Otar Tsereteli

(To the 85th Birthday Anniversary)

This year Doctor of Physical and Mathematical Sciences, Professor, Corresponding Member of the Georgian Academy of Sciences Otar Tsereteli would have turned 85 and would have marked 60 years of his scientific activity. Otar Tsereteli was a brilliant Georgian mathematician, teacher and organizer of science. His scientific works produced an essential impact on the development of the theory of harmonic analysis as well as on a number of other areas of mathematical analysis.

O. Tsereteli was born on November 22, 1926 in the town of Akhaltsikhe (Georgia) and died on April 17, 1991 at the age of 65. But his ideas and results still remain topical and the directions he set up in mathematical science continue to develop.

As a highly skilled mathematician O. Tsereteli was profoundly respected and his results were highly appreciated by the outstanding scientists of modern times, among whom were Anthony Zygmund, Ilayes Stein, Pyotr Ulyanov, Dimitri Menshov, Sergei Stechkin, Nikolai Muskhelishvili, Ilya Vekua and others. Addressing the audience at the defence of O. Tsereteli's doctoral thesis (1975), Academician I. Vekua said, "It will be a big surprise for Anthony Zygmund to learn that today Otar is defending his doctoral thesis because he thinks that he has already been a doctor of science for ages".

O. Tsereteli graduated from the physical-and-mathematical faculty of Tbilisi State University in 1948 and a post-graduate course in 1951. After defending his Candidate of Science dissertation, he began working at A. Razmadze Tbilisi Mathematical Institute of the Georgian Academy of Sciences, where to the last day of his life he was in charge of the function theory and functional analysis department. He was concurrently engaged in the pedagogical activity at I. Javakhishvili Tbilisi State University, was a professor and held the chair of improvement of qualifications in mathematics of teachers working at higher education institutions.

The scientific activities of O. Tsereteli are related to the theory of functions of real and complex variables, the metric function theory. He obtained important results in the theory of Fourier series, the metric theory of conjugate functions and the theory of analytic functions.

O. Tsereteli's published works are distinguished by a clear style and refined manner of exposition, he possessed an ability to explain every detail in a simple transparent way. His convincing arguments evoked a never-failing interest on the part of readers.

O. Tsereteli obtained noteworthy results in various fields of function theory. His earlier papers were dedicated to the theory of integrals and functions of bounded variation. He proposed an original method of characterizing functions of bounded variation from the unique metric standpoint.

O. Tsereteli studied the ergodic properties of internal analytic functions and the boundary values of the Schwartz integral of a Borel measure. In particular he established that the boundary values of an internal function that differs from rotation and vanishes at the point $z = 0$ are a strongly mixing transformation, the ergodic means of any conjugate function with respect to measure-preserving transforms generated by internal functions, converge in measure (but, generally speaking, do not tend a. e. to zero). He studied the uniqueness properties of internal analytic functions. For instance, he proved that a singular nonnegative measure is completely defined by its variation and the Lebesgue set of its conjugate function.

To obtain a function with that property or another, O. Tsereteli proposed a modification of the notion of "improvement" of a function on the set of small measure – on the set of small measure it is allowed to change the function not arbitrarily like in the classical theorems of N. Luzin and D. Menshov, but only to rearrange its values or multiply them by –1 (in that case, the metric class of a function does not change). O. Tsereteli proved that on the set of an arbitrarily small measure the values of an integrable function can be rearranged so that the trigonometric Fourier series of the obtained function will converge almost everywhere. He furthermore showed that on some set of an arbitrarily small measure the sign of an arbitrary integrable function can be changed so that the conjugate of the obtained function will be integrable. Therefore the integrability of the conjugate function does not impose any restrictions on the modulus of an integrable function.

The construction of the metric theory of conjugate functions proposed by O. Tsereteli can be used nearly without any changes for the case of general functional Dirichlet algebras. In particular he obtained a generalization of P. Ulyanov's theorem on the Riesz equality to the case of conjugate functions arising in the theory of uniform Dirichlet algebras.

In the theory of Fourier series, O. Tsereteli obtained a conceptual result on general orthogonal systems. He proved that the values of any nonconstant (nonzero) function from the space L^2 can be rearranged (multiplied by -1) on the set of an arbitrarily small measure so that the Fourier series of the obtained function with respect to a given complete orthonormal system may – after some rearrangement – diverge a.e. This means that there exists no criterion that imposes some restriction on a distribution function and on the modulus of an integrable function which would provide the unconditional convergence a.e. of Fourier series of this function with respect to a given complete orthonormal system.

Most of the results of O. Tsereteli and his followers related to the "improvement" of functions were obtained while solving the following general problem posed by O. Tsereteli: given an equivalence relation R on the set X, characterize a set E from X in terms of R, i.e. define the explicitly largest R-set (i.e. the set which is a union of R-equivalence classes) $\underline{R}(E)$ contained in E and the smallest R-set $\overline{R}(E)$ containing E.

The problem of set characterization with respect to a given equivalence relation as posed by O. Tsereteli is a powerful source of new interesting problems and its application in concrete cases leads to concrete results. Let us present one of the remarkable statements of O. Tsereteli concerning the metric characterization of a set of integrable functions whose conjugates are integrable: if $X = L^1, E = \operatorname{Re} H^1$ (where H^1 is the Hardy class) and fRg means that f and g are equimeasurable (or $|f| = |g|$ a.e.), then $\underline{R}(E) = \mathrm{Lg}^+ L$, and $\overline{R}(E) = Z_1(\overline{R}(E) = L_1$ and, where the class $Z_p,\ p > 0$, introduced by O. Tsereteli is defined as follows: if $f \in L^1$ and $F(t), t > 0$, is an integral of f on

$\{x : |f(x)| > t\}$, then $f \in Z_p$ if and only if the function $|F|^p$ is integrable on $(1, \infty)$ over the measure $t^{-1}dt$. Analogous problems were solved by him for maximal Hardy-Littlewood functions as well.

O. Tsereteli established that a set of A-integrable functions is a metrically invariant set containing a set of all conjugate functions $\tilde{L}$, but is not a minimal metric set containing $\tilde{L}$. More precisely, he constructed an example of an A-integrable function f such that none of the functions g equimeasurable with f on T could not be represented as the conjugate of some integrable function φ, which means that the equality $g = \tilde{\varphi}$, where $\varphi \in L(T)$, is impossible.

O. Tsereteli proved that if f 2π is a periodic function, integrable on $(0, 2\pi)$ and monotone on an open interval $(0, 2\pi)$, then $f \in Re H^1$ if and only if $f \in Z_1$.

His study of the *A*-integral actually summarized the studies which had been carried out previously in the theory of integrals and its applications by Georgian and foreign scientists. O. Tsereteli proved that *A*-integrability is the property not only of a conjugate function, but also of all operators continuous with respect to a measure and commutative with shear. In particular he established that any trigonometric series is a Fourier (*A*) series of some nonzero *A*-integrable function. He obtained generalizations of Titchmarin theorem on the *A*-integrability of conjugate (in the sense of Luzin) functions and showed that the values of any linear operator, continuous with respect to measure, given on the Lebesgue space of Borel functions defined on a compact group with Haar measure and permutable with shears, are *A*-integrable.

O. Tsereteli began his pedagogical carrier in 1952 at I. Javakhishvili Tbilisi State University. There he delivered lectures at the philosophy and psychology faculty and also read a special course for mathematicians who worked at higher education institutions and wished to improve their qualifications. In 1968, O. Tsereteli began reading lectures at the mechanical and mathematical faculty. His lectures covered a wide range of topics related with the theories of measures, Fourier series, holomorphic functions, metric spaces and so on. They were very popular among university students because O. Tsereteli delivered them in an original way of his own, taking into consideration the latest scientific achievements in mathematical disciplines. To make difficult problems be easily understood by the audience, he singled out the main issues from a multitude of mathematical facts and by revealing the intrinsic logic of mathematics he showed its appealing qualities.

In 1966, O. Tsereteli founded a weekly seminar on the function theory at A. Razmadze Mathematical Institute and became its permanent leader. The seminar began to bear scientific fruits from the very start of its existence. It was one of the most important and popular seminars on the function theory in the former Soviet Union and not only Georgian mathematicians but also specialists in the function theory from other Soviet cities and foreign countries used to come to Tbilisi to take part in its work. O. Tsereteli set up the style of the seminar, a speaker was strictly obliged to meet a high standard of material presentation. Productive discussions of the questions arising in the course of the seminar session and the statement of new topical problems made the seminar an excellent school for young researchers.

There are several generations of scientists who were trained and gained experience under the direct guidance of O. Tsereteli. His scientific results stimulated the formation of the mathematical school on the function theory in Georgia. He will always remain the honored and beloved teacher for his former post-graduate students and participants of the seminar who are now working at research centers and universities in Georgia and other countries.

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List of Otar Tsereteli's Scientific Works

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3. On functions of bounded variation. *Soobshch. Akad. Nauk Gruzii*, v. 19, # 2, 1957.
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6. On the Banach indicatrix and some of its applications. *Soobshch. Akad. Nauk Gruzii*, v. 25, # 2, 1960.
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