PSEUDODIFFERENTIAL OPERATORS ON COMPACT MANIFOLDS WTH LIPSCHITZ BOUNDARY

R. Duduchava[(1)], F.-O.Speck

Abstract. Pseudodifferential operators with non-smooth symbols on a manifold $\mathscr{M}$ with Lipschitz boundary are considered. Theorems about order reduction and localization of such operators in Bessel potential $H^s_p(\mathscr{M})$ and Hölder-Zygmund $Z^\alpha_p(\mathbb{R}_n)$ spaces are proved. A pseudodifferential operator $\mathbf{A}$ with locally sectorial matrix symbol is proved to be Fredholm in the space $H^s_2(\mathscr{M})$ and $\operatorname{Ind}\mathbf{A} = 0$ where s depends on $\mathbf{A}$. Application to a boundary value problem for an elastic body with crack is discussed in conclusion.

CONTENTS

1 Introduction **2**

1.1 Preface 2

1.2 Auxiliary material 3

1.3 Bessel potential operators: definition and general properties 5

2 Bessel potential operators for octants **7**

3 Pseudodifferential operators and reduction of order on bessel potential spaces **14**

3.1 Convolution operators on $\mathbb{R}_n$ 14

3.2 Order reduction operators for a canonical lipschitz domain 16

3.3 Operators of local type 18

3.4 Pseudodifferential operators on compact manifolds 20

3.5 Reduction of order for manifolds 24

3.6 Pseudodifferential operators with locally sectorial symbols 27

4 Pseudodifferential operators and reduction of order in hölder-zygmund spaces **31**

4.1 Hölder-zygmund spaces 31

4.2 Multipliers 37

4.3 Bessel potential operators 42

4.4 Pseudodifferential operators 43

5 Application to a crack problem **45**

References **51**

1 INTRODUCTION

1.1 PREFACE

We continue the investigations started in 1985. The first results were published only recently in [40]. There we introduced a Bessel potential operator (BPO in short) for the quarter-plane $\mathbb{R}^+ \times \mathbb{R}^+ \subset \mathbb{R}_2$ which can be also used for a cone in $\mathbb{R}_2$.

In the meantime two papers of R.Schneider [32, 33] appeared which were based on the manuscript [40] and succeeded in constructing BPO's for octants and canonical Lipschitz domains in $\mathbb{R}_n$. Two kinds of BPO's were involved: with non-smooth symbols (cf. [32]) and with smooth symbols from the Hörmander class $S^r_{1,0}(\mathbb{R}_n)$ (cf. [33]). The order reduction operator for a general Lipschitz domain $\Omega \subset \mathbb{R}_n$ was constructed as well (cf. [33]) and applied to the investigation of some strongly elliptic systems of pseudodifferential operators (ΨDOs in short) in $H^s_2(\Omega)$.

The results of [32] concerning BPO's with non-smooth symbols are extended here (cf. § 2). Namely it is proved that operators constructed in [40] for cones $\Omega \subset \mathbb{R}_2$ and in [32] for any Lipschitz domain $\Omega \subset \mathbb{R}_n$ are bounded and invertible in $H^s_p(\Omega)$ spaces for all $1 < p < \infty$ (we recall that in [32] only the case $p = 2$ was considered; cf. Theorems 2.1 and 2.3).

In § 3 ΨDOs with non C^∞-smooth symbols on manifolds with Lipschitz boundary are defined on Bessel potential spaces using operators of local type and some results are obtained: order reduction, Fredholm criteria (in terms of the local representatives). ΨDOs with locally sectorial symbols are introduced and a theorem is proved on their Fredholm property and on the index; the latter results generalize those from [26, 32, 33, 39] and proofs here are more transparent.

A different approach to the order reduction operators is demonstrated in [34].

In § 4 ΨDOs in Hölder-Zygmund spaces $Z^\alpha_p(\mathscr{M})(\alpha > 0,\ 1 \le p \le \infty)$on a manifold $\mathscr{M}$ with Lipschitz boundary are constructed. This section was inspired by the book [41], where BPOs for $Z^\alpha_\infty(\mathbb{R}_n)$ are described. We start with properties of the spaces $Z^\alpha_p(\mathscr{M})$ (with proofs when necessary). Theorems on multipliers in the space $Z^\alpha_p(\mathbb{R}_n)$ follow, which were known, as far as we know, only for $p = \infty$ (we recall here that $Z^\alpha_p(\mathscr{M}) = Z^\alpha_\infty(\mathscr{M})$ for compact $\mathscr{M}$ but they differ for non-compact $\mathscr{M}$). BPOs and order reduction operators for $Z^\alpha_p(\mathscr{M})$ spaces on compact and non-compact manifold $\mathscr{M}$ with Lipschitz boundary are constructed in § 4.3. In § 4.4 ΨDOs on a compact manifold are defined via operators of local type; here the recent results of R.Pöltz [29] were applied. Our approach is the localization principle . This method is not much refined, but it makes possible to investigate ΨDOs with non-smooth symbols. For the case of smooth symbols (of the class $S^r_{1,0}(\Omega \times \mathbb{R}_n)$, for example) and smooth boundary of the manifold it is possible to give almost the full describtion of the Boutet-de-Monvel algebras of the boundary value problems and get results on the spectrum and the resolvent of the operators under investigation. In our case this might be much complicated and we make no attempt to this. But the obtained results are sufficient to investigate the solvability of equations appearing in mechanics and mathematical physics.

Singular integral operators on $Z^\alpha_p(\mathbb{R}_n)$ spaces were investigated in [19, 20].

Applications to crack problems in isotropic elastic media with steady oscillation are exposed in § 5. We refer the reader to this section for the detailed formulations and survey of the earlier results.

It is only for notational convenience that we stick on the scalar case up to Sub§ 3.3 and in § 4 (cf. Remark 3.12). Most of the results remain valid for systems of equations (i.e. for operators with matrix symbols) in vector-spaces of functions.

1.2 AUXILIARY MATERIAL

For the Fourier transform we use the notation

$$\hat{u}(\xi) = Fu(\xi) = \int_{\mathbb{R}_n} exp\{ix\xi\}u(x)dx, \qquad \xi \in \mathbb{R}_n \tag{1.1}$$ e1.1

and F^{-1} is used for the inverse operator.

The well-known Bessel potential operators (cf. AS1, Ca1, SC1, St1 [1, 2, 38, 41])

$$\Lambda^s = F^{-1}\lambda^s F, \quad \lambda^s(\xi) = <\xi>^s = (1+ \mid \xi \mid^2)^{\frac{s}{2}}, \quad s \in \mathbb{R} \tag{1.2}$$ e1.2

generate the Bessel potential spaces

$$H_p^s(\mathbb{R}_n) = \{u \in D'(\mathbb{R}_n) : \|u\|_{sp} = \|\Lambda^s u\|_{L_p(\mathbb{R}_n)}\} < \infty$$

and arrange the isometrical isomorphisms between them

$$\Lambda^r : \ H_p^s(\mathbb{R}_n) \to \ H_p^{s-r}(\mathbb{R}_n), \quad s, r \in \mathbb{R}, \quad \|\Lambda^r u\|_{(s-r)p} = \|u\|_{sp}. \tag{1.3}$$ e1.3

d1.1 **Definition 1.1** *(cf. St1 [41]). An open subset $\Omega \subset \mathbb{R}_n$ is called a canonical Lipschitz domain if*

$$\Omega = \{(x', x_n) \in \mathbb{R}_n : \Phi(x') < x_n\}$$

for some real-valued Lipschitz function

$$\mid \Phi(x') - \Phi(y') \mid \leq L \mid x' - y' \mid, \quad x', y' \in \mathbb{R}_{n-1}. \tag{1.4}$$ e1.4

d1.2 **Definition 1.2** *A compact n-dimensional manifold $\overline{\mathscr{M}} = \mathscr{M} \cup \partial\mathscr{M}$ is said to have a Lipschitz boundary if there exists a covering of the manifold*

$$\overline{\mathscr{M}} = \bigcup_{j=1}^{N} U_j, \qquad \partial\mathscr{M} \subset \bigcup_{j=1}^{N_1} U_j, \qquad N_1 \leq N,$$

a canonical Lipschitz domain $\omega \subset \mathbb{R}_n$ and coordinate C^r-diffeomorphisms $(r \geq 1)$

$$\begin{aligned} &\beta_j : U_j \to V_j \subset \Omega, \qquad j = 1, 2, ..., N, \\ &V_j \cap \partial\Omega \neq \emptyset, \quad j = 1, 2, ..., N_1, \qquad V_j \cap \partial\Omega = \emptyset, \quad j = N_1 + 1, ..., N. \end{aligned} \tag{1.5}$$ e1.5

If $\mathscr{M}$ is a compact domain in $I\!R_n$ and has Lipschitz boundary, it is called Lipschitz domain (cf. [41], VI.3.3).

Let

$$S(I\!R_n) = \{f \in C^\infty(I\!R_n) : \sup\{\langle\xi\rangle^m \mid \partial_\xi^\alpha f(\xi) \mid < \infty : \xi$$

$$m \in I\!N, \alpha \in I\!N_n\}, \qquad \partial_\xi^\alpha f(\xi) = \partial_{\xi_1}^{\alpha_1} ... \partial_{\xi_n}^{\alpha_n} f(\xi)$$

represent the Frechet space of fast decreasing smooth functions. $S'(I\!R_n)$ is used for the space of tempered distributions, the adjoint space to $S(I\!R_n)$.

The Fourier operators $F^{\pm 1}$ and the multiplication operator $aI, a \in S(I\!R_n)$, are well-defined bounded operators in $S(I\!R_n)$ and in $S'(I\!R_n)$.

For a distribution $a \in S'(I\!R_n)$ the *convolution operator* is defined as

$$W_a^0 \varphi = F^{-1} a F \varphi, \qquad \varphi \in S(I\!R_n) \tag{1.7}$$
$$W_a^0 : S(I\!R_n) \to S'(I\!R_n)$$

e1.7

and $a(\xi)$ is called the *symbol* of W_a^0.

The set of functions (symbols) for which W_a^0 has a bounded extension $W_a^0 : L_p(I\!R_n) \to L_p(I\!R_n)$ is denoted by $M_p(I\!R_n)$ $(1 \leq p \leq \infty)$. The function $a \in M_p(I\!R_n)$ is called $L_p - multiplier$. The set $M_p(I\!R_n)$ endoved with the norm $\|a\|_{M_p} = \|W_a^0\|_{L_p(I\!R_n)}$ and pointwise multiplication forms a Banach algebra, since $W_a^0 W_b^0 = W_{ab}^0$ (cf. [17]).

By $M_{p,q}(I\!R_n)$ we denote the algebra $\bigcap_{p<r<q} M_r(I\!R_n)$.

Let further

$$M_p^r(I\!R_n) = \{\lambda^r a : a \in M_p(I\!R_n)\}. \tag{1.8}$$

e1.8

Since $\Lambda^r = W_{\lambda^r}^0$ (cf. (1.2)) it follows that the operator

$$W_a^0 : H_p^s(I\!R_n) \to H_p^{s-r}(I\!R_n), \qquad s, r \in I\!R, \ 1 < p < \infty$$

is bounded if and only if $a \in M_p^r(I\!R_n)$.

For $\alpha, \beta \in I\!N_n$ $\alpha \leq \beta$ denotes the inequalities $\alpha_j \leq \beta_j, (j = 1, 2, ...n)$ and $\alpha < \beta$ is used when $\alpha \leq \beta, \ \alpha \neq \beta$.

t1.3 **Theorem 1.3** *(cf. [35]).*

Let $a \in L_1^{loc}(I\!R_n)$ and the condition

$$\sum_{|\beta| \leq \left[\frac{n}{2}\right]+1, \beta \leq 1} R^{-n} \int_{\frac{R}{2} < |\xi| \leq 2R} \mid \xi^\beta D^\beta a(\xi) \mid^2 d\xi < \infty \tag{1.9}$$

e1.9

hold; then $a \in M_{1,\infty}(I\!R_n)$.

The inequality

$$\sup\{\mid \xi^\beta \partial_\xi^\beta a(\xi) \mid : \beta \in I\!N_n, \ \ \beta \leq 1, \ \ \mid \beta \mid \leq \frac{n}{2} + 1\} < \infty$$

implies (1.9).

$S^r_{1,0}(\Omega\times I\!R_n)$ (or $S^r(\Omega\times I\!R_n)$, $\quad \Omega\subset I\!R_n$) is used for the Hörmander class of functions

$$S^r_{1,0}(\Omega\times I\!R_n)=\{a:\mid\partial^\alpha_x\partial^\beta_\xi a(x,\xi)\mid< C_{\alpha,\beta}\langle\xi\rangle^{r-|\alpha|},\quad \alpha,\beta\in I\!N_n\}. \tag{1.10}$$

e1.10

$S^r(I\!R_n)\subset S^r_{1,0}(\Omega\times I\!R_n)$ denotes the class of functions $a(x,\xi)\equiv a(\xi)$, independent of the variable x. Due to Theorem 1.3 $S^r(I\!R_n)\subset M^r_{1,\infty}(I\!R_n)$.

1.3 BESSEL POTENTIAL OPERATORS: DEFINITION AND GENERAL PROPERTIES

The next two definitions were suggested by F.Speck and R.Duduchava (cf. [40]).

d1.4 **Definition 1.4** *Let $\Omega\subset I\!R_n$ be a Lipschitz domain. A linear operator $B:S(I\!R_n)\to S'(I\!R_n)$ is said to be a Bessel potential operator of order r for Ω $\ (B\in\ BPO(r,\Omega)$ in brief), if B has the following properties:*

(i)B is translation invariant

$$BV_h=V_hB,\qquad V_h\varphi(t)=\varphi(t-h),\quad h\in I\!R_n; \tag{1.11}$$

e1.11

(ii) there exist continuous extensions

$$B:\ H^s_p(I\!R_n)\to\ H^{s-r}_p(I\!R_n) \tag{1.12}$$

e1.12

which are invertible for any $s\in I\!R_n, 1<p<\infty$;

(iii) B and its inverse B^{-1} preserve supports within $\overline{\Omega}$:

$$\operatorname{supp}B^{\pm1}\varphi\subset\overline{\Omega},\quad \text{if}\quad \varphi\in D(I\!R_n)=\ C^\infty_0(I\!R_n),\quad \operatorname{supp}\varphi\subset\overline{\Omega}. \tag{1.13}$$

e1.13

d1.5 **Definition 1.5** *(cf. [40]). $B\in BPO(1,\Omega)$ is said to be a Bessel potential operator for Ω $\ (B\in BPO(\Omega)$ in brief) if it generates a group $\{B^r\}_{r\in\mathbf{R}}$ of operators such that for any $s,r\in I\!R$ the following holds:*

$$B^r\in\ BPO(r,\Omega),\quad B^rB^s=B^{r+s},\quad B^0=I,\quad B^1=B \tag{1.14}$$

e1.14

For a Lipschitz domain $\Omega\subset I\!R_n$ and any $s\in I\!R,\ 1<p<\infty$ the set

$$\tilde{H}^s_p(\Omega)=\{u\in\ H^s_p(I\!R_n):\operatorname{supp}u\subset\overline{\Omega}\} \tag{1.15}$$

e1.15

represents a subspace of $\ H^s_p(I\!R_n)$; in particular $L_p(\Omega)$ is the subspace of $\ L_p(I\!R_n)$.

$H^s_p(\Omega)$ denotes the space

$$H^s_p(\Omega)=\{u=r_\Omega v:v\in\ H^s_p(I\!R_n)\}, \tag{1.16}$$

e1.16

endoved with the factor-norm

$$\|u\|_{sp}=\inf\{\|v\|_{sp}:v\in\ H^s_p(I\!R_n),\quad r_\Omega v=u\},$$

where r_Ω denotes the restriction of a function $v \in S'(I\!R_n)$ to Ω.

For a Lipschitz domain $\Omega \subset I\!R_n$ there exists an extension operator

$$\ell : \; H_p^s(\Omega) \to \; H_p^s(I\!R_n) \tag{1.17}$$ e1.17

which is independent of $s > 0$ and $1 < p < \infty$ (cf. [41], VI.3).

l1.6 **Lemma 1.6** *(cf. [40]. Let $\Omega \subset I\!R_n$ be a Lipschitz domain and $r \in I\!R$. Then $B \in BPO(r, \Omega)$ holds if and only if the following is valid:*

(i) $B = W_\Phi^0$, $\qquad \Phi^{\pm 1} \in M_p^{\pm r}(I\!R_n)$;

(ii) the operator

$$B = W_\Phi^0 : \; \tilde{H}_p^s(\Omega) \to \; \tilde{H}_p^{s-r}(\Omega) \tag{1.18}$$ e1.18

is bounded and invertible by $B^{-1} = W^0_{\Phi^{-1}}$ for any $s \in I\!R_n$, $1 < p < \infty$.

Proof cf. in [40]. ■

l1.7 **Lemma 1.7** *Let $\Omega \subset I\!R_n$ be a Lipschitz domain, $r \in I\!R$ and $B \in BPO(r, \Omega)$. The following holds:*

(i) there exists a generalized (distributional) kernel $k_B \in \mathscr{D}'(I\!R_n)$ of $B = W_\Phi^0$ such that $Bu = k_B * u$, for any $u \in C_0^\infty(I\!R_n)$; if $0 \in \overline{\Omega}$ then $\operatorname{supp} k_B \subset \overline{\Omega}$;

(ii) let $\mathscr{K} \subset I\!R_n$ be another Lipschitz domain and $0 \in \overline{\mathscr{K}}$, $\Omega + \mathscr{K} \subset \overline{\Omega}$; then $BPO(r, \mathscr{K}) \subset BPO(r, \Omega)$ for any $r \in I\!R$.

. **Proof**. Existence of k_B is well known (cf. [17, 18]). To prove the next claim $\operatorname{supp} k_B \subset \overline{\Omega}$ we assume $x_0 \notin \overline{\Omega}$ and nevertheless $x_0 \in \operatorname{supp} k_B$. Then there exist $u_n \in C_0^\infty(I\!R_n)$, $mes \operatorname{supp} u_n \to 0$, $< k_B, u_n > \neq 0 (n = 1, 2, ...)$. Consider any $t_0 \in \Omega, |\; t \;| < dist(x_0, \Omega)$ and $\tilde{u}_n(y) = u_n(x_0 + t_0 - y)$; obviously $\operatorname{supp} \tilde{u}_{n_0} = x_0 + t_0 - \operatorname{supp} u_{n_0} \subset \Omega$ for a large n_0. By the definition of convolution (cf. [18], v.1) $k_B * \tilde{u}_{n0}(x_0 + t_0) = < k_B, V_{x_0+t_0}\tilde{u}_{n0} > = < k_B, u_{n0} > \neq 0$, where $V_z\varphi(t) = \varphi(z - t)$. This contradicts condition (1.13) since $x_0 + t_0 \notin \Omega$ while $\operatorname{supp} \tilde{u}_{n_0} \subset \Omega$.

To prove assertion (ii) we notice that (cf. Theorem 4.1.1 in [18] , v.1) $\operatorname{supp} Bu \subset \operatorname{supp} k_B + \operatorname{supp} u$. Let $B \in BPO(r, \mathscr{K})$. Then $Bu = W_\Phi^0 u = k_B * u$, $\Phi^{\pm 1} \in M_p^{\pm r}(I\!R_n)$ and $\operatorname{supp} k_B \subset \mathscr{K}$ (cf. (i-ii)). Thus to prove that $B \in BPO(r, \Omega)$ only property (1.13) needs to be verified. We have (cf (i)) $\operatorname{supp} Bv \subset \operatorname{supp} k_B + \operatorname{supp} v \subset \overline{\mathscr{K}} + \overline{\Omega} \subset \overline{\Omega}$ for any $v \in \mathscr{D}(I\!R_n) = C_0^\infty(I\!R_n)$, with $\operatorname{supp} v \subset \overline{\Omega}$.

A similar statement holds for the inverse operator B^{-1}. ■

The sufficient part of the next lemma was actually applied in [32, 33] to prove the main assertion (cf. Theorem 3.6 below).

l1.8 **Lemma 1.8** *Let $\Omega \subset I\!R_n$ be a Lipschitz domain, $r \in I\!R$, $Bu = W_\Phi^0 u = k_B * u$, $\Phi^{\pm 1} \in M_p^{\pm r}(I\!R_n)$ where k_B is the distributional kernel.*

Then $B \in BPO(r, \Omega)$ if and only if $\Omega + \operatorname{supp} k_B \subset \Omega$.

Proof. Sufficiency follows from the above mentioned inclusion (cf. Theorem 4.1.1 in [18], v.1) $\operatorname{supp} Bu \subset \operatorname{supp} k_B + \operatorname{supp} u$ and Definition 1.4. The necessity of the condition follows from assertion (ii) of Lemma 1.7 since $\operatorname{supp} k_B \subset \Omega - x$ for any $x \in \Omega$ and BPO(r,Ω)=BPO(r,$\Omega - x$). ■

Well-known Bessel potential operators besides Λ^r (cf. (1.2)-(1.3)) are the following

$$\Lambda^r_\pm = W_{\lambda^r_\pm} \in BPO(r, \mathbb{R}^\pm_n), \qquad \lambda_\pm(\xi) = \xi_n \pm i(1+ \mid \xi' \mid^2)^{\frac{1}{2}}, \tag{1.19}$$ e1.19

$$s, r \in \mathbb{R}, \;\; 1 < p < \infty, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^+_n := \mathbb{R}_{n-1} \times \mathbb{R}^+.$$

For the proof of the next two lemmas we refer to [19] and [8] respectively (cf. also [12]), but one can try to prove it independently.

l1.9 **Lemma 1.9** *Let* $\mathbf{A} : \mathscr{A}_j \to \mathscr{B}_j, \;\; j = 1, 2$ *be a linear bounded operator between Banach spaces and let* $\mathbf{A}$ *admit a left regularizer* $\mathbf{RA} = I + T$ *where* T *is compact in* $\mathscr{A}_1$ *and* $\mathscr{A}_2$.

If the embedding $\mathscr{A}_1 \subset \mathscr{A}_2$ *is dense, then the kernels Ker*$\mathbf{A}$ *of the operator in the spaces* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *coincide.*

If, in addition, $\mathbf{A}$ *is a Fredholm operator and the embedding* $\mathscr{B}_1 \subset \mathscr{B}_2$ *is dense, then the index* $\operatorname{Ind} \mathbf{A}$ *of the operator is independent of the spaces as well.*

l1.10 **Lemma 1.10** *Let* $\mathbf{A} : \mathscr{A}_j \to \mathscr{B}_j$ *be a Fredholm operator between Banach spaces and* $\operatorname{Ind} \mathbf{A}$ *be the same for* $j = 1, 2$.

If the embedding $\mathscr{A}_1 \subset \mathscr{A}_2$ *holds, and the embedding* $\mathscr{B}_1 \subset \mathscr{B}_2$ *is dense, then the kernels Ker*$\mathbf{A}$ *of the operator in the spaces* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *coincide.*

2 BESSEL POTENTIAL OPERATORS FOR OCTANTS

Let

$$\Sigma^n_1 = \{x \in \mathbb{R}_n : x_j > 0, \;\; j = 1, 2, ..., n\} = \underbrace{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{n}$$

denote the first octant in $\mathbb{R}_n$ and consider a function $\psi(\xi)$ with the following properties:

$$\psi(\lambda\xi) \equiv \psi(\xi), \qquad \xi \in \mathbb{R}_n, \qquad \lambda > 0, \tag{2.1}$$ e2.1

$$\psi(x) + 1 \geq 0\,, \qquad \operatorname{supp}[\psi + 1] \subset \Sigma^n_1, \qquad \psi \in C^\infty(S^{n-1}), \tag{2.2}$$ e2.2

$$\int_{S^{n-1}} \psi(\omega) d_\omega S = 0, \qquad S^{n-1} = \{\omega \in \mathbb{R}_n : \mid \omega \mid = 1\}. \tag{2.3}$$ e2.3

A Bessel potential operator for Σ^n_1 is defined as follows (cf. [33])

$$\Lambda^r_{\psi, \Sigma^n_1} = W^0_{a^r}, \quad a^r(\xi) = (1+ \mid \xi \mid^2)^{\frac{r}{2}} \exp[r(F\psi k_o)(\xi)], \tag{2.4}$$

$$k_o(x) = \frac{1}{2(4\pi)^{\frac{n}{2}}} \int_0^\infty t^{-\frac{n}{2}} \exp\left[-\frac{\mid x \mid^2}{4t} - t\right] \frac{dt}{t}.$$

Similar operators are (cf. [32])

$$\Lambda^r_{\psi_+,\Sigma^n_1} = W^0_{g^r}, \quad g^r(\xi) = (1+ \mid \xi \mid^2)^{\frac{r}{2}} \exp[r(F\psi_+ k_o)(\xi)], \tag{2.5}$$

$$\psi_+(\xi) = \prod_{j=1}^{n}(1 + \operatorname{sgn} x_j) - 1,$$

where, due to discontinuity of $\psi_+(\xi)$, the symbol $g^r(\xi)$ is not smooth (in contrast to $a^r(\xi)$; cf. theorems 2.1 and 2.2). This was the reason that the operator $\Lambda_{\psi,\Sigma^n_1} = W^o_{a^r}$ was introduced (cf. [33]), while for $\Lambda_{\psi_+,\Sigma^n_1} = W^o_{g^r}$ the boundedness was proved in the spaces $H^s_p(I\!\!R_n)$ only for the case $p = 2$.

Motivations for the choice of operators (2.4) and (2.5) are the following formulas

$$\Lambda^r_{\psi^+_n,\boldsymbol{R}^+_n} = \Lambda^r_+ = W^0_{\lambda^r_+}, \qquad \psi^+_n(\xi) = \operatorname{sgn} \xi_n, \tag{2.6}$$

$$\lambda^r_+(\xi) = [\xi_n + i(1+ \mid \xi \mid^2)^{\frac{1}{2}}]^r = (1+ \mid \xi \mid^2)^r \exp[r(F\psi^+_n k_o)(\xi)],$$

$$\Lambda^r_{\psi_+,\Sigma^2_1} = \Lambda^r_1 = W^0_{\lambda^r_1}, \tag{2.7}$$

$$\lambda^r_1(\xi) = \langle\xi\rangle^r \exp[r(F\psi_+ k_o)(\xi)]$$

$$= -\left(\frac{\xi_1 - i(1+\xi_2^2)^{\frac{1}{2}}}{\xi_1 + i(1+\xi_2^2)^{\frac{1}{2}}}\right)^r \exp[rI(\xi_1,\xi_2) + rI(\xi_2,\xi_1)]\,,$$

$$I(\xi_1,\xi_2) := -\int_1^\infty \frac{\tau}{\xi^2+\tau^2}\,\frac{2\xi_1}{\pi(\xi_2^2+\tau^2)^{\frac{1}{2}}}\log\frac{\xi_2 + (\xi_1^2+\tau^2)^{\frac{1}{2}}}{\tau}\,d\tau\,,$$

where (2.6) represents a Bessel potential operator for the half-space $I\!\!R^+_n$ (cf. (1.19)) and (2.7) for the quarter-plane $I\!\!R^+ \times I\!\!R^+ \subset I\!\!R_2$ (cf. [40]), respectively.

Several assertions, concerning operators (2.4), (2.5) and proved in [32, 33] are collected in the following theorem.

Theorem 2.1 *Let* $s, r \in I\!\!R$. *Then* $B = \Lambda_{\psi,\Sigma^n_1} \in BPO(\,\Sigma^n_1)$ *and* $B^r =$
$= \Lambda^r_{\psi,\Sigma^n_1} = W^0_{a^r}, \quad a^r \in S^r(I\!\!R_n)$.

The operators

$$B^r_+ = \Lambda^r_{\psi_+,\Sigma^n_1} = W^0_{g^r} : H^s_2(I\!\!R_n) \to H^{s-r}_2(I\!\!R_n) \tag{2.8}$$

are invertible, $B^r_+ B^{-r}_+ = B^o = I \quad (s, r \in I\!\!R)$ *and preserve supports*

$$\operatorname{supp} B^r_+\varphi \subset \Sigma^n_1 \qquad \text{if} \qquad \operatorname{supp}\varphi \subset \Sigma^n_1. \tag{2.9}$$

The next theorem completes the foregoing one.

Theorem 2.2 *Let* $s, r \in I\!R$. *Then* $B_+ = \Lambda_{\psi_+,\Sigma_1^n} \in BPO(\ \Sigma_1^n)$. *The symbol* $g^r(\xi)$ *has radial limits*

$$(g^r)^\infty(\xi) = \lim_{R\to\infty} R^{-r} g^r(R|\xi|^{-1}\xi) = \exp[r(F\psi_+ k^o)(\xi)]\,, \tag{2.10}$$

$$k^o(x) = \frac{\pi^{-\frac{n}{2}}\Gamma(\frac{n}{2})}{2|x|^n}$$

and $(g^r)^\infty \in M_{1,\infty}(I\!R_n)$.

Proof. To prove the first claim $\Lambda_{\psi_+,\Sigma_1^n} \in BPO(\ \Sigma_1^n)$ it suffices to get the inclusion $g^r \in M^r_{1,\infty}(I\!R_n)$ (cf. Theorem 2.1); due to Theorem 1.3 this follows from the estimate

$$\sup\{|\ \xi^\alpha \partial_\xi^\alpha[\langle\xi\rangle^{-r} g^r(\xi)]\ | < \infty : \xi \in I\!R_n\} < \infty \qquad \alpha \in I\!N_n. \tag{2.11}$$

Since $\langle\xi\rangle^{-r} g^r(\xi) = \exp[(F\psi_+ k_o)(\xi)]$, (2.11) is implied by the estimate

$$\sup\{|\ \xi^\alpha \partial_\xi^\alpha (F\psi_+ k_o)(\xi)\ | : \xi \in I\!R_n\} < \infty, \quad \alpha \in I\!N_n. \tag{2.12}$$

To prove (2.12) we introduce the notation

$$g_{\alpha,\beta}(x) = \frac{(-1)^\alpha (ix)^\beta}{2(4\pi)^{\frac{n}{2}}} \psi_+(x) \partial_x^\alpha \left[x^\alpha \exp\left(-\frac{|\ x\ |^2}{4}\right)\right];$$

obviously $g_{\alpha,\beta} \in L_1(I\!R)$ and therefore $\hat{g}_{\alpha,\beta} \in C(\dot{I\!R}_n)$ where $\dot{I\!R}_n = I\!R_n \cup \{\infty\}$ (i.e. $\hat{g}_{\alpha,\beta}(x)$ is continuous and uniformly bounded on $I\!R_n$, and has the same limit however ξ tends to infinity).

Since (cf. (2. 5))

$$\int_{S^{n-1}} \omega^{2\gamma}\psi_+(\omega) d_\omega S = 2^n \int_{S^{n-1}\cap\Sigma_1^n} \omega^{2\gamma} d_\omega S - \int_{S^{n-1}} \omega^{2\gamma} d_\omega S = 0,$$

we get

$$\hat{g}_{\alpha,0}(0) = \frac{1}{2(4\pi)^{\frac{n}{2}}} \int_{\boldsymbol{R}_n} \psi_+(x)\partial_x^\alpha \left[x^\alpha \exp\left(-\frac{|\ x\ |^2}{4}\right)\right] dx =$$

$$= \sum_{\gamma\le\alpha} b_\gamma \int_{\boldsymbol{R}_n} \psi_+(x) x^{2\gamma} \exp\left(-\frac{|\ x\ |^2}{4}\right) dx =$$

$$= \sum_{\gamma\le\alpha} b_\gamma \int_0^\infty R^{n-1} \exp\left(-\frac{R^2}{4}\right) dR \int_{S^{n-1}} \omega^{2\gamma}\psi_+(\omega) d_\omega S = 0. \tag{2.13}$$

In so far as $\partial_\xi^\beta \hat{g}_{\alpha,0}(\xi) = \hat{g}_{\alpha,\beta}(\xi) \in C(\dot{I\!R}_n)$ we have $\hat{g}_\alpha = \hat{g}_{\alpha,0} \in C^\infty(\dot{I\!R}_n)$.

If we recall Lagrange's formula and take into account (2.13), we get

$$\hat{g}_\alpha(\xi) = \hat{g}_\alpha(\xi) - \hat{g}_\alpha(0) = \sum_{j=1}^n \partial_{\xi_j} \hat{g}_\alpha(t_o \xi)\xi_j, \quad 0 \le t_o \le 1, \quad |\, \xi \,| \le 1$$

which implies

$$|\, \hat{g}_\alpha(\xi) \,| \le M \frac{|\, \xi \,|}{1 + |\, \xi \,|}. \tag{2.14}$$

We proceed as follows

$$\begin{aligned} \xi^\alpha \partial_\xi^\alpha \hat{k}_+(\xi) &= \frac{(-1)^\alpha \xi^\alpha}{2(4\pi)^{\frac{n}{2}}} \int_{\boldsymbol{R}_n} \exp(ix\xi) x^\alpha \psi_+(x) dx \int_0^\infty t^{-\frac{n}{2}} \exp\left(-\frac{|\, x \,|^2}{4t} - t\right) \frac{dt}{t} = \\ &= \frac{(-1)^\alpha \xi^\alpha}{2(4\pi)^{\frac{n}{2}}} \int_{\boldsymbol{R}_n} \exp(ix\sqrt{t}\xi) x^\alpha \psi_+(x) dx \int_0^\infty t^{\frac{|\alpha|}{2}} \exp\left(-\frac{|\, x \,|^2}{4} - t\right) \frac{dt}{t} = \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}} \int_0^\infty \exp(-t) \frac{dt}{t} \int_{\boldsymbol{R}_n} [\partial_x^\alpha \exp(ix\sqrt{t}\xi)] x^\alpha \psi_+(x) \exp\left(-\frac{|\, x \,|^2}{4}\right) dx = \\ &= \int_0^\infty \exp(-t) \hat{g}_\alpha(\sqrt{t}\xi) \frac{dt}{t}; \end{aligned} \tag{2.15}$$

here the last integral exists due to (2.14) and an exchange of the order of integration is legal since the integrand is absolutely integrable; the partial integration by x is also allowed here. The last two properties (2.15) and (2.14) imply (2.12).

Now we prove the existence of the radial limits

$$\hat{k}_+^\infty(\xi) = \lim_{R \to \infty} \hat{k}_+(R\xi) = \widehat{\psi_+ k^0}(\xi), \quad \hat{k}_+ \in C^{n-1}(S^{n-1}). \tag{2.16}$$

Let

$$G_\alpha(x) = \frac{1}{2(4\pi)^{\frac{n}{2}}} (ix)^\alpha \psi_+(x) \exp\left(-\frac{|\, x \,|^2}{4}\right), \quad G_\alpha \in L_1(I\!R_n);$$

due to (2.14) there holds

$$|\, \hat{G}_0(\xi) \,| = |\, \hat{g}_0(\xi) \,| \le M \frac{|\, \xi \,|}{1 + |\, \xi \,|}. \tag{2.17}$$

Now we recall some formulas (cf. St1 [41], § V.3.1 and Du1 [10], Lemma 1.35 respectively)

$$FG_0^0(\xi) = \exp(-\,|\, \xi \,|^2), \qquad G_0^0(x) = \frac{1}{2(4\pi)^{\frac{n}{2}}} \exp\left(-\frac{|\, x \,|^2}{4}\right),$$

$$(F\chi_n^+ u)(\xi) = -\frac{1}{2}\hat{u}(\xi) - \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\hat{u}(\xi', t) dt}{\xi_n - t}, \quad \chi_n^+(\xi) = \frac{1}{2}(1 + \operatorname{sgn} \xi_n). \tag{2.18}$$

and proceed as follows

$$\hat{G}_\alpha(\xi) = 2^{-n} \prod_{j=1}^{n} \left[a_j(\xi_j) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a_j(\tau)d\tau}{\tau - \xi_j} \right] - \frac{1}{2} \exp(-|\xi|^2), \qquad (2.19)$$

$$a_j(t) = \partial_t^{\alpha_j} \exp(-t^2).$$

To estimate the integrals we rewrite them in form

$$A_j(t) = \int_{-\infty}^{\infty} \frac{a_j(\tau)d\tau}{\tau - t} = A_{j1}(t) + A_{j2}(t), \qquad A_{j1}(t) = \int_{|t-\tau|>1} \frac{a_j(\tau)d\tau}{\tau - t}, \qquad (2.20)$$

$$A_{j2}(t) = \int_{t-1}^{t+1} \frac{a_j(\tau)d\tau}{\tau - t} = \int_{t-1}^{t+1} \frac{a_j(\tau) - a_j(t)}{\tau - t} d\tau,$$

Using the estimate

$$| a_j(t) - a_j(\tau) | \leq C_1 | t - \tau | \exp\left(-\frac{| t |^2}{2} \right), \quad | t - \tau | \leq 1,$$

we get

$$| A_{j2}(t) | \leq C_1 \exp\left(-\frac{| t |^2}{2} \right) \int_{t-1}^{t+1} dt = 2C_1 \exp\left(-\frac{| t |^2}{2} \right). \qquad (2.21)$$

For $A_{j1}(t)$ in (2.20) we have

$$| A_{j1}(t) | \leq \int_{1}^{\infty} \frac{| a_j(t+\tau) | + | a_j(t-\tau) |}{\tau} d\tau \leq C_2 \int_{1}^{\infty} \exp\left(-\frac{t^2 + \tau^2}{2} \right) \frac{d\tau}{\tau} +$$

$$+ C_2 \exp(-t^2) \int_{1}^{\frac{t}{2}} \frac{d\tau}{\tau} + \frac{2C_2}{t} \int_{1}^{\infty} \exp\left[-\frac{(t-\tau)^2}{2} \right] d\tau \leq \frac{C_3}{t}, \quad t \geq 1. \qquad (2.22)$$

The inequalities (2.20)-(2.22) yield

$$| A_j(t) | \leq C_4 (1 + | t |)^{-1}, \qquad t \in I\!R$$

and therefore (cf. also (2.20))

$$|\hat{G}_0(\xi)| \leq C_5 \frac{| \xi |}{(1 + |\xi|)} \prod_{j=1}^{n} (1 + | \xi_j |)^{-1},$$

$$|\hat{G}_\alpha(\xi)| \leq C_5 \prod_{j=1}^{n} (1 + |\xi_j|)^{-1}, \qquad \xi \in I\!R_n. \qquad (2.23)$$

Since (cf. [33])

$$\begin{aligned} k_+(R\xi) &= \frac{1}{2(4\pi)^{\frac{n}{2}}}\int_{\boldsymbol{R}_n}\exp(iRx\xi)\psi_+(x)dx\int_0^\infty t^{-\frac{n}{2}}\exp\left(-\frac{|x|^2}{4t}-t\right)\frac{dt}{t} \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}}\int_0^\infty t^{-\frac{n}{2}}\exp\left(-\frac{t}{R^2}\right)\frac{dt}{t}\int_{\boldsymbol{R}_n}\exp\left(i\xi y-\frac{|y|^2}{4t}\right)\psi_+(y)dy \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}}\int_0^\infty \exp\left(-\frac{t}{R^2}\right)\hat{G}_0(\sqrt{t}\xi)\frac{dt}{t}; \end{aligned} \tag{2.24}$$

due to (2.23) the limit $R\to\infty$ in (2.24) exists

$$\begin{aligned} \hat{k}_+^\infty(\xi) &= \lim_{R\to\infty}\hat{k}_+(R\xi) = \int_0^\infty \hat{G}_0(\sqrt{t}\xi)\frac{dt}{t} \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}}\int_0^\infty\frac{dt}{t}\int_{\boldsymbol{R}_n}\exp\left(i\sqrt{t}\xi x-\frac{|\,x\,|^2}{4}\right)\psi_+(x)dx \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}}\int_{\boldsymbol{R}_n}\exp(i\xi y)\psi_+(y)dy\int_0^\infty t^{-\frac{n}{2}}\exp\left(-\frac{|\,y\,|^2}{4t}\right)\frac{dt}{t} \\ &= \frac{1}{2(\pi)^{\frac{n}{2}}}\int_{\boldsymbol{R}_n}\exp(i\xi y)\psi_+(y)\frac{dy}{|\,y\,|^n}\int_0^\infty \tau^{n/2}\exp(-\tau^2)\frac{d\tau}{\tau} \\ &= \frac{\Gamma(n/2)}{2\pi^{\frac{n}{2}}}\int_{\boldsymbol{R}_n}\exp(i\xi y)\psi_+(y)\frac{dy}{|\,y\,|^n}\,. \end{aligned}$$

The last formula represents the Fourier transform of the homogeneous (of order $-n$) function $|\,x\,|^{-n}\,\psi_+(x)$ and (2.12) follows from the well-known formula for the symbol of a singular integral operator (cf. [28], X.1.17).

Applying (2.23) again we find the derivatives

$$\partial_\xi^\alpha\hat{k}_+^\infty(\xi) = \int_0^\infty t^{\frac{|\alpha|-2}{2}}\hat{G}_0(\sqrt{t}\xi)dt$$

which exist and are continuous for any $|\,\alpha\,|\le n-1$ due to $G_\alpha\in L_1(I\!\!R_n)$. Due to (2.11), (2.12) and Lemma 3.1 which is proved in the next section, this implies $(g^r)^\infty\in M_{1,\infty}(I\!\!R_n)$. ■

Remark 2.3 *For the function $k_+(x)=\psi_+(x)k_o(x)$ (cf. (2.4) and (2.5)) it can be proved that it represents a singular integral kernel in the sense of [41], II.3:*

$$|\,k_+(x)\,|\le\frac{B}{|\,x\,|^n},\qquad |\,x\,|>0, \tag{2.25}$$

$$\int_{|x|\ge 2|y|}|\,k_+(x-y)-k_+(x)\,|\,dx\le B,\qquad |\,y\,|>0, \tag{2.26}$$

$$\int_{R_1<|x|<R_2}|\,k_+(x)\,|\,dx=0,\qquad 0<R_1<R_2<\infty. \tag{2.27}$$

This implies $\hat{k_+} \in M_{1,\infty}(I\!R_n)$ *and, further,* $a^r \in M^r_{1,\infty}(I\!R_n)$.

Similarly all this holds also for the kernel $k(x) = \psi(x)k_o(x)$ *(cf. (2.1)-(2.4)) and this is easier to verify: in place of (2.26) the stronger inequality*

$$\mid \nabla k(x) \mid \leq B \mid x \mid^{-n-1} . \qquad\qquad \nabla u = \operatorname{grad} u$$

holds and implies (2.26).

Remark 2.4 *Formulas similar to (2.10) hold for the symbol* $a^r(\xi)$ *of the Bessel potential operator* $\Lambda_{\psi,\Sigma^n_1}$ *(cf. (2.1)-(2.4)), namely*

$$\begin{aligned}(a^r)^\infty(\xi) &= \lim_{R\to\infty} R^{-r} a^r(R|\xi|^{-1}\xi) = \exp[r(F\psi k^o)(\xi)] \qquad (2.28)\\ &= \exp\left[r\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{S^{n-1}} \psi(\theta')\left(\ln\frac{1}{\mid \cos(\theta\cdot\theta')\mid} + \frac{i\pi}{2}\operatorname{sgn}(\theta\cdot\theta')\right) d_{\theta'}S \right],\\ &\qquad \theta = \mid \xi \mid^{-1} \xi \in S^{n-1}.\end{aligned}$$

l2.5 **Lemma 2.5** *Let* $r \in I\!R$.*h e radial limits* $(a^r)^\infty(\xi)$ *and* $(g^r)^\infty(\xi)$ *(cf. (2.10) and (2.28)) satisfy the inequalities*

$$\mid \arg(a^r)^\infty(\xi) \mid \leq \frac{\pi|r|}{2}, \qquad \mid \arg(g^r)^\infty(\xi) \mid \leq \frac{\pi \mid r \mid}{2}, \qquad \xi \in I\!R_n, \qquad r \in I\!R. \quad (2.29)$$ e2.29

Proof. Due to (2.28)

$$\begin{aligned}\arg\,(a^r)^\infty(\theta) &= r\Im\left[F\psi k^o(\theta)\right] = r\frac{\Gamma(n/2)}{4\pi^{n/2-1}} \int_{S^{n-1}} \psi(\omega)\operatorname{sgn}(\theta\cdot\omega)\, d_\omega S\\ &= r\frac{\Gamma(n/2)}{4\pi^{\frac{n}{2}-1}} \int_{S^{n-1}} [\psi(\omega)+1]\operatorname{sgn}(\theta\cdot\omega)\, d_\omega S\,, \qquad (2.30)\end{aligned}$$ e2.30

since $\int_{S^{n-1}} \operatorname{sgn}(\theta\cdot\omega)\, d_\omega S = 0$.

Recalling the formulae

$$\int_{S^{n-1}} d_\omega S = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and the properties (2.1)-(2.3) of the indicator function $\psi(x)$, we get

$$\begin{aligned}\int_{S^{n-1}} \big|\psi(\omega)+1\big|\, d_\omega S &= \int_{S^{n-1}} \big[\psi(\omega)+1\big]\, d_\omega S = \int_{S^{n-1}} \psi(\omega)\, d_\omega S\\ &+ \int_{S^{n-1}} d_\omega S = \int_{S^{n-1}} d_\omega S = \frac{2\pi^{n/2}}{\Gamma(n/2)}\,. \qquad (2.31)\end{aligned}$$ e2.31

(2.31) and (2.31) yield

$$|\arg (a^r)^\infty(\theta)| \leq |r| \frac{\Gamma(n/2)}{4\pi^{n/2-1}} \int_{S^{n-1}} \big|\psi(\omega)+1\big|\, d_\omega S \leq \frac{\pi|r|}{2}$$

and the proof of the first inequality in (2.29) is completed. The second inequality is proved similarly ■

3 PSEUDODIFFERENTIAL OPERATORS AND REDUCTION OF ORDER ON BESSEL POTENTIAL SPACES

3.1 CONVOLUTION OPERATORS ON $\mathbb{R}_n$

A function $\psi(\xi)$ on $\mathbb{R}_n$ is called *piecewise-constant* if there exists a partition of $\mathbb{R}_n$ on polyhedral domains by a finite number of (n-1)-dimensional hyperplanes such that $\psi(\xi)$ is constant on each polyhedral domain (bounded or unbounded). The set of such functions (defined for different partitions) will be denoted by $PC^0(\mathbb{R}_n)$. It is known that $PC^0(\mathbb{R}_n) \subset M_{1,\infty}(\mathbb{R}_n)$ (cf., for example, [15]). The subalgebra, generated by $PC^0(\mathbb{R}_n)$ in $M_p(\mathbb{R}_n)$ we denote by $PC_p(\mathbb{R}_n)$ and

$$PC^r_p(\mathbb{R}_n) = \{\langle\xi\rangle^r a(\xi) : a \in PC_p(\mathbb{R}_n)\}.$$

Due to the well-known inequality (cf. [17])

$$\|a\|_{M_2} = \sup\{|\, a(\xi)\,| : \xi \in \mathbb{R}_n\} \leq \|a\|_{M_p}, \qquad 1 \leq p \leq \infty$$

the radial limits exist

$$\begin{gathered} a^x(\xi) \equiv a^x(\omega) = \lim_{\varepsilon\to 0} a(x+\varepsilon\xi), \quad a^\infty(\xi) \equiv a^\infty(\omega) = \lim_{R\to\infty} R^{-r} a(R\xi), \\ a \in PC^r_p(\mathbb{R}_n), \quad x \in \mathbb{R}_n, \quad 0 \neq \xi \in \mathbb{R}_n, \quad \omega = |\,\xi\,|^{-1}\xi \in S^{n-1} \end{gathered} \tag{3.1}$$

since they exist for $a \in PC^0(\mathbb{R}_n)$. The following inclusions hold (cf. [17, 15])

$$\begin{gathered} PC_p(\mathbb{R}_n) = PC_{p'}(\mathbb{R}_n) \subset PC_r(\mathbb{R}_n) \subset PC_2(\mathbb{R}_n), \\ p' = \frac{p}{p-1}, \quad r \in [p,p'] \end{gathered} \tag{3.2}$$

We need also the following subset of $PC^r_p(\mathbb{R}_n)$:

$$\begin{gathered} PC^r_p(\mathbb{R}_n,\alpha) = \{a \in PC^r_p(\mathbb{R}_n) : a^\infty \in M_{p-\varepsilon,p+\varepsilon}(\mathbb{R}_n) \cap C(S^{n-1}) \\ \text{for some } \varepsilon > 0\}, \\ PC_p(\mathbb{R}_n,\infty) \overset{def}{=} PC^0_p(\mathbb{R}_n,\infty), \quad 1 < p < \infty, \quad r \in \mathbb{R}. \end{gathered} \tag{3.3}$$

Lemma 3.1 *Let* $1 < p < \infty,\ s, r \in I\!R$ *and* $a \in\ PC^r_p(I\!R_n)$*. If* $a^\infty \in$
$\in C^{[\frac{n}{2}]+1}(S^{n-1})$*, then* $a \in\ PC^r_p(I\!R_n, \alpha)$*.*

If $n = 2$ *and* $a^\infty \in C^1(S^1)$*, then* $a \in PC^r_p(I\!R_2, \infty)$*.*

Proof. Since $a^\infty(\xi)$ is homogeneous $a^\infty(\lambda\xi) = a^\infty(\xi)\ (\lambda > 0, \xi \in I\!R_n)$, and $a^\infty \in C^{\left[\frac{n}{2}\right]+1}(S^{n-1})$, then $\xi^\alpha \partial^\alpha_\xi a^\infty(\xi)$ is uniformly bounded on $I\!R_n$ for any $\mid \alpha \mid \leq \left[\frac{n}{2}\right] + 1$ which implies $a^\infty \in M_p(I\!R_n),\ 1 < p < \infty$ in virtue of Theorem 1.3.

Let now $n = 2$; then $a^\infty \in C^1(S^1)$ implies that a^∞ has a bounded variation on S^1 which implies again $a^\infty \in M_p(I\!R_2), 1 < p < \infty$ (cf. [15], § 2 and [10], Theorem 2.11). ■

Corollary 3.2 *Let* $a^r(\xi)$ *and* $g^r(\xi)$ *be as in Theorems 2.1 and 2.3; then* $a^r, g^r \in PC^r_p(I\!R_n, \alpha)$ *for any* $r \in I\!R,\ 1 < p < \infty$*.*

Lemma 3.3 *Let* $1 < p < \infty,\ s, r \in I\!R$*. The operator*

$$W^0_a :\ H^s_p(I\!R_n) \to\ H^{s-r}_p(I\!R_n) \tag{3.4}$$

is bounded if and only if $a \in\ M^r_p(I\!R_n)$ *.*

Let further $a \in\ PC^r_p(I\!R_n)$*. Operator (3.4) is invertible if and only if the following inequality holds*

$$\inf\{\langle\xi\rangle^{-r} a(\xi) : \xi \in I\!R_n\} > 0 \tag{3.5}$$

and the inverse reads $W^0_{a^{-1}}$*.*

Proof. (3.4) is equivalent to (cf. (1.2), (1.3))

$$\Lambda^{s-r}\ W^0_a \Lambda^{-s} = W^0_{\lambda^{-r} a} :\ L_p(I\!R_n) \to\ L_p(I\!R_n) \tag{3.6}$$

and this proves the first claim of the lemma.

Applying the local principal (cf. [16], § X.3) to the investigation of the lifted operator (this is possible due to the existence of the radial limits (3.1)), the proof proceeds similarly to the 1-dimensional case (cf. [10], Theorem 2.18). ■

Let $C^k(\dot{I\!R}_n)$ denote the set of functions $b \in C^k(I\!R_n)$ which have a limit $b(\infty) =$
$= \lim_{|\xi|\to\infty} b(\xi)$.

Lemma 3.4 *Let* $a \in\ PC^r_p(I\!R_n),\ \ b \in C^k(\dot{I\!R}_n),\ \ s, r \in I\!R,\quad k \in I\!N,\quad \mid s \mid \leq k,$
$\mid s - r \mid \leq k,\quad 1 < p < \infty$ *. If* $a^\infty(\omega) = 0\ (\omega \in S^{n-1}; cf.(3.1))$ *and* $b(\infty) = 0$*, the operators*

$$b\, W^0_a,\ W^0_a b I :\ H^s_p(I\!R_n) \to\ H^{s-r}_p(I\!R_n)$$

are compact.

Proof. The commutator (cf. (1.2))

$$[bI, \Lambda^\nu] = b\Lambda^\nu - \Lambda^\nu bI : \ H^s_p(I\!R_n) \to H^{s-\nu}_p(I\!R_n), \qquad \nu \in I\!R$$

is compact (cf. [3, 12]). Applying the isomorphisms (1.3) the proof is reduced to the compactness properties of the operators

$$bW^0_{a^r}, W^0_{a^r}bI : \ L_p(I\!R_n) \to \ L_p(I\!R_n), \qquad a_r(\xi) = \langle\xi\rangle^{-r}a(\xi).$$

An appropriate approximation reduces the proof to the case where a and b have compact supports. But $bW^0_{a_r} = bW^0_v W^0_{a_r}$ where $v \in S(I\!R_n), \quad v(\xi)a_r(\xi) \equiv a_r(\xi)$. Since the kernel $b(t)k(t-\tau)$ of bW^0_v has nice integration properties ($k = F^{-1}v \in S(I\!R_n)$ and $\operatorname{supp} b$ is compact), bW^0_v is a compact operator in $L_p(I\!R_n)$. $W^0_a bI$ is compact due to similar arguments.
■

Lemma 3.5 *Let* $a \in \ PC^r_p(I\!R_n, \alpha), \quad b \in C^k(\dot{I\!R}_n), \quad s, r \in I\!R, \quad k \in I\!N,$ $| \, s \, | \le k, \quad | \, s - r \, | \le k, \quad 1 < p < \infty$. *Then the commutator*

$$[bI, \ W^0_a] = b\, W^0_a - \ W^0_a bI : \ H^s_p(I\!R_n) \to \ H^{s-r}_p(I\!R_n)$$

is compact.

Proof. As before the proof is reduced to the case $s = r = 0$. Since

$$b(\xi) = b(\infty) + b_o(\xi), \qquad\qquad a(\xi) = a^\infty(\xi) + a_o(\xi),$$

where b_o and a_o satisfy the conditions of the foregoing lemma, $b(\infty) = const$, we get

$$[bI, \ W^0_a] = [b_o I, W^0_{a^\infty}] + [b_o I, W^0_{a_o}];$$

the second term in the last representation is compact due to Lemma 3.4. The operator $[b_o I, W^0_{a^\infty}]$ in the space $L_2(I\!R_n)$ can be approximated in norm by a similar one with the symbol $b_o \in \ C^\infty_0(I\!R_n), \quad a^\infty \in C^\infty(S^{n-1}), \quad a^\infty(\lambda\xi) \equiv a(\xi), \quad \lambda > 0, \quad \xi \in I\!R_n$. But then $W^0_{a^\infty}$ represents a classical Calderon-Zygmund singular integral operator with the characteristic $f(\theta) \in C^\infty(S^{n-1})$ and the compactness of the operator $[b_o I, W^0_{a^\infty}]$ in the space $L_2(I\!R_n)$ is well-known (cf. [28],XI,7.2).

Compactness in the space $L_p(I\!R_n)$ follows now due to the M.A.Krasnoselskiǐ interpolation theorem of compact operators (cf. [21], Theorem 1.4.1) since the operator is bounded in $L_{p\pm\varepsilon}(I\!R_n)$ spaces ($\varepsilon > 0$) and is compact in $L_2(I\!R_n)$. ■

3.2 ORDER REDUCTION OPERATORS FOR A CANONICAL LIPSCHITZ DOMAIN

Let $\Omega \in I\!R_n$ be a special (canonical) Lipschitz domain with the constant L (cf. Definition 1.1) and

$$\sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & L^{-1} \end{pmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ L & L & \cdots & L & L \end{pmatrix},$$

where σ^{-1} is the inverse matrix to σ.

If σ^T denotes the transposed matrix to σ then $\sigma_*^T a(\xi) = a(\sigma^T\xi)$ defines an operator which leaves invariant the multiplier sets $M_p^r(I\!R_n)$and $PC_p^r(I\!R_n,\alpha)$.

Theorem 3.6 *The operators*

$$B_{0,\Omega}^r = W_{b_o^r}^0, \qquad b_o^r = \sigma_*^T a^r \in PC_p^r(I\!R_n,\alpha)\cap S^r(I\!R_n), \tag{3.7}$$

$$B_{1,\Omega}^r = W_{b_1^r}^0, \qquad b_1^r = \sigma_*^T g^r \in PC_p^r(I\!R_n,\alpha), \qquad r\in I\!R$$

belong to $BPO(r,\Omega)$*; the radial limits read (cf. (2.8), (2.9), (2.12), (3.1))*

$$(b_o^r)^\infty(\xi) = \sigma_*^T(a^r)^\infty(\xi), \qquad\qquad (b_1^r)^\infty(\xi) = \sigma_*^T(g^r)^\infty(\xi). \tag{3.8}$$

Proof(cf. [32, 33]). Easy to verify that $x+\mathscr{K}_\sigma\subset\Omega$ for any $x\in\Omega$ where $\mathscr{K}_\sigma = \sigma^{-1}\,\Sigma_1^n$. On the other hand $B_{0,\Omega}^r,\ B_{1,\Omega}^r\in BPO(r,\mathscr{K})$ since $B_{0,\Omega}^r = \sigma_*^{-1}W_{a^r}^0\sigma_*,\quad B_{1,\Omega}^r = \sigma_*^{-1}W_{b^r}^0\sigma_*$, and therefore (cf Theorem 2.1) supp $B_{1,\Omega}^r u\subset\mathscr{K}_\sigma, j=0,1$. The first claim follows now due to Lemma 1.8.

The remainder follows from Theorem 2.2 and Remark 2.4. ■

Lemma 3.7 *(cf. [40]). Let* $\Omega' = I\!R_n\setminus\Omega$. *The following operators act bijectively for all* $r,s\in I\!R,\ 1<p<\infty:$

$$B_{k,\Omega}^r = W_{b_k^r}^0 : \tilde H_p^s(\Omega)\to\tilde H_p^{s-r}(\Omega),$$
$$\overline{B}_{k,\Omega}^r = W_{\bar b_k^r}^0 : \tilde H_p^s(\Omega')\to\tilde H_p^{s-r}(\Omega'),$$

$$r_\Omega\overline{B}_{k,\Omega}^r\ell = r_\Omega W_{\bar b_k^r}^0\ell \quad : \quad H_p^s(\Omega)\to H_p^{s-r}(\Omega), \qquad k=0,1 \tag{3.9}$$

where ℓ *is any extension operator (from* $H_p^s(\Omega)$ *into* $H_p^s(I\!R_n)$ *) and operator (3.9) is independent of its choice. Operators (3.9) are invertible*

$$(r_\Omega\overline{B}_{k,\Omega}^r\ell)^{-1} = r_\Omega W_{\bar b_k^{-r}}^0\ell.$$

The following equality holds

$$r_\Omega\overline{B}_{k,\Omega}^r\ell r_\Omega W_d^0 = r_\Omega W_{\bar b_k^{-r}d}^0, \qquad d\in M_p^q(I\!R_n). \tag{3.10}$$

From 3.6-3.7 it follows the order reduction theorem.

Theorem 3.8 *Let* $d \in M_p^r(\mathbb{R}_n), \quad s, r \in \mathbb{R}, \quad 1 < p < \infty.$

The operators

$$\begin{aligned} r_\Omega W_d^0 \quad &: \quad \tilde{H}_p^s(\Omega) \to H_p^{s-r}(\Omega), \\ r_\Omega W_{d_k}^0 \quad &: \quad \mathbf{L}_p(\Omega) \to L_p(\Omega), \\ & \quad d_k = \overline{b}_k^{s-r} d b_k^{-s}, \qquad\qquad k = 0, 1 \end{aligned}$$

(cf. (3.7)) are equivalent

$$r_\Omega W_{d_k}^0 = r_\Omega \overline{B}_{k,\Omega}^{s-r} \ell r_\Omega W_d^0 B_{k.\Omega}^{-s}.$$

3.3 Operators of local type

Let $\mathscr{M}$ be a C^l-smooth manifold with Lipschitz boundary $\partial\mathscr{M}$ and $\mathscr{M} = \overline{\mathscr{M}} \setminus \partial\mathscr{M}$.

Let $\text{æ}_j(x)(j = 1, 2, ..., N, x \in \overline{\mathscr{M}})$ represent a partition of the unity subordinated to a covering $U_1, ..., U_N$ (cf. Definition d1.2 1.2)

$$\sum_{j=1}^{N} \text{æ}_j(x) \equiv 1, \qquad \operatorname{supp} \text{æ}_j(x) \subset U_j, \qquad \text{æ}_j \in C^l(\mathscr{M}).$$

The spaces $\tilde{H}_p^s(\mathscr{M})$ and $H_p^s(\mathscr{M})$ can be defined correctly for any $1 \leq p \leq \infty, -l+1 \leq s \leq l$ (cf. [12,44]).

The operators

$$\begin{aligned} \beta_{j*}\varphi(x) = \text{æ}_j(x)\varphi(\beta_j(x)) \quad &: \quad \tilde{H}_p^s(\Omega) \to \tilde{H}_p^s(\mathscr{M}), \\ &: \quad H_p^s(\Omega) \to H_p^s(\mathscr{M}), \\ \beta_{j*}^{-1}\psi(x) = \text{æ}_j(\beta_j^{-1}(x))\psi(\beta_j^{-1}(x)) \quad &: \quad \tilde{H}_p^s(\mathscr{M}) \to \tilde{H}_p^s(\Omega), \\ &: \quad H_p^s(\mathscr{M}) \to H_p^s(\Omega), \end{aligned} \tag{3.11}$$

are correctly defined but are not inverses to each other if not restricted to the subset of $U_j \subset \overline{\mathscr{M}}$ and of $V_j \subset \overline{\Omega}$ respectively.

$\Delta_x(\mathscr{M}), x \in \overline{\mathscr{M}}(\Delta_x(\Omega), x \in \overline{\Omega})$ represents the set of $C^l(\mathscr{M})$ (of $C^\infty(\Omega)$) functions $v(y)$ such that $v(y) = 1$ in some neighbourhood of x which have compact support $\operatorname{supp} v$ in case of a noncompact manifold.

The notations

$$\begin{aligned} q_L(x, \mathbf{A}) &= \inf\{|\!|\!| \, v_x\mathbf{A}|\!|\!|_{sp}^{(r)} : v_x \in \Delta_x\} \\ q_R(x, \mathbf{A}) &= \inf\{|\!|\!| \, \mathbf{A}v_x|\!|\!|_{sp}^{(r)} : v_x \in \Delta_x\} \end{aligned} \tag{3.12}$$

$$|\!|\!| \, \mathbf{B}|\!|\!|_{sp}^{(r)} = \inf\{\|\mathbf{B} + \mathbf{T}\|_{sp}^{(r)} : \mathbf{T} \in \mathscr{K}^r(H_p^s(\mathscr{M}))\}, \tag{3.13}$$

$$\mathbf{B}, \mathbf{T} : \tilde{H}_p^s(\mathscr{M}) \to H_p^{s-r}(\mathscr{M})$$

will appear in the sequel with $\mathscr{K}^r(\,H^s_p(\mathscr{M}))$ representing the set of compact operators between appropriate spaces (cf. (3.12)). If $q_L(x,\mathbf{A}) = q_R(x,\mathbf{A})$ the notation $q(x,\mathbf{A})$ will be used.

Definition 3.9 *An operator*

$$\mathbf{A}:\ \tilde{H}^s_p(\mathscr{M}) \to H^{s-r}_p(\mathscr{M}), \quad -l+1 \le s, s-r \le l, \quad 1 < p < \infty \tag{3.14}$$

is called operator of local type ($\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$ in short) if $v_1\mathbf{A}v_2 \in \mathscr{K}^r(H^s_p(\mathscr{M}))$ for any $v_1, v_2 \in C^l(\mathscr{M}), \operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$.

Obviously $q_L(x,\mathbf{A}) = q_R(x,\mathbf{A})$ for $\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$.

Definition 3.10 *Operators*

$$\mathbf{A}_1, \mathbf{A}_2:\ \tilde{H}^s_p(\mathscr{M}) \longrightarrow H^{s-r}_s(\mathscr{M}) \tag{3.15}$$

are called Δ_x-equivalent at the point $x \in \overline{\mathscr{M}}$ ($\mathbf{A}_1 \overset{\Delta_x}{\sim} \mathbf{A}_2$ in short) if the following holds

$$q(x, \mathbf{A}_1 - \mathbf{A}_2) = 0$$

Theorem 3.11 *If $\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$, then*

$$|||\,\mathbf{A}|||^{(r)}_{sp} = \sup\{q(x,\mathbf{A}) : x \in \overline{\mathscr{M}}\}. \tag{3.16}$$

If further $\mathbf{A} \overset{\Delta_x}{\sim} \mathbf{A}_x,\ x \in \overline{\mathscr{M}},\ \mathbf{A}_x \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$, then

$$|||\,\mathbf{A}|||^{(r)}_{sp} \le \sup\{|||\,\mathbf{A}_x|||^{(r)}_{sp} : x \in \overline{\mathscr{M}}\} \tag{3.17}$$

Proof is similar to the one exposed in [22] for $L_p(\mathscr{M})$ spaces and in [29] for Hölder spaces. Rough estimates can be found in [36, 37]. ∎

Lemma 3.12 $\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$ *if and only if the operators*

$$[vI, \mathbf{A}] = v\mathbf{A} - \mathbf{A}vI:\ \tilde{H}^s_p(\mathscr{M}) \longrightarrow H^{s-r}_s(\mathscr{M}) \tag{3.18}$$

are compact for any $v \in C^l(\mathscr{M})$.

Proof (cf. [37]). If (3.17) is compact then

$$v_1\mathbf{A}v_2I = [v_1I\mathbf{A}]v_2I, \quad v_1, v_2 \in C^l(\overline{\mathscr{M}}), \quad \operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$$

and, therefore, $\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$.

If now $\mathbf{A} \in \mathrm{OLT}^r(\,H^s_p(\mathscr{M}))$, then

$$q(x, [vI, \mathbf{A}]) = q(x, [v(x)I, \mathbf{A}]) = 0, \qquad v \in C^l(\overline{\mathscr{M}})$$

since $q(x, vI) = q(x, v(x)I)$ ($x \in \overline{\mathscr{M}}$ is fixed). Then (3.14) yields $[vI, \mathbf{A}] \in \mathscr{K}^r(\,H^s_p(\mathscr{M}))$. ∎

Remark 3.13 *The relations (3.15) and (3.16) may be derived from the case $r = s = 0$ (i.e. the case of the space $L_p(\mathscr{M})$) with the help of the order reduction operators (cf. (1.3), Theorem 3.8 and Theorem 3.22 below) since all operators involved are of local type (cf. Lemmas 3.5 and 3.12). For the half-space case this was already proved in [30].*

Concluding the present subsection it might be noticed that the notion of a locally continuous family of operators can be introduced similarly to [36] (cf. also [28], § XV.3.1) and existence of the enveloping operator $\mathbf{A} \overset{\Delta_x}{\sim} \mathbf{A}_x,\ x \in \overline{\mathscr{M}}$ for any locally continuous family $\{\mathbf{A}_x\}_{x\in\overline{\mathscr{M}}}$ of operators of local type can be proved.

3.4 PSEUDODIFFERENTIAL OPERATORS ON COMPACT MANIFOLDS

Throughout this section $\mathscr{M}$ will be the same as in the foregoing subsection. Other notation is used also without further comments.

Definition 3.14 *An operator (3.13) is called pseudodifferential operator (with non-smooth symbol) of order r ($\mathbf{A} \in \mathrm{OPC}^r(\ H^s_p(\mathscr{M}))$ in short) if the following holds:*

(i) $\mathbf{A} \in \mathrm{OLT}^r(\ H^s_p(\mathscr{M}))$;

(ii) for any $x \in U_j \subset \overline{\mathscr{M}}$ there exists a function $a_x \in PC^r_p(\mathbb{R}_n, \alpha)$ such that the equivalence

$$\mathbf{A} \overset{\Delta_x}{\sim} \beta_{j*} r_x W^0_{a_x} \beta^{-1}_{j*}, \qquad x \in U_j$$

holds ($r_x = r_\Omega$ for $x \in \partial\mathscr{M}$ and $r_x = I$ for $x \in \mathscr{M}$; cf. (1.5)).

$a_x(\xi) = a(x, \xi)$ is called the *symbol* of $\mathbf{A}$ and the radial limit $a^\infty(x, \xi)$ is called the principal homogeneous symbol of $\mathbf{A}$ (cf. (3.1); $x \in \overline{\mathscr{M}},\ \ \xi \in \mathbb{R}_n$).

Let us introduce the following notation as well

$$\mathrm{OPC}^r(\mathscr{M}) = \bigcap_{s,p} \mathrm{OPC}^r(\ H^s_p(\mathscr{M})).$$

Lemma 3.15 . *The principal homogeneous symbol $a^\infty(x, \xi)$ of any operator $\mathbf{A} \in$ $\in \mathrm{OPC}^r(\ H^s_p(\mathscr{M}))$ is defined uniquely and depends continuously on $x \in \overline{\mathscr{M}}$.*

Proof. The uniqueness follows immediately, since if $a^\infty(x, \xi) \equiv 0$ $(x \in \overline{\mathscr{M}}, \xi \in$ $\in \mathbb{R}_n)$, then $q(x, r_x W^0_{a_x}) \equiv 0$ (cf. Lemma 3.4) and therefore $\mathbf{A}$ is compact (cf.(3.16)).

Let now $x, y \in U_j \subset \overline{\mathscr{M}}$ be points sufficiently close to each other; then due to the homogenity of symbols (cf. (3.2), (3.6) and Lemma 1.16 in [11])

$$\begin{aligned} |a^\infty_x - a^\infty_y\|_\infty &\le \|W^0_{a^\infty_x - a^\infty_y}\|_{sp} \||\ v_y v_x r_x r_y W^0_{a^\infty_x - a^\infty_y}\||_{sp} \le \\ &\le \|B^{-r}_{O,\Omega}\|^{(-r)}_{sp} \{\||\ v_x[r_x W^0_{a^\infty_x} - \beta^{-1}_{j*}\mathbf{A}\beta_{j*}]\||^{(r)}_{sp} \\ &+ \||\ v_y[r_y W^0_{a^\infty_y} - \beta^{-1}_{j*}\mathbf{A}\beta_{j*}]\||^{(r)}_{sp}\} < \varepsilon, \end{aligned}$$

where $v_y \in \Delta_y, \quad v_x \in \Delta_x$ are chosen to satisfy the inequalities

$$\||\, v_x[r_x W^0_{a^\infty_x} - \beta^{-1}_{j*}\mathbf{A}\beta_{j*}]\||^{(r)}_{sp} < \frac{\varepsilon}{2\|B^r_{O,\Omega}\|^{(-r)}_{sp}}$$

$$\||\, v_y[r_y W^0_{a^\infty_y} - \beta^{-1}_{j*}\mathbf{A}\beta_{j*}]\||^{(r)}_{sp} < \frac{\varepsilon}{2\|B^r_{O,\Omega}\|^{(-r)}_{sp}}, \qquad v_x(\xi)v_y(\xi) = v_y(\xi).$$

∎

Let $\mathscr{M}^0$ be another C^l-manifold with the Lipschitz boundary $\partial\mathscr{M}^0$ and let

$$\beta^0_j : U^0_j \longrightarrow V^0_j \subset \Omega \subset I\!R_n, \qquad j = 1, 2, ..., N^0$$

represent a coordinate diffeomorphism.

If the diffeomorphism of manifolds $æ : \overline{\mathscr{M}} \to \overline{\mathscr{M}}^0$ is given we are interested in the operator

$$A_æ = æ^{-1}_* A æ_*, \qquad A \in \mathrm{OPC}^r(\, H^s_p(\mathscr{M})). \tag{3.19}$$

Theorem 3.16 *$A_æ \in \mathrm{OPC}^r_{sp}(\mathscr{M})$ and its principal homogeneous symbol reads*

$$a^\infty_æ(y, \xi) = a^\infty_æ(x, J^T_x \xi), \qquad y = æ(x), \qquad x \in \overline{\mathscr{M}}, \tag{3.20}$$

where J^T_x represents the transposed matrix to the Jacobian

$$J_x = (\beta^0_k \circ æ \circ \beta^{-1}_j)'(t) \equiv æ'_0, \quad x \in U_j, \quad t = \beta_j(x) \in I\!R_n, \quad y = æ(x) \in U^0_k.$$

Proof. Suppose first $a^\infty_x = a^\infty(x, .) \in C^\infty(S^{n-1}),\ x \in \overline{\mathscr{M}}$; then by definition

$$A \overset{\Delta_x}{\simeq} \beta_{j*} r_x W^0_{a_x} \beta^{-1}_{j*}, \qquad A_æ \overset{\Delta_y}{\simeq} \beta^0_{k*} r_y W^0_{a_{æ,y}} (\beta^0_k)^{-1}_*.$$

On the other hand

$$(\beta^0_k)^{-1}_* A_æ \beta^0_{k*} = ((\beta^0_k)^{-1}_* æ^{-1}_* \beta_{j*})(\beta^{-1}_{j*} A \beta_{j*})(\beta^{-1}_{j*} æ_* \beta^0_{k*}) = æ^{-1}_{0*}\beta^{-1}_{j*} A \beta_{j*} æ_{0*} \overset{\Delta_\tau}{\simeq}$$
$$\overset{\Delta_\tau}{\simeq} æ^{-1}_{0*} r_x W^0_{a_x} æ_{0*}, \quad æ_0 = \beta^{-1}_{j*} æ_* \beta^0_{k*} = \beta^0_k \circ æ \circ \beta^{-1}_j \quad \tau = æ^0(t) = \beta^0_k(y) \in I\!R_n.$$

These equivalences yield (cf. Lemma 3.4)

$$r_y W^0_{a_{æ,y}} \overset{\Delta_\tau}{\simeq} æ^{-1}_{0*} r_x W^0_{a_x} æ_{0*} \overset{\Delta_\tau}{\simeq} æ^{-1}_{0*} r_x W^0_{a^\infty_{x,r}} æ_{0*}, \quad a^\infty_{x,r}(\xi) \equiv \,|\, \xi \,|^r\, a^\infty_x(\xi).$$

The well-known formula for the symbol $b(x, \xi) \equiv b_x(\xi)$ of a pseudodifferential operator after a transformation of the variable

$$æ^{-1}_{0*} r_x W^0_{a^\infty_{x,r}} æ_{0*} = r_y W^0_{b_x}$$

reads (cf. [18], vol.3, Theorem 18.1.17)

$$b_x(\xi) = \sum_{\alpha \in \boldsymbol{N}_n} \frac{1}{\alpha!} D^\alpha_\xi [\mid J^T_x(\xi) \mid^r a^\infty_x(J^T_x(\xi)] \, [D^\alpha_z \exp[i\langle \rho_x(z), \xi \rangle]]_{z=t}$$

$$y = æ(x), \qquad \rho_x(z) = æ_0(z) - æ_0(t) - æ_0'(t)(z-t), \qquad t = \beta_j(x) \in I\!\!R_n.$$

Obviously

$$b^\infty_x(\xi) = a^\infty_x(J^T_x \xi)$$

since all other summands in the asymptotic expansion have orders less then r; therefore

$$r_y W_{a^\infty_{æ,y}} \overset{\Delta_\tau}{\simeq} r_y W^0_{\tilde{a}^\infty_x}, \qquad \tilde{a}^\infty_x(\xi) = \tilde{a}^\infty_x(J^T_x \xi)$$

or, written differently (cf. Lemma 3.4)

$$q(\tau, r_y W_{a^\infty_{æ,y} - \tilde{a}^\infty_x}) = 0.$$

Due to the homogenity of the symbol (cf. [11], Lemma 1.15 or [28], Lemma XV.5.1)

$$q(\tau, r_y W_{a^\infty_{æ,y} - \tilde{a}^\infty_x}) = \| W_{a^\infty_{æ,y} - \tilde{a}^\infty_x} \| = \| a^\infty_{æ,y} - \tilde{a}^\infty_x \|_\infty$$

and therefore

$$a^\infty_{æ,y}(\xi) = a^\infty_x(J^T_x \xi).$$

For $a^\infty_x \in C(S^{n-1}) \bigcap M_{p-\varepsilon, p+\varepsilon}(I\!\!R_n),\ a^\infty_x \not\in C^\infty(S^{n-1})$ approximation can be applied.
■

Remark 3.17 *It follows that the principal homogeneous symbol $a^\infty_x(\xi)$ of an operator $A \in \mathrm{OPC}^r(\, H^s_p(\mathscr{M}))$ is unique and correctly defined on the cotangential fibration $\overline{T^*\mathscr{M}}$ (cf. [18], v.3, § 18.1).*

Example 3.18 *Let $\Omega \in I\!\!R_n$ be any compact Lipschitz domain (cf. Definition 1.2.).*

Any convolution operator with coefficients

$$\mathbf{A} = \sum_{j=1}^m b_j r_\Omega W^0_{a_j} c_j I \quad : \quad \tilde{H}^s_p(\Omega) \to H^{s-r}_p(\Omega),$$

$$a_j \in \ PC^r_p(I\!\!R_n, \alpha), \qquad b_j, c_j \in C^k(\Omega), \qquad \mid s \mid, \mid s - r \mid \le k, \qquad 1 < p < \infty$$

represents a pseudodifferential operator $A \in \mathrm{OPC}^r(\, H^s_p(\Omega))$ and the symbols read

$$a(x,\xi) = \sum_{j=1}^k b_j(x) a_j(\xi) c_j(x). \qquad a^\infty(x,\xi) = \sum_{j=1}^k b_j(x) a^\infty_j(\xi) c_j(x).$$

Example 3.19 *The operator* $r_\Omega \mathbf{A}$*, where*

$$\mathbf{A}\varphi(x) = \frac{1}{(2\pi)^n} \int_{\Omega\times\boldsymbol{R}_n} \exp[i\xi(x-y)]a(x,\xi)\varphi(y)dyd\xi$$

with the classical symbol $a \in S^r(\Omega \times I\!R_n)$ *represents a pseudodifferential operator in the sense of Definition 3.14,* $r_\Omega\mathbf{A} \in \mathrm{OPC}^r(\, H^s_p(\Omega))$.

The detailed proof of the last claim will be included in the forthcoming publication [12].

The symbol $a_x(\xi) = a(x,\xi)$ is called *elliptic* if the following holds

$$\inf\{|\, a^\infty(x,\xi)\, | \ : \ x \in \overline{\mathscr{M}}, \quad \xi \in S^{n-1}\} > 0. \tag{3.21}$$

Theorem 3.20 *Let* $1 < p < \infty, \quad -l+1 \le s, s-r \le l$ *and* $\mathbf{A} \in \mathrm{OPC}^r(\, H^s_p(\mathscr{M}))$ *with the symbol* $a(x,.) \in \ PC^r_p(I\!R_n, \alpha),\ x \in \overline{\mathscr{M}}$.

(3.13) is a Fredholm operator if the following holds:

(i) the symbol $a(x,\xi)$ *is elliptic (cf. (3.20));*

(ii) the following convolution operators are invertible either for k=0 or for k=1 (cf.(3.7)) and all $x \in \partial\overline{\mathscr{M}}$

$$r_\Omega W^0_{a^\infty_k(x,.)} \ : \ L_p(\Omega) \to L_p(\Omega), \qquad a_k(x,\xi) \equiv \overline{b^{s-r}_k(\xi)}a(x,\xi)b^{-s}_k(\xi), \tag{3.22}$$

where Ω *is chosen according to Definition 1.2.*

Proof. If we recall Definition 3.14 and the local principal (cf. [9], § 4 or [12]) we find out that (3.13) is a Fredholm operator if and only if the local representatives

$$\begin{array}{rcll} r_\Omega W^0_{a_x} & : & \tilde{H}^s_p(\Omega) \to H^{s-r}_p(\Omega), & x \in \partial\overline{\mathscr{M}} \\ W^0_{a_x} & : & H^s_p(I\!R_n) \to \ H^{s-r}_p(I\!R_n), & x \in \mathscr{M}, \end{array} \tag{3.23}$$

are all locally invertible.

Since an operators $\mathbf{B}$ and $\mathbf{B}+\mathbf{T}$, where $\mathbf{T}$ is compact, are locally equivalent at any finite point $x \in I\!R_n$ (cf. [9], § 4), by virtue of Lemmas 3.4 and 3.8 it follows: (3.22) are locally invertible if and only if the lifted operators

$$\begin{array}{rcll} r_\Omega W^0_{a^\infty_k(x,.)} & : & L_p(\Omega) \to L_p(\Omega), & x \in \partial\overline{\mathscr{M}} \\ W^0_{a^\infty_k(x,.)} & : & L_p(I\!R_n) \to L_p(I\!R_n), & x \in \mathscr{M}, \end{array}$$

are locally invertible at $x \in \overline{\mathscr{M}}$; local invertibility of the operator $W^0_{a^\infty_k(x,.)}$ is equivalent to the ellipticity of $a_k(x,\xi)$ (cf. [9], § 4) and further to the ellipticity of $a^\infty(x,\xi)$, since

$$a_k(x,\xi) \equiv \overline{(b^{s-r})^\infty_k(\xi)}a(x,\xi)(b^{-s})^\infty_k(\xi)$$

and $b^\nu_k(\xi)$ is elliptic (cf. (3.8)). ∎

Remark 3.21 *For the point $x \in \partial\overline{\mathscr{M}}$ for which the tangential cone $\mathscr{K}_y$ to Ω exists at $y = \beta_j(x) \in \partial\overline{\Omega}$ the operator (3.21) in the condition (ii) can be replaced by*

$$r_{\mathscr{K}_y} W^0_{a_k^\infty(x,.)} \quad : \quad L_p(\mathscr{K}_y) \to L_p(\mathscr{K}_y). \tag{3.24}$$

Moreover: if Ω has a tangential cone at any point $y = \beta_j(x) \in \partial\overline{\Omega},\ x \in \partial\overline{\mathscr{M}}$ then the conditions of Theorem 3.20 are necessary as well.

This follows from the equivalence of local invertibility of (3.21) and (3.22) on one side and from the equivalence of invertibility and local invertibility of (3.22) on the other side (cf. [9], § 4 for the last claim).

Remark 3.22 *It was only for notational convenience that we stick to the scalar case. Theorems 3.20 and Remark 3.21 remain valid for systems of pseudodifferential equations (of pseudodifferential operators) with matrix-valued symbols in vector spaces if the ellipticity condition (3.20) is interpreted in a proper way: $a^\infty(x,\xi)$ is replaced with $\det a^\infty(x,\xi)$.*

Remark 3.23 *Theorem 3.20 and Remark 3.21 (the latter only in the sufficient part) remain valid also for Besov spaces $B^s_{p,p}(\mathscr{M})$. This follows with the help of the interpolation theorems (cf. [43], § 2.10) as in [12, 13].*

3.5 Reduction of order for manifolds

The notations used here are mostly from § 3.3 (l denotes, for example the smoothness of a manifold $\mathscr{M}$). In [33] the order reduction operators were constructed

$$\begin{aligned} \mathbf{B}^r_{\mathscr{M}} = b^r_{\mathscr{M}}(x,D) \quad &: \quad \tilde{H}^s_p(\mathscr{M}) \to \tilde{H}^{s-r}_p(\mathscr{M}), \\ \overline{\mathbf{B}}^r_{\mathscr{M}} = \overline{b}^r_{\mathscr{M}}(x,D) \quad &: \quad H^s_p(\mathscr{M}) \longrightarrow H^{s-r}_p(\mathscr{M}), \end{aligned} \tag{3.25}$$

$$s, r \in I\!R, \quad -l+1 \le s, s-r \le l, \quad 1 < p < \infty, \quad b^r_{\mathscr{M}}, \overline{b}^r_{\mathscr{M}} \in S^r(\mathscr{M} \times I\!R_n),$$

but the proof of ellipticity of these operators was incomplete.

Here a modified model of the order reduction operator is suggested. These operators will be needed in the next subsection.

Theorem 3.24 *There exist isomorphisms (3.24) such that the diagram*

$$\begin{array}{ccc} \tilde{H}^s_p(\mathscr{M}) & \stackrel{a(x,D)}{\longrightarrow} & H^{s-r}_p(\mathscr{M}) \\ \uparrow \mathbf{B}^{-s}_{\mathscr{M}} & & \downarrow \overline{\mathbf{B}}^{s-r}_{\mathscr{M}} \\ L_p(\mathscr{M}) & \stackrel{a_0(x,D)}{\longrightarrow} & L_p(\mathscr{M}). \end{array} \tag{3.26}$$

$$a(x,D) \in \mathrm{OPC}^r(\ H^s_p(\mathscr{M})),$$
$$a_0(x,D) = \overline{\mathbf{B}}^{s-r}_{\mathscr{M}} a(x,D) \mathbf{B}^{-s}_{\mathscr{M}} \in \mathrm{OPC}^0(L_p(\mathscr{M}))$$

is commutative and the principal homogeneous symbol of the lifted operator reads

$$a_0^\infty(x,\xi) = (\overline{b}^{s-r}_{\mathscr{M}})^\infty(x,\xi)\, a^\infty(x,\xi)\, (b^{-s}_{\mathscr{M}})^\infty(x,\xi). \tag{3.27}$$

Proof. Let $\mathscr{M}_1$ be any C^l-manifold without boundary, including $\mathscr{M}$ (i.e. $\partial\mathscr{M}_1 = = \emptyset,\ \overline{\mathscr{M}} \subset \mathscr{M}_1$) and suppose the coordinate system $\{(\beta_j^1, U_j^1)\}_{j=1}^{N^1}$ be the extension of the system from $\overline{\mathscr{M}}$: $\quad \beta_j^1|_{\overline{\mathscr{M}}} = \beta_j, \quad U_j^1 \cap \overline{\mathscr{M}} = U_j, \quad j = 1, 2, ..., N, \quad N \leq N^1.$

Let $|\ \nu\ | < 1$ and consider the operator

$$W_{b^\nu} = \mathbf{B}^\nu_{\psi,\Omega\times\mathbb{R}} \in BPO(\nu, \Omega \times \mathbb{R}) \subset BPO(\nu, \mathbb{R}_{n+1}),$$

where $\Omega \times \mathbb{R} \subset \mathbb{R}_{n+1}$ is a canonical Lipschitz domain again (cf. Definition 1.1); we suppose that $b^\nu(\xi, \lambda) \in S^\nu(\mathbb{R}_{n+1})$ (cf. Theorem 3.6; $\xi \in \mathbb{R}_n,\ \lambda \in \mathbb{R}$).

Let $\left\{\psi_j^1(x)\right\}_{j=1}^{N^1}$ be a partition of unity subordinated to the covering $\{U_{0,j}\}_{j=1}^{N^1}$ of $\overline{\mathscr{M}}_0$ and compose the operator

$$\mathbf{B}^\nu_{\mathscr{M}_1,\lambda} = \sum_{j=1}^{N^1} \beta^1_{j*} W_{b^\nu(\cdot,\lambda)} (\beta^1_{j*})^{-1} \psi_j^1 I, \qquad \mathbf{B}^\nu_{\mathscr{M},\lambda} = \sum_{j=1}^{N} \beta_{j*} W_{b^\nu(\cdot,\lambda)} \beta^{-1}_{j*} \psi_j^1 I, \qquad \lambda \in \mathbb{R},$$

where then by the properties of BPO's $W^0_{b^\nu(\cdot,\lambda)}$ (cf. (1.13)) we get

$$r_{\mathscr{M}} \mathbf{B}^\nu_{\mathscr{M}_1,\lambda}\varphi = \mathbf{B}^\nu_{\mathscr{M}_1,\lambda}\varphi = \mathbf{B}^\nu_{\mathscr{M},\lambda}\varphi \qquad\qquad \varphi \in \tilde{H}^s_p(\mathscr{M}); \tag{3.28}$$

therefore the operators

$$\mathbf{B}^\nu_{\mathscr{M},\lambda} :\ \tilde{H}^s_p(\mathscr{M}) \to \tilde{H}^{s-\nu}_p(\mathscr{M}), \qquad\qquad \mathbf{B}^\nu_{\mathscr{M},\lambda} : H^s_p(\mathscr{M}_1) \to H^{s-\nu}_p(\mathscr{M}_1)$$

are bounded.

Obviously $\mathbf{B}^\nu_{\mathscr{M}_1,\lambda} \in \mathrm{OPC}^\nu(\mathscr{M}_1)$ and its symbol reads

$$b^\nu_{\mathscr{M}_1}(x; \xi, \lambda) = \sum_{j=1}^{N^1} \psi_j^1(\beta_j^1(x)) b^\nu([((\beta_j^1)')^\top(x)]^{-1}(x)\xi, \lambda), \tag{3.29}$$

where $[((\beta_j^1)')^\top(x)]^{-1}(x)$ is the transposed Jacoby matrix of the inverse diffeomorphism $[\beta_j^1]^{-1}(x)$.

Let J be any non-degenerate matrix; then (cf. Theorem 3.6 and Lemma 2.5)

$$\arg(b^\nu)^\infty(J\xi, \lambda) = \nu \frac{\Gamma\left(\frac{n+1}{2}\right)}{4\pi^{\frac{n+1}{2}-1}} \int_{S^n} \psi(y) \,\mathrm{sgn}\,[J\xi \cdot y' + \lambda y_{n+1}]\, d_yS = \tag{3.30}$$

$$= \nu \frac{\Gamma\left(\frac{n+1}{2}\right)}{4\pi^{\frac{n-1}{2}}} \int_{S^n \cap \Sigma_1^n} [\psi(y) + 1] \,\mathrm{sgn}\,[J\xi \cdot y' + \lambda y_{n+1}]\, d_yS, \qquad y = (y_1, ..., y_n),$$

since $\operatorname{supp}[\psi(y)+1] \subset \Sigma_1^n$ (cf. (2.2)) and

$$\int_{S^n} \operatorname{sgn}(\eta \cdot y) d_y S = 0$$

for any fixed $\eta \in \mathbb{R}_{n+1}$.

Similarly to (2.31) we get

$$\int_{S^n \cap \Sigma_1^n} [\psi(y)+1]\, d_y S = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

and together with (3.29) this yields

$$|\arg(b_\infty^\nu(J\xi, \lambda)| < \frac{\pi}{2}\nu < \frac{\pi}{2};$$

therefore (cf. (3.28))

$$\inf |(b_{\mathscr{M}_1^\nu})^\infty(x; \xi, \lambda)| > 0.$$

Thus the operator $\mathbf{B}^\nu_{\mathscr{M}_1,\lambda}$ has the parametrix, since it is elliptic and $\mathscr{M}_1$ is closed (cf. [18], vol.3)

$$\mathbf{B}^{-\nu}_{\mathscr{M}_1,\lambda} = \left[b^\nu_{\mathscr{M}_1}(x; D_x, \lambda)\right]^{-1} \in \mathrm{OPC}^{-\nu}(\mathscr{M}_1)$$

such that

$$\mathbf{B}^{-\nu}_{\mathscr{M}_1,\lambda}\mathbf{B}^\nu_{\mathscr{M}_1,\lambda} = I + \mathbf{R}_1(\lambda), \qquad \mathbf{B}^\nu_{\mathscr{M}_1,\lambda}\mathbf{B}^{-\nu}_{\mathscr{M}_1,\lambda} = I + \mathbf{R}_2(\lambda)$$
$$\mathbf{R}_1(\lambda), \mathbf{R}_2(\lambda) \in OPC^{-1}(\mathscr{M}_1)$$

($\mathbf{R}_1(\lambda), \mathbf{R}_2(\lambda) \in OPC^{-\infty}(\mathscr{M}_1)$ if $\mathscr{M}$ is $C^\infty - smooth$) the symbols $\mathbf{R}_j(x; \xi, \lambda)$ of the operators $\mathbf{R}_j(\lambda)$ satisfy the estimations

$$|\partial_x^\beta \partial_\xi^\alpha \mathbf{R}_j(x; \xi, \lambda)| \leq \frac{C_{\alpha,\beta}}{(1+|\xi|+|\lambda|)^{1+|\alpha|}} \leq \frac{C_{\alpha,\beta}(\lambda)}{(1+|\xi|)^{1+|\alpha|}}, \tag{3.31}$$

$$x \in \mathscr{M}_1, \quad \xi \in \mathbb{R}_n \setminus \{0\}, \quad |\beta| \leq \ell, \quad |\alpha| < \infty, \quad j = 1, 2.$$

The functions $C_{\alpha,\beta}(\lambda)$ have limits $\lim_{|\lambda|\to\infty} C_{\alpha,\beta}(\lambda) = 0$ and since the norms of $\mathbf{R}_j(\lambda)$ are estimated by these constants (cf. [18], vol.3) we can get $\|\mathbf{R}_j(\lambda_0)\| < 1,\ j = 1, 2$ for a large $\lambda_0 \in \mathbb{R}$; then operators

$$I + \mathbf{R}_j(\lambda_0) : H_p^{s-\nu}(\mathscr{M}_1) \longrightarrow H_p^{s-\nu}(\mathscr{M}_1), \qquad j = 1, 2$$

are invertible and therefore the operator

$$\mathbf{B}_{\mathscr{M}_1} = \mathbf{B}_{\mathscr{M}_1,\lambda_0} : H_p^s(\mathscr{M}_1) \longrightarrow H_p^{s-\nu}(\mathscr{M}_1)$$

is also invertible. Due to (3.27) this yields the invertibility of the restriction

$$\mathbf{B}_{\mathscr{M}} = \mathbf{B}_{\mathscr{M}_1} \mid_{\tilde{H}^s_p(\mathscr{M})} : \tilde{H}^s_p(\mathscr{M}) \longrightarrow \tilde{H}^{s-\nu}_p(\mathscr{M}) \tag{3.32}$$

If now $a(x, D) \in \mathrm{OPC}^r_{sp}(\mathscr{M}_1)$, then (cf. (3.27))

$$r_{\mathscr{M}} a(x, D)\mathbf{B}^{\nu}_{\mathscr{M}}\varphi = r_{\mathscr{M}} a(x, D)\mathbf{B}^{\nu}_{\mathscr{M}_1}\varphi = r_{\mathscr{M}}(a \circ b^{\nu}_{\mathscr{M}_1})(x, D), \tag{3.33}$$

where $b^{\nu}_{\mathscr{M}_1}(x, \xi) \equiv b^{\nu}_{\mathscr{M}_1}(x, \xi, \lambda_0)$ is the symbol of the operator $\mathbf{B}^{\nu}_{\mathscr{M}_1}$.

The operator $\overline{\mathbf{B}}^{\nu}_{\mathscr{M}} = \overline{b}^{\nu}_{\mathscr{M}}(x, D) = r_{\mathscr{M}}\mathbf{B}^{\nu}_{\mathscr{M}_1}\ell$ (cf. (3.25)) is constructed similarly, where ℓ is any extension operator from $\tilde{H}^s_p(\mathscr{M})$ to $H^s_p(\mathscr{M}_1)$ and the operator $\overline{\mathbf{B}}^{\nu}_{\mathscr{M}}$ is independent of this extension. The constructed operator arranges an isomorphism

$$\overline{\mathbf{B}}^{\nu}_{\mathscr{M}} = \overline{b}^{\nu}_{\mathscr{M}}(x, D) : H^s_p(\mathscr{M}) \longrightarrow H^{s-\nu}_p(\mathscr{M}) \tag{3.34}$$

Obviously

$$\begin{aligned} \overline{\mathbf{B}}^{\nu}_{\mathscr{M}} r_{\mathscr{M}} a(x, D) &= r_{\mathscr{M}}\overline{\mathbf{B}}^{\nu}_{\mathscr{M}_1}\ell r_{\mathscr{M}} a(x, D) \\ &= r_{\mathscr{M}}\overline{\mathbf{B}}^{\nu}_{\mathscr{M}_1} a(x, D) = r_{\mathscr{M}}(\overline{b}^{\nu}_{\mathscr{M}_1} \circ a)(x, D). \end{aligned} \tag{3.35}$$

If now $\mid \nu \mid \geq 1$, then consider the operators

$$\mathbf{B}^{\nu}_{\mathscr{M}_1} = (\mathbf{B}^{\frac{\nu}{m}}_{\mathscr{M}_1})^m, \qquad \overline{\mathbf{B}}^{\nu}_{\mathscr{M}} = (\overline{\mathbf{B}}^{\frac{\nu}{m}}_{\mathscr{M}})^m = r_{\mathscr{M}}(\overline{\mathbf{B}}^{\frac{\nu}{m}}_{\mathscr{M}_1})^m \ell$$

where $m > \mid \nu \mid$ is fixed; the restriction $\mathbf{B}_{\mathscr{M}} = \mathbf{B}_{\mathscr{M}_1}\mid_{\tilde{H}^s_p(\mathscr{M})}$ and $\overline{\mathbf{B}}^{\nu}_{\mathscr{M}}$ represent the desired isomorphisms (3.31) and (3.33) for any $\nu \in \mathbb{R}$. (3.32) and (3.34) are valid as well and imply (3.26). ■

3.6 PSEUDODIFFERENTIAL OPERATORS WITH LOCALLY SECTORIAL SYMBOLS

The next definition is due to I. Spitkovskiĭ, who used it for the investigation of one-dimensional singular integral operators (cf. [25]).

Definition 3.25 *A $m \times m$ matrix symbol $a(x, \xi) \in PC^r_2(\mathbb{R}_n, \infty)$, $(x \in \in \overline{\mathscr{M}},\ r \in \mathbb{R},\ 1 < p < \infty)$ is called α_x -sectorial, $0 < \alpha_x < \pi/2$, if there exists $\theta_x \in [0, 2\pi]$ such that the following inequality*

$$\mathrm{Re}(e^{i(\theta-\theta_x)} a^{\infty}(x, \omega)\eta, \eta) \geq M_0 \mid \eta \mid^2, \qquad M_0 > 0 \tag{3.36}$$

holds for any $\omega \in S^{n-1}$, $\eta \in \mathbb{C}^m$, and $\mid \theta \mid < \pi/2 - \alpha_x$.

Let's notice that a locally $\varepsilon-$sectorial symbol (for some $\varepsilon > 0$) is known as locally strongly elliptic.

The following two conditions are equivalent reformulations of condition (3.35) each:

(i) the Hausdorff set

$$\mathscr{H}(a^{\infty}(x,\omega)) = \{(a^{\infty}(x,\omega)\eta,\eta) \ : \ \omega \in S^{n-1}, \quad \eta \in \mathbb{C}_m, \quad |\eta| = 1\}$$

of the matrix-function $a^{\infty}(x,\omega)$, which includes the spectrum, fits inside the open angle $\{\zeta \in \mathbb{C} \setminus \{0\} \ : \ |\arg\zeta - \theta_x| < \alpha_x\}$;

(ii) the matrix-functions $a^{\infty}_{\mathscr{R}}(x,\omega) \pm \operatorname{ctg}\alpha_x a^{\infty}_{\mathscr{I}}(x,\omega)$ are positive definite; here $a_{\mathscr{R}} = \frac{1}{2}(a + a^*), \quad a_{\mathscr{I}} = \frac{1}{2}(a - a^*)$ and a^* denotes the conjugate matrix.

Theorem 3.26 *Let $\mathscr{M}$ have a Lipschitz boundary, preconditions of Theorem 3.20 hold and the matrix-symbol $a_x(\xi)$ be elliptic (cf. (3.20).*

If $a(x,\xi)$ is α_x- sectorial for any point $x \in \partial\mathscr{M}$ then the pseudodifferential operator

$$a(x,D) \ : \ \tilde{H}_2^{\frac{r}{2}+s}(\mathscr{M}) \longrightarrow H_2^{-\frac{r}{2}+s}(\mathscr{M}) \tag{3.37}$$

is Fredholm for all

$$|s| \leq \frac{1}{2} - \frac{\alpha_a}{\pi}, \qquad \alpha_a = \sup\{\alpha_x \ : \ x \in \partial\overline{\mathscr{M}}\} < \frac{\pi}{2},$$

If, additionally, $a(x,\xi)$ is $(\pi - \varepsilon)-$ sectorial, that means

$$\mathscr{H}\left(\exp(-i\theta_x)a^{\infty}(x,\omega)\right) \bigcap \{z \in \mathbb{C} : z = \operatorname{Re} z \leq 0\} = \emptyset \tag{3.38}$$

for a certain $\theta_x \in [0,2\pi]$and any $x \in \mathscr{M} = \overline{\mathscr{M}} \setminus \partial\mathscr{M}$, then

$$\operatorname{Ind} a(x,D) = 0. \tag{3.39}$$

Proof. It can be assumed that $\theta_x \in C^l(\mathscr{M})$: in according with Lemma 3.15 θ_x depends continuously on $x \in \overline{\mathscr{M}}$; using the approximation by C^l-function $\tilde{\theta}_x$ in $L_\infty-$ norm $a(x,\xi)$ can be supposed to be α_x- sectorial with respect to the approximating function $\tilde{\theta}_x$.

Now the operator $\exp(-i\theta_x)a(x,D)$ can be considered in the same spaces as $a(x,D)$ (cf. (3.36)); since these operators are Fredholm (or are not) simultaneously and their indices coincide, it can be supposed that $\theta_x \equiv 0, \quad x \in \overline{\mathscr{M}}$.

According to Theorem 3.20 the Fredholm property of operator (3.36) is implied by the invertibility of the following operators

$$r_\Omega W^0_{a_0(x,\cdot)} \ : \ L_2(\Omega) \to L_2(\Omega), \qquad x \in \partial\mathscr{M}, \tag{3.40}$$

$$a_0(x,\xi) = \overline{(b_0^{-\frac{r}{2}+s})^\infty(\xi)} a^\infty(x,\xi)(b_0^{-\frac{r}{2}-s})^\infty(\xi) = g_0(\xi)\exp[i\varphi(\xi)]a^\infty(x,\xi),$$
$$g_0(\xi) = \mid (a^{-r})^\infty(\sigma^T\xi) \mid \geq M_0 > 0, \quad \varphi(\xi) = -2\arg a^s(\sigma^T\xi).$$

Due to (2.30) we get

$$\begin{aligned}\varphi(\xi) &= s\frac{\Gamma(\frac{n}{2})}{4\pi^{\frac{n}{2}-1}}\int_{S^{n-1}}\psi(x)\,\mathrm{sgn}(\sigma^T\xi\cdot x)d_xS \\ &= s\frac{\Gamma(\frac{n}{2})}{4\pi^{\frac{n}{2}-1}}\int_{S^{n-1}\cap\Sigma_1^n}[\psi(x)+1]\,\mathrm{sgn}(\sigma^T\xi\cdot x)d_xS,\end{aligned}$$

and therefore (cf. (3.29))

$$\mid \varphi(\xi) \mid \leq \pi \mid s \mid < \frac{\pi}{2} - \alpha_a. \tag{3.41}$$

From (3.39) and (3.40) follows

$$\begin{aligned}\mathrm{Re}(a_0(x,\xi)\eta,\eta) &= \mathrm{Re}\,(\exp\,(i[\varphi(\xi)])\,g_0(\xi)a^\infty(x,\xi)\eta,\eta) \\ &\geq \mathrm{Re}\left(\exp\,(i[\varphi(\xi)])\,a^\infty(x,\xi)\sqrt{g_0(\xi)}\eta, \sqrt{g_0(\xi)}\eta\right) \\ &\geq M_1 \mid \sqrt{g_0(\xi)}\eta \mid^2 \geq M_1 \mid \eta \mid^2 .\end{aligned}$$

Further we proceed with the help of Parseval's equality as follows

$$\begin{aligned}&\mathrm{Re}\left(r_\Omega W^0_{a_0(x,\cdot)}u,u\right) = \mathrm{Re}\left(F^{-1}a_0(x,\cdot)Fu,u\right) \\ &\quad = \frac{1}{(2\pi)^n}\,\mathrm{Re}\,(a_0(x,\xi)Fu,Fu) \geq \\ &\quad \geq \frac{M_1}{(2\pi)^n}\|Fu\|^2_{L_2} = M_1\|u\|^2_{L_2}, \qquad u \in C_0^\infty(\Omega) \subset L_2(I\!R_n).\end{aligned} \tag{3.42}$$

(3.41) yields the invertibility of operator (3.39) and, therefore the Fredholm property of operator (3.36): if $\mathrm{Re}(\mathbf{A}u,u) \geq M_1\|u\|^2$, then the kernels $Ker\mathbf{A}$ and $Ker\mathbf{A}^*$ are trivial (we recall that $(\mathbf{A}u,u) = (u,\mathbf{A}^*u)$; $\mathbf{A}$ has closed range as well, while from $\lim_{n\to\infty}\mathbf{A}u_n = v$ there follow the convergence $u_n \to w$ to a certain w and the equality $\mathbf{A}w = v$.

Thus (3.36) is a Fredholm operator.

Let us prove formula (3.38) under the condition (3.37) (we recall that $\theta_x \equiv 0$).

Let for the beginning $s = 0$. The operator (cf. (3.24), (3.25))

$$a_\lambda(x,D) = (1-\lambda)a(x,D) + \lambda\overline{\mathbf{B}}^{\frac{r}{2}}_{\mathscr{M}}\mathbf{B}^{\frac{r}{2}}_{\mathscr{M}} \;:\; H_p^{\frac{r}{2}}(\mathscr{M}) \longrightarrow H_p^{-\frac{r}{2}}(\mathscr{M}), \quad 0 \leq \lambda \leq 1$$

depends continuously on the parameter λ; the homogeneous symbol of the corresponding lifted operator reads (cf (3.26))

$$a_\lambda(x,\xi) = (1-\lambda)g_0(\xi)a^\infty(x,\xi) + \lambda, \quad 0 \le \lambda \le 1$$

and is α_x- sectorial for any point of the boundary $x \in \partial\mathscr{M}$, $\alpha_x \le \alpha_a \le \pi/2$; due to condition (3.37) it is elliptic as well.

Since $a_0(x,D) = a(x,D)$, we get

$$\operatorname{Ind} a(x,D) = \operatorname{Ind} a_\lambda(x,D) = \operatorname{Ind} a_1(x,D) = 0,$$

because the lifted operator for $a_1(x,D)$ in the space $L_2(\mathscr{M})$ (cf. Theorem 3.24) is the identity

$$\overline{\mathbf{B}}_{\mathscr{M}}^{-\frac{r}{2}} a_1(x,D)\mathbf{B}_{\mathscr{M}}^{-\frac{r}{2}} = I \quad : \quad L_2(\mathscr{M}) \longrightarrow L_2(\mathscr{M}).$$

Let now $\mid s \mid \le 1/2 - \alpha_a/\pi, \;\; s \ne 0$.

The symbol $a_s^0(x,\xi)$ of the lifted operator $a_s^0(x,D) = \overline{\mathbf{B}}_{\mathscr{M}}^{-\frac{r}{2}+s} a(x,D)\mathbf{B}_{\mathscr{M}}^{\frac{r}{2}+s}$ (cf. Theorem 3.24)) reads $a_s^0(x,\xi) = a_0(x,\xi)$ (cf. (3.39)). In according with Theorem 3.24 the indices of operator (3.36) and of the operator

$$a_s^0(x,D) \quad : \quad L_2(\mathscr{M}) \to L_2(\mathscr{M}) \tag{3.43}$$

are equal and therefore we need to prove $\operatorname{Ind} a_s^0(x,D) = 0$.

Operator (3.42) and its symbol $a_s^0(x,\xi)$ depend continuously on s. Due to the proved part of the theorem (3.42) is a Fredholm operator for any $\mid s \mid \le 1/2 - \alpha_a/\pi$; then

$$\operatorname{Ind} a_s^0(x,D) = \operatorname{Ind} a_0^0(x,D) = 0,$$

since $a_0^0(x,D) = a_0(x,D)$ and this operator was already considered. ■

Remark 3.27 *If condition (3.37) fails then formula (3.38) may be violated.*

To prove this claim we will describe the method for calculating the index of operator (3.36) in this case.

Let $\varphi_0(x)$ be a C^l-function on $\overline{\mathscr{M}}$ for which $\operatorname{supp}\varphi_0$ *and the set $\{x \in \overline{\mathscr{M}} : \varphi_0(x) = = 1\}$ are concentrated in a sufficiently small neighbourhood of $\partial\mathscr{M} \subset \overline{\mathscr{M}}$; then the homotopy*

$$\tilde{a}_\lambda(x,D) = \varphi_0(x)\left[(1-\lambda)a(x,D) + \lambda\overline{\mathbf{B}}_{\mathscr{M}}^{\frac{r}{2}}\mathbf{B}_{\mathscr{M}}^{\frac{r}{2}}\right] + [1-\varphi_0(x)]\,a(x,D),$$

$$0 \le \lambda \le 1$$

connects the operator $\tilde{a}_0(x,D) = a(x,D)$ with $\tilde{a}_1(x,D)$, and remains Fredholm in the space $L_2(\overline{\mathscr{M}})$ for all $0 \le \lambda \le 1$; therefore

$$\operatorname{Ind} a(x,D) = \operatorname{Ind} \tilde{a}_1(x,D) = \operatorname{Ind} \tilde{a}_2(x,D),$$

where $\tilde{a}_2(x, D)$ *is the lifted operator (cf. (3.25))*

$$\tilde{a}_2(x, D) = \overline{\mathbf{B}}_{\mathscr{M}}^{-\frac{r}{2}} \tilde{a}_1(x, D) \mathbf{B}_{\mathscr{M}}^{-\frac{r}{2}} \quad : \quad L_2(\mathscr{M}) \longrightarrow L_2(\mathscr{M}).$$

Obviously $\tilde{a}_2(x, \xi) \equiv 1$ *in some neighbourhood of* $\partial\mathscr{M} \subset \overline{\mathscr{M}}$ $(\xi \in \mathbb{R}_n)$ *and if* $a_2(x, \xi)$ *is the extension of* $\tilde{a}_2(x, \xi)$ *on* $\mathscr{M}_1$ *by 1 (we recall that* $\mathscr{M}_1$ *is a closed* C^l-*manifold and* $\mathscr{M} \subset \mathscr{M}_1$*), then the operator*

$$a_2(x, D) \quad : \quad L_2(\mathscr{M}_1) \longrightarrow L_2(\mathscr{M}_1).$$

is Fredholm and

$$\operatorname{Ind} \tilde{a}_2(x, D) = \operatorname{Ind} a_2(x, D).$$

Thus the problem is reduced to the closed manifold case and the Atiyah-Singer index formula can be applied.

Any example of pseudodifferential operator on the closed manifold $\mathscr{M}_1$ *with a non-trivial index can be used to construct an example of a pseudodifferential operator on some open manifold* $\mathscr{M} \subset \mathscr{M}_1$ *with the same (non-trivial) index and the same symbol* $a(x, \xi)$ *for any* $x \in \mathscr{M}$ $(\xi \in \mathbb{R}_n)$.

4 PSEUDODIFFERENTIAL OPERATORS AND REDUCTION OF ORDER IN HÖLDER-ZYGMUND SPACES

4.1 HÖLDER-ZYGMUND SPACES

Let $0 < \alpha < 1, \quad 1 \le p \le \infty$. Then the space $Z_p^\alpha(\mathbb{R}_n)$ consists of functions

$$Z_p^\alpha(\mathbb{R}_n) = \left\{ \varphi \in L_p(\mathbb{R}_n) \quad : \quad \|\varphi\|_{Z^\alpha} = \sup_{x,t\in\mathbb{R}_n} \frac{|\, \varphi(x+t) - \varphi(x)\,|}{|\, t\,|^\alpha} < \infty \right\}$$

and is endowed with the norm (cf. Ka1,Ka2 [19, 20])

$$\|\varphi\|_{Z_p^\alpha} = \|\varphi\|_{L_p} + \|\varphi\|_{Z^\alpha}. \tag{4.1}$$

For a Lipschitz domain $\Omega \subset \mathbb{R}_n$ two different spaces can be defined (cf. e1.15 e1.16 (1.15),(1.16)):

$$\tilde{Z}_p^\alpha(\Omega) = \left\{ u \in Z_p^\alpha(\mathbb{R}_n) \quad : \quad \operatorname{supp} u \subset \Omega \right\},$$

with the norm induced from $Z_p^\alpha(\mathbb{R}_n)$ and the space

$$Z_p^\alpha(\Omega) = \left\{ u = r_\Omega v \quad : \quad v \in Z_p^\alpha(\mathbb{R}_n) \right\}$$

with the norm of the factor-space

$$|\!|\!|\, u |\!|\!|_{Z_p^\alpha} = \inf \left\{ \|v\|_{Z_p^\alpha} \quad : \quad v \in Z_p^\alpha(\mathbb{R}_n), \quad r_\Omega v = u \right\},$$

where r_Ω denotes the restriction operator as before.

To extend the definition of the space $Z_p^\alpha(I\!R_n)$ to the case $\alpha \geq 1$ the Poisson integral is involved (cf. [41], § III.2)

$$P_y\varphi(x) = \int_{\boldsymbol{R}_n} P_y(x-t)\varphi(t)dt = W^0_{a_y} f(x), \qquad P_y(x) = \frac{c_n y}{(\mid x \mid^2 + y^2)^{\frac{n+1}{2}}},$$
$$c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}, \qquad a_y(\xi) = \exp(- \mid \xi \mid y), \quad y > 0, \quad x, \xi \in I\!R_n.$$

$P_y\varphi(x)$ is a harmonic function

$$\left(\frac{\partial^2}{\partial y^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right) P_y\varphi(x) \equiv 0$$

and approximates $\varphi(x)$ (cf. [41], § III.2)

$$\lim_{y\to 0} \|P_y\varphi - \varphi\|_p = 0, \qquad \varphi \in L_p(I\!R_n), \quad 1 < p < \infty. \tag{4.2}$$

Lemma 4.1 *The expression*

$$\|\varphi\|^{(0)}_{Z_p^\alpha} = \|\varphi\|_p + \sup\nolimits_{y>0} y^{1-\alpha} \|D_y P_y \varphi\|_\infty \tag{4.3}$$

defines an equivalent norm in the space $Z_p^\alpha(I\!R_n)$.

Other equivalent norms are

$$\|\varphi\|^{(k)}_{Z_p^\alpha} = \|\varphi\|_p + \sup\nolimits_{y>0} y^{1-\alpha} \|D_{x_k} P_y \varphi\|_\infty, \qquad k = 1, 2, ...n. \tag{4.4}$$

Proof For the case $p = \infty$ cf. [41], § V.4. Let us start with the equivalence of norms (4.4) and (4.3). For this we recall the following inequalities

$$\left\| \frac{\partial P_{x_0}}{\partial x_j} \right\|_1 \leq \frac{C_1}{x_0}, \qquad x_0 > 0, \quad j = 0, 1, ..., n \tag{4.5}$$

which are implied by the following two estimates (they are applied to the integrals over $\mid t \mid \leq x_0$ and over $\mid t \mid > x_0$ respectively)

$$\left| \frac{\partial P_{x_0}(x)}{\partial x_j} \right| \leq \frac{C_2}{x_0^{n+1}}, \qquad \left| \frac{\partial P_{x_0}(x)}{\partial x_j} \right| \leq \frac{C_2}{\mid x \mid^{n+1}},$$
$$x \in I\!R_n, \quad x_0 > 0, \quad j = 0, 1, ..., n.$$

Since $P_y = P_{\frac{y}{2}} * P_{\frac{y}{2}}$ we get

$$\frac{\partial^2 P_{x_0}\varphi(x)}{\partial x_j \partial x_k} = \frac{\partial^2 \left(P_{\frac{x_0}{2}} * P_{\frac{x_0}{2}}\varphi\right)(x)}{\partial x_j \partial x_k}$$

and therefore (cf. (4.3) and (4.5))

$$\left\|\frac{\partial^2 P_{x_0}\varphi}{\partial x_j \partial x_k}\right\|_\infty \le \left\|\frac{\partial P_{\frac{x_0}{2}}}{\partial x_j}\right\|_1 \left\|\frac{\partial P_{\frac{x_0}{2}}\varphi(x)}{\partial x_k}\right\|_\infty \le C_1 \|\varphi\|^{(k)}_{Z^\alpha_p}\left(\frac{x_0}{2}\right)^{\alpha-2}, \qquad j,k = 0,1,...,n. \tag{4.6}$$

Applying the Hölder inequality we get

$$\left\|\frac{\partial P_{x_0}\varphi}{\partial x_k}\right\|_\infty = \left\|\frac{\partial P_{x_0}}{\partial x_k} * \varphi\right\|_\infty \le \left\|\frac{\partial P_{x_0}}{\partial x_k}\right\|_{p'} \|\varphi\|_p = C_3\left(\frac{x_0}{2}\right)^{\frac{n}{p'}-n-1} \|\varphi\|_p\,, \tag{4.7}$$

where $p' = p/(p-1)$ for $1 < p < \infty$ and $p' = \infty$ $(p' = 1)$ for $p = 1$ $(p = \infty)$.

Therefore

$$\lim_{x_0 \to \infty} \frac{\partial P_{x_0}\varphi(x)}{\partial x_k} = 0$$

which yields

$$\frac{\partial P_{x_0}\varphi(x)}{\partial x_j} = \int_{x_j}^\infty \left(\frac{\partial^2 P_{x_0}\varphi(x)}{\partial x_j \partial x_k}\right)_{x_k=\lambda} d\lambda.$$

The last equality together with (4.6) yields

$$\left\|\frac{\partial P_{x_0}\varphi}{\partial x_j}\right\|_\infty \le \frac{2^{2-\alpha}C_1}{\alpha-1}\|\varphi\,\|^{(k)}_{Z^\alpha_p} x_0^{\alpha-1}, \qquad j = 0,1,...,n. \tag{4.8}$$

Hence

$$\|\varphi\|^{(k)}_{Z^\alpha_p} \le \|\varphi\|_p + \frac{2^{2-\alpha}C_1}{\alpha-1}\|\varphi\|^{(k)}_{Z^\alpha_p} \le C_4\,\|\varphi\|^{(k)}_{Z^\alpha_p}\,, \qquad k = 0,1,...,n.$$

St1 Now we prove the equivalence of (4.3) and (4.1). For this we recall the inequalities from [41], § III.2 (the second is implied by the first one)

$$\int_{\boldsymbol{R}_n} P_y(t)dt \equiv 0, \qquad \int_{\boldsymbol{R}_n} \frac{\partial P_y(t)dt}{\partial y} \equiv 0, \qquad y > 0. \tag{4.9}$$

But then

$$\frac{\partial P_y(t)\varphi(x)}{\partial y} = \int_{\boldsymbol{R}_n} \frac{\partial P_y(t)}{\partial y}[\varphi(x-t) - \varphi(x)]dt$$

and further similarly to (4.5)

$$\begin{aligned}\left\|\frac{\partial P_y(t)\varphi}{\partial y}\right\|_\infty &\le \|\varphi\|_{Z_p^\alpha}\int_{\boldsymbol{R}_n}\left|\frac{\partial P_y(t)}{\partial y}\right| \mid t\mid^\alpha dt = C_5\|\varphi\|_{Z_p^\alpha} y^{\alpha-1}, \qquad y>0\\ \|\varphi\|^{(0)}_{Z_p^\alpha} &= \|\varphi\|_p + \sup_{x\in\boldsymbol{R}_n, y>0} y^{1-\alpha}\left|\frac{\partial P_y(t)\varphi(t)}{\partial y}\right|\\ &\le \|\varphi\|_p + C_5\|\varphi\|_{Z_p^\alpha} \le (1+C_5)\|\varphi\|_{Z_p^\alpha}.\end{aligned}$$

To prove the inverse inequality, we apply Lagrange's formula

$$P_y\varphi(x+t) - P_y\varphi(x) = \sum_{k=1}^n t_k\left(\frac{\partial P_y\varphi(z)}{\partial z_k}\right)_{z=x+\theta t}, \qquad 0<\theta<1$$

which yields due to the equivalence of the norms (4.3) and (4.4)

$$\begin{aligned}|P_y\varphi(x+t) - P_y\varphi(x)| &\le \mid t\mid \sum_{k=1}^n \left\|\frac{\partial P_y\varphi}{\partial x_k}\right\|_\infty\\ &\le \mid t\mid^\alpha \sum_{k=1}^n \|\varphi\|^{(k)}_{Z_p^\alpha} \le C_6\|\varphi\|^{(0)}_{Z_p^\alpha} \mid t\mid^\alpha . \qquad (4.10)\end{aligned}$$

Since

$$P_y\varphi(x+z) - \varphi(x+z) = \int_0^y \frac{\partial}{\partial\lambda}P_\lambda\varphi(x+z)d\lambda$$

we get

$$|P_y\varphi(x+z) - \varphi(x+z)| \le \|\varphi\|^{(0)}_{Z_p^\alpha}\int_0^y \lambda^{\alpha-1}d\lambda = \frac{\|\varphi\|^{(0)}_{Z_p^\alpha}}{\alpha}y^{\alpha-1}, \quad y>0. \qquad (4.11)$$

From the identity

$$\varphi(x+t)-\varphi(x) = [\varphi(x+t) - P_{|t|}\varphi(x+t)] + [P_{|t|}\varphi(x+t) - P_{|t|}\varphi(x)] + [P_{|t|}\varphi(x) - \varphi(x)]$$

applying (4.10) and (4.11) (the latter is applied for $y = \mid t\mid, z=t$ and for $z=0$) we get

$$\mid \varphi(x+t)-\varphi(x)\mid \le C_7\|\varphi\|^{(0)}_{Z_p^\alpha} \mid t\mid^\alpha . \qquad x\in I\!R_n, \quad y>0$$

which yields

$$\|\varphi\|_{Z_p^\alpha} \le (1+C_7)\|\varphi\|^{(0)}_{Z_p^\alpha}.$$

■

The foregoing lemma leads to the following definition of the space $Z_p^\alpha(I\!R_n)$ for $0 < \alpha < \infty, \quad 1 \le p \le \infty$:

$$Z_p^\alpha(I\!R_n) = \Big\{\varphi \in L_p(I\!R_n) : \|\varphi\|_{Z_p^\alpha} = \|\varphi\|_p + \sup_{y>0} y^{k-\alpha} \|D_y^k P_y \varphi\|_\infty < \infty, \quad k = [\alpha] + 1\Big\},$$

where $[\alpha]$ denotes the integer part of α.

Lemma 4.2 *Let* $0 < \alpha < \infty, \quad 1 \le p \le \infty.$

For a function $\varphi \in Z_p^\alpha(I\!R_n)$ *the following estimate holds*

$$\|P_y\varphi - \varphi\|_\infty \le C y^\alpha \|\varphi\|_{Z^\alpha}, \qquad y > 0. \tag{4.12}$$

If $\Omega = I\!R_n$ *or* $\Omega \subset I\!R_n$ *is a non-compact Lipschitz domain then*

$$Z_p^\alpha(\Omega) = L_p(\Omega) \cap Z_\infty^\alpha(\Omega), \qquad \tilde{Z}_p^\alpha(\Omega) = L_p(\Omega) \cap \tilde{Z}_\infty^\alpha(\Omega) \tag{4.13}$$

where $\tilde{Z}_p^\alpha(I\!R_n) = Z_p^\alpha(I\!R_n)$.

Moreover: if $\Omega \subset I\!R_n$ *is a compact Lipschitz domain, then*

$$\tilde{Z}_p^\alpha(\Omega) = \tilde{Z}_\infty^\alpha(\Omega), \qquad Z_p^\alpha(\Omega) = Z_\infty^\alpha(\Omega). \tag{4.14}$$

Proof. Let us prove first (4.12). Due to (4.9) we get

$$P_y\varphi(x) - \varphi(x) = \int_{\boldsymbol{R}_n} P_y(t)[\varphi(x-t) - \varphi(t)]dt$$

and further

$$\|P_y\varphi - \varphi\|_\infty \le \|\varphi\|_{Z^\alpha} \int_{\boldsymbol{R}_n} P_y(t) \mid t \mid^\alpha dt = C_1 \|\varphi\|_{Z^\alpha} y^\alpha, \qquad y > 0$$

which is exactly (4.12).

Obviously $L_p(I\!R_n) \cap Z_\infty^\alpha(I\!R_n) \subset Z_p^\alpha(I\!R_n)$.

Let now $\varphi \in Z_p^\alpha(I\!R_n)$; then

$$\|\varphi\|_\infty \le \|P_y\varphi\|_\infty + \|P_y\varphi - \varphi\|_\infty \le \|P_y\|_{p'} \|\varphi\|_p + C_1 y^\alpha \|\varphi\|_{Z^\alpha} < \infty, \quad y > 0$$

which implies $\varphi \in Z_\infty^\alpha(I\!R_n)$ and proves (4.13) for $\Omega = I\!R_n$. For any other $\Omega \subset I\!R_n$ the proof is similar.

(4.14) follows from (4.13) since for a compact $\Omega \subset I\!R_n$ the inclusion $\varphi \in \tilde{Z}_\infty^\alpha(\Omega)$ $(\varphi \in Z_\infty^\alpha(\Omega))$ implies $\varphi \in L_p(\Omega)$. ∎

Lemma 4.3 *Let* $0 < \alpha < 2, \quad 1 \le p \le \infty$. *The expression*

$$\|\varphi\|'_{Z_p^\alpha} = \|\varphi\|_p + \|\varphi\|'_{Z^\alpha}, \tag{4.15}$$

$$\|\varphi\|'_{Z^\alpha} = \sup_{x,t \in \boldsymbol{R}_n, t>0} \frac{\mid \varphi(x+t) + \varphi(x-t) - 2\varphi(x) \mid}{\mid t \mid^\alpha}$$

defines an equivalent norm in the space $Z_p^\alpha(I\!\!R_n)$.

Proof is exposed in [41], § V.4, Proposition 8 for the case $p = \infty$ and is similar for $1 \le p < \infty$. ■

Lemma 4.4 *Let* $\alpha > 1, \quad 1 \le p \le \infty$. *Then* $\varphi \in Z_p^\alpha(I\!\!R_n)$ *if and only if* $D_{x_j}\varphi \in$ $\in Z_p^{\alpha-\varepsilon}(I\!\!R_n), j = 1, 2, ..., n$. *An equivalent norm is the following*

$$\|\varphi\|''_{Z_p^\alpha} = \|\varphi\|_p + \sum_{|\beta|=m \le [\alpha]} \|D^\beta \varphi\|_{Z_p^{\alpha-m}}. \tag{4.16}$$

Proof is similar for any $1 \le p \le \infty$ and is exposed in [41], § V.4, Proposition 9 for the case $p = \infty$. ■

Lemma 4.5 *Let* $\Omega = I\!\!R_n$ *or* $\Omega \subset I\!\!R_n$ *be any special or general Lipschitz domain and* $0 < \gamma < \alpha < \infty, \; 1 < p < \infty$. *Then the following embeddings are continuous*

$$\tilde{Z}_p^\alpha(\Omega) \subset \tilde{Z}_p^\gamma(\Omega), \qquad Z_p^\alpha(\Omega) \subset Z_p^\gamma(\Omega). \tag{4.17}$$

If moreover Ω *is compact then the embeddings (4.17) are compact.*

Proof. The property

$$\|D_y^r P_y \varphi\|_\infty \le C_1 y^{\alpha-r}, \qquad r = 0, 1, ..., \le [\alpha] + 1, \;\; y > 0 \tag{4.18}$$

of the function $\varphi \in L_p(I\!\!R_n)$ is necessary and sufficient for $\varphi \in Z_p^\alpha(\Omega)$ and is important only for $0 < y < 1$ since for $y \ge 1$ the stronger inequality holds (proved similarly to (4.7))

$$\|D_y^r P_y \varphi\|_\infty \le C_2 y^{\frac{n}{p'}-n-r} \|\varphi\|_p, \qquad r = 0, 1, ..., \le [\alpha] + 1, \;\; p' = \frac{p}{p-1};$$

now (4.17) is obvious.

If $\beta < \alpha$ and Ω is compact, we can suppose $p = \infty$ (cf. (4.14)). Let $\{\varphi_j\}_1^\infty \subset Z_\infty^\alpha(\Omega)$ be any bounded set; using the equivalent norm (4.16) and Arzel'a-Ascoli's theorem about compactness in $C(\Omega)$, a subsequence $\{\varphi_{j_k}\}_{k=1}^\infty \subset Z_\infty^\alpha(\Omega)$ can be selected which converges in $C(\Omega)$. If $\varphi = \lim_{k\to\infty} \varphi_{j_k}$, then obviously $\varphi \in Z_\infty^\alpha(\Omega) \subset Z_\infty^\gamma(\Omega)$ and (cf. (4.3),(4,15),(4.16))

$$\lim_k \|\varphi_{j_k} - \varphi\|_{Z_\infty^\gamma} = \lim_k \left[\|\varphi_{j_k} - \varphi\|_\infty + \sup_{y>0} y^{r-\gamma} \|D^\beta P_y(\varphi_{j_k} - \varphi)\|_\infty\right] =$$

$$= \lim_k \|D^\beta P_y(\varphi_{j_k} - \varphi)\|_\infty^{1-\frac{\gamma-r}{\alpha-r}} \sup_{y>0} \left\{y^{r-\alpha} \|D^\beta P_y(\varphi_{j_k} - \varphi)\|_\infty\right\}^{\frac{\gamma-r}{\alpha-r}} = 0,$$

where $\mid \beta \mid = r = [\gamma] \le [\alpha]$. ■

Theorem 4.6 *Let $\Omega \subset I\!\!R_n$ be a Lipschitz domain and either $X^\alpha = Z_p^\alpha(\Omega)$ or $X^\alpha = \tilde{Z}_p^\alpha(\Omega)$. then*

$$\mathscr{L}(X^{\alpha_1}, X^{\gamma_1}) \bigcap \mathscr{L}(X^{\alpha_2}, X^{\gamma_2}) \subset \mathscr{L}(X^\alpha, X^\gamma),$$
$$1 < p < \infty, \;\; 0 < \alpha_j, \beta_j < \infty, \;\; j = 1, 2,$$
$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2, \quad \gamma = (1-\theta)\gamma_1 + \theta\gamma_2, \quad 0 < \theta < 1,$$

where $\mathscr{L}(\mathscr{B}_1, \mathscr{B}_2)$ denotes the set of linear bounded operators $\mathbf{A} : \mathscr{B}_1 \to \mathscr{B}_2$.

Proof is similar for the cases $1 \le p \le \infty$ and is exposed in [43], § 27.2 for the case $p = \infty$. ■

Remark 4.7 *Norms (4.1) and (4.15) are equivalent for $0 < \alpha < 1$ but not for $\alpha = 1$ (cf. [41], Example V.4.3.1).*

The expression

$$\|\varphi\|'''_{Z_p^\alpha} = \|\varphi\|_p + \sum_{|\beta|=[\alpha]^-} \|D^\beta \varphi\|_{Z_p^{\{\alpha\}^+}}, \tag{4.19}$$
$$\alpha - 1 \le [\alpha]^- < \alpha, \quad \{\alpha\}^+ = \alpha - [\alpha]^-$$

where $[\alpha]^-$ is integer, defines an equivalent norm in the space $Z_p^\alpha(I\!\!R_n)$, $1 < p < \infty$, $1 < < \alpha < \infty$.

4.2 MULTIPLIERS

By $M^r(Z_p^\alpha(I\!\!R_n))$ we shall denote the class of multipliers (functions) $a(\xi)$ for which the convolution operator

$$W_a^0 \;\; : \;\; Z_p^\alpha(I\!\!R_n) \longrightarrow Z_p^{\alpha-r}(I\!\!R_n) \tag{4.20}$$

is bounded. If $r = 0$ the notation $M(Z_p^\alpha(I\!\!R_n))$ is used.

Theorem 4.8 *Let $a \in L_1^{loc}(I\!\!R_n)$ and let the condition*

$$\sum_{|\beta| \le \left[\frac{n}{2}\right]+1, \beta \le 1} R^{2|\beta|-n} \int_{\frac{R}{2} < |\xi| \le 2R} |\, D^\beta a(\xi) \,|^2 \, d\xi < \infty \tag{4.21}$$

hold; then $a \in M(Z_p^\alpha(I\!\!R_n))$ for all $0 < \alpha < \infty$, $1 < p < \infty$.

The inequality

$$\sup\{|\, \xi \,|^{|\beta|} |\, D^\beta a(\xi) \,| : \beta \in I\!\!N_n, \;\; \beta \le 1, \;\; |\, \beta \,| \le \frac{n}{2} + 1\} < \infty$$

implies (4.21)

Proof. The identity $D^\beta W_a^0 = W_a^0 D^\beta,\quad \beta \in I\!N_n$ and the equivalent norm (4.19) can be applied to reduce the proof to the case $0 \leq \alpha \leq 1$.

We recall also that (cf. Theorem t1.3 1.3)

$$\| W_a^0 \varphi\|_p \leq C_1 \|\varphi\|_p, \qquad\qquad \varphi \in L_p(I\!R_n) \tag{4.22}$$

Further we follow the proof of Theorem 7.9.6 from Hr2 [18] with slight modifications.

Thus $0 < \alpha \leq 1,\ \varphi \in Z_p^\alpha(I\!R_n)$ can be assumed.

Consider $\chi \in C_0^\infty(I\!R_n),\ \chi(\xi) = \chi(-\xi), \chi(\xi) = 0$ for $\mid \xi \mid \geq 2$ and $\chi(\xi) = 1$ for $\mid \xi \mid \leq 1$. Then

$$\sum_{j=-\infty}^{\infty} \hat{v}(2^{-j}\xi) = 1, \qquad \xi æ 0, \qquad \hat{v}(\xi) = Fv(\xi) = \chi(\xi) - \chi(2\xi).$$

Clearly $v \in S(I\!R_n), \quad v(-\xi) = v(\xi)$ and

$$\varphi = \sum_{-\infty}^{\infty} \varphi_j, \qquad\qquad \hat{\varphi}_j(\xi) = \hat{v}(2^{-j}\xi)\hat{\varphi}(\xi) \tag{4.23}$$

where the explicit expression for $\varphi_j(x)$ reads

$$\begin{aligned} \varphi_j(x) &= \int_{\boldsymbol{R}_n} \varphi(x \pm 2^{-j}y) v(y) dy \\ &= \frac{1}{2} \int_{\boldsymbol{R}_n} \left[\varphi(x + 2^{-j}y) + \varphi(x - 2^{-j}y) - 2\varphi(x)\right] v(y) dy \end{aligned}$$

since

$$\int_{\boldsymbol{R}_n} v(y) dy = \hat{v}(0) = 0.$$

Hence

$$\|\varphi_j\|_\infty \leq C_2 2^{\alpha \cdot j} \|\varphi\|_{Z^\alpha}^{(0)}, \tag{4.24}$$

$$\|\varphi\|_{Z^\alpha}^{(0)} = \sup\nolimits_{x,t \in \boldsymbol{R}_n, |t|>0} \frac{\mid \varphi(x+t) + \varphi(x-t) - 2\varphi(x) \mid}{\mid t \mid^\alpha}.$$

Similarly

$$\begin{aligned} D^\beta \varphi_j(x) &= (-1)^{|\beta|} 2^{|\beta| j} \int_{\boldsymbol{R}_n} \varphi(x \pm 2^{-j}y) D^\beta v(y) dy \\ &= 2^{2j-1} \int_{\boldsymbol{R}_n} \left[\varphi(x + 2^{-j}y) + \varphi(x - 2^{-j}y) - 2\varphi(x)\right] D^\beta v(y) dy, \\ &\qquad\qquad\qquad \beta \in I\!N_n, \quad \mid \beta \mid = 2 \end{aligned}$$

and therefore

$$\|D^\beta \varphi_j\|_\infty \le C_2 2^{(2-\alpha)j}\|\varphi\|^{(0)}_{Z^\alpha}, \qquad |\beta| = 2. \tag{4.25}$$

Let $a_R(\xi) \equiv \hat{v}(\xi)a(R\xi)$; from (4.21) it follows

$$\sum_{|\beta|\le\left[\frac{n}{2}\right]+1} \int_{\boldsymbol{R}_n} |D^\beta a_R(\xi)|^2 \, d\xi \le C_3$$

and further (cf. Hr2 [18], § 7.9)

$$a_R = Fk_R, \qquad k_R \in L_1(I\!R_n), \qquad \sup_R \|k_R\|_1 \le C_R < \infty. \tag{4.26}$$

Since $\operatorname{supp}\hat{\varphi}_j \subset 2^j \operatorname{supp}\hat{v}$ the identity

$$a(\xi)\hat{\varphi}_j\xi = a_R(R^{-1}\xi)\hat{\varphi}_j(\xi), \qquad R = 2^j$$

is valid and we get

$$W^0_a\varphi_j = k_R * \varphi_j = R^n k_R(R\cdot) * \varphi_j \tag{4.27}$$

for their inverse Fourier images.

(4.24)-(4.27) yield

$$\begin{aligned} \| W^0_a\varphi_j\|_\infty &\le \|R^n k_R(R\cdot)\|_1\|\varphi_j\|_\infty \le C_2C_k2^{-\alpha_j}\|\varphi\|^{(0)}_{Z^\alpha}, \\ \| W^0_a D^\beta\varphi_j\|_\infty &\le C_1C_k 2^{2-\alpha_j}\|\varphi\|^{(0)}_{Z^\alpha}, \qquad |\beta| = 2. \end{aligned} \tag{4.28}$$

Applying twice Lagrange's formula for $g \in C^2(I\!R_n)$ we derive

$$g(x+t) + g(x-t) - 2g(x) = \sum_{j.k}\theta_1 t_j t_k D_j D_k g(x+\theta_2 t), \qquad -1 < \theta_1, \theta_2 < 1.$$

Hence

$$\|g\|^{(0)}_{Z^1} \le |t|^2 \sum_{|\beta|=2}\|D^\beta g\|_\infty. \tag{4.29}$$

Estimates (2.28) and (2.29) yield

$$\begin{aligned} & | W^0_a\varphi_j(x+t) + W^0_a\varphi_j(x-t) - 2 W^0_a\varphi_j(x) | \\ &\quad \le \sum_j | W^0_a [\varphi(x+t) + \varphi(x-t) - 2\varphi(x)] | \le \\ &\quad \le C_1C_k\left[3\sum_{2^{-j}<|t|} 2^{-\alpha j} + t^2 \sum_{2^{-j}\ge|t|} 2^{(2-\alpha)j}\right]\|\varphi\|^{(0)}_{Z^\alpha} \\ &\quad \le C_1C_k |t|^\alpha \left[\frac{3}{1-2^{-\alpha}} + \frac{1}{1-2^{2-\alpha}}\right]\|\varphi\|^{(0)}_{Z^\alpha} \le C_4\|\varphi\|^{(0)}_{Z^\alpha} |t|^\alpha, \end{aligned}$$

where C_4 is independent of $\varphi \in Z_p^\alpha(I\!R_n)$.

These inequalities together with (4.22) and Lemma 4.3 yield

$$\| W_a^0\varphi\|_{Z_p^\alpha} \le C_5\| W_a^0\varphi\|'_{Z_p^\alpha} \le C_6\|\varphi\|_{Z_p^\alpha}.$$

■

Lemma 4.9 *Let* $0 < \alpha < \infty, \quad 1 \le p \le \infty, \quad -\infty < r < \infty$ *and* $\alpha - r > 0$. *Then* $\lambda^r(\xi) = (1+ \mid \xi \mid^2)^{\frac{r}{2}} \in M^r(Z_p^\alpha(I\!R_n))$.

Proof(cf. [41], § V.4.4). Let first $r < 0$ and consider

$$G_r(x) = \frac{1}{2^n\Gamma(-\frac{r}{2})(2\pi)^{\frac{3n}{2}}} \int_0^\infty t^{-\frac{r+n}{2}} \exp\left(-t - \frac{\mid x \mid^2}{(4\pi)^2 t}\right) \frac{dt}{t}.$$

It is known (cf. [41], § V.3) that

$$\hat{G}_r(\xi) = FG_r(\xi) = \lambda^r(\xi), \qquad G_r \in L_1(I\!R), \quad r < 0 \tag{4.30}$$

Let

$$G_r(x,y) \overset{\text{def}}{=} P_yG_r(x) \qquad\qquad x \in I\!R_n, \quad y > 0,$$

where P_y denotes the Poisson integral again. Further in [41], Prop.V.5.4 it is proved that

$$\|D_y^m G_r(x,y)\|_\infty \le C_1 y^{-r-m}, \qquad m = [-r]+1, \quad y > 0. \tag{4.31}$$

Since $P_{y_1+y_2} = P_{y_1} * P_{y_2} \;\; (y_1, y_2 > 0)$, we get

$$\begin{aligned} (P_{y_1+y_2}\Lambda^r\varphi)(x) &= P_{y_1+y_2} * G_r * \varphi = P_{y_1} * G_r * P_{y_2} * \varphi \\ &= (G_r(\cdot, y_1) * P_{y_2}\varphi)(x), \qquad \Lambda^r = W_{\lambda^r}^0. \end{aligned}$$

Let $k = [\alpha]+1, \quad m = [-r]+1$. From the last identity and differentiation we get

$$(D_{y_1}^m D_{y_2}^k P_{y_1+y_2}\Lambda^r\varphi)(x) = (D_{y_1}^m G_r(\cdot, y_1) * D_{y_2}^k P_{y_2}\varphi)(x).$$

If $y_1 = y_2 = y/2$ and (4.31) is applied it follows

$$\begin{aligned} \|D_y^{k+m} P_y\Lambda^r\varphi\|_\infty &\le \|D_{\frac{y}{2}}^m G_r\left(\cdot, \frac{y}{2}\right)\|_1 \|D_{\frac{y}{2}}^k P_{\frac{y}{2}}\varphi\|_\infty \\ &\le C_1\|\varphi\|_{Z_p^\alpha}\left(\frac{y}{2}\right)^{\alpha-r-k-m} \end{aligned} \tag{4.32}$$

since (cf. the definition of the space $Z_p^\alpha(I\!R_n)$)

$$\|D_y^k P_y\varphi\|_\infty \le C_1\|\varphi\|_{Z_p^\alpha} y^{\alpha-k}.$$

If $k+m > [\alpha-r]+1$ from (4.32) the similar inequality can be derived for $k+m = [\alpha-r]+1$.

Combining (4.32) with the following (cf. (4.30))

$$\|\Lambda^r\varphi\|_p = \|G_r * \varphi\|_p \leq \|G_r\|_1\|\varphi\|_p$$

we get

$$\|\Lambda^r\varphi\|_{Z_p^{\alpha-r}} \leq C_2\|\varphi\|_{Z_p^\alpha}, \qquad \Lambda^r = W^0_{\lambda^r},$$

which proves the lemma for $r < 0$.

Let now $r > 0$.

Consider any integer $l \in I\!N, 2l > r$. Then the differential operator

$$\Lambda^{2l} = \left(I + \sum_1^2 D_j^2\right)^l \quad : \quad Z_p^{\alpha+2l-r}(I\!R_n) \longrightarrow Z_p^{\alpha-r}(I\!R_n)$$

is continuous (cf. Lemma 4.4). Since

$$\Lambda^r = W^0_{\lambda^r} = W^0_{\lambda^{2l}}W^0_{\lambda^{r-2l}} = \Lambda^{2l}\Lambda^{r-2l}$$

and

$$\Lambda^{r-2l} \quad : \quad Z_p^\alpha(I\!R_n) \longrightarrow Z_p^{\alpha+2l-r}(I\!R_n)$$

is continuous as well, continuity of

$$\Lambda^r \quad : \quad Z_p^\alpha(I\!R_n) \longrightarrow Z_p^{\alpha-r}(I\!R_n) \tag{4.33}$$

follows. ■

Corollary 4.10 *Operators (4.33) represent isomorphisms between the spaces for any* $0 < \alpha \leq \infty, \ \ 1 \leq p \leq \infty, \ \ -\infty < r < \infty, \ \ \alpha - r > 0.$

Corollary 4.11 $M^r(Z_p^\alpha(I\!R_n))$*is independent of* $0 < \alpha \leq \infty$ *and consists of functions* $\lambda^r a, \quad a \in M(Z_p^\alpha(I\!R_n))$.

Let

$$W(I\!R_n) = \{c + Fk(\xi) \quad : \quad k \in L_1(I\!R_n), \ c = \text{const}\}$$

denote the Wiener algebra endowed with the norm $\|a\|_W = \mid c \mid + \|k\|_1$.

Theorem 4.12 $W(I\!R_n) \subset \bigcap_{1\leq p\leq\infty, 0<\alpha<\infty} M(Z_p^\alpha(I\!R_n))$ *and if* $a(\xi) = a(\infty) +$ $+ Fk(\xi), \ \ k \in L_1(I\!R_n)$, *belongs to the Wiener algebra* $a \in W(I\!R_n)$, *then*

$$W_a^0\varphi(t) = a(\infty)\varphi(t) + \int_{\boldsymbol{R}_n} k(t-\tau)\varphi(\tau)d\tau \tag{4.34}$$

and can be approximated in norm by operators of the form

$$W^0_{a_j}\varphi(t) = a(\infty)\varphi(t) + \int_{\boldsymbol{R}_n} k_j(t-\tau)\varphi(\tau)d\tau, \quad a_j = a(\infty) + Fk_j, \quad k_j \in S(I\!R_n). \tag{4.35}$$

Proof. Representation (4.34) is well-known. Since

$$\| W_a^0\varphi\|_q \leq \mid a(\infty) \mid \|\varphi\|_q + \|k * \varphi\|_q \leq (\mid a(\infty) \mid + \|k\|_1) \|\varphi\|_q \\ = \|a\|_W \|\varphi\|_q, \qquad 1 \leq q \leq \infty$$

we get

$$\| W_a^0\varphi\|_{Z_p^\alpha} = \| W_a^0\varphi\|_p + \sup_{x\in\boldsymbol{R}_n, y>0} \mid y^{k-\alpha} D_y^k P_y\, W_a^0\varphi(x) \mid \leq \\ \leq \|a\|_W \|\varphi\|_p + \sup_{y>0} \left\| W_a^0 \left[y^{k-\alpha} D_y^k P_y\, W_a^0\varphi\right] \right\|_\infty \leq \|a\|_W \|\varphi\|_{Z_p^\alpha}. \tag{4.36}$$

Concerning approximation by operators (4.35): any function $k \in L_1(I\!R_n)$ can be approximated by functions $\{k_j\}_1^\infty \subset C_0^\infty(I\!R_n) \subset S(I\!R_n)$ in norm; hence due to (4.36) operators (4.35) approximate operator (4.34) in the operator norm. ■

Remark 4.13 *We refer the reader to [28], § XIII.6 for a wide class of singular integral operators (i.e. of operators* W_a^0 *with a symbol* $a(\xi)$ *homogeneous of order 0) described there, which are bounded in the Hölder-Zygmund spaces. We have to notice only that the assertions, formulated there for the case* $0 < \alpha < 1$, *stay valid for any* $\alpha > 0 (cf. Corrolary 4.11)$.

Theorem 4.14 *Let* $-\infty < r < \infty,\ \ a \in S^r(I\!R_n \times I\!R_n)$ *(cf. (1.10)) and* $a(x,\xi)$ *has compact support in* $x \in I\!R_n$.

If $0 < \alpha < \infty,\ \ 1 < p < \infty,\ \ \alpha - r > 0$ *the operator*

$$a(x,D) = W^0_{a(x,\cdot)} \quad : \quad Z_p^\alpha(I\!R_n) \longrightarrow Z_p^{\alpha-r}(I\!R_n) \tag{4.37}$$

is bounded.

Proof. Since $a_r = a\lambda^r \in S^0(I\!R_n \times I\!R_n)$ the operator $a_r(x,D) = a(x,D)\Lambda^{-r}$ is bounded in $L_p(I\!R_n)$ and in $Z_\infty^\alpha(I\!R_n)$ (cf. [31], § 2.3.2.5 - 2.3.2.6). Hence (4.37) is bounded due to Corollary 4.10 and Lemma 4.2. ■

4.3 Bessel potential operators

Let us recall that Λ^s represent Bessel potential operators on $Z_p^\alpha(I\!R_n)$ spaces for the full $I\!R_n$(cf. Corollary 4.10).

Theorem 4.15 *Let* $\Omega \subset I\!R_n$ *be a special Lipschitz domain,* $\Omega' = I\!R_n \setminus \Omega$ *and* $0 < \alpha < \infty, 1 < p < \infty, \quad -\infty < r < \infty, \quad \alpha - r > 0.$

The operators (cf. (3.7))

$$\begin{aligned} B_{0,\Omega} = W^0_{b_0^r} \quad &: \quad Z_p^\alpha(I\!R_n) \to Z_p^{\alpha-r}(I\!R_n), \\ &: \quad \tilde{Z}_p^\alpha(\Omega) \to \tilde{Z}_p^{\alpha-r}(\Omega), \\ \overline{B}_{0,\Omega} = W^0_{\overline{b}_0^r} \quad &: \quad \tilde{Z}_p^\alpha(\Omega') \to \tilde{Z}_p^{\alpha-r}(\Omega'), \\ \overline{B}_\Omega = r_\Omega W^0_{\overline{b}_0^r}\ell \quad &: \quad Z_p^\alpha(\Omega) \to Z_p^{\alpha-r}(\Omega), \end{aligned} \tag{4.38}$$

represent isomorphisms and the inverses read

$$(B_{0,\Omega})^{-1} = W^0_{b_0^{-r}}, \qquad \left(\overline{B}_{0,\Omega}\right) = W^0_{\overline{b}_0^{-r}} \qquad \left(\overline{B}_{\Omega}\right) = r_\Omega W^0_{\overline{b}_0^{-r}} \ell.$$

Proof. Due to (2.4), (3.7) and Theorem 2.3 from [33] the function $b_\pm = = \lambda^{\mp r} b_0^{\pm r}(\xi)$ meets the conditions of Theorem 4.9 (and moreover $b_\pm \in S^0(I\!R_n)$; therefore $b_0^{\pm r} \in M_p^{\pm r}(\ Z_p^\alpha(I\!R_n))$ (cf. Lemma 4.9).

Since $W^0_{b_0^{\pm r}}$ preserves supports within Ω (cf. Theorem 3.6) and are inverse to each other, the first two assertions in (4.38) are valid. The remainder is proved as in Lemma 3.7. ∎

Remark 4.16 *Despite of the inclusion $b_0^r \in S^r(\Omega \times I\!R_n)$ Theorem 4.14 can not be applied in Theorem 4.15, since the symbol $b_0^r(\xi)$ has not compact support in x.*

Corollary 4.17 *Let $0 < \alpha < \infty, \quad 1 < p < \infty, \quad -\infty < r < \infty, \quad \alpha - r > 0$ and $a \in \ M^r(Z_p^\alpha(I\!R_n))$.*

For a canonical Lipschitz domain $\Omega \subset I\!R_n$ the following diagram is commutative

$$\begin{array}{ccc} \widetilde{Z}_p^\alpha(\Omega) & \overset{r_\Omega\, W_a^0}{\longrightarrow} & Z_p^{\alpha-r}(\Omega) \\ \uparrow W^0_{b_0^{-s}} & & \downarrow W^0_{\overline{b}_0^{s}} \\ \widetilde{Z}_p^{\alpha-s}(\Omega) & \overset{r_\Omega W^0_{\overline{b}^s a b^{-s}}}{\longrightarrow} & Z_p^{\alpha-s-r}(\Omega). \end{array} \tag{4.39}$$

Remark 4.18 *The order reduction operator for $Z_\infty^\alpha(\Omega)$ space and a compact Lipschitz domain Ω, similar to (3.24) is described in [33], Theorem 3.5.But, as in the case of Bessel potential spaces (cf. § 3.4) this model needs some corrections.*

4.4 PSEUDODIFFERENTIAL OPERATORS

To extend Definition 3.14 to the space $Z_p^\alpha(\mathscr{M})$, we need some preliminary information.

For a bounded linear operator

$$\mathbf{A} \quad : \quad Z_p^\alpha(I\!R_n) \longrightarrow \ Z_p^{\alpha-r}(I\!R_n)$$

the factor-norm

$$|||\ \mathbf{A}|||_{Z_p^\alpha} = \inf\left\{\|\mathbf{A} + \mathbf{T}\|_{Z_p^\alpha} \quad : \quad \mathbf{T} \ \text{ is } \ \text{compact}\right\} \tag{4.40}$$

can be defined (cf.(3.11)).

Applying Kuratowski's measure of non-compactness R. Pöltz proved (cf. [29])

$$|||\ aI|||_{Z_p^\alpha} \le C\|a\|_\infty, \qquad\qquad a \in \ Z_p^\alpha(I\!R_n), \tag{4.41}$$

where aI is the multiplication operator in the space $Z_p^\alpha(I\!R_n)$.

The foregoing inequality opens the possibility for localization in Hölder-Zygmund spaces $Z_p^\alpha(\mathscr{M})$, where $\mathscr{M}$ represents a $\nu-$smooth manifold with a Lipschitz boundary $\partial\mathscr{M}$ and $\alpha \leq \nu$.

Further $\mathscr{M}$ is assumed to be compact.

Notations from § 2.3 are used here without further comments; moreover the definitions of the spaces $Z_p^\alpha(\mathscr{M})$, $\tilde{Z}_p^\alpha(\mathscr{M})$, of operators of local type $\mathbf{A} \in \mathrm{OLT}^r(\, Z_p^\alpha(\mathscr{M}))$ and of the equivalence $\mathbf{A} \overset{x}{\sim} \mathbf{B}$ $(\mathbf{A}, \mathbf{B} \;:\; Z_p^\alpha(\mathscr{M}) \to Z_p^{\alpha-r}(I\!\!R_n))$ are similar as in the case of $L_p(\mathscr{M})$-spaces, considered in § 3.3). We recall as well that $\tilde{Z}_p^\alpha(\mathscr{M}) =$
$= \tilde{Z}_\infty^\alpha(\mathscr{M}), \quad Z_p^\alpha(\mathscr{M}) = Z_\infty^\alpha(\mathscr{M})$ due to compactness of $\mathscr{M}$.

Lemma 4.19 $\mathbf{A} \in \mathrm{OLT}^r(\, Z_p^\alpha(\mathscr{M}))$ *if and only if the commutator* $[vI, \mathbf{A}] = v\mathbf{A} -$ $- \mathbf{A}vI \;:\; \tilde{Z}_p^\alpha(\mathscr{M}) \;\to\; Z_p^\alpha(\mathscr{M})$ *is a compact operator for any* $v \in Z_\infty^\mu(\mathscr{M}), \quad \mu \geq \max\{\alpha, \alpha - r\}$.

Proof is similar to Lemma 3.12 and is based on the following equality (which plays a similar role as (3.14) in Lemma 3.12)

$$|\!|\!|\, \mathbf{A}|\!|\!|^{(d)} = \sup\left\{q(x,\mathbf{A}) \;:\; x \in \overline{\mathscr{M}}\right\}, \quad q(x,\mathbf{A}) = \inf\left\{|\!|\!|\, v_x\mathbf{A}|\!|\!|^{(d)} \;:\; v_x \in \Delta_x\right\}; \quad (4.42)$$

here $|\!|\!|\, \mathbf{A}|\!|\!|^{(d)}$ denotes the Kuratovski measure of noncompactness. Inequality is proved by R. Pöltz (cf.[29], Theorem 1) for any $\mathbf{A} \in \mathrm{OLT}^r(Z_\infty^\alpha(\mathscr{M}))$. ∎

Lemma 4.20 *Let* $\Omega \subset I\!\!R_n$ *be a compact Lipschitz domain and* $\alpha > 0, \;\; r \in$ $\in I\!\!R, \;\; \alpha - r > 0$. *Then* $r_\Omega\Lambda^r \in \mathrm{OLT}^r(Z_\infty^\alpha(\Omega))$.

Proof. It is known (cf. [18], vol.3, Theorem 18.1.8) that an operator $b\Lambda^r -$ $- \Lambda^r bI \;:\; Z_\infty^\alpha(I\!\!R_n) \longrightarrow Z_\infty^{\alpha-r+1}(I\!\!R_n)$ is continuous (has the order $r-1$) for any $b \in C_0^\infty(I\!\!R_n)$. Hence by compact embedding (cf. Lemma 4.5) the operator $br_\Omega\Lambda^r - r_\Omega\Lambda^r bI \;:\; \tilde{Z}_\infty^\alpha(\Omega) \to Z_\infty^{\alpha-r}(\Omega)$ is compact. Due to Lemma 4.19 the proof is completed. ∎

Lemma 4.21 *Let* $0 < \alpha < \infty, \;\; -\infty < r < \infty, \;\; \alpha - r > 0, \;\; 1 \leq p \leq \infty, \;\; a_0 \in$ $\in W(I\!\!R_n), \;\; b \in Z_p^\alpha(I\!\!R_n), \;\; d \in Z_p^{\alpha-r}(I\!\!R_n), \;\; \mathrm{supp}\, b, \mathrm{supp}\, d$ *be compact and* $a_0(\infty) = 0$.

If $a = \lambda^r a_0$ *the operators*

$$d\, W_a^0,\; W_a^0 bI \;:\; Z_p^\alpha(I\!\!R_n) \longrightarrow Z_p^{\alpha-r}(I\!\!R_n)$$

are compact.

Proof. It can be supposed $d = b = v \in C_0^\infty(I\!\!R_n)$; in fact: $v \in C_0^\infty(I\!\!R_n)$ can be chosen such that $v(x)d(x) = d(x), \;\; v(x)b(x) = b(x)$ and it remains to prove the compactness of $v\, W_a^0$ and $W_a^0 vI$ only.

Due to Lemmas 4.10 and 4.20 and Theorem 4.12 it suffices to consider only the operator $v\, W_a^0 wI \;:\; Z_p^\alpha(I\!\!R_n) \longrightarrow Z_p^\alpha(I\!\!R_n)$ (i.e. $r = 0$) where $v, w \in C_0^\infty(I\!\!R_n), \;\; v(x)w(x) =$ $= v(x), \;\; a \in S(I\!\!R_n)$. but then $v\, W_a^0 wI$ is an integral operator with the kernel$k(x,y) \in$

$C_0^\infty(\mathbb{R}_n \times \mathbb{R}_n) and \operatorname{supp} k \subset \Omega_0 \times \Omega_0$, where $\Omega_0 = \operatorname{supp} w \supset \operatorname{supp} v$. Hence the operator $v\, W_a^0 wI \ : \ Z_p^\alpha(\Omega) \to Z_p^\beta(\Omega)$ is continuous for any $\beta > \alpha$ and due to compactness of the embedding (cf. (4.17)) $v\, W_a^0 wI \ : \ Z_p^\alpha(\Omega) \longrightarrow Z_p^\alpha(\Omega)$ is compact. ■

Remark 4.22 *If $a \in M^r(Z_p^\alpha(\mathbb{R}_n))$ is any multiplier with compact support* $\operatorname{supp} a \subset \mathbb{R}_n$ *then Lemma 4.21 remains valid. In fact: $W_a^0 = W_u^0\, W_a^0 = W_a^0 W_u^0$, where $u \in S(\mathbb{R}_n) \subset W(\mathbb{R}_n)$ is such that $u \in C_0^\infty(\mathbb{R}_n) \subset S(\mathbb{R}_n) \subset W(\mathbb{R}_n)$, $u(\xi)a(\xi) = a(\xi), \xi \in \mathbb{R}_n$.*

Theorem 4.23 *Let a be a homogeneous function, $a(\lambda\xi) = a(\xi)$ $(\lambda > 0$, $\xi \in \mathbb{R}_n)$ and $a \in Z_\infty^\beta(S^{n-1})$, $\beta > n$. If $\Omega \subset \mathbb{R}_n$ is a compact Lipschitz domain, then $r_\Omega\, W_a^0 \in \mathrm{OLT}^0(Z_\infty^\alpha(\Omega))$.*

Proof. It suffices to prove that $uW_a^0 vI$ is compact in $Z_\infty^\alpha(\mathbb{R}_n)$ for any $u, v \in \in C_0^\infty(\mathbb{R}_n)$, $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$.

The proof can be reduced further (as in the foregoing lemma) to the case $0 < \alpha < \beta - n$ (with the help of the Bessel potential operator $\Lambda^{\alpha-\gamma}$ where $0 < \gamma < \beta - n$).

Let us recall now that W_a^0 is a singular operator with the characteristic $f(\theta) \in \in H_2^{\beta-\frac{n}{2}}(S^{n-1})$ (cf. [28], Theorem X.7.1); by embedding $f \in Z_\infty^{\beta-n}(S^{n-1})$.

The kernel $k(x, y)$ of the operator $uW_a^0 vI$ is sufficiently smooth $(k \in Z_\infty^{\beta-n}(\Omega_0 \times \Omega_0)$, $\Omega_0 = \operatorname{supp} u \cup \operatorname{supp} v)$; obviously $uW_a^0 vI \ : \ Z_\infty^\alpha(\mathbb{R}_n) \to \to Z_\infty^{\beta-n}(S^{n-1})$ is continuous; hence $uW_a^0 vI \ : \ Z_\infty^\alpha(\mathbb{R}_n) \to Z_\infty^\alpha(S^{n-1})$ is compact due to the compact embedding $Z_\infty^\alpha(S^{n-1}) \subset Z_\infty^{\beta-n}(S^{n-1})$ (cf. (4.17)). ■

Let $\Omega \in \mathbb{R}_n$ be a compact Lipschitz domain; the following notation is introduced

$$MZ^r(\Omega) = \left\{ a \in M^r(Z_p^\alpha(\mathbb{R}_n)) \ : \ r_\Omega\, W_a^0 \in \mathrm{OLT}^r(Z_p^\alpha(\Omega)) \right\}.$$

We drop α and p in the notation since the set is independent of these parameters (cf. (4.14) and Corollary 4.10).

The subset $MZ^r(\Omega, \infty)$ of $MZ^r(\Omega)$ consists of functions a such that they have radial limits (cf. (3.1)) $a^\infty \in MZ^r(\Omega)$ and the operator $r_\Omega W_{a-a^\infty}^0 \ : \ \tilde{Z}_\infty^\alpha(\Omega) \longrightarrow Z_\infty^\alpha(\Omega)$ is compact.

Now we are ready to introduce the pseudodifferential operator

$$\mathbf{A} = a(x, D) \ : \ \tilde{Z}_\infty^\alpha(\mathscr{M}) \to Z_\infty^{\alpha-r}(\mathscr{M}) \tag{4.43}$$

of the class $\mathrm{OPC}^r(Z_\infty^r(\mathscr{M}))$. This can be done similarly to the L_p−case (cf. Definition 3.14), replacing there the symbol class $PC_p^r(\mathbb{R}_n, \alpha)$by $MZ^r(\Omega, \infty)$.

5 APPLICATION TO A CRACK PROBLEM

Let us consider the problem of finding the displacement vector $u = (u_1, u_2, u_3)$ in a homogeneous, isotropic, elastic medium, which occupies a Lipschitz domain $\Omega \subset \mathbb{R}_3$ with a crack $\mathscr{M} \subset \Omega$; either boundary data or tractions are prescribed on $\partial\Omega$ and on the both sides of the crack surfaces $\overline{\mathscr{M}}^{\pm}$ (the Dirichlet or the Neumann problems respectively). It is supposed that the crack is interior (i.e. $\partial\Omega \cap \overline{\mathscr{M}} = \emptyset$) and represents a 3-smooth manifold with Lipschitz boundary $\partial\mathscr{M}$.

For the sake of brevity we suppose $\Omega = I\!R_3$.

$\mathscr{M}$ can be extended to a compact closed 3-smooth manifold (the surface) $\mathscr{M}_0 \subset I\!R_3$ which is the boundary of a compact domain $\mathscr{D}^+ \subset I\!R_3$.

Thus we look for a displacement field $u = (u_1, u_2, u_3)$ in $H_2^s I\!R_3 \setminus \overline{\mathscr{M}})$ (a weak solution) which satisfies Lame's system with steady oscillation

$$\Delta^* u(x) + \kappa^2 u(x) = 0, \qquad \kappa^2 = \frac{\rho}{\mu}\omega^2, \qquad x \in I\!R_3 \setminus \overline{\mathscr{M}}, \tag{5.1}$$

where

$$\Delta^* = \Delta + \frac{\lambda + \mu}{\mu} \operatorname{grad} \operatorname{div}$$

is the Lame operator, $\mu > 0, \quad \lambda > -\frac{2}{3}\mu$ are the elastic constants, ρ is the density, ω is the frequence of the oscillation and $\mid s \mid < \frac{1}{2}$. Two different boundary value problems will be considered for equation (5.1): the Dirichlet problem

$$u \mid_{\overline{\mathscr{M}}^{\pm}} = f^{\pm}, \qquad f^{\pm} \in H_2^{\frac{1}{2}+s}(\mathscr{M}), \qquad f_0 = f^+ - f^- \in \tilde{H}_2^{\frac{1}{2}+s}(\mathscr{M}) \tag{5.2}$$

and the Neumann problem

$$\mathscr{T}(\partial_x, n)u \mid_{\overline{\mathscr{M}}^{\pm}} = g^{\pm}, \qquad g^{\pm} \in H_2^{-\frac{1}{2}+s}(\mathscr{M}),$$
$$g_0 = g^+ - g^- \in \tilde{H}_2^{-\frac{1}{2}+s}(\mathscr{M}) \tag{5.3}$$

where $\mathscr{T}(\partial_x, n)$ is the traction operator

$$\mathscr{T}(\partial_x, n)u = \lambda(\operatorname{div} u)n + 2\mu\frac{\partial u}{\partial n} + \mu n \times \operatorname{curl} u,$$

and $n = n(x)$ is the outer normal vector to the surface $\mathscr{M}_0$ at the point $x \in \overline{\mathscr{M}}$.

Particular cases of the boundary value problems (BVP in short) $\{$ (5.1), (5.2)$\}$ and $\{$ (5.1), (5.3)$\}$ were treated in a number of papers. Here we quote some of them, concerning directly our investigations.

For a closed smooth manifold $\mathscr{M}$ the problem is well investigated and results are exposed in CK1, Ku1, RGBB1 [7, 23, 24].

For the statical case (absence of the oscillation) $\omega = 0$ the problem was investigated in CS1, DNS2, Stf1 [5, 13, 42] in the case of the smooth boundary $\partial\mathscr{M}$ and in Sch1, Sch2, Psd1 [32, 33, 44] in the case of the non-smooth $\partial\mathscr{M}$.

For the Helmholz equation, which appears if $\lambda = -\mu, \quad \omega \neq 0$, non-real wave number $\operatorname{Im}\kappa \neq 0$ and $\mathscr{M} = I\!R_2^+ = I\!R^+ \times I\!R^+$ (the quarter-plane case), the explicit solution of the problems was obtained in [26, 40].

The transmission problem for (5.1) was studied in [5, 6, 27, 44].

The case of smooth boundary $\partial\mathscr{M}$ was considered in [14].

The main purpose here is to investigate the existence (and uniqueness) of weak solutions of the formulated problems, using the results of the foregoing sections. The regularity (smoothness) of solutions is not our concern here.

To get the uniqueness of the solution it is necessary to impose some additional conditions on the solution $u(x)$; these conditions are the following:
a) the finite energy norm condition:

$$\int_{\{x\in \boldsymbol{R}_3:|x|<R\}\setminus\mathscr{M}} \left[\mid \operatorname{grad} u(x) \mid^2 + \mid u(x) \mid^2\right] dx < \infty; \tag{5.4}$$

b) the radiation condition at infinity:

$$\begin{aligned} &u(x) = u^{(1)}(x) + u^{(2)}(x), \quad u^{(m)}(x) = o(1), \quad \mid x \mid \to \infty, \quad m = 1, 2 \\ &\frac{\partial u^{(m)}(x)}{\partial \mid x \mid} - i\kappa_m u^{(m)}(x) = o(\mid x \mid^{-1}), \quad \kappa_1^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \quad \kappa_2^2 = \frac{\rho\omega^2}{\mu} \end{aligned} \tag{5.5}$$

The fundamental (Kupradze's) matrix for (5.1) reads (cf. [23, 24])

$$\Gamma(x, \omega) = \left(\sum_{m=1}^{2} \left(\delta_{kj}\alpha_m + \beta_m \partial_{x_k}\partial_{x_j}\right) \frac{\exp\left(i\kappa_m \mid x \mid\right)}{\mid x \mid}\right)_{3\times 3}, \tag{5.6}$$

$$\alpha_m = \frac{\delta_{2m}}{2\pi\mu}, \qquad \beta_m = \frac{(-1)^m}{2\pi\rho\omega^2}.$$

The single layer and the double layer potentials read respectively

$$V(\omega)\varphi(z) = \int_{\mathscr{M}} \Gamma(z-y, \omega)\varphi(y) d_y\mathscr{M}, \qquad\qquad z \in I\!R_3 \setminus \overline{\mathscr{M}},$$

$$U(\omega)\psi(z) = \int_{\mathscr{M}} [\mathscr{T}(\partial_y, n(y))\Gamma(z-y, \omega)]^T \psi(y) d_y\mathscr{M}, \tag{5.7}$$

where $\mathscr{A}^T$ denotes the transposed matrix.

The same operators but considered on the surface (for $z \in \mathscr{M}$) are denoted by $V_{-1}(\omega)$ and $U_0^T(\omega)$ respectively. Let us notice, that the integral in U_0^T exists then in the sense of the Cauchy principal value.

Two more operators, which are necessary for our further investigations, are defined by the formulas

$$\begin{aligned} V_0(\omega)\psi(z) &= \int_{\mathscr{M}} [\mathscr{T}(\partial_y, n(y))\Gamma(z-y,\omega)]\psi(y)d_y\mathscr{M}, \\ V_1(\omega)\psi(z) &= \int_{\mathscr{M}} \mathscr{T}(\partial_x, n(x))[\mathscr{T}(\partial_y, n(y))\Gamma(z-y,\omega)]^T\psi(y)d_y\mathscr{M} \end{aligned} \tag{5.8}$$

Operators $V_{-1}(\omega), V_0(\omega), V_0^T(\omega)$ and $V_1(\omega)$ are all pseudodifferential operators and have orders -1, 0, 0 and 1, respectively (for detailed proofs cf. [12]).

Theorem 5.1 *The boundary value problem { (5.1), (5.2), (5.4), (5.5) } has the solution*

$$u(z) = V(\omega)f_0(z) - U(\omega)g_0(z), \qquad z \in I\!R_3 \setminus \overline{\mathscr{M}}, \tag{5.9}$$

where $f_0 = f^+ - f^-$ is defined in (5.2) and $g_0 \in \tilde{H}_2^{-\frac{1}{2}+s}(\mathscr{M})$ is a solution of the pseudo-differential equation

$$V_{-1}(\omega)g_0(x) = f_1(x), \qquad f_1 = \frac{1}{2}[f^+ + f^-] + V_0^T f_0, \quad x \in \overline{\mathscr{M}}. \tag{5.10}$$

Equation (5.10) has the unique solution $g_0 \in \tilde{H}_2^{-\frac{1}{2}+s}(\mathscr{M})$ for any $f_1 \in H_2^{\frac{1}{2}+s}(\mathscr{M})$, $|\, s \,| < \frac{1}{2}$.

Theorem 5.2 *The boundary value problem { (5.1), (5.3)-(5.5) } has the solution of the form (5.9), where $g_0 = g^+ - g^-$ is defined in (5.3) and $f_0 \in \tilde{H}_2^{\frac{1}{2}+s}(\mathscr{M})$ is the solution of the pseudodifferential equation*

$$V_1(\omega)f_0(x) = -g_1(x), \qquad g_1 = \frac{1}{2}[g^+ + g^-] - V_0 g_0, \quad x \in \overline{\mathscr{M}}. \tag{5.11}$$

Equation (5.11) has the unique solution $f_0 \in \tilde{H}_2^{\frac{1}{2}+s}(\mathscr{M})$ for any $g_1 \in H_2^{-\frac{1}{2}+s}(\mathscr{M})$, $|\, s \,| < \frac{1}{2}$.

PROOFS. The standard procedure is used to prove, that any solution of the problem { (5.1), (5.4) } is represented by formula (5.9) with

$$f_0 = u\,|_{\overline{\mathscr{M}^+}} - u\,|_{\overline{\mathscr{M}^-}}, \qquad g_0 = \mathscr{T}(\partial_x, n)u\,|_{\overline{\mathscr{M}^+}} - \mathscr{T}(\partial_x, n)u\,|_{\overline{\mathscr{M}^-}}$$

(cf. [23], § III.2,[4], § 2,[12]). Inserting the boundary data (either (5.2) or (5.3)) in (5.9) and applying the well-known properties of the layer potentials (cf. [23], § V.3), we derive equation (5.10) or (5.11) (cf. [4, 12]).

Now we are about to prove the uniqueness of the solutions of (5.10) and (5.11).

The principal symbol $V_{-1}(\omega,x,\xi)$ $(\omega>0,\ \ x\in\overline{\mathscr{M}},\ \ \xi\in I\!R_2)$ of the operator $V_1(\omega)$ reads

$$V_{-1}(\omega,x,\xi)=V_{-1}(\omega,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega,\xi,\xi_3)d\xi_3,\qquad \xi\in I\!R_2,\tag{5.12}$$

where $G(\omega,\tilde\xi)$ is the Fourier transform of the fundamental matrix

$$G(\omega,\tilde\xi)=\int_{\boldsymbol{R}_3}e^{i\tilde\xi\cdot y}\Gamma(y,\omega)dy\qquad \tilde\xi=(\xi,\xi_3)\in I\!R_3\tag{5.13}$$

(for the detailed proofs cf. DNS1 [12]).

Since $\Gamma(x,\omega)$ is the fundamental solution of (5.1) we get

$$G(\omega,\tilde\xi)=\mathbf{C}^{-1}(\omega,\tilde\xi)\quad\text{if } |\ \tilde\xi\ |>N,\quad \mathbf{C}(\omega,\tilde\xi)F\varphi(\tilde\xi)=F(\Delta^*+\kappa^2)\varphi(\tilde\xi),\\ \varphi\in S(I\!R_3),$$

where N is sufficiently large. Since

$$\mathbf{C}(\omega,\tilde\xi)=-\left(|\ \tilde\xi\ |^2\ \delta_{jk}+\frac{\lambda+\mu}{\mu}\tilde\xi_j\tilde\xi_k-\kappa^2\right)_{3\times 3},\qquad \kappa^2=\frac{\omega^2\rho}{\mu},$$

the integral in (5.13) converges absolutely for all $\tilde\xi$ and the radial limits exist (cf. (3.1))

$$V_{-1}^{\infty}(\omega,\xi)=\lim_{R\to\infty}RV_{-1}(\omega,R\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C}^{-1}(0,\xi,t)dt=$$
$$=-\frac{1}{2æ\ |\ \xi\ |^3\ \mu}\begin{pmatrix} |\ \xi\ |^2+æ\xi_1^2 & -æ\xi_1\xi_2 & 0\\ -æ\xi_1\xi_2 & |\ \xi\ |^2+æ\xi_2^2 & 0\\ 0 & 0 & |\ \xi\ |^2\end{pmatrix},$$
$$\xi=(\xi_1,\xi_2)\in I\!R_2,\qquad 0<æ=\frac{\lambda+\mu}{\lambda+3}<1.$$

The matrix-function $-V_{-1}^{\infty}(\omega,\xi)$ is self-adjoint and positiv definite. Therefore

$$\mathrm{Re}(e^{i(\theta-\pi)}V_{-1}^{\infty}(\omega,\xi)\eta,\eta)=(-V_{-1}^{\infty}(\omega,\xi)\eta,\eta)\,\mathrm{Re}\,e^{i\theta}\geq C_0\cos\theta\ |\ \eta\ |^2,\\ C_0>0,\qquad \eta\in\mathbb{C}_3,\qquad |\ \theta\ |<\frac{\pi}{2}$$

and Theorem 3.26 implies the Fredholm property of the operator

$$V_{-1}(\omega)\ \ :\ \tilde H_2^{-\frac{1}{2}+s}(\mathscr{M})\to H_2^{\frac{1}{2}+s}(\mathscr{M}),\qquad |\ s\ |<\frac{1}{2}.\tag{5.14}$$

and the index formula

$$\mathrm{Ind}\,V_{-1}=0.\tag{5.15}$$

Due to dense embeddings

$$\tilde{H}_2^{\nu}(\mathscr{M}) \subset \tilde{H}_2^{r}(\mathscr{M}), \quad H_2^{\nu}(\mathscr{M}) \subset H_2^{r}(\mathscr{M}) \quad r \leq \nu,$$

as well as the Fredholm property of (5.14) and the independence of the index from s, from Lemma 1.10 we get that the kernel $\mathrm{Ker} V_{-1}(\omega)$ is independent of s.

In a standard way (cf. [23, 12]) it can be proved that the equation

$$V_{-1}(\omega)\varphi = 0, \qquad \varphi \in \tilde{H}_2^{-\frac{1}{2}}(\mathscr{M})$$

has only the trivial solution $\varphi = 0$, which implies $\mathrm{Ker} V_{-1} = \{0\}$ for any $\mid s \mid < 1/2$.

Thus (5.14) is an invertible operator and equation (5.10) has a unique solution.

The solvability and the uniqueness for equation (5.11)) is proved similarly. ∎

BIBLIOGRAPHY

AS1 [1] N. Aronszajn and K.T. Smith. Theory of Bessel potentials , I. *Ann. Inst. Fourier*, 11:385–475, 1961.

Ca1 [2] A.P. Calderon. Lebesgue spaces of differentiable functions and distributions. In C.B. Morrey, editor, *Partial Differential Equations*, volume 5 of *Proceedings of a Simposium in Pure Mathematics*, pages 33–49. Amer. Math. Soc., Providence, 1961.

Cor1 [3] H.O. Cordes. *Elliptic Pseudodifferential Operators - an Abstract Theory*, volume 756 of *Lecture Notes in Mathematics*. Springer, New York, 1979.

Cos2 [4] M. Costabel. Boundary integral operators on Lipschitz-domains: Elementary results. *SIAM J. Math. Anal.*, 19:613 – 626, 1988.

CS1 [5] M. Costabel and E. Stephan. An improved boundary element Galerkin method for three-dimensional crack problems.*Integral Equations and Operator Theory*, 10:367 – 413, 1987.

CS3 [6] M. Costabel and E. Stephan. Strongly elliptic boundary integral equations for electromagnetic transmission problems. *Proceedings of the Royal Society of Edinburgh*, 109A:271–296, 1988.

CK1 [7] D.Colton and Kress R. *Integral Equation Methods in Scattering Theory*. Wiley, New York, 1983.

Du2 [8] R. Duduchava. On Nötherian theorems for singular integral equations. In *Proceedings of Symposium on Continuum Mechanics and Related Problems of Analysis, vol. I*, pages 19–52. Metzniereba, Tbilisi, 1973. In Russian.

Du4 [9] R. Duduchava. Convolution integral operators on a quadrant with discontinuous symbols. *Mathematics USSR, Izvestia*, 10:371–392, 1976.

Du1 [10] R. Duduchava. *Integral equations with fixed singularities*. Teubner, Leipzig, 1979.

Du3 [11] R. Duduchava. On multidimensional singular integral operators, I-II. *Journal of Operator Theory*, 11:41–76, 199–214, 1984.

DNS1 [12] R. Duduchava, D. Natroshvili, and E. Shargorodsky. Pseudodifferential operators on manifolds with boundary and their applications. To appear.

DNS2 [13] R. Duduchava, D. Natroshvili, and E. Shargorodsky. On the continuity of generalized solutions of boundary value problems of the mathematical theory of cracks. *Bulletin of the Georgian Academy of Sciences*, 135:497–500, 1989.

DNS3 [14] R. Duduchava, D. Natroshvili, and E. Shargorodsky. Boundary value problems of the mathematical theory of cracks. In *Proceedings of I.Vekua Institute of Applied Mathematics, Some problems of Elasticity Theory*, volume 39, pages 68–84. I. Vekua Institute of Applied Mathematics, Tbilisi University, Tbilisi University Press, 1990.

DS1 [15] R. Duduchava and R. Schneider. The algebra of non-classical singular integral operators on a half-space. *Integral Equations and Operator Theory*, 10:531–553, 1987.

GK2 [16] I. Gohberg and N. Krupnik. *Einführung in die Theorie der Eindimensionalen Singulären Integraloperatoren*. BirkhäuserVerlag, Basel, 1979.

Hr1 [17] L. Hörmander. Estimates for translation invariant operators on L^p-spaces. *Acta Mathematica*, 104:33–140, 1960.

Hr2 [18] L. Hörmander. *The Analysis of Linear Partial Differential Operators*. Springer, Berlin, 1985.

Ka1 [19] R. Kapanadze. On some properties of singular integral operators in normed spaces. *Transactions of the Tbilisi State University*, 129:17–26, 1968.

Ka2 [20] R. Kapanadze. On the compactness property of some integral operators. *Transactions of the Institute of Applied Mathematics, Tbilisi State University*, 3:109–114, 1972.

KPS1 [21] S. Kreǐn, J. Petunin, and E. Semenov. *Interpolation of Linear Operators*. Nauka, Moscow, 1978. (in Russian).

Kru1 [22] N. Krupnik. *Banach Algebras with Symbol and Singular Integral Operators*. Birkhäuser Verlag, Basel, 1987.

Ku1 [23] V. Kupradze. *Randwertaufgaben der Schwingungsteorie und Integralgleichungen*. Akademie Verlag, Berlin, 1956.

KGBB1 [24] V. Kupradze, T. Gegelia, M. Bashaleishvili, and T. Burchuladze. *Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. North Holland, Amsterdam, 1979.

LS1 [25] G. Litvinchuk and I. Spitkovsky. *Factorization of Measurable Matrix-Functions*. Birkhäuser Verlag, Basel, 1987.

MS1 [26] E. Meister and F.-O. Speck. A contribution to the quarter-plane problem in diffraction theory. *Journal of Mathematical Analysis and Applications*, 130:223–236, 1988.

MS3 [27] E. Meister and F.-O. Speck. The explicit solution of elastodynamic diffraction problems by symbol factorization. *Zeitschrift für Analysis und ihre Anwendungen*, 8(4):307–328, 1989.

MP1 [28] S.G. Michlin and S. Prössdorf. *Singular Integral Operators*. Springer Verlag, Berlin, 1986.

Plt1 [29] R. Pöltz. Operators of local type in spaces of Hölder functions. *Seminar Analysis, Operator Equations and Numerical Analysis 1986/87, Zentralinstitut für Mathematik und Mechanik, Berlin,*, pages 107–122, 1987.

Ra1 [30] V. Rabinovič. Pseudodifferential operators on a class of noncompact manifolds. *Mathem. USSR Sbornik*, 18:45–59, 1972.

RS1 [31] S. Rempel and B.W. Schulze. *Index Theory of Elliptic Boundary Value Problems*. Akademie Verlag, Berlin, 1982.

Sch1 [32] R. Schneider. Bessel potential operators for canonical Lipschitz-domains. *Mathematische Nachrichten*, 150:277–299, 1991.

Sch2 [33] R. Schneider. Reduction of order for pseudodifferential operators on Lipschitz domains. *Commun. in Partial Differential Equations*, 16:1263–1286, 1991.

Su1 [34] B.V. Schulze. Corner Mellin operators and reduction of orders. ZIMM-preprint, Zentralimstitut für Matematik und Mechanik, Berlin, 1988.

Sh1 [35] E. Shamir. A remark on the Michlin-Hörmander multiplier theorem. *Mathematical Analysis and Applications*, 16:104–107, 1966.

Si1 [36] I. Simonenko. A new general method of investigating linear operator equations of the type of singular integral equations I-II. *Izvestia Akademii Nauk SSSR, ser. Matemat.*, 29:567–586, 757–782, 1965.

SiC1 [37] I. Simonenko and Chin Ngok Min. *Local Method in the Theory of One-dimensional Singular Integral Equations with Piecewise-continuous Coefficients. Ntherity*. Rostov University Press, Rostov, 1986. (In Russian).

SC1 [38] C. Sodosky and M. Cotlar. On quasi-homogeneous Bessel potential operators. In A.P. Calderon, editor, *Singular Integrals*, volume 11 of *Proceedings of Symposium in Pure Mathematics*, pages 275–287. AMS, Providence, 1967.

Sp1 [39] F.-O. Speck. *General Wiener-Hopf Factorization Methods*. Pitman, London, 1985.

SD1 [40] F.-O. Speck and R.V. Duduchava. Bessel potential operators for the quarter-plane. *Preprint Nr. 25, Univers. Stuttgart, Math. Institute*, 1990. To appear in Applicable Analysis.

St1 [41] E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, 1970.

Stf1 [42] E. Stephan. A boundary integral equation method for three-dimensional crack problems in elasticity. *Math. Meth. in the Applied Sciences*, 8:609–623, 1986.

Tr1 [43] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Amsterdam, 1978.

Psd1 [44] T. von Petersdorff. *Randwertprobleme der Elastizitätstheorie für Polyeder-Singularitäten und Approximation mit Randelementmethoden*. PhD thesis, Techn. Hochsch. Darmstadt, 1989.

Roland Duduchava
A.Razmadze Mathematical Institute,
Academy of Sciences of Georgia,
Z.Ruchadze str.1,
380093 TBILISI,
REPUBLIC OF GEORGIA

Frank-Olme Speck,
Departamento de Matemática,
Instituto Superior Técnico,
Avenida Rovisco Pais,
1096 LISBOA CODEX,
PORTUGAL