Singular integral equations in special weighted spaces

R. Duduchava, F.–O. Speck

Abstract

We prove boundedness of the Cauchy singular integral operator the special weighted Sobolev $\mathbb{KW}^m_p(\Gamma,\rho)$ and Hölder–Zygmund $\mathbb{KZ}^0_{m+\mu}(\Gamma,\rho)$ spaces for large values of the smoothness parameter, which is integer $m\in\mathbb{N}_0$, when the underlying contour is piecewise–smooth with angular points and even with cusps. We obtain Fredholm criteria and an index formula for singular integral equations with piecewise–continuous coefficients and complex conjugation in the spaces $\mathbb{KW}^m_p(\Gamma,\rho)$ and $\mathbb{KZ}^0_{m+\mu}(\Gamma,\rho)$, provided the underlying contour has no cusps, but only angular points. The Fredholm property and the index turn out to be independent of the smoothness parameter $m\in\mathbb{N}_0$.

Let $\Omega^+\subset\mathbb{C}$ be a bounded domain in the complex plane with a piecewise–smooth boundary $\Gamma=\partial\Omega^+$ and let $\Omega^-=\overline{\Omega}^+$ be the complementary outer domain. Let $t_j\in\Gamma$, $j=1,\dots,n$ be all knots on the boundary $\Gamma=\partial\Omega^+$ with the angles $\pi\gamma_j$, $0\leq\gamma_j\leq 2$, $j=1,\dots,n$. The boundary curve, which is simple (i.e. without self–intersection), might contain cusps ($\gamma_j=0,2$) corresponding to an outward (for $\gamma_j=0$) and an inward (for $\gamma_j=2$) peak of the domain Ω^+. The arcs between the knots $\Gamma_j:=\overset{\frown}{t_jt_{j+1}}$, $j=1,\dots,n$, $t_{n+1}=t_1$, have only endpoints in common and they are sufficiently smooth, say the parameterizations

$$\omega_j(x)\ :\ \mathscr{I}:=[0,1]\longrightarrow\Gamma_j\,,\qquad \omega_j(0)=t_j\,,\quad \omega_j(1)=t_{j+1} \tag{1}$$

are m–smooth $\omega_j\in\mathbb{C}^m(\mathscr{I})$, $j=1,\dots,n$, $m\in\mathbb{N}_0:=\{0,1,\dots\}$.

The Sobolev space $\mathbb{W}^\ell_p(\mathscr{I})$ for an integer $\ell\in\mathbb{N}_0$ on the unit interval is defined as follows

$$\mathbb{W}^\ell_p(\mathscr{I}):=\left\{\varphi\in\mathbb{L}_p(\mathscr{I})\ :\ \partial^k\varphi\in\mathbb{L}_p(\mathscr{I}),\ \ k=0,\dots,m\right\}\,,$$

and is endowed with a natural norm

$$\|\varphi\mid\mathbb{W}^\ell_p(\mathscr{I})\|:=\left(\sum_{k=0}^{\ell}\|\partial^k_x\varphi\mid\mathbb{L}_p(\mathscr{I})\|^p\right)^{\frac{1}{p}}=\left(\sum_{k=0}^{\ell}\int\limits_0^1|\partial^k_x\varphi(x)|^p dx\right)^{\frac{1}{p}}\,,$$

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$$\partial_x^k \varphi(x) := \frac{\partial^k \varphi(x)}{\partial x^k}\,.$$

Applying the parameterization $\omega_j(x)$ we can define the Sobolev space $\mathbb{W}^\ell_p(\Gamma_j)$ on the smooth arcs Γ_j for all $1 < p, q < \infty$ provided $\ell \leq m$. For the entire piecewise–smooth curve Γ the space $\mathbb{W}^\ell_p(\Gamma)$ can be defined only for $\ell = 0, 1$ since any parameterization $\omega(x) \,:\, \mathscr{I} \to \Gamma$ has a piecewise–continuous first derivative $\omega \in \mathbb{PC}(\mathscr{I})$ and $(\partial_t\varphi)(\omega(x)) = \partial_x\varphi(\omega(x))[\omega'(x)]^{-1}$ for $t = \omega(x)$.

The second derivative of ω, which participates in the definition of the space $\mathbb{W}^2_p(\Gamma)$, might be a generalized function. In fact, for a piecewise–continuous function $g \in \mathbb{PC}^m(\Gamma, t_1, \ldots, t_n)$, which might have jumps only at $t_1, \ldots, t_n$ but a continuous m–th derivative on the closed arcs $\partial^m_t g \in \mathbb{C}(\Gamma_j)$ for all $j = 1, \ldots, n$, the first derivative is a distribution

$$g(t) = g_0(t) + \sum_{j=1}^{n} [g(t_j + 0) - g(t_j - 0)]\delta(t - t_j)\,, \qquad g_0(t \pm 0) = g'(t \pm 0) \quad (2)$$

where $g_0(t)$ is piecewise–continuous, more precisely $g_0 \in \mathbb{PC}^{m-1}(\Gamma, t_1, \ldots, t_n)$ and

$$\langle \delta(\cdot - t_j), \psi\rangle := \psi(t_j)\,, \qquad \psi \in \mathbb{C}^1(\Gamma)\,, \quad j = 1, \ldots, n\,. \quad (3)$$

is a δ–function. To prove (2) we use the transformation to the unit interval $\omega_*\varphi(x) := \varphi(\omega(x))$ via parameterization $\omega(x) \,:\, \mathscr{I} \to \Gamma$ and extend functions continuously from $\mathscr{I}$ to the entire real axis $\mathbb{R}$. The problem of representation (2) is thus we reduced to the case $\Gamma = \mathbb{R}$. Here it suffices to note that a function $g \in \mathbb{PC}^m(\mathbb{R}, x_1, \ldots, x_n)$ can be represented as follows

$$g(x) = g_0(x) + \sum_{j=1}^{n} [g(x_j + 0) - g(x_j - 0)]\chi_+(x - x_j)$$

with a continuous $g_0 \in \mathbb{PC}(\mathbb{R})$, $g_0'(x \pm 0) = g'(x \pm 0)$ and the Heaviside function $\chi_+(x) = 0$ for $x < 0$, $\chi_+(x) = 1$ for $x > 0$. There remains to note that $\chi_+' = \delta$ in the since of distributions.

It is clear, due to (3), that the multiplication operator

$$gI \;:\; \mathbb{W}^\ell_p(\Gamma) \longrightarrow \mathbb{W}^\ell_p(\Gamma)\,, \qquad g \in \mathbb{PC}^m(\Gamma, t_1, \ldots, t_n) \quad (4)$$

is bounded only if $\ell = 0$ (i.e. in the Lebesgue space $\mathbb{L}_p(\Gamma)$ only).

Furthermore, we like to consider singular integral operators in such spaces, which requires boundedness of the Cauchy singular integral operator (see Theorem 3 below) To solve all three problems (namely, to define relevant Sobolev spaces for large $m > 1$, to ensure boundedness of multiplication operators (4) and of Cauchy singular integral operator there) we suggest to consider a special Sobolev space $\mathbb{KW}^m_p(\Gamma,\rho)$ with a power weight

$$\rho(t) := \prod_{j=1}^{n}(t-t_j)^{\alpha_j}\,, \qquad \alpha_j\in\mathbb{C}\,, \quad 1<p<\infty\,. \tag{5}$$

Namely, the space is defined as follows

$$\mathbb{KW}^\ell_p(\Gamma,\rho) := \left\{\varphi\in\mathbb{L}_p(\mathscr{I},\rho)\ :\ \partial^k\varphi\in\mathbb{L}_p(\mathscr{I},\rho^{(k)}),\ \ k=0,\ldots,m\right\}\,, \tag{6}$$
$$\rho^{(k)}(t) := \prod_{j=1}^{n}|t-t_j|^{\alpha_j+k}\,.$$

The space is endowed with a natural norm

$$\begin{aligned}\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\| &:= \left(\sum_{k=0}^{m}\|\partial^k_t\varphi \mid \mathbb{L}_p(\Gamma,\rho^{(k)})\|^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{k=0}^{m}\int_\Gamma|\rho^{(k)}(t)\partial^k_t\varphi(t)|^p|dt|\right)^{\frac{1}{p}}\,,\end{aligned} \tag{7}$$

which makes it a Banach space. It can be verified straightforwardly that the following norms are equivalent to the original norm in (7):

$$\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\|_1 := \left(\sum_{k=0}^{m}\|(\vartheta\partial_t)^k\varphi \mid \mathbb{L}_p(\Gamma,\rho)\|^p\right)^{\frac{1}{p}}\,,\quad \vartheta(t) := \prod_{j=1}^{n}(t-t_j)\,, \tag{8}$$

$$\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\|_2 := \left(\sum_{k=0}^{m}\|\partial^k_t\vartheta^k\varphi \mid \mathbb{L}_p(\Gamma,\rho)\|^p\right)^{\frac{1}{p}}\,. \tag{9}$$

Lemma 1 *The space $\mathbb{KW}^m_p(\Gamma,\rho)$ is correctly defined, i.e. it is independent of the choice of a parameterization $\omega(x)\ :\ \mathscr{I}\to\Gamma$ of the curve Γ.*

The multiplication operator gI is bounded in $\mathbb{KW}^m_p(\Gamma,\rho)$ for any function $g\in\mathbb{PC}^m(\Gamma,t_1,\ldots,t_n)$ and all $\ell=0,1,\ldots,m$.

Proof. From the definition of the δ–function (3) it is clear that

$$(t-t_j)\delta(t-t_j)=0\,,\quad j=1,\ldots,n \tag{10}$$

and, therefore, $\vartheta g' = \vartheta g'_0$ (see (2)). Thus, dealing with functions from the space $\mathbb{KW}^m_p(\Gamma,\rho)$ we can ignore the δ–functions while taking derivatives. Therefore,

$$g' = g'_0\in\mathbb{PC}^{m-1}(\Gamma,t_1,\ldots,t_n)\,.$$

For higher derivatives we obtain similarly: $g^{(k)} := \partial_t^k g \in \mathbb{PC}^{m-1}(\Gamma, t_1, \ldots, t_n)$ for all $k = 2, \ldots, m$ and, at the knots,

$$g^{(k)}(t_j \pm 0) := \lim_{t \to t_j \pm 0} \partial_t^k g(t) \quad \text{for all} \quad j = 1, \ldots, n\,, \quad k = 0, 1, \ldots, m\,. \tag{11}$$

This yields

$$\vartheta^k \partial_t^k (g\varphi) = \sum_{j=0}^{k} \left[\begin{array}{c} j \\ k \end{array} \right) \vartheta^{k-j} g^{(k-j)} \vartheta^j \partial_t^j \varphi\,, \quad k = 0, 1, \ldots, m\,. \tag{12}$$

and boundedness of the multiplication operator (4) follows.

The independence of the space $\mathbb{KW}_p^m(\Gamma, \rho)$ of the choice of a parameterization of Γ is a consequence of the boundedness property (4) proved above, because

$$\vartheta \partial_x \varphi(\omega) = \vartheta(\partial_t \varphi)(\omega) \partial_x \omega\,, \quad \vartheta^2 \partial_x^2 \varphi(\omega) = \vartheta^2 (\partial_t^2 \varphi)(\omega)(\partial_x \omega)^2 + \vartheta(\partial_t \varphi)(\omega) \partial_x \vartheta \partial_x^2 \omega$$

etc. Since multiplication by the function $(\vartheta^{k-1} \partial_x^k \omega)^\ell \in \mathbb{PC}^{m-k}(\Gamma, t_1, \ldots, t_n)$ is bounded in $\mathbb{KW}_p^{m-k}(\Gamma, \rho)$, we find that the transformation

$$\omega_* \; : \; \mathbb{KW}_p^m(\Gamma, \rho) \longrightarrow \mathbb{KW}_p^m(\mathscr{I}, \rho_0)\,, \qquad \omega_* \varphi(x) := \varphi(\omega(x))\,, \tag{13}$$
$$\omega(1-0) = t_1 - 0, \;\; \omega(0+0) = t_1 + 0\,, \;\; \omega(x_j) = t_j\,, \;\; j = 2, \ldots, n\,,$$
$$\rho_0(x) := x^{\alpha_1}(x-1)^{\alpha_1} \prod_{j=2}^{n} (x - x_j)^{\alpha_j}$$

is a homeomorphism. ■

Let us consider the weighted Hölder–Zygmund space

$$\mathbb{Z}_\mu^0(\Gamma, \rho) := \{\varphi_0 := \rho\varphi \in \mathbb{Z}_\mu(\Gamma) \; : \; \varphi_0(t_j) = 0\,, \;\; k = 0, \ldots, m\}\,, \quad 0 < \mu \le 1\,,$$

which is endowed with a natural norm

$$\|\varphi \mid \mathbb{Z}_\mu^0(\Gamma, \rho)\| = \|\rho\varphi \mid \mathbb{Z}\mu(\Gamma)\|\,, \tag{14}$$

$$\|\psi \mid \mathbb{Z}\mu(\Gamma)\| := \sup_{t \in \Gamma} |\psi(t)| + \sup_{\substack{0 < s \le \ell \\ h > 0}} h^{-\mu} |\psi(t(s+h)) - 2\psi(t(s)) + \psi(t(s-h))| \Bigg]\,,$$

where

$$t(s) \; : \; [0, \ell] \to \Gamma$$

is the natural parameterization of Γ with the help of the arc length parameter $0 < s \le \ell$. Equivalent norms can be written as in (8) and (9).

If $0 < \mu < 1$ the second difference $|\varphi_k(t(s+h)) - 2\varphi_k(t(s)) + \varphi_k(t(s-h)|$ in the definition of the norm in (14) can be replaced by the first difference $|\varphi_k(t(s+h)) - \varphi_k(t(s))|$ (see [St1]) which means that the weighted Hölder–Zygmund space $\mathbb{KZ}^0_\mu(\Gamma,\rho)$ coincides with the weighted Hölder space $H^0_\mu(\Gamma,\rho)$ (the case considered in [Du1, Du2, Du4]). For $\mu = 1$ the spaces $H^0_1(\Gamma,\rho)$ and $\mathbb{Z}^0_1(\Gamma,\rho)$ differ essentially (see [St1])..

If we define straightforward the Hölder–Zygmund space $\mathbb{Z}^0_{m+\mu}(\Gamma,\rho)$ for $m \geq 1$ we need $m+\mu$–smooth contour Γ. If Γ is piecewise–smooth and we like to have a space which has properties similar to those presented in Lemma 1, we suggest the following special weighted Hölder–Zygmund space

$$\begin{aligned}\mathbb{KZ}^0_{m+\mu}(\Gamma,\rho) := \{\rho\varphi \in \mathbb{Z}_\mu(\Gamma) \ : \ \varphi_k := \rho^{(k)}\partial^k\varphi \in \mathbb{Z}_\mu(\Gamma),\\ \varphi_k(t_j) = 0\,, \ \ k = 0,\dots,m\,, \ j = 0,\dots,n\}\end{aligned} \tag{15}$$

and endow it with a natural norm (cf. (14)).

$$\|\varphi \mid \mathbb{KZ}^0_{m+\mu}(\Gamma,\rho)\| := \sum_{k=0}^{m-1} \sup_{t\in\Gamma} |\varphi_k(t)| + \|\varphi_m \mid \mathbb{Z}^0_\mu(\Gamma)\|\,.$$

Lemma 2 *The space $\mathbb{KZ}^0_{m+\mu}(\Gamma,\rho)$ is correctly defined.*

A multiplication operator

$$gI \ : \ \mathbb{KZ}^0_{m+\mu}(\Gamma,\rho) \longrightarrow \mathbb{KZ}^0_{m+\mu}(\Gamma,\rho) \tag{16}$$

is bounded for every piecewise–Hölder–Zygmund function $g \in \mathbb{PZ}^{m+\mu}(\Gamma,t_1,\dots,t_n)$, which might have jumps only at $t_1,\dots,t_n$ provided the m–th derivative is Hölder–Zygmund continuous on the closed arcs, i.e. $\partial^m_t g \in \mathbb{Z}^\mu(\Gamma_j)$ for all $j = 1,\dots,n$.

Proof. The proof follows word in word the proof of the preceding Lemma 1 with obvious modifications. ■

Theorem 3 *The Cauchy singular integral operator with weight*

$$S_{\Gamma,w}\varphi(t) := \frac{1}{\pi i}\int_\Gamma \frac{w(t)}{w(\tau)}\frac{\varphi(\tau)d\tau}{\tau - t}\,, \qquad w(t) := \prod_{j=1}^n (t-t_j)^{\beta_j} \tag{17}$$

is bounded in the spaces $\mathbb{KW}^m_p(\Gamma,\rho)$ provided the following conditions hold

$$-\frac{1}{p} < \alpha_j + \beta_j < 1 - \frac{1}{p}\,, \quad j = 1,\dots,n\,, \quad 1 < p < \infty\,, \quad m \in \mathbb{N}_0\,. \tag{18}$$

Furthermore, $S_{\Gamma,w}$ is bounded in the space $\mathbb{KZ}^0_{m+\mu}(\Gamma,\rho)$ if the parameters satisfy

$$\mu < \alpha_j + \beta_j < \mu + 1\,, \quad j = 1,\dots,n\,, 1 < \mu \leq 1\,, \quad m \in \mathbb{N}_0\,. \tag{19}$$

Proof. Obviously,

$$\begin{aligned}
\vartheta^k \partial_t^k S_{\Gamma,w}\varphi &= \sum_{j=0}^{k} c_{1j} w \partial_t^j (\vartheta^j S_\Gamma w^{-1}\varphi) \\
&= \sum_{j=0}^{k} c_{2j} w \partial_t^j (S_\Gamma \vartheta^j w^{-1}\varphi) + \sum_{j+r\le(m-1)k-1}^{k} c_{3jr} t^j w (B_\Gamma t^r \varphi) \\
&= \sum_{j=0}^{k} c_{2j} w (S_\Gamma \partial_t^j \vartheta^j w^{-1}\varphi) + \sum_{j+r\le(m-1)k-1}^{k} c_{3jr} t^j w (B_\Gamma t^r \varphi) \\
&= \sum_{j=0}^{k} c_{4j} (S_\Gamma \vartheta^j \partial_t^j \varphi) + \sum_{j+r\le(m-1)k-1}^{k} c_{3jr} t^j w (B_\Gamma t^r \varphi)\,, \qquad (20)
\end{aligned}$$

where

$$B_\Gamma \psi := \frac{1}{\pi i} \int_\Gamma \frac{\psi(\tau)}{w(\tau)} d\tau\,, \quad B_\Gamma \,:\, \mathbb{L}_p(\Gamma,\rho) \longrightarrow \mathbb{C}$$

is a functional. Due to the conditions (18) the singular integral operator

$$S_{\Gamma,w} \,:\, \mathbb{L}_p(\Gamma,\rho) \longrightarrow \mathbb{L}_p(\Gamma,\rho) \qquad (21)$$

is bounded (which is the same as boundedness of $S_\Gamma := S_{\Gamma,1}$ for $w(t) \equiv 1$ in the space $\mathbb{L}_p(\Gamma,\rho w)$; see [GK1, Kh1]). Since $w \in \mathbb{L}_p(\Gamma,\rho)$ (see (18)) and, by definition, $\vartheta^j \partial_t^j \varphi \in \mathbb{L}_p(\Gamma,\rho)$, $j = 0, \dots, m$, due to (21) we obtain

$$\|S_{\Gamma,w} \vartheta^j \partial_t^j \varphi \mid \mathbb{L}_p(\Gamma,\rho)\| \le C_j \|\vartheta^j \partial_t^j \varphi \mid \mathbb{L}_p(\Gamma,\rho)\|\,. \qquad (22)$$

From (20) and (22) we get the final result for the weighted Sobolev space $\mathbb{KW}_p^m(\Gamma,\rho)$:

$$\begin{aligned}
\|S_{\Gamma,w} \vartheta^j \partial_t^j \varphi \mid \mathbb{W}_p^m(\Gamma,\rho)\| &\le \left(\sum_{j=0}^{m} \|\vartheta^j \partial_t^j \varphi \mid \mathbb{L}_p(\Gamma,\rho)\| \right)^{\frac{1}{p}} \\
&\le C \left(\sum_{j=0}^{m} \|\vartheta^j \partial_t^j \varphi \mid \mathbb{L}_p(\Gamma,\rho)\| \right)^{\frac{1}{p}} = C \|vf \mid \mathbb{W}_p^m(\Gamma,\rho)\|\,.
\end{aligned}$$

For weighted Hölder–Zygmund spaces $\mathbb{KZ}_{m+\mu}^0(\Gamma,\rho)$ the proof is absolutely similar and uses the boundedness result

$$S_{\Gamma,w} \,:\, \mathbb{Z}_\mu^0(\Gamma,\rho) \longrightarrow \mathbb{Z}_\mu^0(\Gamma,\rho)\,, \qquad (23)$$

proved in [Du1, Du2] for the case $0 < \mu < 1$. In the case $\mu = 1$ the boundedness of (23) is proved similarly, based on the boundedness of the singular integral operator S_Γ (without weights) in the Hölder–Zygmund space $\mathbb{Z}_1(\Gamma)$ (see [DS1, St1]). ∎

Remark 4 *Spaces* $\mathbb{H}_p^{(s,\nu),m}(\mathscr{M})$ *similar to* $\mathbb{KW}_p^m(\Gamma,\rho)$ *were introduced in* $[CD1]$ *for a multidimensional case (called there anisotropic Bessel potential spaces). In that paper* $\mathscr{M} = \mathbb{R}_+^n$ *or* $\mathscr{M}$ *is a manifold with smooth boundary. The authors proved boundedness of a certain class of pseudodifferential operators and obtained Fredholm criteria for them. Spaces* $L^{p,m}(\mathbb{R}^+)$ *and* $X_\rho^{p,m}(\mathbb{R}^+)$*, also similar to* $\mathbb{KW}_p^m(\Gamma,\rho)$*, were defined by J.Elschner and applied for spline approximation for solutions to convolution equations (see* $[Pr1, Ch..5]$*).*

Theorem 3 enables us to establish the Fredholm property and an index formula for a singular integral operator with complex conjugation

$$\begin{gathered}A\varphi := a\varphi + bS_\Gamma\varphi + dVS_\Gamma V\varphi = f\,, \quad V\varphi(t) := \overline{\varphi(t)}\,, \\ a,b,d \in \mathbb{PC}^m(\Gamma,t_1,\dots,t_n) \quad \text{for} \quad \varphi,\ f \in \mathbb{KW}_p^m(\Gamma,\rho)\,, \\ a,b,d \in \mathbb{PH}^{m+\mu}(\Gamma,t_1,\dots,t_n) \quad \text{for} \quad \varphi,\ f \in \mathbb{KZ}_{m+\mu}^0(\Gamma,\rho)\,.\end{gathered} \tag{24}$$

Although the coefficients of the operator A are $N\times N$ matrix–functions and the equation (24) is considered in the weighted N–vector spaces, we use the same notation for spaces and classes of functions as in the scalar case $N=1$ for the sake of simplicity.

The weight function $\rho(t)$ is defined in (5) and we assume

$$-\frac{1}{p} < \alpha_j < 1-\frac{1}{p}\,, \quad j=1,\dots,n \quad \text{for} \quad \mathbb{KW}_p^m(\Gamma,\rho)\,, \tag{25}$$

$$\mu < \alpha_j < \mu+1\,, \quad j=1,\dots,n \quad \text{for} \quad \mathbb{KZ}_{m+\mu}^0(\Gamma,\rho)\,. \tag{26}$$

Let $\mathbb{X}(\Gamma)$ denote the space $\mathbb{KW}_p^m(\Gamma,\rho)$ or $\mathbb{KZ}_{m+\mu}^0(\Gamma,\rho)$ with the corresponding conditions (25) and (26). The symbol of operator A (see (24)) in the space $\mathbb{X}(\Gamma)$, when Γ has no cusps $0<\gamma_j<2$, $j=1,\dots,n$, is defined as follows

$$\mathscr{A}_{\mathbb{X}(\Gamma)}(t,\xi) ::= \widetilde{a}(t) + \widetilde{b}(t)S_{\mathbb{X}(\Gamma)}(t,\xi) + \widetilde{d}(t)\overline{S_{\mathbb{X}(\Gamma)}(t,-\xi)}\,, \tag{27}$$

where

$$\widetilde{g}(t) := \begin{bmatrix} g(t+0) & 0 \\ 0 & g(t-0) \end{bmatrix}, \qquad g \in (\mathbb{PC}^m)^{N\times N}(\Gamma,t_1,\dots,t_n)\,, \quad t\in\Gamma\,,$$

$$S_{\mathbb{X}(\Gamma)}(t,\xi) := \begin{bmatrix} \coth\pi(i\beta_t+\xi) & -\dfrac{e^{\pi(\gamma_t-1)(i\beta_t+\xi)}}{\sinh\pi(i\beta_t+\xi)} \\ \dfrac{e^{\pi(1-\gamma_t)(i\beta_t+\xi)}}{\sinh\pi(i\beta_t+\xi)} & -\coth\pi(i\beta_t+\xi) \end{bmatrix}, \quad \xi\in\mathbb{R}\,, \tag{28}$$

$$\beta_t := \begin{cases} \frac{1}{p} & \text{if } t \neq t_1, \ldots, t_n\,, \qquad \mathbb{X}(\Gamma) = \mathbb{KW}^m_p(\Gamma, \rho)\,, \\ \frac{1}{2} & \text{if } t \neq t_1, \ldots, t_n\,, \qquad \mathbb{X}(\Gamma) = \mathbb{KZ}^0_{m+\mu}(\Gamma, \rho)\,, \\ \frac{1}{p} + \alpha_j & \text{if } t = t_j\,, \qquad \mathbb{X}(\Gamma) = \mathbb{KW}^m_p(\Gamma, \rho)\,, \\ \alpha_j - \mu & \text{if } t = t_j\,, \qquad \mathbb{X}(\Gamma) = \mathbb{KZ}^0_{m+\mu}(\Gamma, \rho)\,, \end{cases}$$

$$\gamma_t := \begin{cases} 1 & \text{if} \quad t \neq t_1, \ldots, t_n\,, \\ \gamma_j & \text{if} \quad t = t_j\,, \end{cases} \qquad \chi_\pm(\lambda) := \frac{1}{2}(1 + \text{ sign } \lambda)\,.$$

Due to assumptions (25), (26) we have $0 < \beta_t < 1$ for all $t \in \Gamma$ and the symbol $\mathscr{A}_{\mathbb{X}(\Gamma)}(t, \lambda, \xi)$ represents a piecewise–continuous uniformly bounded function of all variables.

Theorem 5 *Let Γ have no cusps, i.e. $0 < \gamma_j < 2$, $j = 1, \ldots, n$ and let $\mathbb{X}^m(\Gamma) = \mathbb{KW}^m_p(\Gamma, \rho)$ or $\mathbb{X}^m(\Gamma) = \mathbb{KZ}^0_{m+\mu}(\Gamma, \rho)$ with appropriate condition in* (25), (26) *being satisfied. The equation* (24) *is Fredholm in the space $\mathbb{X}^m(\Gamma)$ if and only if*

$$\inf_{t\in\Gamma,\ \lambda,\xi\in\mathbb{R}} \left| \det \mathscr{A}_{\mathbb{X}^m(\Gamma)}(t, \xi) \right| > 0\,. \tag{29}$$

If condition (29) *holds, then*

$$\text{Ind } A = -\frac{1}{2\pi} \left\{ \left[\arg \det \mathscr{A}_{\mathbb{X}^m(\Gamma)}(t, +\infty)\right]_\Gamma + \sum_{j=1}^n \left[\arg \det \mathscr{A}_{\mathbb{X}^m(\Gamma)}(t_j, \lambda, 0)\right]_{\mathbb{R}\setminus\{0\}} \right\}\,.$$

If, in particular, $c = 0$ and the operator $A = aI + bS_\Gamma$ has scalar coefficients ($N = 1$), A is invertible in $\mathbb{X}^m(\Gamma)$ either from the left provided Ind A ≥ 0 *or from the right provided* Ind A ≤ 0.

Proof. To prove the theorem we apply results on quasilocalization (see [DLS1] for the case $L_p(\Gamma, \rho)$ spaces, [GK1] for the case $c = 0$ and $L_p(\Gamma, \rho)$ spaces and [Du4] for the case of weighted Hölder spaces $H^0_\mu(\Gamma, \rho)$; see also [Du1, Du2] for the case of weighted Hölder spaces). We will not go into details of the proof and restrict ourselves only by some comments.

I. All singular integral operators are bounded in $\mathbb{X}^m(\Gamma)$ if they are bounded in $\mathbb{X}^0(\Gamma)$ (see Theorem 3). This is valid also for any inverse operator and any regularizer to the canonical operator $A = aI + bS_\Gamma$, because they have the form $cI + dS_{\Gamma,w}$. The same proposition on boundedness holds if $\Gamma = \mathbb{R}$ and $\rho(x) \equiv 1$ and for $\Gamma = \mathbb{R}^+$ and $\rho(x) = x^\alpha$.

   Similar simultaneous boundedness property for all values of the parameter $m \in \mathbb{N}_0$ holds for a Mellin convolution operators $\mathfrak{M}^0_g$ in the space $\mathbb{X}^m(\mathbb{R}^+, x^\alpha)$ (for boundedness of a Mellin convolution operators see also J. Elschner's results in [Pr1, Ch. 5]).

II. A local representative A_{t_0} of A at $t_0 \in \Gamma$ (including knots $t_0 = t_1, \ldots, t_n$) is the following Mellin convolution operator

$$A \overset{M_{t_0}}{\sim} A_{t_0} := \widetilde{a}(t_0)I + \widetilde{b}(t_0)\mathfrak{M}^0_{S_{\mathbb{X}^m(\Gamma)}(t_0,\cdot)} + \widetilde{c}(t_0)\mathfrak{M}^0_{\overline{S_{\mathbb{X}^m(\Gamma)}(t_0,-\cdot)}} = \mathfrak{M}^0_{\mathscr{A}_{\mathbb{X}^m(\Gamma)}(t_0,\cdot)}$$

in the space $\mathbb{X}^m(\mathbb{R}^+, x^{\alpha_{t_0}})$, where $\alpha_{t_0} = 0$ for $t_0 \neq t_1, \ldots, t_n$ and $\alpha_{t_j} = \alpha_j$ (see [DLS1, Du4]). This operator, as we already noted, is invertible in $\mathbb{X}^m(\mathbb{R}^+, x^{\alpha_{t_0}})$ if and only if it is invertible in $\mathbb{X}^0(\mathbb{R}^+, x^{\alpha_{t_0}})$ (i.e. either in $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0}})$ or in $\mathbb{Z}^0_\mu(\mathbb{R}^+, x^{\alpha_{t_0}})$ – the cases considered in [DLS1, Du4]).

III. The symbol of the operator A_0 defined in [DLS1] and in [Du4] (see also [Du1, Du2]) has a block–diagonal form

$$\begin{bmatrix} (\mathscr{A}_0)_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) & 0 \\ 0 & \overline{(\mathscr{A}_0)_{\mathbb{X}(\Gamma)}(t,-\lambda,-\xi)} \end{bmatrix}$$

and it suffices to consider only the first block as a symbol of A_0. Due to this change the index formula carries the factor $\dfrac{1}{2}$.

Let us note that the symbol would be a full matrix–function if the corresponding operator contains terms VS_Γ, VaI, aV or $S_\Gamma V$ (see Remark 6).

IV. If $\mathscr{B}_{\mathbb{X}^m(\Gamma)}(t,\lambda,\xi)$ is the symbol of B, the symbol of VBV reads as follows

$$(\mathscr{V}\mathscr{B}\mathscr{V})_{\mathbb{X}^m(\Gamma)}(t,\lambda,\xi) = \overline{\mathscr{B}(t,-\lambda,-\xi)} \tag{30}$$

(see [DLS1, § 1]). ■

Remark 6 *From Theorem* 5 *we find that the Fredholm properties and the index of the operator* A *(see* (24)*) in the space* $\mathbb{X}^0(\Gamma)$ *are independent of the smoothness parameter* $m \in \mathbb{N}_0$. *That means that if the equation* $A\varphi = f$ *for* $f \in \mathbb{X}^m(\Gamma,\rho)$ *has a solution* $\varphi \in \mathbb{X}^0(\Gamma,\rho)$*, then automatically* $\varphi \in \mathbb{X}^m(\Gamma,\rho)$.

Remark 7 *Equations more general than* (24)

$$\widetilde{A}\varphi := a\varphi + bV\varphi + cS_\Gamma\varphi + dVS_\Gamma\varphi + eS_\Gamma V\varphi + gVS_\Gamma V\varphi = f\,, \tag{31}$$

are linear in the space $\mathbb{X}^m(\Gamma)$ *over the field of real numbers* $\mathbb{R}$. *After "doubling" the equation by adding the composition* $V\widetilde{A}\varphi = Vf$ *and introducing new vector–functions* $\Phi := (\varphi, V\varphi)$, $F := (f, Vf)$, *we get the following equation*

$$\begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}\Phi + \begin{bmatrix} c & e \\ \overline{d} & \overline{g} \end{bmatrix} S_\Gamma\Phi + \begin{bmatrix} g & d \\ \overline{e} & \overline{c} \end{bmatrix} VS_\Gamma V\Phi = F\,, \tag{32}$$

which is linear (the same as in (24)*) and can be treated in the space* $\mathbb{X}^m(\Gamma)$ *over the field of complex numbers* $\mathbb{C}$ *(see [DL1, li1]). We will only indicate the symbol of the operator* $\widetilde{A}$ *because the corresponding Fredholm properties and the index are defined by the symbol as in Theorem* (5) *(note, that we does not need to double the size of the symbol of the operator* $\widetilde{A}$ *as this was done for the operator A). Namely,*

$$\begin{aligned}\mathscr{A}_{\mathbb{X}^m(\Gamma)}(t,\xi) &:= \widetilde{a}(t) + \widetilde{b}(t)\mathscr{V} + \widetilde{c}(t)S_{\mathbb{X}^m(\Gamma)}(t,\xi) \\ &+\widetilde{d}(t)\mathscr{V}S_{\mathbb{X}^m(\Gamma)}(t,\xi)\widetilde{c}(t) + \widetilde{e}(t)S_{\mathbb{X}^m(\Gamma)}(t,\xi)\mathscr{V} + \widetilde{g}\mathscr{V}S_{\mathbb{X}^m(\Gamma)}(t,\xi)\mathscr{V}\,, \qquad (33)\end{aligned}$$

where, in addition to (28)*, we have to indicate the symbol* $\mathscr{V} = \mathscr{V}_{\mathbb{X}^m(\Gamma)}$ *of the complex conjugate operator:*

$$\mathscr{V} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is independent of the point $t \in \Gamma$ *and the space* $\mathbb{X}^m(\Gamma)$ *(note, that* $\mathscr{V}S_{\mathbb{X}^m(\Gamma)}(t,\xi)\mathscr{V} = \overline{S_{\mathbb{X}^m(\Gamma)}(t,-\xi)}$*; cf.* (31)*).*

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Roland Duduchava
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M. Alexidze str.
TBILISI 93
GEORGIA
E–mail: duduch@rmi.acnet.ge

Frank–Olme Speck
Departamento de Matématica
Instituto Superior Técnico, U.T.L.
Avenida Rovisco Pais
1049–001 LISBOA
PORTUGAL
Email fspeck@math.ist.utl.pt