; the first plot corresponds to $L = 1$ and the second one to $L = 4$. Some slight oscillation can be seen in the case $L = 1$, but what is most striking is that after a short while, the kinetic entropy is very close to the total entropy: an indication that the solution evolves basically in a spatially homogeneous way (contrary to the intuition of the hydrodynamic regime). On the contrary, in the case $L = 4$, the oscillations are much more important in frequence and amplitude (note that this is a logarithmic plot): the solution “hesitates” between states where it is very close to hydrodynamic, and states where it is not at all. Further note that the equilibration is much more rapid when the box is small, and that the convergence seems to be exponential.

Such simulations lead to tricky numerical problems, and certainly more experiments should be performed in this direction.

VII.2. Quasi-equilibria and conformal mappings. The H-theorem identifies local Maxwellians with distribution functions for which the entropy production vanishes. The computations in Subsect. III.2 show that if one looks at the variation of entropy production in time, then these states are not all equivalent:

– In general, a smooth solution of the Boltzmann equation starting at time t_0 from a local equilibrium satisfies, for t close to t_0, $\| f - M^f_{\rho\, u\, T}\|^2_{L^2} \simeq K(t - t_0)^2$, and accordingly the entropy production is formally of about the

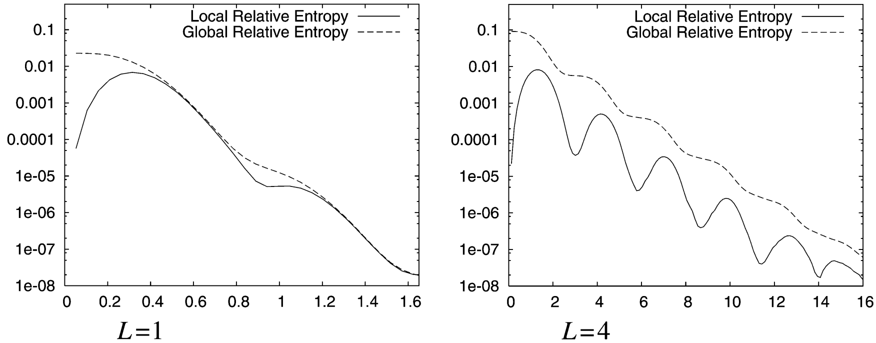

Fig. 5. Time-evolution of the H functional

same order (more rigorously, it is at least $K(t-t_0)^\alpha$ for all $\alpha > 2$). This can be interpreted by saying that the flow of the Boltzmann equation usually goes transversally through the "manifold" $\mathcal{M}$ of local Maxwellians.

– However, there are certain particular local Maxwellians such that if a smooth solution starts from one of these states at time t_0, then $(d^2/dt^2)\|f-M^f_{\rho\,u\,T}\|^2_{L^2}=0$, whence formally $\|f-M^f_{\rho\,u\,T}\|^2_{L^2}=O((t-t_0)^4)$, and formally the entropy production is about the same order (typically $O((t-t_0)^\alpha)$ for all $\alpha < 4$, if the solution satisfies some very stringent decay conditions at large velocities). This can be interpreted by saying that the flow of the Boltzmann equation goes tangentially through $\mathcal{M}$ at those particular states. We shall call them "quasi-equilibria", in the sense that they are associated with "particularly low" production of entropy. According to Subsect. III.2 they are defined by the equations

$$\nabla T \equiv 0, \qquad \{\nabla u\} = 0. \tag{113}$$

It is therefore natural to try to identify all admissible vector fields u satisfying the second equation, which is equivalent to the requirement that

$$\nabla^{\mathrm{sym}} u \quad \text{is pointwise proportional to } I_N. \tag{114}$$

As we shall see, solutions are automatically smooth. Their classification heavily depends on the boundary conditions. For periodic boundary conditions, solutions are all trivial (constant vector fields), as shown by the computations in Subsect. IV.1. For specular boundary conditions, i.e. $u\cdot n = 0$ on $\partial\Omega_x$, Grad [30] "proves" that u has to be identically 0 if Ω_x is not axisymmetric. But his argument relies on an identity (see the first equation after eq. (5.13) in that reference) which seems erroneous; and the conclusion in fact does not always hold true.

In fact, the following can be proven:

Proposition 13. *Let* $u : \Omega_x \to \mathbb{R}^N$ *be a smooth vector field satisfying* $\{\nabla u\} = 0$. *Then,*

a) if $\Omega_x = \mathbb{T}^N$, then u is constant;
b) if Ω_x is a smooth connected open set of $\mathbb{R}^N$ and $u = 0$ on $\partial\Omega_x$, then $u \equiv 0$;
c) if Ω_x is a smooth connected open set of $\mathbb{R}^N$ ($N = 2$ or 3) and $u \cdot n = 0$ on $\partial\Omega_x$, then u can be non-zero if and only if Ω_x is conformally equivalent to an open set admitting a nontrivial group of isometries. In this case u belongs to the Lie algebra $TC(\Omega_x)$ of the Lie group $C(\Omega_x)$ made of conformal transforms preserving Ω_x. In particular,
  - *if $N = 2$, then $TC(\Omega_x)$ is three-dimensional if Ω_x is simply connected, one-dimensional if Ω_x is doubly connected, and trivial in all other cases;*
  - *if $N = 3$, then $TC(\Omega_x)$ has dimension 6 if Ω_x is a ball, 3 if Ω_x is the inner space between two concentric balls, 1 if it is conformally equivalent to a "cylinder" admitting only one axis of symmetry, and 0 in all the other cases.*

Remarks. 1. The problem of identifying those domains Ω_x for which a nontrivial solution of (114) exists, amounts to identifying those domains for which $C(\Omega_x)$ (or more precisely the connected component of the identity in $C(\Omega_x)$) is nontrivial. This is closely related to some famous theorems in conformal geometry, like the Liouville or the Obata-Ferrand theorems.

2. For the case $N = 2$ of case c), recall that any simply connected domain (distinct from the plane, which is obviously our case) is conformally equivalent to the 2-dimensional ball; while any doubly connected domain is conformally equivalent to an annulus. Of course this case can be treated in terms of holomorphic mappings; the equation (114) means that $f(x_1 + ix_2) = u_1(x_1, x_2) + iu_2(x_1, x_2)$ is a holomorphic function. In the case of the 2-dimensional ball, the Lie algebra $TC(\Omega_x)$ can be identified with the vector space generated by the three particular holomorphic functions

$$f(z) = iz, \qquad f(z) = i(1 + z^2), \qquad f(z) = -1 + z^2;$$

or, in terms of vector fields,

$$u(x) = (-x_2, x_1),\ u(x) = \big(-2x_1x_2, 1 + x_1^2 - x_2^2\big),$$
$$u(x) = \big(-1 + x_1^2 - x_2^2, 2x_1x_2\big).$$

3. In the case when Ω_x is a ball, then the conformal group $C(\Omega_x)$ coincides with the conformal group of the sphere S^N, leaving invariant an N-dimensional sphere (or spherical cup) B^N of S^N. It can be identified with the isometries of the hyperbolic ball, leaving invariant some spherical cup. But hyperbolic isometries of B^{N+1} induce hyperbolic isometries of B^N by restriction. We conclude that $C(\Omega_x)$ can be identified with the group of hyperbolic isometries of B^N, or equivalently the conformal group of S^{N-1}, which has dimension $N(N + 1)/2$ (dimension 6 when $N = 3$).

Sketch of proof of Proposition 13. Point c) is an essentially well-known theorem in geometry; we thank E. Ghys for showing us a rather elementary proof. Part a) is a consequence of the Korn-like inequality proven in Subsect. IV.1 below. Next, we consider part b) in dimension $N = 2$. The equation (114) can be recast as

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2} = 0 \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} = 0, \end{cases}$$

which means, as we said above, that $f(x_1+ix_2) = u_1(x_1, x_2)+iu_2(x_1, x_2)$ is a holomorphic function. Since it has to vanish on $\partial\Omega_x$ (identified as a subset of $\mathbb{C}$), $f \equiv 0$.

Now, let u be a solution of (114) in $\Omega_x \subset \mathbb{R}^N$, where $N \geq 2$ is arbitrary. For any choice of $x_3^0, \dots, x_N^0$ we see that $(x_1, x_2) \longmapsto u(x_1, x_2, x_3^0, \dots, x_N^0)$ is a solution of (114) in $\Omega_x{}' = \{(x_1, x_2);\ (x_1, x_2, x_3^0, \dots, x_N^0) \in \Omega_x\}$, and it vanishes on $\partial\Omega_x{}'$, so it has to be 0. This shows that $u \equiv 0$. □

VII.3. Comments on the proof of Theorem 2. We now make a few further comments on our method of proof and possible improvements left for future research.

- In our proof, we did not take advantage of the deviator tensor $\{\nabla u\}$ appearing in the right-hand side of (43), because in general it is not sufficient to control the full deformation tensor $\nabla^{\mathrm{sym}} u$. In the case of periodic, or bounce-back boundary conditions however, it can be proven that

$$\int_{\Omega_x} |\{\nabla u\}|^2\, dx \geq K \int_{\Omega_x} |\nabla^{\mathrm{sym}} u|^2\, dx. \tag{115}$$

As we discussed in Subsect. VII.2, this is also true for generic domain, if specular boundary conditions are enforced and the dimension N is at least 3. Whenever such an inequality is available, one can devise a slightly shorter proof of Theorem 2, with improved constants, using only equations (43) and (45) (not (44)), and the Poincaré-like inequality

$$\int_{\Omega_x} |\nabla T|^2\, dx \geq K_T \mathcal{H}(T|1) - C_T \mathcal{H}(u|0).$$

However, as shown by the discussion in Subsect. VII.2, in some cases $\{\nabla u\}$ is just *not* sufficient for specular boundary conditions; this is the case in particular in dimension $N = 2$ when the domain is simply connected. Moreover, even in cases when (115) holds, it looks quite difficult to establish this inequality with explicitable constants.

Since our three-steps argument is anyway more natural in certain respects, we did not try to make progress in this direction. However, it would be interesting to obtain explicit estimates on the constants in (115) in three dimensions, for instance to make more quantitative the plausible guess that convergence to equilibrium might be more efficient in dimension 3 than in dimension 2.

• For simplicity, we did not consider here the degenerate case in which Ω_x admits an axis of (continuous) symmetry and specular reflection is enforced. However, it is quite likely that this case can be treated by a variant of our proof, involving corresponding variants of Korn's inequality, as established in [22].

• We also wish to insist on one important advantage of our method – possibly also its main drawback: our way of putting together information about the x and about the v variables makes hardly any use of the structure of the Boltzmann collision operator. We relied crucially on this structure in order to establish the quantitative H Theorem (Theorem 4), and therefore equation (42); but all the rest of the system in Subsect. II.4 was established by using extremely little of the Boltzmann operator, just the fact that it has some continuity properties and that it leaves Maxwellian distributions invariant. This is a sign that the method is very robust. For instance, the proof as it stands here also works verbatim for the BGK approximation, up to replacing (28) by its (much simpler) analogue in that context.

This robustness property was one of the features that we looked for when designing our strategy. On the other hand it might also be one of the most serious limitations of the method in the end, since it does not enable one to take advantage of possible "positive interplay" between the collision and the transport operator. An example of such interplay is given by the algebraic structure revealed in the beautiful recent series of works by Hérau, Nier and Helffer [36, 37] about the linear kinetic Fokker-Planck equation. Although past experience seems to indicate that the Boltzmann operator and the transport operator do interact very badly, it might be possible that such interplays are still waiting to be unveiled. We however think that this is most likely to occur in a linearized regime, and this is not the topic of the present paper.

• In the same line of ideas, one could think of more efficient approaches by using more information about the collision operator Q. For instance (following a suggestion by D. Serre), it might be very helpful if we could continuously "invert" Q in some regime, by means of an ad hoc implicit function theorem. For the moment, absolutely nothing is known about such a possibility.

• Speaking about linearized regime, one could also think that it might be possible to consider the linearized Boltzmann operator around a *local* Maxwellian, and in this way treat the instability property more cleverly.

All these remarks suggest that the present paper is certainly not a final answer to the problem of convergence to equilibrium, and that a lot remains to be explored in the field, even in the linear or linearized regime, which are still far from complete understanding – this direction of research may in fact deserve priority now...

VII.4. Bibliographical notes. The topics addressed in this paper are related to the contributions and ideas of many authors. In addition to the references already mentioned within the text, we can quote in particular the following.

Details about the controversy triggered by Boltzmann's work, around Poincaré's recurrence theorem, Zermelo's paradox, irreversibility etc. can be found in [7, pp. 203–207] and in [14, pp. 100 sqq.]. About Poincaré's feelings it is particularly interesting to consult [46].

It seems that Kac [39] was the first to address the problem of finding explicit estimates on the rate of convergence, back in the fifties. His attempts to solve it in a particular spatially homogeneous case, via the study of many-particle systems, gave birth to the so-called Kac problem about spectral gaps in large dimension [23, 38, 13, 60], to one of the first mathematical treatments of continuous mean-field limit, and to the crucial concept of propagation of chaos [50, 49], which was studied by so many authors since then.

But the problem of convergence to equilibrium did not make substantial progress before the nineties – with the exception of just two contributions which appeared around 1965: the first was McKean's brilliant insight [42], about an analogy between this problem and the central limit theorem, and the second was a rather obscure but inspired paper by Grad [30]. McKean was focusing on the Boltzmann equation from the point of view of information theory, while Grad was looking at it from the point of view of fluid mechanics; in some sense, both viewpoints are reconciled in the present work.

McKean's paper was dealing with a one-dimensional caricature of the Boltzmann equation, called the Kac kinetic model. It was immediately noticed by the physicist community, but for a misleading reason: in his paper, McKean proved that the entropy is a convex function of time for the Kac model, and this led people to wild conjectures about the "super-H-theorem", according to which the entropy would be a completely monotone function of time. That guess turned out to be wrong: consult the references in [58, Chap. 4, Sect. 4.3] for more details. On the other hand, McKean's introduction of the Fisher information in kinetic theory turned out to be extremely fruitful (see [56] and the references therein). More importantly, at the beginning of the nineties, Carlen and Carvalho [11, 12] took back the study of McKean and brilliantly extended part of his analysis to the Boltzmann equation, establishing new, quantitative entropy production estimates. Independently, the first author [17] established other estimates by a strategy inspired from Boltzmann's own work. Both approaches were unified and improved in more recent developments by Toscani and both authors [20, 52], culminating with the second author's close-to-optimal results [60] which were used in the present paper. We refer to the detailed introduction of [60] for more information.

Positivity estimates in the style of (21) are as old as the mathematical study of the Boltzmann equation, since they take their roots in Carleman's seminal work [10]. In the case of spatial homogeneity, cutoff hard potentials, the contributions by A. Pulvirenti and Wennberg [47] provide essentially optimal results. Much more general situations are considered by Mouhot in [43]; this paper can also be consulted for further references and explanations.

Before the present study, convergence to equilibrium for the Boltzmann equation had been investigated by many authors in various contexts. The list is too long to be given here and we refer to our survey papers [19, 58] for (tentative) exhaustive records. Among all results which do not try to obtain explicit rates of convergence, the works initiated by Arkeryd [3] and continuated by Wennberg [62, 63] in the spatially homogeneous case are certainly the most precise. Together with Nouri, Arkeryd also obtained many results about diffuse boundary conditions, with uniform temperature or not; of particular interest is their theorem of convergence for uniform temperature in a non-perturbative setting [4].

When one considers the Boltzmann equation in the whole space, without any confinement, counterexamples by Pitteri [45] and Toscani [51] show that there is in general no trend to equilibrium, even locally. See [41] for developments of Toscani's results. Rather than the equilibration, it is the dispersion at infinity which seems to be the relevant qualitative issue in that context.

The Korn-like inequality established by the authors in [22] takes its roots in a false but inspiring estimate derived by Grad [30]. Moreover, we showed how one can estimate explicitly the constants in this inequality by using some results from the theory of optimal transportation [61]. All of this is precisely discussed in [22].

Acknowledgements. The results in this paper constituted the subject of a series of lectures by the second author, at Institut Henri Poincaré, in Winter 2001, on the invitation of Stefano Olla. Besides him, we should thank all the attendents for their interest and constructive remarks. It is also a pleasure to thank Etienne Ghys, Jean-Claude Sikorav and Ghani Zeghib for their explanations about conformal mappings. Seiji Ukai provided encouragements for undertaking this work, and Yoshio Sone helped us with some critical remarks. Dave Levermore kindly supplied references [46] and [7]. Francis Filbet performed the numerical tests which were briefly discussed within the paper. Finally, the anonymous referees are thanked for their constructive comments, which helped improve the presentation of the paper. Our research was partly supported by the European TMR "Asymptotic methods in kinetic theory", contract ERB FMBX-CT97-0157; and partly by the European network "Hyperbolic and Kinetic Equations", contract HPRN-CT-2002-00282.

References

1. Alexandre, R., Villani, C.: On the Boltzmann equation for long-range interactions. Commun. Pure Appl. Math. **55**, 30–70 (2002)
2. Alexandre, R., Villani, C.: On the Landau approximation in plasma physics. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **21**, 61–95 (2004)
3. Arkeryd, L.: Stability in L^1 for the spatially homogeneous Boltzmann equation. Arch. Ration. Mech. Anal. **103**, 151–167 (1988)
4. Arkeryd, L., Nouri, A.: Boltzmann asymptotics with diffuse reflection boundary conditions. Monatsh. Math. **123**, 285–298 (1997)

5. Baranger, C., Mouhot, C.: Explicit spectral gap estimates for the Boltzmann and Landau operators. To appear in Rev. Mat. Iberoam.
6. Bardos, C., Golse, F., Levermore, D.: Fluid dynamical limits of kinetic equations, II: Convergence proofs for the Boltzmann equation. Commun. Pure Appl. Math. **46**, 667–753 (1993)
7. Brush, S.: Kinetic Theory, Vol. 2: Irreversible Processes. Oxford: Pergamon Press 1966
8. Cáceres, M.-J., Carrillo, J.-A., Goudon, T.: Equilibration rate for the linear inhomogeneous relaxation-time Boltzmann equation for charged particles. Commun. Partial Differ. Equations **28**, 969–989 (2003)
9. Caflisch, R.: The Boltzmann equation with a soft potential. I. linear, spatially-homogeneous. II. nonlinear, spatially-periodic. Commun. Math. Phys. **74**, 71–95, 97–109 (1980)
10. Carleman, T.: Sur la théorie de l'equation intégrodifférentielle de Boltzmann. Acta Math. **60**, 369–424 (1932)
11. Carlen, E., Carvalho, M.: Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. J. Stat. Phys. **67**, 575–608 (1992)
12. Carlen, E., Carvalho, M.: Entropy production estimates for Boltzmann equations with physically realistic collision kernels. J. Stat. Phys. **74**, 743–782 (1994)
13. Carlen, E., Carvalho, M., Loss, M.: Determination of the spectral gap for Kac's master equation and related stochastic evolutions. Acta Math. **191**, 1–54 (2003)
14. Cercignani, C.: Ludwig Boltzmann, the man who trusted atoms. New York: Oxford University Press 1998
15. Cercignani, C.: Rarefied gas dynamics. From basic concepts to actual calculations. Cambridge: Cambridge University Press 2000
16. Degond, P., Pareschi, L., Russo, G. (eds.): Modeling and computational methods for kinetic equations. Birkhäuser 2003
17. Desvillettes, L.: Entropy dissipation rate and convergence in kinetic equations. Commun. Math. Phys. **123**, 687–702 (1989)
18. Desvillettes, L.: Convergence to equilibrium in large time for Boltzmann and BGK equations. Arch. Ration. Mech. Anal. **110**, 73–91 (1990)
19. Desvillettes, L.: Convergence to equilibrium in various situations for the solution of the Boltzmann equation. In: Nonlinear kinetic theory and mathematical aspects of hyperbolic systems (Rapallo, 1992), pp. 101–114. River Edge, NJ: World Sci. Publishing 1992
20. Desvillettes, L., Villani, C.: On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. Commun. Partial Differ. Equations **25**, 261–298 (2000)
21. Desvillettes, L., Villani, C.: On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. Commun. Pure Appl. Math. **54**, 1–42 (2001)
22. Desvillettes, L., Villani, C.: On a variant of Korn's inequality arising in statistical mechanics. ESAIM, Control Optim. Calc. Var. **8**, 603–619 (2002) (electronic). A tribute to J.L. Lions
23. Diaconis, P., Saloff-Coste, L.: Bounds for Kac's master equation. Commun. Math. Phys. **209**, 729–755 (2000)
24. DiPerna, R., Lions, P.-L.: On the Cauchy problem for the Boltzmann equation: Global existence and weak stability. Ann. Math. (2) **130**, 312–366 (1989)
25. Fellner, K., Neumann, L., Schmeiser, C.: Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes. Monatsh. Math. **141**, 289–299 (2004)
26. Filbet, F., Russo, G.: High order numerical methods for the space non-homogeneous Boltzmann equation. J. Comput. Phys. **186**, 457–480 (2003)
27. Filbet, F.: Quelques résultats numériques sur l'équation de Boltzmann non homogène. Preprint 2004

28. Gallay, T., Wayne, C.: Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$. Arch. Ration. Mech. Anal. **163**, 209–258 (2002)
29. Golse, F., Saint-Raymond, L.: The Navier-Stokes limit of the Boltzmann equation: Convergence proof. Invent. Math. **155**, 81–161 (2004)
30. Grad, H.: On Boltzmann's H-theorem. J. Soc. Indust. Appl. Math. **13**, 259–277 (1965)
31. Guo, Y.: The Landau equation in a periodic box. Commun. Math. Phys. **231**, 391–434 (2002)
32. Guo, Y.: Classical solutions to the Boltzmann equation for molecules with an angular cutoff. Arch. Ration. Mech. Anal. **169**, 305–353 (2003)
33. Guo, Y.: The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. **153**, 593–630 (2003)
34. Guo, Y.: Private communication (December 2003)
35. Gustafsson, T.: L^p-estimates for the nonlinear spatially homogeneous Boltzmann equation. Arch. Ration. Mech. Anal. **92**, 23–57 (1986)
36. Helffer, B., Nier, F.: Hypoellipticity and spectral theory for Fokker-Planck operators and Witten Laplacians. Preprint 03-25, Université de Rennes (2003). Available at http://name.math.univ-rennes1.fr/francis.nier/recherche/liste.html
37. Hérau, F., Nier, F.: Isotropic hypoellipticity and trend to the equilibrium for the Fokker-Planck equation with high degree potential. Arch. Ration. Mech. Anal. **171**, 151–218 (2004)
38. Janvresse, E.: Spectral gap for Kac's model of Boltzmann equation. Ann. Probab. **29**, 288–304 (2001)
39. Kac, M.: Foundations of kinetic theory. In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 171–197. Berkeley, Los Angeles: University of California Press 1956
40. Lions, P.-L., Masmoudi, N.: From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II. Arch. Ration. Mech. Anal. **158**, 173–193, 195–211 (2001)
41. Lu, X.: Spatial decay solutions of the Boltzmann equation: converse properties of long time limiting behavior. SIAM J. Math. Anal. **30**, 1151–1174 (1999)
42. McKean, H. J.: Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas. Arch. Ration. Mech. Anal. **21**, 343–367 (1966)
43. Mouhot, C.: Quantitative lower bounds for the full Boltzmann equation. Preprint 2003
44. Mouhot, C., Villani, C.: Regularity theory for the spatially homogeneous Boltzmann equation with cutoff. Arch. Ration. Mech. Anal., in press
45. Pitteri, M.: On the asymptotic behaviour of Boltzmann's H function in the kinetic theory of gases. Rend. Sc. Fis. Mat. e Nat. **67**, 248–251 (1979)
46. Poincaré, H.: Le mécanisme et l'expérience. Revue de Métaphysique et de Morale **I**, 534–537 (1893)
47. Pulvirenti, A., Wennberg, B.: A Maxwellian lower bound for solutions to the Boltzmann equation. Commun. Math. Phys. **183**, 145–160 (1997)
48. Shizuta, Y., Asano, K.: Global solutions of the Boltzmann equation in a bounded convex domain. Proc. Japan Acad., Ser. A **53**, 3–5 (1977)
49. Spohn, H.: Large scale dynamics of interacting particles. Texts and Monographs in Physics. Berlin: Springer 1991
50. Sznitman, A.-S.: Topics in propagation of chaos. In: École d'Été de Probabilités de Saint-Flour XIX–1989, pp. 165–251. Berlin: Springer 1991
51. Toscani, G.: H-theorem and asymptotic trend to equilibrium of the solution for a rarefied gas in the vacuum. Arch. Ration. Mech. Anal. **100**, 1–12 (1987)
52. Toscani, G., Villani, C.: Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. Commun. Math. Phys. **203**, 667–706 (1999)
53. Toscani, G., Villani, C.: On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds. J. Stat. Phys. **98**, 1279–1309 (2000)
54. Truesdell, C., Muncaster, R.: Fundamentals of Maxwell's kinetic theory of a simple monoatomic gas. New York: Academic Press 1980

55. Ukai, S.: On the existence of global solutions of mixed problem for non-linear Boltzmann equation. Proc. Japan Acad. **50**, 179–184 (1974)
56. Villani, C.: Fisher information bounds for Boltzmann's collision operator. J. Math. Pures Appl. **77**, 821–837 (1998)
57. Villani, C.: Limites hydrodynamiques de l'équation de Boltzmann (d'après C. Bardos, F. Golse, C. D. Levermore, P.-L. Lions, N. Masmoudi, L. Saint-Raymond). Astérisque **282** (2002), Exp. No. 893, ix, 365–405. Séminaire Bourbaki, Vol. 2000/2001
58. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Handbook of mathematical fluid dynamics, Vol. I, pp. 71–305. Amsterdam: North-Holland 2002
59. Villani, C.: On the Boltzmann equation with singular kernel. Unpublished notes
60. Villani, C.: Cercignani's conjecture is sometimes true and always almost true. Commun. Math. Phys. **234**, 455–490 (2003)
61. Villani, C.: Topics in Optimal Transportation, vol. 58 of Graduate Series in Mathematics. Providence: American Mathematical Society 2003
62. Wennberg, B.: Stability and exponential convergence in L^p for the spatially homogeneous Boltzmann equation. Nonlinear Anal. **20**, 935–964 (1993)
63. Wennberg, B.: Stability and exponential convergence for the Boltzmann equation. Arch. Ration. Mech. Anal. **130**, 103–144 (1995)