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BOUNDARY VALUE PROBLEMS IN DOMAINS WITH PEAKS

Abstract. We obtain criteria of solvability of the DIRICHLET and the NEUMANN boundary value problems (BVPs) for the Laplacian in 2D domains with angular points and peaks on the boundary. We start with the correct formulation of BVPs and modify it for domains with outward peaks (classical conditions are incorrect). Boundary integral equations (BIEs), obtained by the indirect potential method, turn out to be equivalent to the corresponding BVPs only when inward peaks are absent. BIEs on boundary curve with angular points are investigated in different weighted function spaces. If boundary curve has a cusp, corresponding to an inward or an outward peak, equations are non-FREDHOLM in usual spaces and we should impose restrictions on the right-hand sides. The conditions are defined by some mean–value integrals. We consider also equivalent reduction to boundary pseudo-differential equations (BPsDEs) of orders ± 1 by the direct potential method. Crucial role in our investigations of BVPs and of corresponding BIEs, PsDOs belongs to the equivalent reduction of BVPs to the RIEMANN–HILBERT problem for analytic functions on the unit disk. The latter problem can be investigated thoroughly, even when peaks are present and equations have non-closed image by invoking results on convolution equations with vanishing symbols.

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Introduction

Let $\Omega^+ \subset \mathbb{C}$ be a bounded domain in the complex plane with a piecewise-smooth boundary $\Gamma = \partial\Omega^+$ and $\Omega^- = \mathbb{C} \setminus \Omega^+$ be the complementary outer domain. Let $t_j \in \Gamma$, $j = 1, \dots, n$, be all knots on the boundary $\Gamma = \partial\Omega^+$ with the angles $\pi\gamma_j$, $0 \le \gamma_j \le 2$, $j = 1, \dots, n$. Boundary curve might contain cusps $\gamma_j = 0, 2$ corresponding to an outward (for $\gamma_j = 0$) and an inward (for $\gamma_j = 2$) peaks of the domain Ω^+. By $\vec{\nu}(t) = (\nu_1(t), \nu_2(t))$ we denote the outer unit normal vector to Γ (with respect to Ω^+).

As a model we consider the DIRICHLET $u^\pm(t) = g(t)$ (and the NEUMANN $\partial_{\vec{\nu}(t)} u^\pm(t) = f(t)$, $t \in \Gamma$) BVPs for harmonic functions

$$\Delta u(x) = 0\,, \quad x \in \Omega^\pm \tag{0.1}$$

and look for the solution, as common, in the SOBOLEV space

$$u \in W_2^1(\overline{\Omega^+}) \quad \text{or} \quad u \in W_{2,loc}^1(\overline{\Omega^-})\,, \quad u(x) = O(1)\,, \quad \text{as} \quad |x| \to \infty\,. \tag{0.2}$$

Applying the potential method, based on the GREEN formula and its consequence-representation of solution by layer potentials, invoking the PLEMELJI formulae (see § 1) we get boundary integral equations (BIEs) of logarithmic potential

$$\pm\frac{1}{2}\varphi(t) + \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec{\nu}(\tau)} \log|t-\tau|\varphi(\tau)|d\tau| = g(t)\,, \tag{0.3}$$

$$\pm\frac{1}{2}\psi(t) + \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec{\nu}(t)} \log|t-\tau|\psi(\tau)|d\tau| = f(t)\,, \quad t \in \Gamma, \tag{0.4}$$

which are conjugate to each-other (the **indirect method**; see [Ma1]). It is rather a classical result, that (0.3) and (0.4) are FREDHOLM equations provided Γ is smooth. Note that reduction to the BIEs (0.3) and (0.4) is equivalent.

When Γ has angular points, equations (0.3) and (0.4) have fixed singularities in the kernels (i.e., they are MELLIN convolution equations) and are FREDHOLM except some discrete values of parameters of spaces they are treated in (see Theorems 1.23, 1.24 and cf. [Du1, Du3, Ma1]). It is important that in both mentioned cases equivalence of BVPs with corresponding BIEs still hold.

Piecewise-smooth domains without peaks are particular cases of LIPSCHITZ domains and BVPs for second order equations in such domains were thoroughly investigated recently (mostly in the Hilbert spaces L_2 and W_2^1) even for domains in $\mathbb{R}^n$, $n > 2$. For details of these profound investigations as well as for exhaustive survey of vast literature in this field we recommend recent publications [Ke1, MMP1, MMT1, MT1].

Situation changes completely if domain $\Omega^\pm$ has peaks. There arise three principal problems.

- If a single outward peak occurs constraints (0.2) become incorrect. Namely, if we look for solution of BVP in the SOBOLEV space $W_p^1(\overline{\Omega^+})$ for arbitrary fixed value of $p \in (1, \infty)$, there exists a compact domain $\Omega_{2\gamma+3} \subset \mathbb{C}^+$ with outward peak at $0 \in \partial\Omega_{2\gamma+3}$ in the first quadrant $\mathbb{R}^+ + i\mathbb{R}^+ \subset \mathbb{C}$ of the complex plane such that the analytic function z^γ, $z \in \Omega$, with arbitrary $0 < \gamma < \infty$ belongs to the space $W_p^1(\overline{\Omega_{2\gamma+3}})$ (details see below in Example 1.2). Therefore in classical formulation $u \in W_2^1(\overline{\Omega^+})$ solution to BVP might have non-integrable singularity on the boundary and it is necessary to change constraints on harmonic functions in the domain. Moreover, due to complicated relations between traces of functions on different faces of outward peaks (see, e.g., [Ia1]) it is almost impossible to investigate corresponding BIEs.

- If a single inward peak occurs, equivalence of BVPs (0.1), (0.2) with the corresponding BIEs (0.3), (0.4) fail completely. Such reduction is connected with a representation of harmonic function of the SMIRNOV class by the CAUCHY integral with real valued density. This turned out to be possible if and only if the RIEMANN–HILBERT BVPs for analytic functions is surjective in the same SMIRNOV space but for the complementary domain (see Lemmata 1.1 and 1.13). If the domain has an inward peak, the complementary domain has an outward peak and the RIEMANN–HILBERT BVP is not normally solvable (see Lemma 1.11).

- If a single peak (outward or inward) occurs solvability property of BIEs (0.3) and (0.4) change dramatically: symbols of these convolution-type equations vanish and equations can not be FREDHOLM in any $L_p(\Gamma)$ or any other space with weight or without (see [MS1]–[MS8] and § 1.6 below). For the space of continuous functions this was noticed already by J.RADON [Ra1].

We start with investigations of correct formulation of the BVPs. Namely, we look for solutions in the weighted SMIRNOV–LEBESGUE space $e_p(\overline{\Omega^\pm}, \rho)$ (see § 1.2) of harmonic functions written as the real part of analytic functions represented by the Cauchy integrals with densities in the Lebesgue spaces with weight $L_p(\Gamma, \rho)$ (plus constanta for the unbounded domain Ω^-). The choice of constraints is justified in the following sense: looking for solutions in more narrow SMIRNOV–SOBOLEV space $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$ is the same as the common (classical) constraint $u \in W_2^1(\overline{\Omega^\pm})$ provided the domain $\Omega^\pm$ has no outward peaks (see Lemma 1.2). Moreover, to raise flexibility of the method we suggest to look for solutions in some other SMIRNOV spaces: weighted SMIRNOV–SOBOLEV $w_p^s(\overline{\Omega^\pm}, \rho)$, $0 \le s \le 1$, SMIRNOV–HÖLDER $h_{m+\mu}^0(\overline{\Omega^\pm}, \rho)$ etc. (see § 1.2).

If the boundary curve has cusps (i.e., the domain has peaks) equations (0.3) and (0.4) have non-closed images. Same is true for the DIRICHLET and the NEUMANN BVPs for (0.1) when inward peaks are present. MAZ'YA

and V.Solov'ev in [MS1]–[MS4] suggested to study BIEs (0.3), (0.4) directly. Namely, they have found conditions on the right-hand sides which ensure existence of solutions and have established properties (smoothness, asymptotic) of such solutions. The method is based on the corresponding results for boundary value problems in domains with peaks, obtained with the help of conformal mappings (see [Wa1, Wa2] for properties of such conformal mappings). In more recent investigations [MS5]–[MS8] for curves with cusps of order $\mu \in \mathbb{R}^+$ they have found pairs of Banach spaces where BIEs (0.3), (0.4) are surjective.

Different approach (transformation of the underlying domain which maintains the structure of BVPs) was exploited in [RST1, RST2]. The authors obtained solvability results for BVPs in domains with special cusps when the right-hand sides and solutions are in special weighted spaces.

Essential role in our investigations play an equivalent reduction of the Dirichlet and the Neumann BVPs for (0.1) to the Riemann–Hilbert BVPs for analytic functions on the unit circumference, using the conformal mapping. Namely, we apply the approach exposed in [Mu1, Ch. III] and contributed by I.Vekua in [Ve1]. Obtained BVPs are reduced further to equivalent Cauchy singular integral equations on the unit circumference.

The same method was applyed by V.Kokilashvili and V.Paatashvili. Namely, they look for solutions of BVPs in the Smirnov–Lebesgue space $e_p(\overline{\Omega^{\pm}})$, $1 < p < \infty$. Although the motivation for the choice of constraints, ensuring equivalent reduction to the Riemann–Hilbert problem, was clear justification for the change of conditions in [KKP1] is missing.

For the investigation of the Cauchy singular integral equations on the unit circumference, which arise as an equivalent equation, we apply localization to 2×2 systems of convolution equations on the real semi-axes. Local representatives at cusps have vanishing symbols and, by applying results on convolution equations with vanishing symbols of integer order (see [Pr1, § 5.2] and § 3.1 below), we describe the image space by mean–value integrals and find the criteria for the data which ensures unique solvability of the Dirichlet and the Neumann BVPs for (0.1).

Further we prove equivalence of BVPs and of corresponding BIEs (0.3) and (0.4) if inward peaks are absent (see Theorems 1.12 and 1.14). If the boundary curve has no cusps, obtained BIEs are particular cases of general equations studied in § 4 by invoking results from [DLS1]. They are Fredholm with rare exceptions for the parameters of the space. Although such investigations were carried out earlier (see survey in [Ma1]) some results of the present paper are new: we prove boundedness of harmonic (the double and the single) layer potentials and obtain criteria for Fredholm property of equations (0.3) and (0.4) in the spaces of continuous and piecewise-continuous functions $C(\Gamma, \varkappa)$ and $PC(\Gamma, \varkappa)$ (in some cases also in $PC^1(\Gamma, \varkappa)$; see § 1.7) with exponential weight $\varkappa(t) = \prod_{j=1}^{n} |t - c_j|^{\alpha_j}$, $0 \leq$

$\alpha_j < 1$.

If inward peaks are present equivalence with BVPs fail (see Lemma 1.13) and equations (0.3), (0.4) are investigated by localization. The localization enables replacement of inward peaks by outward ones (see § 5.4). Solvability criteria of equations (0.3) and (0.4) are summarized in Theorems 1.23 and 1.24, which are proved in § 5.4.

Let $\mathscr{T}_{ow}$, $\mathscr{T}_{iw}$ be the discrete sets of all outward, all inward peaks and $\mathscr{T}_{pk} = \mathscr{T}_{ow} \cup \mathscr{T}_{iw}$ be the set of all peaks of Ω^+. We define the spaces

$$L_p(\Gamma, \rho, \mathscr{T}_{ow}), L_p(\Gamma, \rho, \mathscr{T}_{iw}) \subset L_p(\Gamma, \rho, \mathscr{T}_{pk}) \subset L_p(\Gamma, \rho)\,,$$

with the help of the integrals (see (1.76)), where $\rho(t) = \prod_{j=1}^{n} |t - c_j|^{\alpha_j}$, $-\frac{1}{p} < \alpha_j < 1 - \frac{1}{p}$, $1 < p < \infty$. It is proved that equations (0.3) and (0.4) are FREDHOLM between spaces $L_p(\Gamma, \rho) \longrightarrow L_p(\Gamma, \rho, \mathscr{T}_{pk})$ provided the conditions $\frac{1}{p} + \alpha_j \neq \min\left\{\frac{1}{\gamma_j}, \frac{1}{2-\gamma_j}\right\}$ holds for all $t_j \notin \mathscr{T}_{pk}$. Moreover, if the inequalities $\frac{1}{p} + \alpha_j < \min\left\{\frac{1}{\gamma_j}, \frac{1}{2-\gamma_j}\right\}$ hold, the mappings are isomorphic.

As for solvability of the DIRICHLET BVP for Ω^+ (for Ω^-) it suffices to restrict the data $g \in L_p(\Gamma, \rho, \mathscr{T}_{ow})$ (respectively, $g \in L_p(\Gamma, \rho, \mathscr{T}_{iw})$) and the solution is unique provided $\frac{1}{p} + \alpha_j < \min\left\{\frac{1}{\gamma_j}, \frac{1}{2-\gamma_j}\right\}$ for all $t_j \notin \mathscr{T}_{pk}$ (note, that inward peaks of $\Omega^{\pm}$ have no impact on the corresponding DIRICHLET BVP). Similar holds for the NEUMANN BVPs.

In Lemma 1.22 we formulate sufficient conditions for the inclusion $\varphi \in L_p(\Gamma, \rho, \mathscr{T}_{iw})$, which involves the conformal mapping $\zeta(z) : \Omega^+ \longrightarrow \mathscr{D}_1$ of the domain Ω^+ onto the unit disk $\mathscr{D}_1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. It is possible to write more transparent and explicit condition, but for these we need asymptotic behaviour of the conformal mapping $\zeta(z)$ in the vicinity of outward peak. This we leave for a forthcoming paper.

In our investigations we apply the CISOTTI formula, which represents the derivative of the conformal mapping $\omega : \mathscr{D}_1 \to \Omega^+$ (see [LS1, Ch. III, § 1, n^o. 44, Example 5]):

$$\omega'(z) = \omega'(0) \exp\left[\frac{1}{\pi} \int\limits_{|\tau|=1} \frac{\beta(\tau) d\tau}{\tau - z} - \frac{1}{\pi} \int\limits_{|\tau|=1} \beta(\tau) \frac{d\tau}{\tau}\right], \quad z \in \mathscr{D}_1\,. \tag{0.5}$$

Here $\beta(\tau) := \arg \vec{\nu}(\omega(\tau)) - \arg\tau$ and $\arg \vec{\nu}(\omega(\tau))$ stands for the argument of the outer unit normal vector to the curve $\Gamma = \partial\Omega^+$ at the point $\tau = e^{i\vartheta} \in \Gamma_1 := \partial\mathscr{D}_1$. The formula was rediscovered in [PK1] for a piecewise-smooth boundary (see also [KKP1]). We return to the classical approach in [LS1]) which is, above all, very simple and prove the CISOTTI formula (0.5) in § 5.1 for a domain with rectifiable JORDAN boundary.

Although the conformal mapping is participating implicitly, representation (0.5) simplifies proofs of some classical theorems on conformal mappings[1)] (see [KKP1, Ch. III] and § 5.1 for the proofs of LINDELÖF'S, KELLOGG'S, WARSCHAWSKY'S theorems). Moreover, using the CISOTTI formula we generalize the KELLOGG theorem for the ZYGMUND space (see Theorem 5.9).

In [Po1, Theorem 3.15] the Cisotti formula is rediscovered for a so-called regulated domain, i.e., for a domain for which the inclination $\alpha(t)$, $t \in \Gamma$ of the tangent vector to the boundary has limits $\alpha(t \pm 0)$ everywere on the boundary $t \in \Gamma$.

V.KOKILASHVILI and V.PAATASHVILI had applied formula (0.5) to find discontinuities of the coefficient, but the obtained RIEMANN–HILBERT problems they have found "non-solvable in $L_p(\Gamma)$ spaces in general" when outward cusps are present (see [KKP1, Ch. IY]) and have written sufficient condition of solvability as well as explicit formula for solutions provided the solvability conditions hold.

Applying the representation of solution by layer potentials and the **direct method** we obtain boundary pseudo-differential equation

$$\frac{1}{2\pi}\int\limits_{\Gamma} \log|t-\tau|\varphi(\tau)|d\tau| = g_*(t)\,, \quad t \in \Gamma\,, \tag{0.6}$$

$$g_*(t) := -\frac{1}{2}g(t) - \frac{1}{2\pi}\int\limits_{\Gamma} \partial_{\vec{\nu}(\tau)} \log|t-\tau|g(\tau)|d\tau|\,,$$

of order -1 for the DIRICHLET problem for the Laplacian (0.1) and the boundary pseudo-differential equation

$$\frac{1}{2\pi}\int\limits_{\Gamma} \partial_{\vec{\nu}(t)}\partial_{\vec{\nu}(\tau)} \log|t-\tau|\varphi(\tau)|d\tau| = f_*(t)\,, \quad t \in \Gamma\,, \tag{0.7}$$

$$f_*(t) := \frac{1}{2}f(t) + \frac{1}{2\pi}\int\limits_{\Gamma} \partial_{\vec{\nu}(t)} \log|t-\tau|f(\tau)|d\tau|\,,$$

of order $+1$ for the NEUMANN problem. We can formulate criteria of solvability of equations (0.6) and (0.7) based on full equivalence with corresponding BVPs (see Theorems 1.19, 1.20).

All principal theorems on solvability of boundary value problems and boundary integral equations are formulated in § 1.7. Some of them are proved later, mostly in § 5.

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[1)] See [Ga1] for a survey on application of linear and non-linear integral equations in conformal mappings.

We thank also V.Vlasov (Moscow, Computing Center of the Acad. of Sciences) who draw our attention to the CISOTTI formula in [LS1, Ch. III, § 1, n^o. 44, Example 5].

1 Boundary value problems

In the present section we formulate the DIRICHLET and the NEUMANN boundary value problems for the Laplacian in domains with angular points and peaks; discuss their equivalent reduction to boundary integral equations (the direct potential method), to boundary pseudo-differential equations (the indirect potential method) and to singular integral equations on the unit circumference (MUSKHELISHVILI–VEKUA method); we expose properties of harmonic potentials appearing in the method and formulate all principal results.

1.1 spaces

We start by rigorous definitions of domains and spaces which are necessary for our considerations.

Let Γ be a closed, oriented, simple (i.e., without self-intersection), piecewise-LJAPUNOV curve on the complex plane $\mathbb{C}$, circumventing a domain Ω^+ and having knots at $t_1, \dots, t_n \in \Gamma$, i.e.,

$$\Gamma = \bigcup_{j=1}^{n} \Gamma_j \,, \quad \Gamma_j = t_j t_{j+1} \,, \; t_{n+1} := t_1 \,, \quad j = 1, \dots, n \,; \tag{1.1}$$

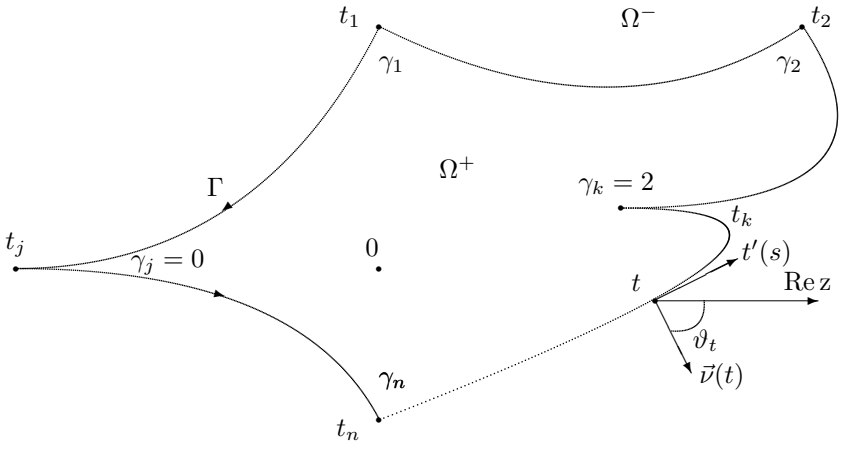

Fig. 1

here Γ_j are $\nu-$smooth, $\nu > 1$, oriented curves connecting knots t_j and t_{j+1}. Let $\pi\gamma_j$ be the angle at t_j between Γ_{j-1} and Γ_j, measured from Ω^+, $0 \le \gamma_j \le 2$, $j = 1, \dots, n$. When $\gamma_j = 0$ or $\gamma_j = 2$ the domain Ω^+ has an outward or an inward peak, respectively or, what is the same, the boundary curve Γ has a cusp (see Fig. 1).

We use the following standard notation for spaces:

$C^m(\Gamma)$ for the space of functions $\varphi(t)$, $t \in \Gamma$ with continuous derivatives up to the order m

$$\partial_t^k \varphi \in C(\Gamma)\,, \quad k = 0, 1, \dots, m\,, \quad \partial_t := \frac{d}{dt}\,, \quad m \in \mathbb{N}_0 := \{0, 1, \dots\}.$$

Let us note that invariant (with respect to a parametrisation of the underlying curve Γ) definition of the space $C^m(\Gamma)$ can be provided iff Γ is m-smooth. Therefore for piecewise-smooth curves (with angular points or cusps) we can define only $C(\Gamma) := C^0(\Gamma)$.

$H_\mu(\Gamma)$ for the space of HÖLDER continuous functions $\psi(t)$, $t \in \Gamma$ with the following finite norm

$$\|\psi\big|H_\mu(\Gamma)\| := \|\psi\big|C(\Gamma)\| + \sup_{t_1 \ne t_2} \frac{|\psi(t_2) - \psi(t_1)|}{|t_2 - t_1|^\mu}\,, \quad 1 < \mu \le 1\,.$$

$PC(\Gamma)$ for the space of functions $\varphi(t)$ which are continuous on each closed arc between knots $t_1, \dots, t_n$ and might have jumps at these knots.

$PC^m(\Gamma)$ for the space of functions $\varphi \in C^{m-1}(\Gamma)$ which have piecewise-continuous last derivative $\partial_t^m \varphi \in PC(\Gamma)$ with possible jumps at knots $t_1, \dots, t_n$.

Both, the spaces $C^m(\Gamma)$ and $PC^m(\Gamma)$ are endowed with the uniform norm

$$\|\varphi\big|PC^m(\Gamma)\| := \sum_{k=1}^{m} \sup\left\{|\partial^k \varphi(t)| \;:\; t \in \Gamma\right\}\,,$$

which makes them into BANACH spaces.

Let

$$\rho(t) = \prod_{j=1}^{n} |t - t_j|^{\alpha_j} \tag{1.2}$$

be a weight function and $C^m(\Gamma, \rho) \subset PC^m(\Gamma, \rho)$ denote the weighted spaces of functions:

$$C^m(\Gamma, \rho) := \left\{\varphi \in C^{m-1}(\Gamma) \;:\; \rho\partial^m \varphi \in C(\Gamma)\right\}\,,$$
$$PC^m(\Gamma, \rho) := \left\{\varphi \in C^{m-1}(\Gamma) \;:\; \rho\partial^m \varphi \in PC(\Gamma)\right\}\,.$$

These spaces both can be endowed with the weighted norm $\|\varphi\big|PC^m(\Gamma, \rho)\| = \|\varphi\big|C^{m-1}(\Gamma))\| + \|\rho\partial^m\varphi\big|PC(\Gamma))\|$.

We will write $C(\Gamma)$, $PC(\Gamma, \rho)$ etc. when $m = 0$.

$H^0_{m+\mu}(\Gamma,\rho)$, $0<\mu<1$, $m=0,1,\ldots$, for the weighted function space

$$H^0_{m+\mu}(\Gamma,\rho) := \big\{\varphi\in C^{m-1}(\Gamma)\ :\ \widetilde{\varphi}^{(m)} := \rho\partial^m\varphi\in H_\mu(\Gamma),\\ \widetilde{\varphi}^{(m)}(t_1)=\ldots=\widetilde{\varphi}^{(m)}(t_n)=0\big\},$$
$$\|\varphi\big|H^0_{m+\mu}(\Gamma,\rho)\| := \|\varphi\big|C^{m-1}(\Gamma)\| + \|\rho\partial^m\varphi\big|H_\mu(\Gamma)\|,$$

provided $\Gamma\setminus\{t_1,\ldots,t_n\}$ is $C^{m+\mu}$-smooth, while Γ itself is PC^{m-1}-smooth. Note, that for piecewise-smooth curve Γ definition is correct only for $m=0,1$.

$L_p(\Gamma,\rho)$ for the weighted Lebesgue space endowed with the norm

$$\|\varphi\big|L_p(\Gamma,\rho)\| := \left(\int\limits_\Gamma |\rho(t)\varphi(t)|^p\,|dt|\right)^{\frac{1}{p}}.$$

$W^m_p(\Gamma,\rho)$ for the Sobolev space

$$W^m_p(\Gamma,\rho) := \big\{\varphi\ :\ \varphi,\partial^k\varphi\in L_p(\Gamma,\rho),\ \ k=0,\ldots,m\big\},$$

$$\|\varphi\big|W^m_p(\Gamma,\rho)\| := \sum_{k=0}^m \|\partial^k\varphi\big|L_p(\Gamma,\rho)\|.$$

$W^s_p(\Gamma,\rho)$, $s\in\mathbb{R}$, for the weighted Sobolev–Slobodetski space which for $s\geq 0$ can be defined by the complex interpolation (see [Tr1]) between the spaces $W^m_p(\Gamma,\rho)$ and $W^0_p(\Gamma,\rho):=L_p(\Gamma,\rho)$ ($s\leq m\in\mathbb{N}$, while for negative $s<0$ can be defined as the dual space to $W^{-s}_{p'}(\Gamma,\rho^{-1})$, $p':=p/(p-1)$.

Since multiplication by a piecewise-continuous function is a bounded operator in $W^s_{p,loc}(\mathbb{R})$ only for $s<1/p$, the space $W^s_p(\Gamma,\rho)$ on piecewise-smooth curve Γ can be defined correctly only for $|s|<1+1/p$.

$\mathscr{E}_p(\overline{\Omega^+},\rho)$ will stand for the Smirnov–Lebesgue space of analytic functions: if $\omega\ :\ \mathscr{D}_1\to\Omega^+$ denotes the conformal mapping of the unit disk $\mathscr{D}_1:=\{\zeta\in\mathbb{C}:|\zeta|<1\}$ onto the domain Ω^+, the norm of $\psi\in\mathscr{E}_p(\overline{\Omega^+},\rho)$ is defined as follows

$$\|\psi\big|\mathscr{E}_p(\overline{\Omega^+},\rho)\| := \sup_{0<r<1}\left(\int\limits_{\Gamma^{(r)}}|\rho(\tau)\psi(\tau)|^p\,d\tau\right)^{\frac{1}{p}},$$

where $\Gamma^{(r)}:=\{z=\omega(\zeta)\ :\ |\zeta|=r\}$ are the images of the concentric circumferences of the radius r.

Similarly is defined the Smirnov–Lebesgue space $\mathscr{E}_p(\overline{\Omega^-},\rho)$ for the outer domain Ω^-.

An equivalent definition of the Smirnov–Lebesgue spaces $\mathscr{E}_p(\overline{\Omega^\pm},\rho)$ is the following: $u\in\mathscr{E}_p(\overline{\Omega^\pm},\rho)$ iff $u(z)$ is represented by the Cauchy integral

as follows

$$\Phi(z) = c_0 + C_\Gamma\varphi(z)\,, \quad c_0 = \text{ const }\,, \quad \varphi \in \mathrm{L_p}(\Gamma,\rho)\,,$$
$$C_\Gamma\varphi(z) := \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}\,, \quad z \in \Omega^\pm \tag{1.3}$$

(cf. [Pv1] and [Go1, Ch.X, § 5]). In particular, for the compact domain $\Omega^+ \subset \mathbb{C}$ representation (1.3) can be written also as follows $\Phi(z) = C_\Gamma\varphi_0(z)$, $\varphi_0(t) = c_0 + \varphi(t)$, $t \in \Gamma$, while for Ω^- we have $c_0 = \Phi(\infty)$.

Taking the advantage of the last definition we will introduce the following new spaces, suited for our purposes.

$\mathscr{W}_p^s(\overline{\Omega^\pm},\rho)$–the weighted Smirnov–Sobolev space of functions $\Phi(z)$ represented as in (1.3) with a density $\varphi \in W_p^s(\Gamma,\rho)$. Note, that the restriction for piecewise-smooth contour Γ is $|s| < 1 + \frac{1}{p}$.

The space $\mathscr{W}_p^1(\overline{\Omega^\pm},\rho)$ consists of functions $\Phi(z)$ which belong to $\mathscr{E}_p(\overline{\Omega^\pm},\rho)$ together with their derivatives $\Phi, \partial\Phi \in \mathscr{E}_p(\overline{\Omega^\pm},\rho)$. This is easy to check with a partial integration.

Due to Theorem 1.8 proved below, we get $\mathscr{W}_2^s(\overline{\Omega^\pm}) \subset W_{2,\mathrm{loc}}^{s+\frac{1}{2}}(\overline{\Omega^\pm})$. If outward peaks are absent $0 < \gamma_j \le 1$, the following inverse is also true: traces of functions from $W_{2,\mathrm{loc}}^{s+\frac{1}{2}}(\overline{\Omega^\pm})$ belong to $W_2^s(\Gamma)$. In case of outward peaks the last assertion fails as shown in Example 1.3 (see also [Ia1]). Note that formulated theorem on traces remain valid even in the presence of inward peaks (with interior angle 2π).

$\mathscr{H}_{m+\mu}^0(\overline{\Omega^\pm},\rho)$, $\mathscr{C}^m(\overline{\Omega^\pm},\rho)$ and $\mathscr{PC}^m(\overline{\Omega^\pm},\rho)$ are used for the weighted Smirnov–Hölder etc. spaces of functions $\Phi(z)$ represented as in (1.3) with a density φ in appropriate spaces $H_{m+\mu}^0(\Gamma,\rho)$, in $C^m(\Gamma,\rho)$ (with the restriction $m \le 1$ for a piecewise-smooth contour Γ) or in $PC^m(\Gamma,\rho)$, respectively (with the restriction $m \le 2$ for a piecewise-smooth contour Γ).

$e_p(\overline{\Omega^\pm},\rho) = w_p^0(\overline{\Omega^\pm},\rho)$, $w_p^s(\overline{\Omega^\pm},\rho)$, $h_{m+\mu}^0(\overline{\Omega^\pm},\rho)$, $c^m(\overline{\Omega^\pm},\rho)$ and $pc^m(\overline{\Omega^\pm},\rho)$ is used for the spaces of harmonic functions represented as real $u(z) = \mathrm{Re}\,\Phi(z)$ (or the imaginary $u(z) = \mathrm{Im}\,\Phi(z)$ parts of functions $\Phi(z)$ from $\mathscr{E}_p(\overline{\Omega^\pm},\rho) = \mathscr{W}_p^0(\overline{\Omega^\pm},\rho)$, $\mathscr{W}_p^s(\overline{\Omega^\pm},\rho)$, $\mathscr{H}_{m+\mu}^0(\overline{\Omega^\pm},\rho)$, $\mathscr{C}^m(\overline{\Omega^\pm},\rho)$ and, respectively, from $\mathscr{PC}^m(\overline{\Omega^\pm},\rho)$. We use $e_p(\overline{\Omega^\pm}$ etc for the space $e_p(\overline{\Omega^\pm},1)$ etc.

It is important to have representations of functions (1.3) with a pure real or a pure imaginary density $\varphi(t)$. Next lemma provides the condition for such representation. Similar considerations can be found in [Mu1, §§ 62–66].

Lemma 1.1 *Let $\mathbb{X}(\Gamma)$ be one of the following spaces: $W_p^s(\Gamma,\rho)$, $H_{m+\mu}^0(\Gamma,\rho)$, $C^m(\Gamma,\rho)$ or $PC^m(\Gamma,\rho)$ and $\mathscr{X}(\overline{\Omega^\pm})$–the corresponding* Smirnov *space $\mathscr{W}_p^s(\Gamma,\rho)$, $\mathscr{H}_{m+\mu}^0(\Gamma,\rho)$ etc.*

The function $\Phi \in \mathscr{X}(\overline{\Omega^\pm})$ can be represented by the Cauchy *integral as in* (1.3) *with a pure real $\varphi = \mathrm{Re}\,\varphi \in \mathbb{X}(\Gamma)$ or a pure imaginary $\varphi =$*

$i\,\mathrm{Im}\,\varphi \in \mathbb{X}(\Gamma)$ *density if and only if the* RIEMANN–HILBERT *problem for the complementary domain* $\Omega^{\mp} = \mathbb{C} \setminus \Omega^{\pm}$

$$\mathrm{Re}\,\Psi^{\mp}(t) = g(t), \quad t \in \Gamma\,, \quad g \in \mathbb{X}(\Gamma)\,, \quad \Psi(z) \to 0 \quad as \quad |z| \to \infty$$

is surjective, i.e., has solution for all right-hand sides in $\mathscr{X}(\overline{\Omega^{\mp}})$.

For the domain Ω^{+} *the same conditions provide the representation* $\Phi(z) = C_{\Gamma}\varphi_0(z)$, $z \in \Omega^{+}$ *with a real valued density* $\varphi_0 = \mathrm{Re}\,\varphi_0$.

We postpone the proof of the formulated Lemma until Subsection 2.3.

Let us conclude this subsection by the following agreements which we will hold on in the sequel.

I. $\mathbb{X}^s(\Gamma,\rho)$ (or more simple $\mathbb{X}(\Gamma)$) is used to denote the spaces $W_p^s(\Gamma,\rho)$, $H_s^0(\Gamma,\rho)$ $C^s(\Gamma,\rho)$ or $PC^s(\Gamma,\rho)$, where the weight function $\rho(t)$ is defined in (1.2) and $\mathscr{X}^s(\overline{\Omega^{\pm}},\rho)$, $x^s(\overline{\Omega^{\pm}},\rho)$–for the corresponding SMIRNOV spaces of analytic and harmonic functions.

For the parameters there hold the following constraints:

$$\begin{aligned}
&|s| \le 1\,,\ -\frac{1}{p} < \alpha_j < 1 - \frac{1}{p}\,,\ 1 < p < \infty && \text{for} && W_p^s(\Gamma,\rho)\,,\\
&\begin{cases} m + s\,,\ m = 0, 1\,,\ 0 < s < 1\,, \\ s < \alpha_j < s + 1\,,\ \Gamma \setminus \{t_1, \dots, t_n\} \in C^{m+s} \end{cases} && \text{for} && H_s^0(\Gamma,\rho)\,, \qquad (1.4)\\
&s = m \in \mathbb{N}_0\,,\ 0 < \alpha_j < 1 && \text{for} && PC^m(\Gamma,\rho)\\
& && \text{and for} && C^m(\Gamma,\rho)
\end{aligned}$$

for $j = 0, \dots, n$. Conditions (1.4) are necessary and sufficient for boundedness of the CAUCHY singular integral operator

$$S_{\Gamma}\varphi(t) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)d\tau}{\tau - t}\,, \quad t \in \Gamma \qquad (1.5)$$

(the integral in (1.5) is understood in the CAUCHY mean value sense) in the spaces $W_p^m(\Gamma,\rho)$ ([GK1, Go2, Kh1, Kh2] and $H_{m+\mu}^0(\Gamma,\rho)$ (see [Du1, Du6, Du7, GK1]) and of operators with fixed singularities in the kernel (see § 3.2) in all four spaces $W_p^m(\Gamma,\rho)$, $H_{m+\mu}^0(\Gamma,\rho)$, $C^m(\Gamma,\rho)$ and in $PC^m(\Gamma,\rho)$ (see [Du1] and § 3.2 below; S_{Γ} is not bounded in $C^m(\Gamma,\rho)$ and in $PC^m(\Gamma,\rho)$).

II. For a space with weight $W_p^m(\Gamma,\rho)$, $H_{m+\mu}^0(\Gamma,\rho)$ or $PC^m(\Gamma,\rho)$, if not otherwise stated, the weight function is defined in (1.1) and the exponents satisfy the appropriate conditions (1.4).

1.2 Boundary value problems

For a real valued harmonic function

$$\Delta u(x) = 0\,, \quad x \in \Omega^{\pm}, \tag{1.6}$$

we consider the DIRICHLET

$$u^{\pm}(t) = g(t)\,, \quad g \in \mathbb{X}^s(\Gamma, \rho)\,, \quad 0 \le s \le 1\,, \quad t \in \Gamma\,, \tag{1.7}$$

and the NEUMANN

$$(\partial_{\vec{\nu}(t)} u)^{\pm}(t) = f(t)\,, \quad f \in \mathbb{X}^{s-1}(\Gamma, \rho)\,, \quad 0 \le s \le 1\,, \quad t \in \Gamma, \tag{1.8}$$

boundary value problems, with some real valued data [2] $\operatorname{Im} g(t) \equiv \operatorname{Im} f(t) \equiv 0$, where $\partial_{\vec{\nu}(t)} := \nu_1(t)\partial_{t_1} + \nu_2(t)\partial_{t_2}$, $t = (t_1, t_2) \in \Gamma$ denotes the normal derivative. We hold on the agreement about spaces and weights made in conclusion of § 1.1.

We look for solutions of problem (1.6), (1.7) (of (1.6), (1.8)) in the SMIRNOV class

$$u \in x^s(\overline{\Omega^{\pm}}, \rho)\,, \qquad 0 \le s \le 1\,. \tag{1.9}$$

Let us note that by definition of the SMIRNOV class a function $u \in x^s(\overline{\Omega^-}, \rho)$ automatically possesses a finite limit at the infinity: $u(x) = O(1)$ for $x \in \Omega^-$ as $|x| \to \infty$ (see (1.3)).

Next Lemma and example are a certain justification of the choice of constraints (1.9) instead of (0.2) which is common for domains with a LIPSCHITZ boundary (see [Ke1, Ma1, MT1]).

Lemma 1.2 *If* (0.2) *holds,* $\Omega^{\pm}$ *has no outward peak and* $u(z)$ *is a harmonic function (i.e.,* $u(z)$ *solves* (1.6)*). Then*

$$u \in w_2^{\frac{1}{2}}(\overline{\Omega^{\pm}})\,. \tag{1.10}$$

Vice versa, $u \in w_2^{\frac{1}{2}}(\overline{\Omega^{\pm}}) \subset e_2(\overline{\Omega^{\pm}})$ *implies* (0.2) *and* $u(z)$ *is a harmonic function, also for domains* $\Omega^{\pm}$ *with outward peaks.*

We postpone the proof of the formulated Lemma until Subsection 2.3.

Next example shows that under constraints (0.2) solution $u(x)$ of BVPs (1.6), (1.7) and (1.6), (1.8) might have non-integrable trace $u^+(t)$ on the boundary $\Gamma = \partial\Omega^+$ as soon as $\Omega^{\pm}$ has a single outward peak.

Example 1.3 *Let* $0 < \sigma < \infty$, $\gamma > 0$ *and*

$$\Omega_\sigma^+ := \left\{ x_1 + ix_2 \,:\, 0 \le x_1 \le 1\,,\ \ 0 \le x_2 \le x_1^{\sigma+1} \right\}. \tag{1.11}$$

[2] If we admit complex-valued data $\operatorname{Im} g \not\equiv 0$ in (1.6) and $\operatorname{Im} f \not\equiv 0$ in (1.8) but then we have to look for a complex-valued solution $u = u_r + iu_i$, $u_r, u_i \in x^s(\overline{\Omega^{\pm}}, \rho)$ in (1.9).

Then, choosing the branch of the analytic function $\varphi_\gamma(z) := z^{-\gamma}$ *appropriately, for the harmonic function* $\varphi_\gamma(z) := \operatorname{Re} z^{-\gamma}$ *we get* $\Delta\varphi_\gamma = 0$ *in* Ω_γ *and* $\varphi_\gamma \in W_p^1(\overline{\Omega_\sigma^+})$ *provided* $\sigma - (\gamma+1)p > -2$.

In particular, $\varphi_\gamma \in W_2^1(\overline{\Omega_{2\gamma+3}^+})$.

In fact,

$$\|\varphi_\gamma|W_p^1(\Omega_\sigma^+)\|^p = \int_0^1 dx_1 \int_0^{x_1^{\sigma+1}} \left[(x_1^2+x_2^2)^{-\frac{\gamma}{2}p} + \gamma(x_1^2+x_2^2)^{-\frac{\gamma+1}{2}p}\right] dx_2$$

$$\le C_1 \int_0^1 dx_1 \int_0^{x_1^{\sigma+1}} (x_1+x_2)^{-(\gamma+1)p} dx_2 = C_2 \int_0^1 x_1^{-(\gamma+1)p+1} dx_1$$

$$\times \int_0^{x_1^\sigma} (1+t)^{-(\gamma+1)p} dt \le C_3 \int_0^1 x_1^{\sigma-(\gamma+1)p+1} dx_1 = C_4 < \infty\,.$$

1.3 Representation of solutions and layer potentials

Applying the Gauss formula on divergence (on "partial integration")

$$\int_{\Omega^\pm} \partial_j u(y)\overline{v(y)}dy = \int_{\Omega^\pm} u(y)\overline{\partial_j v(y)}dy \pm \oint_\Gamma \nu_j(\tau)u(\tau)\overline{v(\tau)}d\tau\,, \tag{1.12}$$

we readily obtain two well-known Green formulae

$$\int_{\Omega^\pm} \Delta u(y)\overline{v(y)}dy = \sum_{j=1}^2 \int_{\Omega^\pm} \partial_j u(y)\overline{\partial_j v(y)}dy \pm \oint_\Gamma \partial_{\vec\nu(\tau)} u(\tau)\overline{v(\tau)}d\tau\,, \tag{1.13}$$

$$\int_{\Omega^\pm} \left[\Delta u(y)\overline{v(y)} - u(y)\overline{\Delta v(y)}\right] dy = \oint_\Gamma \left[\partial_{\vec\nu(\tau)} u(\tau)\overline{v(\tau)} - u(\tau)\overline{\partial_{\vec\nu(\tau)} v(\tau)}\right] d\tau\,, \quad u, v \in C^\infty_{\rm com}(\overline{\Omega^\pm})\,. \tag{1.14}$$

Invoking the fundamental solution of equation (1.6)

$$F_\Delta(z) := \frac{1}{2\pi}\log|z|\,, \qquad \Delta F_\Delta(z) = \delta(z)\,, \quad z \in \mathbb{R}^2\,,$$

where δ is Kroneker's delta function, we can easily derive from (1.14) the following representation formula for a harmonic function $u(x)$ which meets condition (1.9)

$$\chi_+(x)u(x) = W_\Gamma u^+(x) - V_\Gamma(\partial_{\vec\nu}u)^+(x)\,,$$

$$\chi_-(x)u(x) = u_\infty - W_\Gamma u^-(x) + V_\Gamma(\partial_{\vec\nu}u)^-(x)\,, \tag{1.15}$$

$$x \in \mathbb{R}^2 \setminus \Gamma = \Omega^- \cup \Omega^+$$

(see [Ma1, Ch.1, § 1.2]), where u_∞ =const, $\chi_\pm$ is the characteristic function of the domain $\Omega^\pm$ and

$$\begin{aligned} W_\Gamma\varphi(x) &= \frac{1}{2\pi}\int\limits_\Gamma \varphi(\tau)\partial_{\vec\nu(\tau)}\log|\tau-x|ds\,, \quad ds=|d\tau|\,,\\ V_\Gamma\varphi(x) &= \frac{1}{2\pi}\int\limits_\Gamma \varphi(\tau)\log|\tau-x|ds\,, \qquad x\in\Omega^\pm, \end{aligned} \tag{1.16}$$

are the double and the single layer potentials (known as the harmonic or the logarithmic potentials as well).

Let us note, that constants are included into the class of harmonic functions in unbounded domains Ω^- (see (1.3) and the second formulae in (1.15)) only in 2-dimensional case (see, e.g., [Ma1, p.216], [Vl1, p.333]).

For the direct values of harmonic potentials (1.16) on Γ we use the notation $W_{\Gamma,0}$ and $V_{\Gamma,-1}$ where the additional subscript indices indicate the order of these operators, treated as pseudo-differential operators on the manifold Γ (see Theorem 1.5 below). According this rule we have also $S_\Gamma := C_{\Gamma,0}$ (see (1.3) and (1.5)).

Lemma 1.4 *The following holds:*

$$\begin{aligned} W_{\Gamma,0}\varphi(t) &= \frac{1}{4}\left(S_\Gamma+\mathscr{V}S_\Gamma\mathscr{V}\right)\varphi(t) = \frac{1}{4\pi i}\int\limits_\Gamma \varphi(\tau)d\log\frac{\tau-t}{\overline\tau-\overline t}\\ &= \frac{1}{4\pi i}\int\limits_\Gamma \varphi(\tau)\left[\frac{d\tau}{\tau-t}-\frac{d\overline\tau}{\overline\tau-\overline t}\right], \end{aligned} \tag{1.17}$$

$$\begin{aligned} W^*_{\Gamma,0}\varphi(t) &= \frac{1}{4}\left(hS_\Gamma\overline h+\mathscr{V}hS_\Gamma\overline h\mathscr{V}\right)\varphi(t)\\ &= \frac{1}{4\pi i}\int\limits_\Gamma \varphi(\tau)\left[\frac{h(t)}{h(\tau)}\frac{d\tau}{\tau-t}-\frac{\overline{h(t)}}{\overline{h(\tau)}}\frac{d\overline\tau}{\overline\tau-\overline t}\right], \end{aligned} \tag{1.18}$$

$$\begin{aligned} \partial_t V_{\Gamma,-1}\varphi(t) &= \frac{i}{4}\left(S_\Gamma-\mathscr{V}S_\Gamma\mathscr{V}\right)\varphi(t)\\ &= \frac{1}{4\pi}\int\limits_\Gamma \varphi(\tau)\left[\frac{d\tau}{\tau-t}+\frac{d\overline\tau}{\overline\tau-\overline t}\right], \quad t\in\Gamma\,, \end{aligned} \tag{1.19}$$

where

$$\mathscr{V}\varphi(t) := \overline{\varphi(t)}\,, \quad h(t) := ie^{i\vartheta_t} \tag{1.20}$$

and ϑ_t denotes the inclination to the abscissa axes of the outer unit normal vector $\vec\nu(t)$ ($t\in\Gamma\setminus\{t_1,\dots,t_n\}$; see Fig. 1).

Proof (see [Mu1, §§ 12,14]). Let us consider the natural parametrisation of the curve Γ by the arc length parameter

$$\tau(s)\ :\ [0,\ell]\longrightarrow\Gamma\,, \quad \tau(0)=\tau(\ell)\,.$$

Easy to ascertain that if the derivative $\tau'(s)$ exists, coincides with the unit tangent vector to Γ. We have

$$\vec{\nu}(\tau) = (\cos\vartheta_\tau, \sin\vartheta_\tau)\,,$$
$$d\tau = \left[\cos\left(\tfrac{\pi}{2}+\vartheta_\tau\right) + i\sin\left(\tfrac{\pi}{2}+\vartheta_\tau\right)\right]|d\tau| = h(\tau)ds \tag{1.21}$$

(see Fig. 1 and (1.20)). Therefore

$$\begin{aligned}\frac{1}{2\pi}\partial_{\vec{\nu}(\tau)}[\log|\tau-t|]ds &= \frac{1}{2\pi|\tau-t|}\left[\frac{d|\tau-t|}{d\operatorname{Re}\tau}\cos\vartheta_\tau + \frac{d|\tau-t|}{d\operatorname{Im}\tau}\sin\vartheta_\tau\right]ds\\ &= \frac{\operatorname{Re}(\tau-t)\cos\vartheta_\tau + \operatorname{Im}(\tau-t)\sin\vartheta_\tau}{2\pi|\tau-t|^2}ds = \frac{-(\tau-t)d\overline{\tau}+(\overline{\tau}-\overline{t})d\tau}{4\pi i}\\ &= \frac{1}{4\pi i}\left[\frac{d\tau}{\tau-t} - \frac{\overline{d\tau}}{\overline{\tau}-\overline{t}}\right] = \frac{1}{4\pi i}d_\tau\log\frac{\tau-t}{\overline{\tau}-\overline{t}}\,,\end{aligned}$$

which gives (1.17).

Formula (1.18) follows from (1.17) since the adjoint operator S_Γ^* to S_Γ in (1.5) with respect to the sesquilinear form

$$\langle\varphi,\psi\rangle := \int_\Gamma \varphi(\tau)\overline{\psi(\tau)}|d\tau|$$

reads

$$S_\Gamma^* = \mathscr{V}hS_\Gamma h^{-1}\mathscr{V} = h^{-1}\mathscr{V}S_\Gamma\mathscr{V}hI\,. \tag{1.22}$$

In fact, since $d\tau = h(\tau)|d\tau|$ and $\overline{h(\tau)} = h^{-1}(\tau)$ (see (1.20), (1.21)), we get

$$\begin{aligned}\langle S_\Gamma\varphi,\psi\rangle &:= \int_\Gamma S_\Gamma\varphi(\tau)\overline{\psi(\tau)}|d\tau| = \int_\Gamma S_\Gamma\varphi(\tau)\overline{h(\tau)\psi(\tau)}d\tau\\ &= -\int_\Gamma \varphi(t)S_\Gamma\overline{h(\tau)\psi(\tau)}dt = \int_\Gamma \varphi(t)\overline{\mathscr{V}(S_\Gamma\mathscr{V}h\psi)(t)}h(t)|dt|\\ &= \int_\Gamma \varphi(t)\overline{\mathscr{V}h(t)(S_\Gamma h^{-1}\mathscr{V}\psi)(t)}|dt|.\end{aligned}$$

To prove formula (1.19) we proceed as follows:

$$\begin{aligned}\partial_t V_{\Gamma,-1}\varphi(t) &= \frac{1}{2\pi}\int_\Gamma \varphi(\tau)\partial_\tau\log|\tau-t|ds\\ &= \frac{1}{2\pi}\int_\Gamma \varphi(\tau)\frac{-\operatorname{Re}(\tau-t)\sin\vartheta_\tau + \operatorname{Im}(\tau-t)\cos\vartheta_\tau}{|\tau-t|^2}ds\\ &= \frac{1}{2\pi}\int_\Gamma \varphi(\tau)\frac{\operatorname{Re}(\tau-t)\operatorname{Re}d\tau + \operatorname{Im}(\tau-t)\operatorname{Im}d\tau}{|\tau-t|^2}\\ &= \frac{1}{4\pi}\int_\Gamma \varphi(\tau)\left[\frac{d\tau}{\tau-t} + \frac{\overline{d\tau}}{\overline{\tau}-\overline{t}}\right].\end{aligned}$$

■

Next two theorems deal with boundedness properties of layer potentials. They are based on Lemma 1.4 and justify constraints (1.4) on the weight function $\rho(t)$.

Theorem 1.5 *$W_{\Gamma,0}$ is bounded in the spaces $W_p^s(\Gamma,\rho)$ for $0 \le s \le 1$ and in $H^0_{m+\mu}(\Gamma,\rho)$ for $m = 0, 1$.*

The operator $W^*_{\Gamma,0}$ is bounded in the spaces $W_p^s(\Gamma,\rho)$ for $-1 \le s \le 0$ and in $H^0_\mu(\Gamma,\rho)$.

$V_{\Gamma,-1}$ is bounded from $L_p(\Gamma,\rho)$ to $W_p^1(\Gamma,\rho)$ and from $H^0_\mu(\Gamma,\rho)$ to $H^0_{1+\mu}(\Gamma,\rho)$.

Theorem 1.6 *The operator $W_{\Gamma,0}$ is bounded in $C(\Gamma,\rho)$ and in $PC^m(\Gamma,\rho)$ for $m = 0, 1$.*

The operator $W^*_{\Gamma,0}$ is bounded in $PC(\Gamma,\rho)$.

We postpone the proofs of the formulated theorems until Subsection 2.3. Here we will prove the following corollary.

Corollary 1.7 *Let $\rho(t)$ be defined by* (1.2) *and* (1.4). *If Γ is smooth (contains no cusps and no angular points $\gamma_1 = \cdots = \gamma_n = 1$) operators $W_{\Gamma,0}$ and $W^*_{\Gamma,0}$ have weak singular kernels and are compact in the spaces $L_p(\Gamma,\rho)$ and $PC(\Gamma,\rho)$.*

Operator $W_{\Gamma,0}$ is compact also in spaces $W_p^s(\Gamma,\rho)$ for $-1 \le s \le 1$, in $C(\Gamma,\rho)$ and in $PC^1(\Gamma,\rho)$.

Proof. It suffices to prove compactness of $W_{\Gamma,0}$, since $W^*_{\Gamma,0}$ is the adjoint operator and would have weak singular kernel if $W_{\Gamma,0}$ has.

If $\Gamma = \mathbb{R}$ or $\Gamma \subset \mathbb{R}$ then $K_1 = W_{\Gamma,0} = 0$ as it is clear from representations (1.17) and (1.18).

If $\Gamma = \Gamma_1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circumference then $\vartheta_t \equiv \vartheta$, $h(t) = e^{i\vartheta}$ and $\tau = e^{i\vartheta}$, $t = e^{i\lambda}$ $(0 \le \vartheta, \lambda \le 2\pi)$ inserted into (1.17) gives

$$K_1 = W_{\Gamma_1,0}\varphi(\lambda) = \frac{1}{4\pi}\int_0^{2\pi} \varphi(\vartheta)d\vartheta\,; \tag{1.23}$$

therefore $W_{\Gamma_1,0}$ is one dimensional and compact.

If Γ is arbitrary smooth curve and $\omega : \Gamma \to \Gamma'$ is a corresponding diffeomorphism where either $\Gamma' \subset \mathbb{R}$ or $\Gamma' = \Gamma_1$, then

$$W_{\Gamma,0} = K_\omega - K^*_\omega + \omega_*^{-1}K_1\omega_*\,,$$
$$K_\omega := \omega_*^{-1}S_\Gamma\omega_* - S_\Gamma\,, \quad \omega_*\varphi(t) = \varphi(\omega(t))\,, \quad t \in \Gamma\,,$$

with $\omega^{-1} : \Gamma \to \Gamma'$ standing for the inverse diffeomorphism and K^*_ω–for the adjoint to K_ω. the integral operator K_ω has a weak singular kernel (see [DLS1, § 3.5] or [Kh1, GK1]). As for K_1, either $K_1 = 0$ or it is a one dimensional operator (see (1.23)). ∎

To accomplish boundedness properties of potential operators and their direct values on the curve we formulate the next result. For a general assertion (layer potentials for partial differential operators with variable coefficient and arbitrary order in $\mathbb{R}^n$, provided they have a fundamental solution) we quote [Du10, Theorem 3.2] (for LIPSCHITZ domains see also [MMP1, MMT1, MT1]).

Theorem 1.8 *Let $s \in \mathbb{R}$ and the boundary $\Gamma = \partial\Omega^{\pm}$ be m-smooth, where $m \in \mathbb{N}_0$, $m \geq |s|$.*

The potential operators[3)]

$$\begin{aligned} C_\Gamma &: W_2^s(\Gamma) \longrightarrow W_{2,\mathrm{com}}^{s+\frac{1}{2}}(\overline{\Omega^{\pm}})\,, \\ W_\Gamma &: W_2^s(\Gamma) \longrightarrow W_{2,\mathrm{com}}^{s+\frac{1}{2}}(\overline{\Omega^{\pm}})\,, \\ V_\Gamma &: W_2^s(\Gamma) \longrightarrow W_{2,\mathrm{com}}^{s+\frac{3}{2}}(\overline{\Omega^{\pm}}) \end{aligned} \tag{1.24}$$

(*see* (1.3) *and* (1.16)) *are bounded*[4)].

In particular, if Γ is piecewise-smooth we should restrict $-1 \leq s \leq 1$.

Proof. For a smooth $\Gamma = \partial\Omega^{\pm}$ see [Du10, Theorem 3.2].

Let Γ have knots $t_1, \ldots, t_n$ (see (Fig. 1) and consider $C_\Gamma\varphi(z)$. The operator C_Γ is of the local type, i.e., if

$$v_1 \in C_0^\infty(\overline{\Omega^+})\,, \quad v_2 \in L_\infty(\Gamma)\,, \quad \text{supp } \mathrm{v}_1 \cap \text{supp } \mathrm{v}_2 = \emptyset\,,$$

then $v_1 C_\Gamma v_2\varphi \in C_0^\infty(\overline{\mathbb{C}})$. Therefore it suffices to establish continuity (1.24) for $vC_\Gamma uI$, where $v \in C_0^\infty(\overline{\Omega^+})$, $u \in C(\Gamma)$ are cut-off functions, equal 1 in some small neighbourhood of a knot t_j and vanishing outside another one; in particular, $v(t_k) = u(t_k) = 0$ for $j \neq k$.

We can suppose that

$$\varphi = \varphi_1 + \varphi_2\,, \quad \varphi_k := u_k\varphi \in W_2^s(\Gamma)\,, \quad u_k := \chi_k u\,, \quad k = 1, 2\,,$$

where $\chi_1(t)$ and $\chi_2(t)$ are characteristic functions of the left and right neighbourhoods of $t_j \in \Gamma$ and $[\chi_1(t) + \chi_1(t)]u(t) = u(t)$. Since $\chi_k(t)$, $k = 1, 2$ have discontinuities at t_j, for the claimed inclusions $\varphi_k \in W_2^s(\Gamma)$ we need $\varphi(t_j) = 0$ if $s \geq \frac{1}{2}$. The latter can be provided since $(C_\Gamma 1)(z) \equiv 1$ for $z \in \Omega^+$ and

$$C_\Gamma\varphi(z) = C_\Gamma\varphi_0(z) + \varphi(t_j)\,.$$

Thus,

$$v(z)C_\Gamma u\varphi(z) = u(z)C_\Gamma\varphi_1(z) + u(z)C_\Gamma\varphi_2(z) = u(z)C_{\Gamma_1}\varphi_1(z) + u(z)C_{\Gamma_2}\varphi_2(z),$$

[3)] For a compact domain we define $W_{2,\mathrm{com}}^{\mu}(\overline{\Omega^{\pm}}) = W_2^{\mu}(\overline{\Omega^{\pm}})$.

[4)] We have formulated only a particular result–the case $p = 2$. The general result for $1 < p < \infty$ in [Du10] states boundedness between the BESSEL potential and the BESOV spaces.

where Γ_1 and Γ_2 are smooth and closed contours which have in common either only the point $\{t_j\}$ when $\gamma_j < 1$, or two points (one of them $\{t_j\}$) when $1 < \gamma_j < 2$, or some arc $\Gamma' = t_j z_o$ when $\gamma_j = 2$. We can assume $\varphi_k \in W_2^s(\Gamma_k)$ extending functions to $\Gamma_k \setminus (\Gamma \cap \Gamma_k)$ by 0 ($k = 1, 2$). As noted above, due to smoothness of Γ_k we get $vC_{\Gamma_k}\varphi_k \in W_2^{s+\frac{1}{2}}(\overline{\Omega_k^+})$, where Ω_k^+ is the inner domain for Γ_k, $k = 1, 2$. On the other hand,

$$\operatorname{supp} v \cap \overline{\Omega^+} = \overline{\Omega_1} \cup \overline{\Omega_2}\,, \quad \Omega_k := (\operatorname{supp} v \cap \Omega^+) \cap \Omega_k^+\,, \quad k = 1, 2\,.$$

Then $vC_\Gamma\varphi_k = vC_{\Gamma_k}\varphi_k \in W_2^{s+\frac{1}{2}}(\overline{\Omega_1} \cup \Gamma_2)$ since on the common boundary $\overline{\Omega_1} \cap \overline{\Omega_2} = \Gamma_1 \cap \Gamma_2 \subset \Omega^+ \cup \{t_j\}$, except t_j, functions are C^∞ smooth. Therefore, $vC_\Gamma u\varphi = vC_\Gamma\varphi_1 + vC_\Gamma\varphi_2 \in W_2^{s+\frac{1}{2}}(\overline{\Omega^+})$.

The inclusion $vC_\Gamma u\varphi \in W_{2,\mathrm{com}}^{s+\frac{1}{2}}(\overline{\Omega^-})$ and other results in (1.24) are proved similarly. ∎

To proceed further we need the Plemelji formulae (the jump relations) for layer potentials, which we formulate next.

Let $\Phi \in C(\overline{\Omega^\pm})$. By $\Phi^\pm(t)$, $t \in \Gamma = \partial\Omega^\pm$ is denoted, as usual, non-tangential boundary values $\Phi^\pm(t) = \lim\limits_{z \in \Omega^\pm,\, z \to t} \Phi(z)$.

Lemma 1.9 *Let $1 < p < \infty$, $-1 \le s \le 1$ and $\varphi \in W_p^s(\Gamma, \rho)$, where $\rho(t)$ is defined in* (1.2), (1.4). *Then*

$$(W_\Gamma\varphi)^\pm(t) = \pm\frac{1}{2}\varphi(t) + W_{\Gamma,0}\varphi(t)\,, \quad (\partial_{\vec\nu}V_\Gamma\varphi)^\pm(t) = \mp\frac{1}{2}\varphi(t) + W_{\Gamma,0}^*\varphi(t)\,,$$

$$(\partial_{\vec\nu}W_\Gamma\varphi)^+(t) = (\partial_{\vec\nu}W_\Gamma\varphi)^-(t)\,, \quad (C_\Gamma\varphi)^\pm(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2}S_\Gamma\varphi(t)\,, \quad (1.25)$$

for almost all $t \in \Gamma$ (for all $t \in \Gamma \setminus \{t_1, \dots, t_n\}$ provided $s > \frac{1}{2}$ or $\varphi \in H_\mu^0(\Gamma, \rho)$).

Proof. The proof can be found e.g. in [Mu1, §§15,16]) (see the survey [Ma1]). See also [MT1, Appendix C] for the case of Lipschitz domains and [Du10, §6.4] for much more general operators. ∎

If Γ is a compact curve and $\varphi \in L_1(\Gamma)$ then

$$W_\Gamma\varphi(x) = \mathrm{O}\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty\,. \quad (1.26)$$

As for the single layer potential,

$$V_\Gamma\varphi(x) = \mathrm{O}(1) \quad \text{as} \quad |x| \to \infty \quad \text{iff} \quad \int_\Gamma \varphi(\tau)|d\tau| = 0 \quad (1.27)$$

and then

$$V_\Gamma\varphi(x) = \mathrm{o}(1) \quad \text{as} \quad |x| \to \infty\,. \quad (1.28)$$

In fact,

$$V_\Gamma\varphi(x) = \int\limits_\Gamma \varphi(\tau)\log\frac{r}{|x|}|d\tau| + \log|x|\int\limits_\Gamma \varphi(\tau)|d\tau|$$

$$= \mathrm{o}(1) + \log|x|\int\limits_\Gamma \varphi(\tau)|d\tau| \quad \text{as} \quad |x|\to\infty\,.$$

and (1.27), (1.28) follow.

If

$$\int\limits_\Gamma (\partial_{\vec\nu(\tau)}u)^-(\tau)|d\tau| = 0, \tag{1.29}$$

then in (1.15) we have

$$u_\infty = u(\infty) = W_\Gamma u^-(0) - V_\Gamma(\partial_{\vec\nu}u)^-(0)\,. \tag{1.30}$$

In fact, the first equality $u_\infty = u(\infty)$ follows from (1.15), and (1.26)–(1.28) since (1.29) holds.

Passing to the limit $x\to t\in\Gamma$, $x\in\Omega^-$, in the representation formula (1.15) and applying the appropriate Plemelji formulae (1.25) we find:

$$u^-(t) = u_\infty + \frac{1}{2}u^-(t) - W_\Gamma u^-(t) + V_\Gamma(\partial_{\vec\nu}u)^-(t)\,,\quad t\in\Gamma\,.$$

The obtained formula can be rewritten as follows

$$u_\infty = \frac{1}{2}u^-(t) + W_\Gamma u^-(t) - V_\Gamma(\partial_{\vec\nu}u)^-(t)\,,\quad t\in\Gamma\,.$$

Therefore, the trace of the harmonic function

$$w(x) = W_\Gamma u^-(x) - V_\Gamma(\partial_{\vec\nu}u)^-(x)\,,\quad x\in\Omega^+,$$

on the boundary $\Gamma=\partial\Omega^+$

$$w^+(t) = \frac{1}{2}u^-(t) + W_\Gamma u^-(t) - V_\Gamma(\partial_{\vec\nu}u)^-(t) = u_\infty\,,\quad t\in\Gamma\,.$$

(see the appropriate Plemelji formulae (1.25)) is constant. This implies $w(x)\equiv$=const for the entire domain $x\in\overline{\Omega^+}$ and, therefore, $u_\infty = w(0) = W_\Gamma u^-(0) - V_\Gamma(\partial_{\vec\nu}u)^-(0)$.

The integral $W_\Gamma 1(x)$ is known as the Gaussian integral and can be written explicitly:

$$W_\Gamma 1(x) = \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(\tau)}\log|\tau-x||d\tau| = \begin{cases} 1 & \text{if} \quad x\in\Omega^+\,,\\ 0 & \text{if} \quad x\in\Omega^-\,,\\ \frac{1}{2} & \text{if} \quad x\in\Gamma \end{cases} \tag{1.31}$$

(see [Ma1, Chapter I, § 1.1]).

Remark 1.10 *The homogeneous equation*

$$-\frac{1}{2}\mathfrak{r} + W^*_{\Gamma,0}\mathfrak{r} = 0\,, \tag{1.32}$$

has a unique linearly independent solution $\mathfrak{r}_0 \not\equiv 0$ *in* $L_2(\Gamma)$ *such that*

$$\begin{aligned} &V_\Gamma \mathfrak{r}_0(x) \equiv 1\,, \quad \int\limits_\Gamma \mathfrak{r}_0(\tau)|d\tau| \equiv 1\,, \\ &V_\Gamma \mathfrak{r}_0(x) = O\left(\log|x|\right) \quad \text{for } \ x \in \Omega^- \quad \text{as } \ |x| \to \infty\,. \end{aligned} \tag{1.33}$$

The solution $\mathfrak{r}_0 \in W^1_\infty(\Gamma)$ *is known as the* Robin *function* (*or the density of the* Robin *potential; see* [Ma1, § 2.2]).

The homogeneous equation

$$-\frac{1}{2}\psi(t) + W_{\Gamma,0}\psi(t) = 0$$

has, due to (1.31), *the solution* $\psi(t) \equiv 1$, *which is a unique linearly independent solution of this equation in* $L_2(\Gamma)$ (*see* [Ma1, § 2.2]).

Lemma 1.11 *The* Riemann–Hilbert *problem*

$$\operatorname{Re} \Psi^{\pm}(t) = g(t)\,, \quad t \in \Gamma, \tag{1.34}$$

has a solution $\Psi \in \mathscr{E}_p(\overline{\Omega^{\pm}}, \rho)$ *for all right-hand sides* $g \in L_p(\Gamma, \rho)$ *(i.e., is surjective under asserted conditions) if and only if:*

$$i.\ \ \frac{1}{p} + \alpha_j \neq \begin{cases} \dfrac{1}{\gamma_j} & for \quad \Omega^+\,, \\ \dfrac{1}{2-\gamma_j} & for \quad \Omega^-\,, \end{cases} \tag{1.35}$$

$$ii.\ \text{the domain has no inward peaks:} \quad \begin{cases} 0 \le \gamma_j < 2 & for \quad \Omega^+\,, \\ 0 < \gamma_j \le 2 & for \quad \Omega^- \end{cases} \tag{1.36}$$

for all $j = 1, \dots, n$.

Moreover, (1.34) *is* Fredholm *if and only if* (1.36) *holds and then the index of the corresponding operator reads*

$$\operatorname{Ind} A = \sum_{\left(\frac{1}{p}+\alpha_j\right)\gamma_j > 1} 1 \quad \text{for} \quad \Omega^+\,,$$

$$\operatorname{Ind} A = \sum_{\left(\frac{1}{p}+\alpha_j\right)(1-\gamma_j) > 1} 1 \quad \text{for} \quad \Omega^-\,.$$

Proof. The proof will be given in § 5.2.

1.4 Reduction to boundary integral equations (the indirect method)

Theorem 1.12 *Let conditions* (1.35) *and* (1.36) *hold for the complementary domain* $\Omega^{\mp}$. *A harmonic function* $u \in e_p(\overline{\Omega^{\pm}}, \rho)$ *solves the* DIRICHLET *problem* (1.6), (1.7) *if and only if*

$$u(x) = \chi_-(x) g_0 + W_\Gamma \varphi_\pm(x)\,, \qquad x \in \Omega^\pm\,, \tag{1.37}$$

where

$$g_0 := \int_\Gamma g(\tau)\mathfrak{r}_0(\tau)|d\tau|\,, \tag{1.38}$$

$\mathfrak{r}_0(\tau)$ *is the* ROBIN *function* (*see Remark* 1.10) *and* $\varphi_\pm = \operatorname{Re}\varphi_\pm \in L_p(\Gamma, \rho)$ *is some real valued solution of the corresponding boundary integral equation* (*written separately for the domains* Ω^+ *and* Ω^-, *respectively*)

$$A_+\varphi_+(t) := \frac{1}{2}\varphi_+(t) + W_{\Gamma,0}\varphi_+(t) = g(t)\,, \quad t \in \Gamma\,, \tag{1.39}$$

$$A_-\varphi_-(t) := -\frac{1}{2}\varphi_-(t) + W_{\Gamma,0}\varphi_-(t) = g(t) - g_0\,, \quad t \in \Gamma\,. \tag{1.40}$$

Proof. Easy to ascertain that formulae (1.17) and (1.19) hold for the corresponding potential operators as well

$$\begin{aligned} W_\Gamma\varphi(z) &= \frac{1}{2}\left(C_\Gamma + \mathscr{V}C_\Gamma\mathscr{V}\right)\varphi(z) = \operatorname{Re}\left[C_\Gamma \operatorname{Re}\varphi(z)\right] + i\operatorname{Re}\left[C_\Gamma \operatorname{Im}\varphi(z)\right], \\ \partial_z V_\Gamma\varphi(z) &= \frac{i}{2}\left(C_\Gamma - \mathscr{V}C_\Gamma\mathscr{V}\right)\varphi(z) = -\operatorname{Im}\left(C_\Gamma \operatorname{Re}\varphi\right)(z) + i\operatorname{Im}\left(C_\Gamma \operatorname{Im}\varphi\right)(z) \\ &= \operatorname{Re}\left(C_\Gamma i \operatorname{Re}\varphi\right)(z) - i\operatorname{Re}\left(C_\Gamma i \operatorname{Im}\varphi\right)(z),, \quad z \in \Omega^\pm \end{aligned} \tag{1.41}$$

(see (1.3)).

Conditions (1.35), (1.36) provide representation of a solution $u \in e_p(\overline{\Omega^{\pm}}, \rho)$, by the real part of the CAUCHY integral with a real valued density

$$u(x) = \chi_-(x) g_0 + \operatorname{Re}\left[C_\Gamma\varphi_\pm(x)\right]\,, \quad \varphi \in L_p(\overline{\Omega^{\pm}}, \rho)\,, \quad x \in \Omega^\pm$$

(see Lemmata 1.1, 1.13 and 1.11) and, due to (1.41) the latter can be rewritten in the form (1.37).

Passing to the limit $x \to t \in \Gamma$, $x \in \Omega^\pm$ in the representation formula (1.37), applying the appropriate PLEMELJI formulae (1.25) and inserting $u^\pm(t) = g(t)$ we get equations (1.39) for the density φ_+ and (1.40) for the density φ_-, respectively.

The constant $u(\infty)$ in (1.37) is chosen in the form (1.38) to justify the orthogonality condition

$$\int_\Gamma [g(\tau) - g_0]\mathfrak{r}_0(\tau)|d\tau| = \int_\Gamma g(\tau)\mathfrak{r}_0(\tau)|d\tau| - g_0 = 0 \tag{1.42}$$

(see (1.33)) which is necessary and sufficient for the existence of the solution of equation (1.40) provided the equation is FREDHOLM (see § 1.6, Theorem 1.23).

Vice versa, let $\varphi_+, \varphi_- + c_0 \in L_p(\Gamma, \rho)$, c_0 =const, be solutions of (1.39), (1.40), respectively (we remind, that homogeneous equation (1.40) has constants as solutions; see Remark 1.10); let $u(\infty) = g_0$ =const be defined by (1.38). $u(x)$ in (1.37) solves equation (1.6); passing to the limit $x \to t \in \Gamma$, $x \in \Omega^{\pm}$ and invoking the appropriate PLEMELJI formulae (1.25) due to equalities (1.39), (1.40) we get

$$u^+(t) = \frac{1}{2}\varphi_+(t) + W_{\Gamma,0}\varphi_+ = g(t)\,, \quad t \in \Gamma\,,$$

$$u^-(t) = \chi_-(t)g_0 - [(W_\Gamma\varphi_- + c_0)(x)]^- = \chi_-(t)g_0 - [(W_\Gamma\varphi_-)(x)]^-$$

$$= \chi_-(t)g_0 - \frac{1}{2}\varphi_-(t) + W_{\Gamma,0}\varphi_- = g(t)\,, \quad t \in \Gamma$$

since $(W_\Gamma c_0)(x) \equiv 0$ for $x \in \Omega^-$ (see Remark 1.10) and the boundary condition (1.7) holds. ∎

Let us note that representation (1.37) (and, later, a similar one (1.43)) can not be used if inward peak is present. Namely, there holds the following.

Lemma 1.13 *The function $u \in e_p(\overline{\Omega^{\pm}}, \rho)$ ($u \in w_p^s(\overline{\Omega^{\pm}})$) can be represented by the double layer potential* (1.37) *with a density $\varphi \in L_p(\Gamma, \rho)$ (in $W_p^s(\Gamma)$) if and only if the Riemann–Hilbert problem for the complementary domain $\Omega^{\mp}$* (1.34) *is surjective* (*see Lemma* 1.11).

Proof. The proof follows from Lemma 1.11. In fact, let $\Phi(z) = u(z) + iv(z)$, $\Phi \in \mathscr{E}_p(\overline{\Omega^{\pm}}, \rho)$ ($\Phi \in \mathscr{W}_p^s(\overline{\Omega^{\pm}}, \rho)$) be the analytic function in the same domain $\Omega^{\pm}$. Since, due to (1.41), $W_\Gamma = \frac{1}{2}(C_\Gamma + \mathscr{V}C_\Gamma\mathscr{V})$, representation (1.37) follows if the representation of the analytic function $\Phi(z)$ by the Cauchy integral (1.3) with a pure real $\varphi = \operatorname{Re}\varphi$ density in $L_p(\Gamma, \rho)$ (in $W_p^s(\Gamma)$) holds.

Vice versa, let $\varphi = \operatorname{Re}\varphi$ and $u = W_\Gamma\varphi = \operatorname{Re} C_\Gamma\varphi$; since $\Phi = u + iv$ is defined by $u(z)$ uniquelly modulo a pure imaginary aditive constant ic_0, we find $\Phi(z) = ic_0 + C_\Gamma\varphi(z)$ (cf. (1.3)) with the same density $\varphi = \operatorname{Re}\varphi$. ∎

Theorem 1.14 *Let conditions* (1.35) *and* (1.36) *hold for the complementary domain $\Omega^{\mp}$. A harmonic function $u \in e_p(\overline{\Omega^{\pm}}, \rho)$ solves the* NEUMANN *problem* (1.6), (1.8) *if and only if*

$$u(x) = c_0 + V_\Gamma\psi_\pm(x)\,, \quad x \in \Omega^{\pm}\,,$$
$$\int_\Gamma \psi_-(\tau)|d\tau| = 0\,, \tag{1.43}$$

where $\psi_\pm \in w_p^{-1}(\Gamma,\rho)$ are solutions of equations (written separately for the domains Ω^+ and Ω^-, respectively)

$$B_+\psi_+(t) := -\frac{1}{2}\psi_+(t) + W^*_{\Gamma,0}\psi_+(t) = f(t)\,, \quad t \in \Gamma\,, \tag{1.44}$$

$$B_-\psi_-(t) := \frac{1}{2}\psi_-(t) + W^*_{\Gamma,0}\psi_-(t) = f(t)\,, \quad t \in \Gamma\,, \tag{1.45}$$

and c_0 is arbitrary constant.

Proof. Since solution belongs to the Smirnov space $u \in e_p(\overline{\Omega^\pm},\rho)$ and conditions of Lemmata 1.1, 1.11 hold, we have the following representation

$$u(x) = c_0 + \text{ Im } \left[C_\Gamma \psi^0_\pm(x)\right]\,, \quad \text{Im } \psi^0_\pm = 0\,, \quad \psi^0_\pm \in L_p(\Gamma,\rho)\,, \quad x \in \Omega^\pm$$

(see (1.3)). Due to (1.41) the latter can be rewritten in the form (1.43)

$$u = \text{ Im } C_\Gamma\psi^0_\pm = i\partial_z V_\Gamma \psi^0_\pm = V_\Gamma[\partial_\tau \psi^1_\pm] = V_\Gamma \psi_\pm\,, \quad \psi_\pm := \partial_t \psi^1_\pm = i\partial_t \psi^0_\pm$$

and $\psi_\pm \in w_p^{-1}(\Gamma,\rho)$ since $\psi^0_\pm \in e_p(\Gamma,\rho)$.

Applying the normal derivative $\partial_{\vec\nu(x)}$ to the representation (1.43), passing to the limit $x \to t \in \Gamma$, $x \in \Omega^\pm$ with the help of appropriate Plemelji formulae (1.25) and inserting $u^\pm = g$ we get equations (1.44) for the density $\psi_+(t)$ and (1.45) for the density ψ_-, respectively.

The second condition in (1.43) provides $u(x) = c_0 + o(1)$ for $x \in \Omega^-$ as $|x| \to \infty$ (see (1.26), (1.28)).

Vice versa, let $\psi_\pm \in w_p^{-1}(\Gamma,\rho)$ be solutions of (1.44), (1.45). Then $u(x)$ in (1.43) solves equation (1.6) and has the asymptotic $u(x) = c_0 + o(1)$ as $|x| \to \infty$. Applying the normal derivative $\partial_{\vec\nu(x)}$, passing to the limit $x \to t \in \Gamma$, $x \in \Omega^\pm$ and invoking the appropriate Plemelji formulae (1.25) due to equalities (1.44), (1.45) we get

$$(\partial_{\vec\nu} u)^\pm(t) = \mp\frac{1}{2}\psi_\pm(t) + W^*_{\Gamma,0}\psi_\pm = f(t)\,, \quad t \in \Gamma,$$

and the boundary condition (1.8) holds as well. ∎

Lemma 1.15 *The homogeneous* Dirichlet *BVP* (1.6), (1.7) *with $g = 0$ and $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$ has a unique solution.*

The homogeneous Neumann *BVP* (1.6), (1.8) *with $f = 0$ and $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$ has only a constant solution $u(x) \equiv$ const .*

Proof. The proof is based on the Green formula (1.13) and is standard. In fact, if $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$ then on the boundary $u^\pm \in W_2^{\frac{1}{2}}(\Gamma)$. Due to Theorem 1.8 this yields $u \in W_2^1(\overline{\Omega^\pm})$ and $\partial_{x_j} u \in L_2(\overline{\Omega^\pm})$, $\Delta u \in W_2^{-1}(\overline{\Omega^\pm})$, $j = 1, 2$.

Now if $\Delta u(x) = 0$ in $\Omega^{\pm}$ and $u^{\pm}(t) = 0$ on Γ (see (1.6), (1.7)) by assuming $v(x) = u(x)$ in (1.13) we get

$$\sum_{j=1}^{2} |\partial_j u(x)|^2 \equiv 0 \quad \text{for} \quad x \in \Omega^{\pm}\,.$$

Therefore $u(x)$ =const on entire domain and since $u(t) = 0$ on the boundary, $u(x) = 0$ everywhere.

For the NEUMANN BVP (1.6), (1.8) the proof is similar. ∎

1.5 Reduction to Cauchy singular integral equations on the circumference

In the present subsection we reduce the DIRICHLET (1.6), (1.7) and the NEUMANN (1.6), (1.7) BVPs to RIEMANN–HILBERT BVPs for analytic functions on the unit circumference Γ_1 or, what is equivalent, to CAUCHY singular integral equations (SIEs) on Γ_1. Theorems on the FREDHOLM and the solvability properties for the obtained SIEs will be formulated in § 1.6.

The method goes back to N.MUSKHELISHVILI (see [Mu1, Ch. III]) and I.Vekua [Ve1]; they investigated BVPs in HÖLDER spaces when domain has smooth boundary (see [Mu1, §§ 41,43,75]) and for domains with finite number of cuts (see [Mu1, § 109]). In [Kh1] B.Khvedelidze treated similar problems in the LEBESGUE spaces and in [KKP1, Ch. IV] the method was applied to the same BVPs on domains with angular points and cusps in the SMIRNOV–LEBESGUE space $e_p(\overline{\Omega^{\pm}})$ without weight. For the weighted space see [Me1].

Let $\Omega^{\pm}$, $t_1 \dots, t_j \in \Gamma = \partial\Omega^{\pm}$ be as in § 1.1 and

$$\omega \;:\; \mathscr{D}_1 \longrightarrow \Omega^{\pm}\,, \quad \omega(\zeta_j) = t_j\,, \;\; j = 1, \dots, n, \tag{1.46}$$

be a conformal mapping of the unit disk

$$\mathscr{D}_1 = \mathscr{D}_1^{+} := \{z \in \mathbb{C} \;:\; |z| < 1\}$$

onto the domain $\Omega^{\pm}$ ($\omega(0) = 0, \omega'(0) = 1$ for the domain Ω^{+} and $\omega(0) = \infty, \omega'(0) = 1$ for the domain Ω^{-}; see § 5 for further details). By $\zeta(x)$ we denote the inverse mapping

$$\zeta \;:\; \Omega^{\pm} \longrightarrow \mathscr{D}_1\,, \quad \zeta(\omega(z)) \equiv z\,, \;\; \omega(\zeta(x)) \equiv x\,. \tag{1.47}$$

Then

$$\begin{gathered}\omega'(\zeta(z)) = [\zeta'(z)]^{-1}\,,\\ \zeta(0) = 0, \quad \zeta'(0) = 1 \;\text{ for }\; \Omega^{+}\,,\\ \zeta(\infty) = 0, \quad \zeta'(\infty) = 1 \;\text{ for }\; \Omega^{-}\,.\end{gathered}$$

Let $\mathscr{D}_1^- := \{z \in \mathbb{C} \ : \ |z| > 1\}$ be the domain outer to the unit disk $\mathscr{D}_1 = \mathscr{D}_1^+$ and

$$\rho_0(z) := \prod_{j=1}^{n} (z - t_j)^{\alpha_j} \quad \text{for } \ z \in \Omega^+ \subset \mathbb{C} \tag{1.48}$$

denote the analytic function in the domain Ω^+, which is the extension of the weight function; namely, $\rho_0(x)$ is analytic in the complex plane $\mathbb{C}$ cut along some curves connecting knots $t_1, \ldots, t_n \in \partial\Omega^\pm$ with infinity and do not crossing the domain Ω^+.

Theorem 1.16 *A harmonic function $u \in e_p(\overline{\Omega^\pm}, \rho)$ solves the* Dirichlet *problem* (1.6), (1.7) *if and only if*

$$u(x) = \text{Re } \left[\frac{[\zeta'(x)]^{\frac{1}{p}}}{2\pi\rho_0(x)} \left\{ \int\limits_{|\tau|=1} \frac{\varphi(\tau)d\tau}{\tau - \zeta(x)} - \frac{i}{2} \int\limits_{-\pi}^{\pi} \varphi(e^{i\vartheta})d\vartheta \right\} \right] \tag{1.49}$$

for $x \in \Omega^\pm$,. where $\zeta(x)$ is the conformal mapping from (1.47). *$\varphi = \text{Re } \varphi \in L_p(\Gamma_1)$ in* (1.49) *is a real-valued solutions of the following singular integral equation on the unit circumference*

$$\begin{gathered} A\varphi(\zeta) := P^+_{\Gamma_1}\varphi(\zeta) + G(\zeta)P^-_{\Gamma_1}\varphi(\zeta) + \frac{G(\zeta) - 1}{2} K\varphi = g_0(\zeta)\,, \quad \zeta \in \Gamma_1\,, \\ K\varphi := \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi(e^{i\vartheta})d\vartheta\,, \quad P^\pm_{\Gamma_1} := \frac{1}{2}(I \pm S_{\Gamma_1})\,, \end{gathered} \tag{1.50}$$

where the coefficient $G \in PC(\Gamma_1)$ (see § 5.2) and the right-hand side $g_0 \in L_p(\Gamma_1)$ are defined as follows:

$$\begin{gathered} G(\zeta) := -\frac{\rho_0(\omega(\zeta))}{\overline{\rho_0(\omega(\zeta))}} \left[\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}} \right]^{\frac{1}{p}}\,, \\ g_0(\zeta) := 2\rho_0(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}} g(\omega(\zeta))\,, \quad \zeta \in \Gamma_1\,. \end{gathered} \tag{1.51}$$

The solution has the following asymptotic at infinity

$$u(\infty) = \text{Re } \left[(2\pi)^{-1} \prod_{j=1}^{n} (-t)^{\alpha_j} \int\limits_{-\pi}^{\pi} \varphi(e^{i\vartheta})d\vartheta \right]. \tag{1.52}$$

Proof. The Dirichlet problem (1.6), (1.7) can be written as follows

$$\begin{gathered} \text{Re } [\Psi^\pm(t)] = g(t)\,, \quad t \in \Gamma_1\,, \\ u(x) = \text{Re } \Psi(x)\,, \quad \Psi \in \mathscr{E}_p(\overline{\Omega^\pm}, \rho_0)\,, \quad x \in \Omega^\pm\,. \end{gathered} \tag{1.53}$$

Then for the analytic function

$$\Phi(z) := \begin{cases} \rho_0(\omega(z))[\omega'(z)]^{\frac{1}{p}}\Psi(\omega(z)) & \text{for} \quad |z|<1\,, \\ \overline{\rho_0\left(\omega\left(\frac{1}{\overline{z}}\right)\right)}\left[\overline{\omega'\left(\frac{1}{\overline{z}}\right)}\right]^{\frac{1}{p}}\overline{\Psi\left(\omega\left(\frac{1}{\overline{z}}\right)\right)} & \text{for} \quad |z|>1 \end{cases} \tag{1.54}$$

(see [Mu1, § 39] and [KKP1, Ch. IV, § 1]), where $\rho_0(x)$ is defined in (1.48), boundary condition (1.53) acquires the form

$$\text{Re}\,[\Psi^{\pm}(\omega(\zeta))] = \frac{1}{2}\left[\frac{\Phi^+(\zeta)}{\rho_0(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}}} + \frac{\Phi^-(\zeta)}{\overline{\rho_0(\omega(\zeta))}[\overline{\omega'(\zeta)}]^{\frac{1}{p}}}\right] = g(\omega(\zeta))\,,$$

which can also be written as follows

$$\Phi^+(\zeta) - G(\zeta)\Phi^-(\zeta) = g_0(\zeta)\,, \quad \zeta\in\Gamma_1\,, \tag{1.55}$$

with $G(\zeta)$ and $g_0(\zeta)$ defined in (1.51). Since $\Phi\in\mathscr{E}_p(\overline{\mathscr{D}_1})$ it is represented by the Cauchy integral

$$\Phi(z) = -K\varphi + C_{\Gamma_1}i\varphi(z) = -\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\varphi(e^{i\vartheta})d\vartheta + \frac{1}{2\pi}\int\limits_{|\tau|=1}\frac{\varphi(\tau)d\tau}{\tau - z} \tag{1.56}$$

for all $|z|\neq 1$ with a pure imaginary density $i\varphi$, $\varphi\in L_p(\Gamma_1)$. If we apply the Plemelji formulae for the Cauchy integral (see (1.25)), we get

$$\Phi^{\pm}(\zeta) = -K\varphi + \frac{i}{2}\left[\pm\varphi(\zeta) + S_{\Gamma_1}\varphi(\zeta)\right] = -K\varphi \pm iP^{\pm}_{\Gamma_1}\varphi(\zeta)$$

for $\zeta\in\Gamma_1$ and inserting this into (1.55) we get equation (1.50) for the density $\varphi\in L_p(\Gamma_1)$.

Let us remind that we need only the real-valued solution $\varphi = \text{Re}\,\varphi$ of (1.50). To this end let us check that if $\psi\in L_p(\Gamma_1)$ is a solution, than $\overline{\psi}$ is a solution as well. In fact, applying the relations

$$\overline{\zeta} = \frac{1}{\zeta}\,,\ \ \overline{\tau} = \frac{1}{\tau}\,,\ \ d\overline{\tau} = \frac{d\tau}{\tau^2}\,,\ \ \frac{d\tau}{\tau} = id\vartheta \ \text{ for } \ \tau = e^{i\vartheta},\ |\zeta|=1,\ \ -\pi<\vartheta<\pi$$

we find that

$$\begin{aligned} &\overline{G(\zeta)} = G^{-1}(\zeta)\,, \quad \overline{g_0(\zeta)} = G^{-1}(\zeta)g_0(\zeta)\,, \quad \overline{g} = g\,, \\ &\overline{P^{\pm}_{\Gamma_1}\psi(\zeta)} = \frac{1}{2}\overline{\psi(\zeta)} \mp \frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{\overline{\psi(\tau)}\overline{d\tau}}{\overline{\tau}-\overline{\zeta}} = \frac{1}{2}\overline{\psi(\zeta)} \mp \frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{\zeta}{\tau}\frac{\overline{\psi(\tau)}d\tau}{\tau-\zeta} \\ &\qquad = P^{\mp}_{\Gamma_1}\overline{\psi}(\zeta) \pm \frac{1}{2\pi i}\int\limits_{|\tau|=1}\overline{\psi(\tau)}\frac{d\tau}{\tau} = P^{\mp}_{\Gamma_1}\overline{\psi}(\zeta) \pm K\overline{\psi}\,. \end{aligned} \tag{1.57}$$

Now, if $\psi \in L_p(\Gamma_1)$ is a solution of equation (1.50), taking the complex conjugate and invoking (1.57) we get the same equality for $\overline{\psi}$:

$$G(\zeta)\overline{A\varphi(\zeta)} := P^+_{\Gamma_1}\overline{\psi}(\zeta) + G(\zeta)P^-_{\Gamma_1}\overline{\psi}(\zeta) + \frac{G(\zeta)-1}{2}K\overline{\psi} = g_0(\zeta)\,, \quad \zeta \in \Gamma_1.$$

Therefore, the real-valued function $\varphi := \text{Re}\ \psi = \frac{1}{2}(\psi + \overline{\psi})$ is a solution we look for.

With a solution $\varphi := \text{Re}\ \varphi$ of (1.50) at hand we find $\Phi(z)$ from (1.56), but the latter might have the following symmetry property

$$\Phi_*(z) := \overline{\Phi\left(\frac{1}{\overline{z}}\right)} = \Phi(z)\,, \quad z \in \Omega^+ \cup \Omega^-\,,$$

as it follows from the definition (1.54). This property is proved similarly to (1.57):

$$\Phi(z) = \overline{\Phi\left(\frac{1}{\overline{z}}\right)} = \frac{i}{2}K\varphi + \frac{1}{2\pi}\int\limits_{|\tau|=1}\frac{\varphi(\tau)\overline{d\tau}}{\overline{\tau}-\frac{1}{z}} = \frac{i}{2}K\varphi + \frac{1}{2\pi}\int\limits_{|\tau|=1}\frac{z}{\tau}\frac{\varphi(\tau)d\tau}{\tau - z}$$
$$= -\frac{i}{2}K\varphi + \frac{1}{2\pi}\int\limits_{|\tau|=1}\frac{\varphi(\tau)d\tau}{\tau - z} = -\frac{i}{2}K\varphi + iC_{\Gamma_1}\varphi(z) = \Phi(z)\,. \qquad (1.58)$$

Inserting $\Phi(z)$ in (1.54) we find first $\Psi(z)$ and afterwards $u = \text{Re}\ \Psi$. The result is written in (1.49).

Vice versa, if $\varphi(\zeta)$ is a solutions of (1.58) we easily ascertain that $\Psi(z)$ found in (1.56) and (1.54) solves BVP (1.53) and $u(x)$ (see (1.49)) solves BVP (1.6), (1.7).

The asymptotic (1.52) result from (1.47)–(1.49) and from the following asymptotic of the weight function

$$\overline{\rho_0\left(\omega\left(\frac{1}{\overline{z}}\right)\right)} = \prod_{j=1}^{n}(-t)^{-\alpha_j} + O(|z|^{-1}) \quad \text{as} \quad |z| \to \infty\,. \qquad \blacksquare$$

Theorem 1.17 *A harmonic function $u \in w^1_p(\overline{\Omega^{\pm}}, \rho)$ solves the* Neumann *problem* (1.6), (1.8) *if and only if*

$$u(x) = c_0 + \text{Re}\ \left\{\int\limits_{x_0}^{x}\frac{[\zeta'(y)]^{\frac{1}{p}}}{2\pi\rho_0(y)}\int\limits_{|\tau|=1}\frac{\psi^-(\tau)d\tau}{\tau-\zeta(y)}dy\right\}, \qquad (1.59)$$

$$u(x) = c_0 + \text{Re}\ \left\{\int\limits_{x_0}^{x}\frac{[\zeta'(y)]^{\frac{1}{p}}}{2\pi\rho_0(y)}\left[\int\limits_{|\tau|=1}\frac{\psi^+(\tau)d\tau}{\tau-\zeta(y)} - \frac{i}{2}\int\limits_{-\pi}^{\pi}\psi^+(e^{i\vartheta})d\vartheta\right]dy\right\}$$

for $x \in \Omega^-$ and $x \in \Omega^+$, respectively; $x_0 \in \Omega^\pm$ is some fixed point, $c_0 \in \mathbb{R}$ is a real constant and $\zeta(x)$ is the conformal mapping from (1.46). *$\psi^\pm = \operatorname{Re}\,\psi^\pm \in L_p(\Gamma_1)$ are real-valued solutions of the following singular integral equations on the unit circumference*

$$B^-\psi^-(\zeta) := \begin{cases} P^+_{\Gamma_1}\psi^-(\zeta) + F(\zeta)P^-_{\Gamma_1}\psi^-(\zeta) = f_0(\zeta)\,, \quad \zeta \in \Gamma_1\,, \\ K\psi^- = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \psi^-(e^{i\vartheta})d\vartheta = 0\,, \end{cases} \tag{1.60}$$

$$B^+\psi^+(\zeta) := P^+_{\Gamma_1}\psi^+(\zeta) + F(\zeta)P^-_{\Gamma_1}\psi^+(\zeta) + \frac{F(\zeta)-1}{2}K\psi^+ = f_0(\zeta)\,.$$

The coefficient $F \in PC(\Gamma_1)$ (see § 5.2*) and the right-hand side $f_0 \in L_p(\Gamma_1)$ are defined as follows:*

$$\begin{aligned} F(\zeta) &:= \frac{\rho_0(\omega(\zeta))}{\overline{\rho_0(\omega(\zeta))}}\left[\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\right]^{\frac{1}{p}-1}\,, \\ f_0(\zeta) &:= 2\rho_0(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}} f(\omega(\zeta))\,, \quad \zeta \in \Gamma_1\,. \end{aligned} \tag{1.61}$$

Proof. The Neumann problem (1.6), (1.8) can be written as follows (see [Mu1, § § 74,75])

$$\begin{aligned} \operatorname{Re}\,\left[e^{i\vartheta_t}(\Psi')^\pm(t)\right] &= f(t)\,, \quad t \in \Gamma_1\,, \\ u(x) = \operatorname{Re}\,\Psi(x)\,, &\quad \Psi \in \mathscr{W}^1_p(\overline{\Omega^\pm}, \rho_0) \quad x \in \Omega^\pm\,, \end{aligned} \tag{1.62}$$

where $\vartheta(\zeta) = \vartheta_t$ denotes the inclination of the outer unit normal vector $\vec{\nu}(t)$ to the abscissa axes at $t = \omega(\zeta) \in \Gamma \setminus \{t_1, \dots, t_n\}$ (see Fig. 1). In fact, since

$$\begin{aligned} &\Psi = u + iv \in \mathscr{W}^1_p(\overline{\Omega^\pm}, \rho_0)\,, \quad \Psi' := \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \in \mathscr{E}_p(\overline{\Omega^\pm}, \rho_0)\,, \\ &\partial_{\vec{\nu}(t)}u(t) = \cos\vartheta_t\frac{\partial u}{\partial x} + \sin\vartheta_t\frac{\partial u}{\partial y}\,, \quad \cos\vartheta_t + i\sin\vartheta_t = e^{i\vartheta_t} \end{aligned} \tag{1.63}$$

(see (1.21)), we get

$$\operatorname{Re}\,\left[e^{i\vartheta_t}(\Psi')^\pm(t)\right] = \cos\vartheta_t\left[\frac{\partial u}{\partial x}\right]^\pm + \sin\vartheta_t\left[\frac{\partial u}{\partial y}\right]^\pm = (\partial_{\vec{\nu}(t)}u)^\pm(t)$$

and (1.62) follows.

Similarly to (1.54) (see also [Mu1, § 39] and [KKP1, Ch. IV, § 2]) for the analytic function

$$\Phi(z) := \begin{cases} \rho_0(\omega(z))[\omega'(z)]^{\frac{1}{p}}\Psi'(\omega(z)) & \text{for} \quad |z| < 1, \\ \dfrac{}{\overline{\rho_0\left(\omega\left(\dfrac{1}{\overline{z}}\right)\right)}}\left[\overline{\omega'\left(\dfrac{1}{\overline{z}}\right)}\right]^{\frac{1}{p}}\dfrac{}{\overline{\Psi'\left(\omega\left(\dfrac{1}{\overline{z}}\right)\right)}} & \text{for} \quad |z| > 1, \end{cases} \tag{1.64}$$

which belongs to the space $\mathscr{E}_p(\overline{\mathscr{D}_1})$, we get the following BVP:

$$\Phi^+(\zeta) - F(\zeta)\Phi^-(\zeta) = f_0(\zeta)\,, \quad \zeta \in \Gamma_1\,, \tag{1.65}$$

where $f_0(\zeta)$ is defined in (1.61) and

$$F(\zeta) := -e^{-2\vartheta(\zeta)i}\frac{\rho_0(\omega(\zeta))}{\overline{\rho_0(\omega(\zeta))}}\left[\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\right]^{\frac{1}{p}} = e^{-2\alpha(\zeta)i}\frac{\rho_0(\omega(\zeta))}{\overline{\rho_0(\omega(\zeta))}}\left[\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\right]^{\frac{1}{p}}$$

with $\alpha(\zeta) = \vartheta(\zeta) + \frac{\pi}{2}$ denoting the inclination of the tangent to Γ vector to the abscissa axes at $t = \omega(\zeta) \in \Gamma \setminus \{t_1, \dots, t_n\}$ (see Fig. 1). Let us recall that $\omega'(z)$ has an angular (i.e., non-tangential) boundary limits $\overset{\wedge}{\to}$ for almost all $\zeta \in \Gamma_1$ and

$$\omega'(\zeta) = e^{\alpha(\zeta)i}|\omega'(\zeta)| \tag{1.66}$$

(see, e.g., [Go1, p.p. 405–411] and [Ks1, Ch. I, II]). Therefore

$$e^{-2\alpha(\zeta)i} = \left(\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\right)^{-1}$$

and by inserting this into the foregoing formula we get $F(\zeta)$ as written in (1.61). In (1.64), (1.65) $\Phi \in \mathscr{E}_p(\overline{\mathscr{D}_1})$ and, therefore, it can be represented by the Cauchy integral (cf. (1.56))

$$\Phi(z) := -K\psi^+ + i(C_{\Gamma_1}\psi)^+(x) = -\frac{i}{2\pi}\int\limits_{-\pi}^{\pi}\psi^+(e^{i\vartheta})d\vartheta + \frac{1}{2\pi}\int\limits_{|\tau|=1}\frac{\psi^+(\tau)d\tau}{\tau - z}\,. \tag{1.67}$$

For the domain Ω^- we should require in addition (see the condition in (1.60))

$$K\psi^- = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\psi^-(e^{i\vartheta})d\vartheta = 0\,.$$

To justify the latter we remind that $\Psi \in \mathscr{W}_p^1(\overline{\Omega^\pm}, \rho_0)$ and, due to representation (1.3) the derivative should vanish at the infinity $\Psi'(\infty) = 0$; therefore (see (1.64), (1.67))

$$\int\limits_{-\pi}^{\pi}\psi^-(e^{i\vartheta})d\vartheta = 2\pi\Phi(0) = 2\pi\rho_0(\omega(0))[\omega'(0)]^{\frac{1}{p}}\Psi'(\omega(0)) = 0$$

because $\omega(0) = \infty$ (see (1.46)–(1.47)).

Let us note that $\Phi(z)$ in (1.67) would have the symmetry property $\Phi_*(z) = \Phi(z)$(cf. (1.58)).

Since we need only real-valued solutions $\psi^{\pm} = \operatorname{Re}\psi^{\pm}$ of (1.60), we check, based on the properties similar to (1.57) that along with $\psi^{\pm}$ equations (1.60) have solutions $\overline{\psi^{\pm}}$. Therefore the real-valued solutions $\psi^{\pm} = \operatorname{Re}\psi^{\pm} = \frac{1}{2}(\psi^{\pm} + \overline{\psi^{\pm}})$ are those we look for.

Vice versa, if $\psi^{\pm} = \operatorname{Re}\psi^{\pm}$ are real-valued solutions of (1.59), (1.60), we find easily that $\Phi(z)$, defined in (1.64) solves BVP (1.65), which implies that $u(x)$ in (1.59) solves BVP (1.6), (1.8). ∎

Remark 1.18 *Equivalent reduction of the* Dirichlet (1.6), (1.7) *and the* Neumann (1.6), (1.8) *BVPs to BIEs* (1.50) *and* (1.60) *can be carried out in the spaces of continuous* $C(\overline{\Omega^{\mp}}, \rho)$ *and piecewise-continuous* $PC(\overline{\Omega^{\mp}}, \rho)$ *functions. Transition to the unit disk is clear and smooth, but is senseless because the* Cauchy *SIO is unbounded in these spaces, even on the unit circumference.*

Solvability results we possess e.g. for the Hölder *spaces with weight* $h^0_{\mu}(\overline{\mathscr{D}_1}, \rho)$ *on the unit disk* (*see* § 4), *but transformation of the* Riemann–Hilbert *problem for* Ω^+ *to the unit disk* (*similar to* (1.53)–(1.59)) *is not implemented so far.*

1.6 Reduction to boundary pseudo-differential equations (the direct method)

Theorem 1.19 *Let* $\mathbb{X}^s(\Gamma, \rho)$ *stand for one of the following spaces:* $W^s_p(\Gamma, \rho)$ *with* $0 \le s \le 1$ *or for* $H^0_{\mu+1}(\Gamma, \rho)$, $PC^1(\Gamma, \rho)$. $x^s(\overline{\Omega^{\pm}}, \rho)$ *is used for the corresponding* Smirnov *space of harmonic functions.* $\rho(t)$ *is defined in* (1.2) *and inequalities* (1.4) *hold.*

A harmonic function $u \in x^s(\overline{\Omega^{\pm}}, \rho)$ *solves the* Dirichlet *problem* (1.6), (1.7) *if and only if*

$$u(x) = \chi_-(x)[W_\Gamma g(0) - V_\Gamma \varphi_-(0)] \pm W_\Gamma g(x) \mp V_\Gamma \varphi_\pm(x)\,, \tag{1.68}$$

where $\varphi \in \mathbb{X}^{s-1}(\Gamma, \rho)$ *is a solution of the following pseudo-differential equation of order* -1 (*written separately for the domains* Ω^+ *and* Ω^-, *respectively*)

$$V_{\Gamma,-1}\varphi_+(t) := \frac{1}{2\pi}\int\limits_\Gamma \log|t-\tau|\varphi_+(\tau)|d\tau| = g_+(t)\,, \quad t \in \Gamma, \tag{1.69}$$

$$\begin{cases} V_{\Gamma,-1}\varphi_-(t) := \dfrac{1}{2\pi}\displaystyle\int\limits_\Gamma \log\left|\frac{t-\tau}{\tau}\right|\varphi_-(\tau)|d\tau| = g_-(t)\,, \quad t \in \Gamma\,, \\ \displaystyle\int\limits_\Gamma \varphi_-(\tau)|d\tau| = 0 \end{cases} \tag{1.70}$$

and

$$g_+(t) := -\frac{1}{2}g(t) - \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(\tau)} \log|t-\tau| g(\tau)|d\tau|\,, \quad t\in\Gamma\,,$$

$$g_-(t) := \frac{1}{2}g(t) - \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(\tau)} \log\left|\frac{t-\tau}{\tau}\right| g(\tau)|d\tau|\,, \quad t\in\Gamma\,.$$

Proof. Solution $u(x)$ of the the DIRICHLET problem (1.6), (1.7) has the form (1.68) (see (1.15) and (1.30)). Taking the trace on Γ from $\Omega^\pm$, invoking the PLEMELJI formulae (1.25), inserting $u^\pm(t) = g(t)$ from (1.7) and choosing the function $\varphi_\pm(t) := (\partial_{\vec\nu(t)}u)^\pm(t)$ for an unknown, we get equations (1.69) and the first equation in (1.70), because

$$W_\Gamma\varphi(x) - W_\Gamma\varphi(0) = \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(\tau)} \log\left|\frac{x-\tau}{\tau}\right| \varphi(\tau)|d\tau|\,,$$

$$V_\Gamma\varphi(x) - V_\Gamma\varphi(0) = \frac{1}{2\pi}\int\limits_\Gamma \log\left|\frac{x-\tau}{\tau}\right| \varphi(\tau)|d\tau|\,, \quad x\in\Omega^\pm\,.$$

The second equation in system (1.70) is necessary for boundedness of solution (1.68) at infinity (see (1.27), (1.28)).

Vice versa, if $u(x)$ is written in the form (1.68), it is obviously harmonic and $u\in x^s(\overline{\Omega^\pm},\rho)$. In fact, $\varphi_\pm = \operatorname{Re}\varphi_\pm \in \mathbb{X}^{s-1}(\Gamma,\rho)$ and, due to (1.41),

$$u(x) = \chi_-(x)u(\infty) \pm W_\Gamma g(x) \mp V_\Gamma\varphi_\pm(x) = \chi_-(x)u(\infty) + \operatorname{Re} C_\Gamma[\pm g \mp i\varphi_\pm](x).$$

Further, $u(x)$ has finite limit $u(\infty) = W_\Gamma g(0) - V_\Gamma\varphi_-(0)$ at infinity and it remains to check the boundary condition (1.7). To this end it suffices to take the trace in (1.68), applying the PLEMELJI formulae (1.25), and remember that equations (1.69) and (1.70) hold. We easily get:

$$\begin{aligned} u^\pm(x) = \chi_-(x)[W_\Gamma g(0) - V_\Gamma\varphi_-(x)] + \frac{1}{2}g(t) \\ \pm W_{\Gamma,0}g(t) \mp V_\Gamma\varphi_\pm(t) = g(t). \end{aligned}$$

∎

Theorem 1.20 *Let $\mathbb{X}^s(\Gamma,\rho)$ stand for one of the following spaces: $W_p^s(\Gamma,\rho)$ with $0\le s\le 1$ or for $H^0_{\mu+1}(\Gamma,\rho)$, $PC^1(\Gamma,\rho)$. $x^s(\overline{\Omega^\pm},\rho)$ is used for the corresponding* SMIRNOV *space of harmonic functions. $\rho(t)$ is defined in* (1.2) *and inequalities* (1.4) *hold.*

A harmonic function $u\in x^s(\overline{\Omega^\pm},\rho)$ solves the NEUMANN *problem* (1.6), (1.8) *if and only if*

$$u(x) = C_0 \pm W_\Gamma\psi_\pm(x) \mp V_\Gamma f(x)\,, \tag{1.71}$$

where C_0 is arbitrary constant, $\psi \in \mathbb{X}^s(\Gamma,\rho)$ is a solution of the following pseudo-differential equation of order $+1$

$$D_{\Gamma,+1}\psi_\pm(t) := \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(t)}\partial_{\vec\nu(\tau)} \log|t-\tau|\psi_\pm(\tau)|d\tau| = f_\pm(t),\ \ t\in\Gamma, \tag{1.72}$$

and

$$f_\pm(t) := \pm\frac{1}{2}f(t) + \frac{1}{2\pi}\int\limits_\Gamma \partial_{\vec\nu(t)} \log|t-\tau|f(\tau)|d\tau|\,, \quad t\in\Gamma\,.$$

For the outer domain problem Ω^- the data $f(t)$ should meet the additional constraint

$$\int\limits_\Gamma f(\tau)|d\tau| = 0\,. \tag{1.73}$$

Proof. Solution $u(x)$ of the the NEUMANN problem (1.6), (1.8) has the form (1.71) (see (1.15)) and to be bounded in the outer domain condition (1.73) should hold (see (1.27), (1.28)). Taking the trace on Γ from $\Omega^\pm$, invoking the PLEMELJI formulae (1.25), inserting $(\partial_{\vec\nu(t)}u)^\pm(t) = f(t)$ from (1.8) and anounceing $\psi_\pm(t) := u^\pm(t)$ as an unknown function, we get equations (1.72).

Vice versa, if $u(x)$ is written in the form (1.71), it is obviously harmonic and $u \in x^s(\overline{\Omega^\pm},\rho)$. In fact, $\psi_\pm = \mathrm{Re}\ \psi_\pm \in \mathbb{X}^s(\Gamma,\rho)$ and, due to (1.41),

$$u(x) = C_0 \pm W_\Gamma\psi_\pm(x) \mp V_\Gamma f(x) = C_0 + \mathrm{Re}\, C_\Gamma[\pm\psi_\pm(x) \mp if](x)\,.$$

Further, $u(x)$ has finite limit $u(\infty) = C_0$ at infinity (see (1.73) and recall (1.27), (1.28)). It remains to check the boundary condition (1.8). To this end it suffices to take the trace in (1.71), applying the PLEMELJI formulae (1.25), and remember that equations (1.72) hold. We easily get:

$$(\partial_{\vec\nu(t)}u)^\pm(x) = \pm D_{\Gamma,+1}\psi_\pm(t) + \frac{1}{2}g(t) \mp V_\Gamma f(t) = f(t)\,. \qquad \blacksquare$$

1.7 Statement of the principal results

In the present subsection we formulate principal results on BIEs (1.39), (1.40), (1.44), (1.45), (1.50), (1.60) (see Theorems 1.26 and 1.29), which we prove later in § 5.3–§ 5.4. We also formulate (and prove) their immediate consequences-solvability results for corresponding BVPs (see Theorems 1.28 and 1.30). Theorems are formulated separately for the case of absence of cusps because in such a case equations can be studied directly and not only the weighted LEBESGUE space $L_p(\Gamma,\rho)$, but in the weighted spaces of continuous, piecewise-continuous and HÖLDER functions. Moreover, equations are FREDHOLM in usual spaces, in contrast to the case of cusps, when we have to introduce special image spaces to make operators FREDHOLM. The approaches to the cases are substantially different (cf. § 5.3 and § 5.4).

Before formulating theorems on solvability of boundary integral equations and boundary value problems let us recall [DNS1, Lemma 19] which will be quoted later and which is useful in establishing additional smoothness properties of solutions to BVPs (e.g., HÖLDER continuity with weight).

A pair of BANACH spaces $\{\mathbb{X}_0, \mathbb{X}_1\}$ embedded in some topological space $\mathbb{E}$ is called an interpolation pair. For such a pair we can introduce the following two spaces $\mathbb{X}_{\min} = \mathbb{X}_0 \cap \mathbb{X}_1$ and $\mathbb{X}_{\max} = \mathbb{X}_0 + \mathbb{X}_1 := \big\{x \in \mathbb{E} : x = x_0 + x_1,\ x_j \in \mathbb{X}_j,\ j = 0, 1\big\}$; $\mathbb{X}_{\min}$ and $\mathbb{X}_{\max}$ become BANACH spaces if they are endowed with the norms

$$\|x|\mathbb{X}_{\min}\| = \max\big\{\|x|\mathbb{X}_0\|,\ \|x|\mathbb{X}_1\|\big\},$$
$$\|x|\mathbb{X}_{\max}\| = \inf\big\{\|x_0|\mathbb{X}_0\| + \|x_1|\mathbb{X}_1\| : x = x_0 + x_1,\ x_j \in \mathbb{X}_j,\ j = 0, 1\big\},$$

respectively.

Besides, we have the continuous embedding

$$\mathbb{X}_{\min} \subset \mathbb{X}_0,\ \mathbb{X}_1 \subset \mathbb{X}_{\max}.$$

For any interpolation pairs $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ the space $\mathscr{L}(\{\mathbb{X}_0\mathbb{X}_1\}, \{\mathbb{Y}_0\mathbb{Y}_1\})$ consists of all linear operators from $\mathbb{X}_{\max}$ into $\mathbb{Y}_{\max}$ whose restrictions to $\mathbb{X}_j$ belong to $\mathscr{L}(\mathbb{X}_j, \mathbb{Y}_j)$ $(j = 0, 1)$. The notation $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ is used for the space of all linear bounded operators $A : \mathbb{X} \to \mathbb{Y}$.

Lemma 1.21 (*see* [DNS1, Lemma 19]). *Assume* $\{\mathbb{X}_0, \mathbb{X}_1\}$ *and* $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ *are interpolation pairs and the embedding* $\mathbb{X}_{\min} \subset \mathbb{X}_{\max}$, $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$ *to be dense. Let an operator* $A \in \mathscr{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathscr{L}(\mathbb{X}_1, \mathbb{Y}_1)$ *have a common regularizer:* $R \in \mathscr{L}(\mathbb{Y}_0, \mathbb{X}_0) \cap \mathscr{L}(\mathbb{Y}_1, \mathbb{X}_1)$ *and* $RA - I \in \mathscr{L}(\mathbb{X}_0\mathbb{X}_0) \cap \mathscr{L}(\mathbb{X}_1, \mathbb{X}_1)$ *be compact. Then*

$$A : \mathbb{X}_{\min} \to \mathbb{Y}_{\min}, \quad A : \mathbb{X}_{max} \to \mathbb{Y}_{\max}$$

are FREDHOLM *operators and*

$$\mathrm{Ind}\,{}_{\mathbb{X}_{\min}\to\mathbb{Y}_{\min}}\mathrm{A} = \mathrm{Ind}\,{}_{\mathbb{X}_{\max}\to\mathbb{Y}_{\max}}\mathrm{A} = \mathrm{Ind}\,{}_{\mathbb{X}_\mathrm{j}\to\mathbb{X}_\mathrm{j}}\mathrm{A}, \quad \mathrm{j} = 0, 1.$$

If $y \in \mathbb{Y}_j$, *then any solution* $x \in \mathbb{X}_{\max}$ *of the equation* $Ax = y$ *belongs to* $\mathbb{X}_j$. *In particular,*

$$\mathrm{Ker}\,{}_{\mathbb{X}_{\min}}\mathrm{A} = \mathrm{Ker}\,{}_{\mathbb{X}_\mathrm{j}}\mathrm{A} = \mathrm{Ker}\,{}_{\mathbb{X}_{\max}}\mathrm{A}\,, \quad \mathrm{j} = 0, 1. \qquad \blacksquare$$

Let

$$\begin{aligned} \mathscr{T} &:= \{t_1, \dots t_n\}\,, \quad \mathscr{T}_{pk} := \mathscr{T}_{ow} \cup \mathscr{T}_{iw}\,, \\ \mathscr{T}_{ow} &:= \{t_j \in \mathscr{T} \ :\ \gamma_j = 0\}\,, \quad \mathscr{T}_{iw} := \{t_j \in \mathscr{T} \ :\ \gamma_j = 2\} \end{aligned} \tag{1.74}$$

be the collections of all knots, of all peaks, of all outward and all inward peaks on Γ.

Let us define the following mean value integral on the contour Γ (cf. (1.90))

$$\mathscr{V}_{t_j}\varphi(t) := \int\limits_{\widetilde{t_j t}} \left(\frac{\zeta'(t)}{\zeta'(\tau)}\right)^{\frac{1}{p}} \left(\frac{\zeta(\tau)-\zeta(t_j)}{\zeta(t)-\zeta(t_j)}\right)^{\frac{1}{p}} \frac{\rho(\tau)}{\rho(t)} \frac{\varphi(\tau)\zeta'(\tau)d\tau}{\zeta(\tau)-\zeta(t_j)}\,. \qquad (1.75)$$

Let $t_j \in \mathscr{T}_{pk}$ be a peak and $t \in \Gamma$. The points

$$t = \omega(\zeta) \quad \text{and} \quad t^*_{t_j} := \omega(\overline{\zeta(t)}\zeta_j^2)\,, \quad \omega(\zeta_j) = t_j$$

are the images of equidistant points $|\zeta(t)-\zeta_j| = |\overline{\zeta(t)}\zeta_j^2 - \zeta_j|$ on the unit circumference under the conformal mapping (1.46). Points $t \in \Gamma$ and $t^*_{t_j} \in \Gamma$ are on different sides from the outward peak $t_j \in \widetilde{t^*_{t_j} t}$. Let $\Gamma_{t_j} \subset \Gamma$ be, similarly to $\Gamma_{1\zeta_j} \subset \Gamma_1$, a sufficiently small fixed neighbourhood of $t_j \in \Gamma$ such that $\Gamma_{t_k} \cap \Gamma_{t_j} = \emptyset$ (and, therefore, $t_k \notin \Gamma_{t_j}$) for $k \neq j$. Let $\Gamma_{t_j} = \Gamma^-_{t_j} \cup \Gamma^+_{t_j}$ be the decomposition of the neighbourhood of t_j into the semi-closed left and right neighbourhoods and χ_{t_j} be the characteristic function of Γ_{t_j}. We define the space

$$L_p(\Gamma,\rho,\mathscr{T}_{pk}) := \left\{\varphi \in L_p(\Gamma,\rho) \;:\; \widetilde{\mathscr{V}}_{t_j}\varphi \in L_p(\Gamma,\rho)\,,\;\; t_j \in \mathscr{T}_{pk}\right\}\,, \quad (1.76)$$

$$\widetilde{\mathscr{V}}_{t_j}\varphi := \mathscr{V}_{t_j}\varphi^*_{t_j}\,, \quad \varphi^*_{t_j}(t) := \varepsilon_j\varphi(t) - \varphi(\omega(\overline{\zeta(t)}\zeta_j^2))\,,, \quad \varepsilon_j := e^{\frac{\pi}{p}(\gamma_j-1)i}\,,$$

$$\|\varphi\big|L_p(\Gamma,\rho,\mathscr{T}_{pk})\| := \|\varphi\big|L_p(\Gamma,\rho)\| + \sum_{t_j\in\mathscr{T}_{pk}} \|\widetilde{\mathscr{V}}_{t_j}\chi_{t_j}\varphi\big|L_p(\Gamma^+_{t_j},(t-t_j)^{\alpha_j})\|$$

and [5] $\varepsilon_j = e^{-\frac{\pi}{p}i}$ for $t_j \in \mathscr{T}_{ow}$, $\varepsilon_j = e^{\frac{\pi}{p}i}$ for $t_j \in \mathscr{T}_{iw}$. Similarly is defined the space $L_p(\Gamma,\rho,\mathscr{T}_{ow}) \subset L_p(\Gamma,\rho,\mathscr{T}_{pk})$.

Lemma 1.22 *Let Γ be a piecewise-Ljapunov curve. If $\psi \in L_p(\Gamma,\rho)$ and $\log[\zeta(t)-\zeta(t_j)]\psi^*_{t_j} \in L_p(\Gamma,\rho)$ for all $t_j \in \mathscr{T}_{pk}$, then $\psi \in L_p(\Gamma,\rho,\mathscr{T}_{pk})$.*

Let $a \in L_\infty(\Gamma)$ and

$$a(t) = a(t_j) + O\left(|\log[\zeta(t)-\zeta(t_j)]|^{-1}\right) \qquad (1.77)$$

for all $t_j \in \mathscr{T}_{pk}$ as $t \to t_j$. Then the operators

$$\begin{aligned} aI \quad &: \quad L_p(\Gamma,\rho,\mathscr{T}_{pk}) \longrightarrow L_p(\Gamma,\rho,\mathscr{T}_{pk})\,, \\ [a-a_0(t)]I \quad &: \quad L_p(\Gamma,\rho) \longrightarrow L_p(\Gamma,\rho,\mathscr{T}_{pk}) \end{aligned} \qquad (1.78)$$

are bounded, where $a_0(t) := \sum\limits_{t_j\in\mathscr{T}_{pk}} a(t_j)\chi_j(t)$ and $\chi_j(t)$ denotes the characteristic function of Γ_{t_j}.

[5] Non-equal rights of left and right neighbourhoods and differences for outward and inward peaks in the definition of the space $L_p(\Gamma,\rho,\mathscr{T}_{pk})$ are explained in Remark 5.12.

Proof. The proof is an easy consequence of Lemmata 1.25 and 1.27. ∎

Note, that if Γ has no cusps, $0 < \gamma_j < 2$ for all $j = 11, \dots, n$, than

$$\log[\zeta(t) - \zeta(t_j)] \ \sim \ \log[t - t_j]\,, \quad t \in \Gamma$$

(see Corollary 5.10). For a curve with cusps this is not valid any more.

Theorem 1.23 *Let $\mathscr{T}_{pk} = \emptyset$ and $\mathbb{X}^m(\Gamma, \rho)$ be one of the following spaces $W_p^m(\Gamma, \rho)$, $H^0_{\mu+m}(\Gamma, \rho)$, $C^m(\Gamma, \rho)$ or $PC(\Gamma, \rho)$, $m = 0, 1$.*

Equations (1.39) *and* (1.40) *are* FREDHOLM *in the space $\mathbb{X}^m(\Gamma, \rho)$ if and only if*

$$\beta_j \neq \begin{cases} \gamma_j^0 & \text{if} \quad m = 0\,, \\ 1 - \gamma_j^0 & \text{if} \quad m = 1\,, \end{cases} \qquad \gamma_j^0 := \min\left\{\frac{1}{\gamma_j}, \frac{1}{2 - \gamma_j}\right\}$$

for all $j = 1, \dots, n$, where

$$\beta_j := \begin{cases} \dfrac{1}{p} + \alpha_j & \text{for} \quad \mathbb{X}^m(\Gamma, \rho) = W_p^m(\Gamma, \rho)\,, \\ \alpha_j & \text{for} \quad PC^m(\Gamma, \rho)\,, \quad C(\Gamma, \rho), \\ \alpha_j - \mu_j & \text{for} \quad H^0_{\mu+m}(\Gamma, \rho)\,. \end{cases} \tag{1.79}$$

If $\mathscr{T}_{pk} \neq \emptyset$ or $\beta_j = \gamma_j^0$ when $m = 0$, $\beta_j = 1 - \gamma_j^0$ when $m = 1$, then the operators $A_\pm$ in (1.39), (1.40) *have non-closed images in $W_p^m(\Gamma, \rho)$.*

Equations (1.39) *and* (1.40) *with $\varphi \in L_p(\Gamma, \rho)$, $g \in L_p(\Gamma, \rho, \mathscr{T}_{pk})$ are Fredholm, i.e., the operators*

$$A_\pm \ : \ L_p(\Gamma, \rho) \to L_p(\Gamma, \rho, \mathscr{T}_{pk}) \tag{1.80}$$

are bounded and are Fredholm if and only if $\beta_j \neq \gamma_j^0$ for all $t_j \notin \mathscr{T}_{pk}$; the following formulae hold for the index, kernel and cokernel in the space $\mathbb{X}^m(\Gamma, \rho)$ when $\mathscr{T}_{pk} = \emptyset$ or in the pairs (1.80) *when $\mathscr{T}_{pk} \neq \emptyset$*

$$\operatorname{Ind}_{\mathbb{X}^0(\Gamma,\rho)} \mathrm{A}_\pm = \sum_{\substack{t_j \notin \mathscr{T}_{pk} \\ \beta_j > \gamma_j^0}} 1\,, \qquad \operatorname{Ind}_{\mathbb{X}^1(\Gamma,\rho)} \mathrm{A}_\pm = -\sum_{\substack{t_j \in \mathscr{T} \\ \beta_j < 1 - \gamma_j^0}} 1\,, \tag{1.81}$$

$$\dim_{\mathbb{X}^0(\Gamma,\rho)} \operatorname{Ker} \mathrm{A}_\pm = \varepsilon_\pm + \operatorname{Ind}_{\mathbb{X}^0(\Gamma,\rho)} \mathrm{A}_\pm\,, \quad \dim \operatorname{Coker}_{\mathbb{X}^0(\Gamma,\rho)} \mathrm{A}_\pm = \varepsilon_\pm\,,$$

$$\dim \operatorname{Ker}_{\mathbb{X}^1(\Gamma,\rho)} \mathrm{A}_\pm = \varepsilon_\pm\,, \quad \dim \operatorname{Coker}_{\mathbb{X}^1(\Gamma,\rho)} \mathrm{A}_\pm = \varepsilon_\pm - \operatorname{Ind}_{\mathbb{X}^1(\Gamma\rho)} \mathrm{A}_\pm$$

with $\varepsilon_+ = 0$ and $\varepsilon_- = 1$.

In particular, if

$$0 < \beta_j < \gamma_j^0 \quad \text{for all} \quad t_j \notin \mathscr{T}_{pk}\,, \tag{1.82}$$

then equations (1.39) *and* (1.40) *have solutions for all right-hand sides* $g(t)$ *in* $L_p(\Gamma,\rho,\mathscr{T}_{pk})$ *(in* $C(\Gamma,\rho)$ *and in*[6]) $H^0_\mu(\Gamma,\rho)$ *when* $\mathscr{T}_{pk}=\emptyset$*), while for*

$$\mathscr{T}_{pk}=\emptyset\,,\quad 1-\gamma_j^0<\beta_j<1\quad\text{for all}\quad t_j\in\mathscr{T} \tag{1.83}$$

they have solutions in $W^1_p(\Gamma,\rho)$*, in* $PC^1(\Gamma,\rho)$ *and in* $H^0_{\mu+1}(\Gamma,\rho)$ *for the right hand sides in the same spaces. Equation* (1.39) *has a unique solution in these spaces, while homogeneous equation* (1.40)*,* $g(t)\equiv 0$ *has a single linearly independent solution* $\varphi_-(t)\equiv 1$*.*

Proof. The proof is postponed to § 5.4.

Theorem 1.24 *Let* $\mathscr{T}_{pk}=\emptyset$ *and* $\mathbb{X}^m(\Gamma,\rho)$ *be either* $W^m_p(\Gamma,\rho)$ *(*$m=0,-1$*) or* $H^0_\mu(\Gamma,\rho)$*.*

Equations (1.44) *and* (1.45) *are* Fredholm *in* $\mathbb{X}(\Gamma,\rho)$ *if and only if*

$$\beta_j\neq\begin{cases}1-\gamma_j^0 & \text{for}\quad L_p(\Gamma,\rho),\ \ H^0_\mu(\Gamma,\rho)\,,\\ \gamma_j^0 & \text{for}\quad W_p^{-1}(\Gamma,\rho)\,,\end{cases}\qquad \gamma_j^0:=\min\left\{\frac{1}{\gamma_j},\frac{1}{2-\gamma_j}\right\}$$

for all $j=1,\dots,n$*, where* β_j *is defined in* (1.79)*.*

If either $\mathscr{T}_{pk}\neq\emptyset$ *or* $\beta_j=\gamma_j^0$ *when* $m=0$*,* $\beta_j=1-\gamma_j^0$ *when* $m=1$ *then the operators* $B_\pm$ *in* (1.44)*,* (1.45) *have non-closed images in* $W^m_p(\Gamma,\rho)$*.*

Equations (1.44) *and* (1.45) *with* $\psi\in L_p(\Gamma,\rho)$*,* $f\in L_p(\Gamma,\rho,\mathscr{T}_{pk})$ *are Fredholm, i.e., the operators*

$$B_\pm\ :\ L_p(\Gamma,\rho)\to L_p(\Gamma,\rho,\mathscr{T}_{pk}) \tag{1.84}$$

are bounded and are Fredholm if and only if $\beta_j\neq 1-\gamma_j^0$ *for all* $t_j\notin\mathscr{T}_{pk}$*; the following formulae hold for the index, kernel and cokernel in the space* $\mathbb{X}^m(\Gamma,\rho)$ *when* $\mathscr{T}_{pk}=\emptyset$ *or in the pairs* (1.84) *when* $\mathscr{T}_{pk}\neq\emptyset$

$$\operatorname{Ind}{}_{\mathbb{X}^0(\Gamma,\rho)}\mathrm{B}_\pm=\sum_{\substack{t_j\notin\mathscr{T}_{pk}\\ \beta_j>1-\gamma_j^0}}1\,, \tag{1.85}$$

$$\operatorname{Ind}{}_{\mathbb{X}^1(\Gamma,\rho)}\mathrm{B}_\pm=-\sum_{\substack{t_j\in\mathscr{T}\\ \beta_j<\gamma_j^0}}1\,, \tag{1.86}$$

$$\dim\operatorname{Ker}{}_{\mathbb{X}^0(\Gamma,\rho)}\mathrm{B}_\pm=\varepsilon_\pm+\operatorname{Ind}{}_{\mathbb{X}^0(\Gamma,\rho)}\mathrm{B}_\pm\,,\quad \dim\operatorname{Coker}{}_{\mathbb{X}^0(\Gamma,\rho)}\mathrm{B}_\pm=\varepsilon_\pm\,,$$

$$\dim\operatorname{Ker}{}_{\mathbb{X}^{-1}(\Gamma,\rho)}\mathrm{B}_\pm=\varepsilon_\pm\,,\quad \dim\operatorname{Coker}{}_{\mathbb{X}^{-1}(\Gamma,\rho)}\mathrm{B}_\pm=\varepsilon_\pm-\operatorname{Ind}{}_{\mathbb{X}^{-1}(\Gamma,\rho)}\mathrm{B}_\pm\,,$$

where $\varepsilon_+=0$ *and* $\varepsilon_-=1$*.*

[6] Absence of additional solvability condition for equation (1.40) under constraints (1.82) and (1.83), which are inevitable since dim *Coker* $A_-=1$ (see Remark 1.10), is due to the special right-hand side $g(t)-g_0$, which already satisfies the orthogonality condition (1.42).

In particular, if

$$0 < \beta_j < 1 - \gamma_j^0 \quad \text{for all} \quad t_j \notin \mathscr{T}_{pk}\,, \tag{1.87}$$

then equation (1.45) *has solution for all right-hand sides* $f(t)$ *in* $L_p(\Gamma, \rho, \mathscr{T}_{pk})$ *(in* $H_\mu^0(\Gamma, \rho)$ *when* $\mathscr{T}_{pk} = \emptyset$*), if and only if* (1.73) *holds. If*

$$\mathscr{T}_{pk} = \emptyset\,, \quad \gamma_j^0 < \beta_j < 1 \quad \text{for all} \quad t_j \in \mathscr{T} \tag{1.88}$$

then, again, equation (1.45) *has a solution for all right-hand sides* $f(t)$ *in* $W_p^{-1}(\Gamma, \rho)$*, while equation* (1.44) *has a solution if and only if condition* (1.73) *holds.*

Equation (1.45) *has a unique solution in these spaces, while homogeneous equation* (1.44)*,* $f(t) \equiv 0$ *has a single linearly independent solution* $\psi_- = \mathfrak{r}_0$ *(see Remark* 1.10*).*

Proof. For the cases $\mathscr{T}_{pk} = \emptyset$ and $W_p^m(\Gamma, \rho)$ $(m = 0, -1)$ the proof follows from the foregoing Theorem 1.23 because equations (1.39), (1.40) in $W_p^m(\Gamma, \rho)$ $(m = 0, 1)$ and equations (1.45), (1.44) in $W_{p'}^{-m}(\Gamma, \rho^{-}1)$ are pairwise conjugate.

As for equations (1.44) and (1.45) in the HÖLDER spaces $H_\mu^0(\Gamma, \rho)$ and the case $\mathscr{T}_{pk} \neq \emptyset$ (see (1.84)), the assertion is proved word to word as Theorem 1.23 (see § 5.4). ∎

Let

$$\Xi := \{\zeta_1, \dots, \zeta_n\} \subset \Gamma_1\,, \quad \Xi_{pk} := \Xi_{ow} \cup \Xi_{iw}\,, \tag{1.89}$$

$$\Xi_{ow} := \{\zeta_j = \{\omega^{-1}(t_j) \,:\, t_j \in \mathscr{T}_{ow}\}\,,$$
$$\Xi_{iw} := \{\zeta_j = \{\omega^{-1}(t_j) \,:\, t_j \in \mathscr{T}_{iw}\}$$

be the images on the unit circumference of the discrete sets $\mathscr{T}$, $\mathscr{T}_{pk}$, $\mathscr{T}_{ow}$ $\mathscr{T}_{iw}$ (see (1.74)) under the inverse conformal mapping $\omega^{-1}(\zeta)$ in (1.46)–(1.47). We define the following mean value integral

$$\mathscr{V}_{\zeta_j}\varphi(\zeta) := \int\limits_{\overset{\frown}{\zeta_j\zeta}} \left(\frac{\tau - \zeta_j}{\zeta - \zeta_j}\right)^{\frac{1}{p}} \varphi(\tau)\frac{d\tau}{\tau - \zeta_j}\,. \tag{1.90}$$

Let us fix a neighbourhood $\Gamma_{1\zeta_j} \subset \Gamma_1$ of $\zeta_j \in \Gamma_1$ such that $\Gamma_{1\zeta_k} \cap \Gamma_{1\zeta_j} = \emptyset$ (which implies $\zeta_k \notin \Gamma_{1\zeta_j}$) for $k \neq j$ and decompose $\Gamma_{1\zeta_j}$ into the left and the right neighbourhoods $\Gamma_{1\zeta_j} = \Gamma_{1\zeta_j}^- \cup \Gamma_{1\zeta_j}^+$. χ_{ζ_j} be the characteristic function of $\Gamma_{1\zeta_j}$. We define the space (see (1.89))

$$L_p(\Gamma_1, \Xi_{ow}) := \left\{\varphi \in L_p(\Gamma_1) \,:\, \widetilde{\mathscr{V}}_{\zeta_j}\varphi \in L_p(\Gamma_{1\zeta_j}^+)\,,\ \zeta_j \in \Xi_{ow}\right\}\,, \tag{1.91}$$

$$\widetilde{\mathscr{V}}_{\zeta_j}\varphi := \mathscr{V}_{\zeta_j}\varphi_{\zeta_j}^*\,, \quad \varphi_{\zeta_j}^*(\zeta) := e^{-\frac{\pi}{p}i}\varphi(\zeta) - \varphi(\overline{\zeta}\zeta_j^2)\,,$$

$$\|\varphi\big|L_p(\Gamma_1, \Xi_{ow})\| := \|\varphi\big|L_p(\Gamma_1)\| + \sum_{\zeta_j \in \Xi_{ow}} \|\widetilde{\mathscr{V}}_{\zeta_j}\chi_{\zeta_j}\varphi\big|L_p(\Gamma_{1\zeta_j}^+)\|\,.$$

Below, in Lemma 1.22, there is given a sufficient condition for the inclusion $\varphi \in L_p(\Gamma_1, \Xi_{ow})$ and for the boundedness of a multiplication operator aI).

Let us note, that if $\zeta \in \Gamma^{\pm}_{1\zeta_j}$, then the point $\overline{\zeta}\zeta_j^2$ belongs to the different half-neighbourhood $\overline{\zeta}\zeta_j^2 \in \Gamma^{\mp}_{1\zeta_j}$ (i.e., points are on different sides of $\zeta_j \in \Gamma_{1\zeta_j}$), but are equidistant from ζ_j: $|\zeta - \zeta_j| = |\overline{\zeta}\zeta_j^2 - \zeta_j|$.

Lemma 1.25 *If $\psi \in L_p(\Gamma_1)$ and $\log(\zeta - \zeta_j)\psi^*_{\zeta_j} \in L_p(\Gamma_1)$ for all $\zeta_j \in \Xi_{pk}$, then $\psi \in L_p(\Gamma_1, \Xi_{pk})$.*

Let $a \in L_\infty(\Gamma_1)$ and

$$a(t) = a(t_j) + O\left(|\log(\zeta - \zeta_j)|^{-1}\right) \quad \text{for all} \quad \zeta_j \in \Xi_{pk} \quad \text{as} \quad \zeta \to \zeta_j\,.$$

Then the operators

$$\begin{aligned} aI \quad &: \quad L_p(\Gamma_1, \Xi_{pk}) \longrightarrow L_p(\Gamma_1, \Xi_{pk})\,, \\ [a - a_0(\zeta)]I \quad &: \quad L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{pk}) \end{aligned} \tag{1.92}$$

are bounded, where $a_0(t) := \sum\limits_{\zeta_j \in \Xi_{pk}} a(\zeta_j)\chi_j(t)$ and $\chi_j(t)$ denotes the characteristic function of Γ_{ζ_j}.

Proof will be given later at the end of § 3.3.

Theorem 1.26 *The operator*

$$A \;:\; L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{ow}) \tag{1.93}$$

(see (1.50)*) is bounded and is* Fredholm *(i.e. equation* (1.50) *is* Fredholm *if $g_0 \in L_p(\Gamma_1, \Xi_{ow})$ and we look for a solution $\varphi \in L_p(\Gamma_1)$) if and only if*

$$\nu_j := \left(\frac{1}{p} + \alpha_j\right)\gamma_j \neq 1 \quad \text{for all} \quad \zeta_j \notin \Xi_{ow}\,. \tag{1.94}$$

If conditions (1.94) *hold,*

$$\begin{aligned} \text{Ind A} &= \sum_{\substack{\zeta_j \notin \Xi_{ow} \\ \nu_j > 1}} 1\,, \\ \dim \text{Ker A} &= \text{Ind A}\,, \quad \dim \text{Coker A} = 0\,, \end{aligned} \tag{1.95}$$

Proof. The proof is postponed to § 5.3.

Let (cf. (1.64))

$$\mathscr{Z}_\omega \varphi(\zeta) := \rho(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}}\varphi(\omega(\zeta))\,. \tag{1.96}$$

Lemma 1.27 *$\mathscr{Z}_\omega$ defines an isomorphism of spaces*

$$\begin{aligned} \mathscr{Z}_\omega \;&:\; L_p(\Gamma,\rho) \longrightarrow L_p(\Gamma_1)\,, \\ &:\; L_p(\Gamma,\rho,\mathscr{T}_{ow}) \longrightarrow L_p(\Gamma_1,\Xi_{ow}), \\ &:\; L_p(\Gamma,\rho,\mathscr{T}_{pk}) \longrightarrow L_p(\Gamma_1,\Xi_{pk}) \end{aligned} \tag{1.97}$$

and the inverse operator reads

$$\mathscr{Z}_\omega^{-1}\psi(t) := \rho^{-1}(t)[(\omega^{-1})'(t)]^{\frac{1}{p}}\psi(\omega^{-1}(t))\,. \tag{1.98}$$

The integral operators $\mathscr{V}_{t_j}$ in (1.75)*, $\widetilde{\mathscr{V}}_{t_j}$ in* (1.76) *and $\mathscr{V}_{\zeta_j}$ in* (1.90)*, $\widetilde{\mathscr{V}}_{\zeta_j}$ in* (1.91) *are related as follows*

$$\mathscr{Z}_\omega\chi_{t_j}\mathscr{V}_{t_j}\chi_{t_j}\mathscr{Z}_\omega^{-1} = \mathscr{V}_{\zeta_j}\,, \qquad \mathscr{Z}_\omega\chi_{t_j}\widetilde{\mathscr{V}}_{t_j}\chi_{t_j}\mathscr{Z}_\omega^{-1} = \widetilde{\mathscr{V}}_{\zeta_j}\,. \tag{1.99}$$

Proof. The proof is direct and follows from the definitions. ∎

Theorem 1.28 *The* Dirichlet *problem* (1.6)*,* (1.7) *with*

$$u \in e_p(\overline{\Omega^+},\rho) \quad \text{and} \quad g \in L_p(\Gamma,\rho,\mathscr{T}_{ow}) \tag{1.100}$$

is Fredholm if and only if the conditions

$$\nu_j := \left(\frac{1}{p}+\alpha_j\right)\gamma_j \neq 1 \quad \text{for all} \quad \zeta_j \notin \mathscr{T}_{iw}\,. \tag{1.101}$$

hold. If this is the case, the problem has solution for each right hand-side in (1.100) *and the homogeneous problem has exactly*

$$\varkappa := \sum_{\nu_j>1} 1 \tag{1.102}$$

solutions (i.e., the index of the corresponding operator is $\varkappa$). In particular, if conditions

$$\nu_j = \nu_j^0 := \left(\frac{1}{p}+\alpha_j\right)\gamma_j < 1 \quad \text{for all} \quad \zeta_j \notin \mathscr{T}_{iw} \tag{1.103}$$

hold, the problem has a unique solution.

Moreover, if $\mathscr{T}_{ow} = \emptyset$ the Dirichlet *problem* (1.6)*,* (1.7) *with*

$$\begin{aligned} &u \in w_p^1(\overline{\Omega^+},\rho) && \text{and} && g \in W_p^1(\Gamma,\rho)\,,\ \ \nu_j^1 := \left(\tfrac{1}{p}+\alpha_j-1\right)\gamma_j\,, \\ &u \in pc^m(\overline{\Omega^+},\rho) && \text{and} && g \in PC^m(\Gamma,\rho)\,, \\ & && && \nu_j^m := (\alpha_j-m)\gamma_j\,, \quad m=0,1\,, \\ &u \in c(\overline{\Omega^+},\rho) && \text{and} && g \in C(\Gamma,\rho)\,, \quad \nu_j^0 := \alpha_j\gamma_j\,, \\ &u \in h_{\mu+m}^0(\overline{\Omega^+},\rho) && \text{and} && g \in H_{\mu+m}^0(\Gamma,\rho)\,, \\ & && && \nu_j^m := (\alpha_j-\mu_j-m)\gamma_j\,, \ \ m=0,1, \end{aligned} \tag{1.104}$$

is Fredholm if and only if the condition

$$\nu_j^m \neq (-1)^m \tag{1.105}$$

holds for all $j = 1, \dots, n$. *If this is the case, the problem has the following index*

$$\text{Ind A} := \sum_{|\nu_j^{\mathrm{m}}|>1} (-1)^{\mathrm{m}}. \tag{1.106}$$

and either the kernel (*when* Ind $\mathrm{A} \leq 0$) *or the cokernel* (*when* Ind $\mathrm{A} \geq 0$) *is trivial. For* Ind $\mathrm{A} = 0$ *both kernel and cokernel are trivial and the problem has a unique solution for all right-hand sides* (*see* (1.103)).

The same holds for the domain Ω^- *with the obvious replacements:* $\mathscr{T}_{ow}$ *by* $\mathscr{T}_{iw}$ *and* γ_j *by* $1 - \gamma_j$.

Proof. The first part of the theorem (1.100)–(1.103) follows from Theorems 1.16 and 1.26.

The second half of the theorem, when $\mathscr{T}_{ow} = \emptyset$, follows from equivalence of the DIRICHLET problem and of the corresponding singular integral equation (1.39)–(1.40) in appropriate space, which can be proved as in Theorem 1.12, and from appropriate assertions on singular integral equations in § 4. ∎

Theorem 1.29 *The operator*

$$B^+ \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{ow}) \tag{1.107}$$

(*see* (1.60)) *is bounded and is* FREDHOLM *(i.e., equation* (1.60) *is* FREDHOLM *if* $f_0 \in L_p(\Gamma_1, \Xi_{ow})$ *and we look for a solution* $\varphi \in L_p(\Gamma_1)$*) if and only if the conditions*

$$\mu_j := \left(1 - \frac{1}{p} - \alpha_j\right) \gamma_j \neq 1 \quad \text{for all} \quad t_j \notin \mathscr{T}_{ow}\,. \tag{1.108}$$

hold. If conditions (1.107) *hold,*

$$\begin{aligned} \text{Ind } \mathrm{B}^+ &= -1 + \sum_{\substack{\zeta_j \notin \Xi_{ow} \\ \mu_j > 1}} 1\,, \\ \dim \text{ Ker } \mathrm{B}^+ &= \text{Ind } \mathrm{B}^+\,, \quad \dim \text{ Coker } \mathrm{B} = 1^+\,. \end{aligned} \tag{1.109}$$

Proof. The proof follows word in word the proof of Theorem 1.26 (see § 5.3) with obvious modifications (including substitution of $\frac{1}{p}$ by $\frac{1}{p} - 1$, as seen from (1.51) and (1.61)). The only difference which we have found worth explaining is the appearance of "-1" and "1" in the index formulae (1.106): the second condition in (1.60) obviously increases dim Coker B^+ by 1 and diminishes Ind B^+ also by 1. ∎

Theorem 1.30 *The* NEUMANN *problem* (1.6), (1.8) *for* Ω^+ *has solutions* $u(x) + c_0$, *where* $c_0 =$*const is arbitrary and,* $u \in w_p^1(\overline{\Omega^+}, \rho)$ *for* $f \in L_p(\Gamma, \rho, \mathscr{T}_{ow})$ *if and only if conditions* (1.108) *hold and the solution is unique modulo a constant if* $\mu_j < 1$ *for* $t_j \notin \mathscr{T}_{ow}$. *The index of the problem is given by the formula*

$$\text{Ind } \mathrm{B}^+ := 1 + \sum_{\mu_j > 1} 1. \tag{1.110}$$

If $\mathscr{T}_{ow} = \emptyset$ *the* NEUMANN *problem* (1.6), (1.8) Ω^+ *with*

$$\begin{aligned} &u \in pc^1(\overline{\Omega^+}, \rho) \qquad \text{and} \quad g \in PC^m(\Gamma, \rho)\,, \quad \mu_j := (1 - \alpha_j)\gamma_j\,, \\ &u \in h^0_{\mu+1}(\overline{\Omega^+}, \rho) \qquad \text{and} \quad g \in H^0_{\mu+1}(\Gamma, \rho)\,, \\ &\qquad\qquad\qquad\qquad\quad \mu_j := (1 - \alpha_j + \mu_j)\gamma_j\,, \quad m = 0, 1 \end{aligned} \tag{1.111}$$

is Fredholm if and only if the condition $\mu_j \neq 1$ *holds for all* $j = 1, \ldots, n$. *If this is the case, the problem has the same index* (1.110).

The same holds for the domain Ω^- *with the obvious replacements:* $\mathscr{T}_{ow}$ *by* $\mathscr{T}_{iw}$ *and* γ_j *by* $1 - \gamma_j$.

Proof. The first part of the theorem (1.110) follows from Theorems 1.17 and 1.29.

The second half of the theorem, when $\mathscr{T}_{ow} = \emptyset$, follows from equivalence of the NEUMANN problem to the corresponding singular integral equation (1.43)–(1.44) in appropriate space, which can be proved as in Theorem 1.14, and from appropriate assertions on singular integral equations in § 4. ∎

Remark 1.31 *Fredholm and solvability properties of pseudodifferential equations* (1.69), (1.70), (1.72) *can easily be derived from Theorems* 1.28 *and* 1.30 *(see Theorems* 1.19 *and* 1.20*). To save the space we leave this to readers.*

2 Convolutions with elliptic symbols

2.1 Boundedness properties

$C_0^\infty(\mathbb{R})$ denotes the FRECHET space of all infinitely differentiable functions on $\mathbb{R} := (-\infty, \infty)$ with compact supports supp φ and $D'(\mathbb{R})$ – the dual space of distributions.

The convolution operator W_a^0 with a symbol $a \in L_\infty(\mathbb{R})$ is defined as follows

$$W_a^0 \varphi := \mathscr{F}^{-1} a \mathscr{F} \varphi\,, \quad \varphi \in C_0^\infty(\mathbb{R}), \tag{2.1}$$

where

$$\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}} e^{i\xi x} \varphi(x) dx \qquad \text{and}$$

$$\mathscr{F}^{-1}\psi(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}} e^{-ix\xi}\psi(\xi)d\xi, \quad x,\xi \in \mathbb{R}^n, \tag{2.2}$$

are the FOURIER transforms.

$\mathbb{M}_p(\mathbb{R})$ denotes, as usual (see [Du1, Hr1], the class of FOURIER L_p-multipliers, i.e., the class of all those symbols $a(\lambda) \in L_\infty(\mathbb{R})$ for which the operator W_a^0 admits a bounded extension

$$W_a^0 \ : \ L_p(\mathbb{R}) \ \to \ L_p(\mathbb{R}) \tag{2.3}$$

for all $1 < p < \infty$ (see [BS1, Du1, RS1]).

In particular, if symbol $a(\lambda)$ has: one of the following properties:

i. bounded total variation $a \in V_1(\mathbb{R})$ (B.STECHKIN theorem),

ii. if

$$a \in C^1(\mathbb{R} \setminus \{0\}), \quad |a(t)| \le M_0 < \infty, \quad |ta'(t)| \le M_0 < \infty \tag{2.4}$$

(J.MARCINKIEVICZ theorem),

iii. belongs to the Wiener algebra

$$a \in W(\mathbb{R}) := \{a(\lambda) = c + \mathscr{F}k(\lambda) \ : \ k \in L_1(\mathbb{R})\},$$

then $a \in \mathbb{M}_p(\mathbb{R})$. Moreover, in the case (iii.) W_a^0 is written as an integral convolution

$$W_a^0\varphi(x) = c\varphi(x) + \int\limits_{-\infty}^{\infty} k(x-y)\varphi(y)dy,$$

while in general case convolution has distributional kernel (see [Du1, Hr1, St1] for details).

Let $\dot{\mathbb{R}}$ and $\ddot{\mathbb{R}}$ denote one point and two point compactifications of the real axes

$$\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \quad \text{or} \quad \ddot{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

respectively and $PC(\dot{\mathbb{R}})$ denote the space of all piecewise-continuous functions on $\dot{\mathbb{R}}$, i.e., the space of all functions $a(\lambda)$ on $\mathbb{R}$ which have finite limits $a(\lambda \pm 0)$ for all $\lambda \in \dot{\mathbb{R}}$. The space $PC(\dot{\mathbb{R}})$ coincides with the closure of all piecewise-constant functions on $\dot{\mathbb{R}}$ with respect to the uniform norm (in $L_\infty(\dot{\mathbb{R}})$; see [Du1]). Let $PC_p(\dot{\mathbb{R}})$ be the same closure of all piecewise-constant functions with respect to the multiplier norm $\|a\big|\mathbb{M}_p(\mathbb{R})\| := \|W_a^0\big|\mathscr{L}(L_p(\mathbb{R}))\|$. Then

$$PC_2(\dot{\mathbb{R}}) = PC(\dot{\mathbb{R}}), \quad V_1(\mathbb{R}), W(\mathbb{R}) \subset \bigcap_{1<p<\infty} PC_p(\dot{\mathbb{R}}).$$

For a matrix symbol $a \in PC_p^{N\times N}(\dot{\mathbb{R}})$ invertibility criteria of the operator W_a^0 in $L_p(\mathbb{R})$ space reads

$$\inf_{\lambda\in\mathbb{R}} |\det \mathrm{a}(\lambda)| > 0\,, \tag{2.5}$$

which yields $a^{-1} \in PC_p^{N\times N}(\dot{\mathbb{R}})$ and the inverse operator is $W_{a^{-1}}^0$ (see [Du1, Hr1] for these and other properties of multipliers).

Moreover, we can take $1 \le p \le \infty$ and involve new spaces. Namely W_a^0 has bounded extensions in the following spaces of smooth functions: $C_0^m(\dot{\mathbb{R}})$, $C^m(\dot{\mathbb{R}})$, $C_0^m(\ddot{\mathbb{R}})$ for all $m = 0, 1, \dots$ (see [Kr1]). These spaces are defined as follows.

Let $\mathbb{X}$ be either one point or two point compactifications of the real axes and $C^m(\mathbb{X})$ denote the Banach space of continuous functions on the compact Hausdorff set $\mathbb{X}$, which have continuous derivatives up to the order m and is endowed with the appropriate uniform norm

$$\|\varphi\| = \sum_{k=0}^{m} \sup_{t\in\mathbb{R}} \left| \left(\frac{d^k}{dt^k}\varphi\right)(t) \right|$$

($C^m(\mathbb{X})$ is even a Banach algebra with pointwise multiplication). Note, that a function $\varphi \in C^m(\dot{\mathbb{R}})$ and its derivatives have equal limits at infinity $(d^k/dt^k)\varphi(\infty) = (d^k/dt^k)\varphi(\pm\infty)$ while function $\varphi \in C^m(\ddot{\mathbb{R}})$ might have two different limits $(d^k/dt^k)\psi(\pm\infty)$ for all $k = 0, 1, \dots, m$.

$C_0^m(\dot{\mathbb{R}})$ denotes the subspace (the sub-algebra) of $C^m(\dot{\mathbb{R}})$ of those functions $\varphi(x)$ which vanish at infinity with all derivatives up to the order m:

$$C_0^m(\dot{\mathbb{R}}) := \left\{ \varphi \in C^m(\dot{\mathbb{R}}) \;:\; \varphi(\infty) = \cdots = \left(\frac{d^m}{dt^m}\varphi\right)(\infty) = 0 \right\}.$$

Let

$$W_a\varphi := r_+ W_a^0 \ell_0 \varphi\,, \qquad \varphi \in C_0^\infty(\mathbb{R}^+)\,, \tag{2.6}$$

where r_+ denotes the restriction to $\mathbb{R}^+$ from $\mathbb{R}$, while ℓ_0–the right inverse to r_+ which extends functions by 0 from $\mathbb{R}^+$ to $\mathbb{R}$. Let $L_p(\mathbb{R}^+, \rho)$, $\rho(x) \ge 0$, denote the weighted Lebesgue space endowed with the standard norm $\|\varphi | L_p(\mathbb{R}^+, \rho)\| := \|\rho\varphi | L_p(\mathbb{R}^+)\|$.

Lemma 2.1 (see [Du1, Sc1]). *Let $a \in V_1(\mathbb{R})$ and*

$$1 < p < \infty\,, \quad -\frac{1}{p} < \alpha, \gamma < 1 - \frac{1}{p}\,. \tag{2.7}$$

Then

$$W_a \;:\; L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \tag{2.8}$$

is continuous. ∎

Let $\beta \in \mathbb{R}$ and

$$L_1^{\langle\beta\rangle}(\mathbb{R}) := \begin{cases} L_1(\mathbb{R}^+) \cap L_1(\mathbb{R}^-, (1-x)^{-\beta}) & \text{for } \beta < 0\,, \\ L_1(\mathbb{R}^-) \cap L_1(\mathbb{R}^+, (1+x)^{\beta}) & \text{for } \beta > 0\,, \end{cases}$$

where $\mathbb{R}^- := (-\infty, 0]$. Let further

$$W_\beta(\mathbb{R}) := \left\{a(\lambda) = c + \mathscr{F}k(\lambda) : c = \text{ const}\,,\ k \in L_1(\mathbb{R}, (1+|x|)^{|\beta|})\right\}, \tag{2.9}$$

$$W_1^{\langle\beta\rangle}(\mathbb{R}) := \left\{a(\lambda) = c + \mathscr{F}k(\lambda) : c = \text{ const}\,,\ k \in L_1^{\langle\beta\rangle}(\mathbb{R})\right\} \tag{2.10}$$

and endow them with the appropriate norms

$$\|a\big|W_\beta(\mathbb{R})\| := |c| + \|k\big|L_1(\mathbb{R}, (1+|x|)^{|\beta|})\| \quad \text{for} \quad \pm\beta > 0\,,$$

$$\|a\big|W^{\langle\beta\rangle}(\mathbb{R})\| := |c| + \|k\big|L_1^{\langle\beta\rangle}(\mathbb{R})\| = |c| + \|k\big|L_1(\mathbb{R}^{\mp})\|$$
$$+\|k\big|L_1(\mathbb{R}^{\pm}, (1+|x|)^{|\beta|})\| \quad \text{for} \quad \pm\beta > 0$$

provided $a(\lambda) = c + \mathscr{F}k(\lambda)$. Obviously, $W_\beta(\mathbb{R}) \subset W^{\langle\beta\rangle}(\mathbb{R})$.

Let $C(\dot{\mathbb{R}}^+)$ denote the restriction of the space $C(\dot{\mathbb{R}})$ to the semi-axes $\mathbb{R}^+$ and $C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ denote the weighted space of functions $\varphi(x)$ on the semi-axes $\mathbb{R}^+$ for which $(1+x)^\beta\varphi(x)$ belong to $C(\dot{\mathbb{R}}^+)$.The space is endowed with the appropriate weighted norm $\|\varphi\big|C(\dot{\mathbb{R}}^+, (1+x)^\beta)\| := \|(1+x)^\beta\varphi(x)\big|C(\dot{\mathbb{R}}^+)\|$.

Lemma 2.2 *Let $a \in W^{\langle\beta\rangle}(\mathbb{R})$ and $\beta \in \mathbb{R}$. Then the operator*

$$W_a \ :\ C(\dot{\mathbb{R}}^+, (1+x)^\beta) \longrightarrow C(\dot{\mathbb{R}}^+, (1+x)^\beta) \tag{2.11}$$

is continuous[7)] *and*

$$\lim_{x\to\infty}(1+x)^\beta W_a\varphi(x) = a(0)\lim_{x\to\infty}(1+x)^\beta\varphi(x)\,, \tag{2.12}$$

$$\|W_a\big|C(\dot{\mathbb{R}}^+, (1+x)^\beta\| \le \|a\big|W^{\langle\beta\rangle}(\mathbb{R})\|\,. \tag{2.13}$$

Proof. For $a(\lambda) = c$ we have $W_a = cI$ and the assertion is trivial. Thus, we can take $a = \mathscr{F}k,\ \ k \in L_1^{\langle\beta\rangle}(\mathbb{R})$.

The integral

$$W_a^0 = \int_{-\infty}^{\infty} k(x-y)\varphi(y)dy = \int_{-\infty}^{\infty} k(y)\varphi(x-y)dy$$

[7)] See similar assertions in [GF1, Pr1, PS1].

is continuous function for a continuous $\varphi \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ and we should check only (2.12)–(2.13).

Obviously,

$$\|W_a\big|C(\dot{\mathbb{R}}^+, (1+x)^\beta)\| \le K_\beta\,,$$
$$K_\beta = \sup_{x\in\mathbb{R}^+} \int\limits_0^\infty \left(\frac{1+x}{1+y}\right)^\beta |k(x-y)|dy\,. \tag{2.14}$$

If $\beta < 0$, applying the inequality

$$1 + x \le (1 + |x-y|)(1+y)\,, \quad x, y \in \mathbb{R}^+\,, \tag{2.15}$$

we proceed as follows

$$\begin{aligned} K_\beta &\le \sup_{x>0}\left[\int\limits_0^x \left(\frac{1+x}{1+y}\right)^\beta |k(x-y)|dy + \int\limits_x^\infty \left(\frac{1+y}{1+x}\right)^{-\beta} |k(x-y)|dy\right] \\ &\le \sup_{x>0}\left[\int\limits_0^\infty |k(x-y)|dy + \int\limits_x^\infty (1+|x-y|)^{-\beta}|k(x-y)|dy\right] \\ &\le \int\limits_0^\infty |k(t)|dt + \int\limits_{-\infty}^0 (1+|t|)^{-\beta}|k(t)|dt = \|a\big|W^{\langle\beta\rangle}(\mathbb{R})\|\,. \end{aligned}$$

Now let $\beta > 0$. Similarly to the foregoing case we find (see (2.14) and (2.15))

$$\begin{aligned} K_\beta &\le \sup_{x>0}\left[\int\limits_0^x \left(\frac{(1+x}{1+y}\right)^\beta |k(x-y)|dy + \int\limits_x^\infty \left(\frac{(1+x}{1+y}\right)^\beta |k(x-y)|dy\right] \\ &\le \sup_{x>0}\left[\int\limits_0^x (1+|x-y|)^\beta |k(x-y)|dy + \int\limits_x^\infty |k(x-y)|dy\right] \\ &\le \sup_{x>0}\left[\int\limits_0^x (1+t)^\beta |k(t)|dt + \int\limits_{-\infty}^0 |k(t)|dt\right] = \|a\big|W_\beta(\mathbb{R})\|\,. \end{aligned}$$

To prove (2.12) (for arbitrary $\beta \in \mathbb{R}$) we represent

$$\varphi_\beta(x) := (1+x)^\beta\varphi(x) = \varphi_\beta(\infty) + \varphi_\beta^0(x)\,, \quad \varphi_\beta^0(\infty) = 0$$

and suppose that both $\varphi_\beta^0(x)$ and $k(t)$ have compact supports

$$\operatorname{supp}\varphi_\beta^0 \subset [0, c_1]\,, \qquad \operatorname{supp} k \subset [-c_2, c_2]\,.$$

Since such functions are dense in appropriate spaces, this does not restricts generality. Then

$$\lim_{x\to\infty}(1+x)^{\beta}W_a\varphi(x)=\lim_{x\to\infty}\int\limits_0^{\infty}\left(\frac{1+x}{1+y}\right)^{\beta}k(x-y)\varphi_{\beta}(y)dy$$

$$=\lim_{x\to\infty}\int\limits_0^{c_1}\left(\frac{1+x}{1+y}\right)^{\beta}k(x-y)\varphi^0_{\beta}(y)dy$$

$$+\varphi_{\beta}(\infty)\lim_{x\to\infty}\int\limits_0^{\infty}\left(\frac{1+x}{1+y}\right)^{\beta}k(x-y)dy=a(0)\varphi_{\beta}(\infty)\,,$$

since

$$k(x-y)\varphi^0_{\beta}(y)=0\quad\text{if}\quad x\geq c_1+c_2,$$

$$\lim_{x\to\infty}\int\limits_0^{\infty}\left(\frac{1+x}{1+y}\right)^{\beta}k(x-y)dy=\lim_{x\to\infty}\int\limits_{-c_2}^{c_2}\left(\frac{1+x}{1+x-t}\right)^{\beta}k(t)dt$$

$$=\int\limits_{-c_1}^{c_2}k(t)dt=\int\limits_{-\infty}^{\infty}k(t)dt=a(0)\,.$$

This accomplishes the proof. ∎

2.2 Fredholm properties

Lemma 2.3 *Let $\beta\in\mathbb{R}$. Then $W_{\beta}(\mathbb{R})\subset W_0(\mathbb{R})=W(\mathbb{R})\subset C(\dot{\mathbb{R}})$ is an inverse closed* Banach *algebra in $C(\dot{\mathbb{R}})$, which reads: the element $a\in W_{\beta}(\mathbb{R})$ is invertible if and only if it is invertible in $C(\dot{\mathbb{R}})$, i.e., iff $\inf\limits_{\lambda\in\mathbb{R}}|a(\lambda)|>0$, and then $a^{-1}\in W_{\beta}(\mathbb{R})$.*

Proof. The proof see in [GRS1, § 18]. ∎

Let for a matrix-function $a=[a_{jk}]_{N\times N}$ with entries $a_{jk}\in\mathfrak{A}$ use the same notation $a\in\mathfrak{A}$.

Lemma 2.4 *Let $\beta\in\mathbb{R}$ and a matrix-function $a\in W_{\beta}(\mathbb{R})$ be elliptic*

$$\inf_{\lambda\in\mathbb{R}}|\det a(\lambda)|>0\,. \tag{2.16}$$

Then $a(\lambda)$ has the following factorization

$$a(\lambda)=a_-(\lambda)\,\mathrm{diag}\left\{\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_1},\dots,\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_N}\right\}a_+(\lambda) \tag{2.17}$$

where the matrix-functions $a_-^{\pm} \in W_\beta(\mathbb{R})$ *and* $a_+^{\pm} \in W_\beta(\mathbb{R})$ *have uniformly bounded analytic extensions* $a_-^{\pm}(\lambda - i\sigma)$ *and* $a_+^{\pm}(\lambda + i\sigma)$ *in the lower and upper* $\sigma > 0$ *complex half-planes, respectively. The integers* $\varkappa_1, \dots, \varkappa_N$ *are known as the partial indices of the factorization* (2.17).

Proof. For the algebra $W(\mathbb{R}) = W_0(\mathbb{R})$ the proof is well-known (see, e.g., [GF1]) and we follow the same scheme: if all rational functions are dense in $W_\beta(\mathbb{R})$ (a rationally dense algebra) and the HILBERT transform

$$S_{\mathbb{R}}\varphi(x) = \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi(y)dy}{y-x} = -W^0_{\operatorname{sign}\lambda}\,\varphi(x)\,, \quad x \in \mathbb{R} \tag{2.18}$$

(see [Du1, Lemma 1.35]) is bounded there (a decomposable algebra), then according to the general theorem proved in [BG1] (see also [CG1, GF1]) all invertible elements of $W_\beta(\mathbb{R})$ would possess factorization (2.17). Invertibility of $a \in W_\beta(\mathbb{R})$ under condition (2.16) is provided by Lemma 2.3.

Rational density of $W_\beta(\mathbb{R})$ follows since the LAGUERRE polynomials are dense in $L_1(\mathbb{R}, (1+|x|)^{|\beta|})$ (see, e.g., [GF1, § 8]).

$W_\beta(\mathbb{R})$ is a decomposable because $\mathscr{F}S_{\mathbb{R}}\mathscr{F}^{-1}\psi(\lambda) = -\operatorname{sign}\lambda\,\psi(\lambda)$ (see (2.18)) is a bounded operator in $L_1^{\langle\beta\rangle}(\mathbb{R})$ and $W_\beta(\mathbb{R}) = \text{const} + \mathscr{F}L_1(\mathbb{R}, (1+|x|)^{|\beta|}))$ (see (2.8)). ∎

Let us consider $a = c + \mathscr{F}k \in W_\beta(\mathbb{R})$ and the corresponding equation

$$W_a^0\varphi(x) = c\,\varphi(x) + \int\limits_{-\infty}^{\infty} k(x-y)\varphi(y)dy = f(x)\,, \quad x \in \mathbb{R}^+ \tag{2.19}$$

(cf. (2.5)).

Theorem 2.5 *Equation* (2.19) *in the space* $C(\dot{\mathbb{R}}^+, (1+x)^\beta)$, $\beta \in \mathbb{R}$ *is* FREDHOLM *if and only if the symbol* $a(\lambda)$ *is elliptic (see* (2.16)*). If this is the case, then*

$$\operatorname{Ind} W_a = -\operatorname{ind} a\,.$$

If, in addition, (2.19) *is a scalar equation* $N = 1$*, then:*

i. *equation* (2.19) *is uniquely solvable for all* $f \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ *provided* $\operatorname{ind} a = 0$*;*

ii. *if* $\varkappa = \operatorname{ind} a < 0$ *equation* (2.19) *has a solution* $\varphi \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ *for all* $f \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ *and the homogeneous equation* $f = 0$ *has exactly* $-\varkappa$ *linearly independent solutions;*

iii. *if* $\varkappa = \operatorname{ind} a > 0$ *equation* (2.19) *has a solution* $\varphi \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ *only for those right-hand sides* $f \in C(\dot{\mathbb{R}}^+, (1+x)^\beta)$ *for which*

$$\int\limits_0^{\infty} f(y)\overline{g_j(y)}dy = 0\,, \qquad j = 1, \dots, \varkappa\,,$$

where $g_1, \dots, g_\varkappa$ are all solutions to the dual homogeneous equation

$$c\, g(x) + \int_{-\infty}^{\infty} k(y-x) g(y) dy = 0 \tag{2.20}$$

in the dual space $C(\dot{\mathbb{R}}^+, (1+x)^{-\beta})$.

If the solution exists it is unique.

Proof. The proof is standard and based on Lemmata 2.3, 2.4 (see [Du1, GF1, GK1, Kr1] for similar proofs, except the last claim).

Concerning the last claim–we replaced the adjoint space $C^*(\dot{\mathbb{R}}^+, (1+x)^{\beta})$ by the dual one $C(\dot{\mathbb{R}}^+, (1+x)^{-\beta})$; this is possible since the equation (2.20) has the same solutions in these two spaces (see [Du5] for a similar assertion).

The last claim follows also from Lemma 1.21. which states that equation has the same solutions in two spaces $\mathfrak{B}_1 \subset \mathfrak{B}_2$ provided the embedding is dense and the equation has a common regularizer in $\mathfrak{B}_1$ and in$\mathfrak{B}_2$. ∎

Now let $a \in V_1(\mathbb{R})$; then W_a can be written as integral convolution (2.19) only conventionally–the kernel $k(t)$ is a distribution. If $a(\lambda)$ possesses a single jump, operator W_a is not bounded in $C(\dot{\mathbb{R}}^+, (1+x)^{\beta})$ because the HILBERT transform (2.18) is not bounded in these space.

Thus, we should consider equation (2.19) with $a \in PC_p(\mathbb{R})$in the LEBESGUE space $L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ with weight under conditions (2.7). With equation (2.19) we associate the symbol

$$a_\omega(\lambda, \xi) := \frac{1 + \coth \pi[i\beta(\lambda) + \xi]}{2} a(\lambda - 0)$$

$$+ \frac{1 - \coth \pi[i\beta(\lambda) + \xi]}{2} a(\lambda + 0)\,, \quad \lambda \in \dot{\mathbb{R}}\,, \qquad \xi \in \mathbb{R}\,, \tag{2.21}$$

where (note, that $a \in PC_p(\mathbb{R})$ has limits $a(\lambda \pm 0)$, $\lambda \in \dot{\mathbb{R}}$ including infinity $a(\infty \pm 0) := a(\mp\infty)$). $\omega := (p, \alpha, \gamma)$ reminds the space and

$$\beta(\lambda) := \begin{cases} \frac{1}{p}, & \text{if} \quad \lambda \neq 0, \infty\,, \\ \frac{1}{p} + \alpha, & \text{if} \quad \lambda = 0, \\ \frac{1}{p} + \gamma, & \text{if} \quad \lambda = \infty\,. \end{cases}$$

Theorem 2.6 *Let $a \in PC_p(\mathbb{R})$; the weight $\rho(t)$ be defined by* (1.2) *and satisfy appropriate (namely the first) condition in* (1.4).

Equation (2.32) *is* FREDHOLM *in the space $L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ if and only if the symbol $a_\omega(\lambda, \xi)$ is elliptic*

$$\inf_{\lambda \in \dot{\mathbb{R}},\ \xi \in \mathbb{R}} |\det\, a_\omega(\lambda, \xi)| > 0\,.$$

If this is the case, then

$$\text{Ind } W_a = -\frac{1}{2\pi i}\sum_{j=1}^{\infty}\left\{[\arg a(\lambda)]_{\lambda\in[\lambda_j,\lambda_{j+1}]} + [\arg a_\omega(\lambda_j,\xi)]_{\xi\in\mathbb{R}}\right\}, \quad (2.22)$$

where $\{\lambda_j\}_{j=1}^{\infty} \subset \dot{\mathbb{R}}$ *denotes the set of all points where* $a \in V_1(\mathbb{R})$ *has jumps* $a(\lambda_j - 0) \neq a(\lambda_j + 0)$ *and*[8] $[\arg g(t)]_{t\in\mathscr{I}}$ *denotes the increment of any continuous branch of* $\arg g(t)$ *as* t *ranges through* $\mathscr{I}$ *in the positive direction.*

If, in addition, (2.19) *is a scalar equation* $N = 1$*, then:*

i. *equation* (2.19) *is unuquelly solvable for all* $f \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *provided* Ind $W_a = 0$*;*

ii. *if* $\varkappa =$ Ind $W_a > 0$ (2.19) *has a solution* $\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *for all* $f \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *and the homogeneous equation* $f = 0$ *has exactly* $\varkappa$ *linearly independent solutions;*

iii. *if* $\varkappa =$ ind $W_a < 0$ (2.19) *has a solution* $\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *only for those right-hand sides* $f \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *for which*

$$\int_0^\infty f(y)\overline{g_j(y)}dy = 0\,, \qquad j = 1,\dots,-\varkappa\,,$$

where $g_1,\dots,g_{-\varkappa}$ *are all solutions of the dual homogeneous equation*

$$c\,g(x) + \int_{-\infty}^{\infty} k(y-x)g(y)dy = 0 \quad (2.23)$$

in the dual space $L_{p'}(\mathbb{R}^+, x^{-\alpha}(1+x)^{-\gamma+\alpha})$ *with* $p' := \dfrac{p}{p-1}$.

If solution exists it is unique.

Proof. For the proof we quote [Du1] (the case $\alpha = \gamma = 0$) and [Sc1] (the case $\alpha \neq 0,\ \beta \neq 0$). ∎

2.3 Some proofs

Proof of Lemma 1.1. Let, for definiteness, consider the domain Ω^+. Since $\Phi \in \mathscr{X}(\overline{\Omega^+})$ we have

$$\Phi(z) = c_0 + \frac{1}{2\pi i}\int_\Gamma \frac{\Phi_0^+(\tau)d\tau}{\tau - z}\,, \quad z \in \Omega^+\,,$$

[8] The set $\{\lambda_j\}_{j=1}^{\infty}$ is at most countable and the sum in (2.22) is convergent (see, e.g., [Du1]).

where $\Phi_0^+ \in \mathbb{X}(\Gamma)$ is the trace of $\Phi_0(z) := \Phi(z) - c_0$ on Γ from Ω^+. On the other hand

$$\Phi(z) = c_0 + \frac{1}{2\pi i} \int\limits_\Gamma \frac{\varphi(\tau) d\tau}{\tau - z}\,, \quad z \in \Omega^+\,,$$

for some $\varphi \in \mathbb{X}(\Gamma)$ (see (1.3)) and we get

$$\int\limits_\Gamma \frac{\Phi_0^+(\tau) - \varphi(\tau)}{\tau - z} d\tau \equiv 0\,, \quad z \in \Omega^+\,.$$

The obtained equality means

$$\Phi_0^+(t) - \varphi(t) = \Psi^-(t)\,, \quad t \in \Gamma\,, \tag{2.24}$$

where $\Psi \in \mathscr{X}(\overline{\Omega^-})$; therefore,

$$\Psi(z) = -C_\Gamma \varphi(z) := -\frac{1}{2\pi i} \int\limits_\Gamma \frac{\varphi(\tau) d\tau}{\tau - z}\,, \quad z \in \Omega^-,$$

and $\Psi(z) \to 0$ as $|z| \to \infty$, since $C_\Gamma \Phi_0^+(z) \equiv 0$ for $z \in \Omega^-$. In fact, $P_\Gamma^+ \Phi_0^+ = \Phi_0^+$ yields $P_\Gamma^- \Phi_0^+ = 0$ (we remind that $P_\Gamma^- + P_\Gamma^+ = I$; see (1.50)). On the other hand, due to the Plemelji formula for $C_\Gamma \varphi$ in (1.25) $(C_\Gamma \Phi_0^+)^- = P_\Gamma^- \Phi_0^+ = 0$ and the analytic function $C_\Gamma \Phi_0^+(z)$, $z \in \Omega^-$ vanishing on the boundary vanishes everywhere in Ω^-.

(2.24) can be written as follows

$$\operatorname{Re}(-i\Psi^-)(t) = \operatorname{Im}\, \Psi^+(t) = \operatorname{Im}\, \Phi_0^+(t)\,, \quad t \in \Gamma$$

if $\varphi(t) = \operatorname{Re}\, \varphi(t) = \operatorname{Re}\, \Phi_0^+(t) - \operatorname{Re}\, \Psi^-(t)$ is pure real and

$$\operatorname{Re}\, \Psi^-(t) = \operatorname{Re}\, \Phi_0^+(t)\,, \quad t \in \Gamma,$$

if $\varphi(t) = i \operatorname{Im}\, \varphi(t) = i \operatorname{Im}\, \Phi_0^+(t) - i \operatorname{Im}\, \Psi^-(t)$ is pure imaginary. Since $\Phi_0^+(t)$ is known, solvability of the obtained Riemann–Hilbert problems is equivalent to the claimed representations. ∎

Proof of Lemma 1.2. If (0.2) holds and $\Omega^\pm$ has no outward peak ($\mathscr{T}_{ow} = \emptyset$), $u^\pm \in W_2^{\frac{1}{2}}(\Gamma)$ due to theorem on traces (see, e.g., [Tr1])). Although $\partial_{\overline{\nu}} u \in W^0_{2,\mathrm{com}}(\Omega^\pm)$, we can not claim $(\partial_{\overline{\nu}} u)^\pm \in W_2^{-\frac{1}{2}}(\Gamma)$ because the trace does not exists. But u is harmonic $\Delta u(z) = 0$ in $\Omega^\pm$ and from the Green formula (1.13) we get

$$\oint\limits_\Gamma \partial_{\overline{\nu}(\tau)} u(\tau) \overline{v(\tau)} d\tau = \pm \sum_{j=1}^{2} \int\limits_{\Omega^\pm} \partial_j u(y) \overline{\partial_j v(y)} dy\,. \tag{2.25}$$

Taking arbitrary $v \in W^1_{2,\mathrm{com}}(\overline{\Omega^\pm})$, which implies $v^\pm \in W_2^{\frac{1}{2}}(\Gamma)$ due to theorem on traces, by duality of spaces from (2.25) follows $(\partial_{\vec{\nu}} u)^\pm \in W_2^{-\frac{1}{2}}(\Gamma)$.

Since $u(z)$ is a harmonic function, due to representation formula (1.15)

$$\begin{aligned} u(z) = \ & \mathrm{Re}\ u(z) = \chi_-(z)u(\infty) \pm W_\Gamma u^\pm(z) \mp V_\Gamma(\partial_{\vec{\nu}} u)^\pm(x) \\ & = \chi_-(z)u(\infty) \pm W_\Gamma u^\pm(z) \mp \mathrm{Re}\,(\partial_z V_\Gamma)v^\pm(z) \qquad (2.26) \\ & = \chi_-(z)u(\infty) + \mathrm{Re}\,(C_\Gamma u_\pm)(z)\,, \quad u_\pm(t) := \pm u^\pm(t) + i\,\mathrm{Re}\ v^\pm(t)\,, \\ & v^\pm(t) := \int\limits_{\breve{ct}} (\partial_{\vec{\nu}(\tau)} u)^\pm(\tau)d\tau\,, \quad z \in \Omega^\pm\,, \quad t \in \Gamma\,. \end{aligned}$$

From (2.26) we get the inclusion into the SMIRNOV class $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$ with the complex valued density $u_\pm \in W_2^{\frac{1}{2}}(\Gamma)$ because $u^\pm, v^\pm \in W_2^{\frac{1}{2}}(\Gamma)$.

Vice versa, $u \in w_2^{\frac{1}{2}}(\overline{\Omega^\pm})$, also for $\Omega^\pm$ with peaks, implies the representation

$$u(z) = u(\infty) + \mathrm{Re}\ C_\Gamma\varphi(z)\,, \qquad z \in \Omega^\pm\,, \qquad \varphi \in W_2^{\frac{1}{2}}(\Gamma)\,.$$

Then $u(z)$ is harmonic in $\Omega^\pm$ and $u(z) = u(\infty) + O((1+|z|)^{-1})$ as $|z| \to \infty$ and, due to Theorem 1.8, $u \in W_2^1(\overline{\Omega^\pm})$. ∎

Proof of Theorem 1.5. The first and the second claims for $s = m = 0$ follows from representations (1.17), (1.18) and boundedness of the singular integral operator S_Γ (see (1.5)) in $L_p(\Gamma,\rho)$ (see, e.g., [GK1, Kh1, Pr1]) and in $H^0_\mu(\Gamma,\rho)$ (see [Du6, Du7] and also [Du3, Du5]).

The operators

$$\begin{aligned} W^{(k)}_{\Gamma,0}\varphi(t) &:= \frac{1}{4}\left(S_\Gamma + \mathscr{V} h^k S_\Gamma h^{-k}\mathscr{V}\right)\varphi(t) \\ &= \frac{1}{4\pi i}\int\limits_\Gamma \varphi(\tau)\left[\frac{d\tau}{\tau - t} - \frac{\overline{h^k(t)}}{\overline{h^k(\tau)}}\frac{d\overline{\tau}}{\overline{\tau} - \overline{t}}\right] \end{aligned} \qquad (2.27)$$

are bounded in $L_p(\Gamma,\rho)$ and in $H^0_\mu(\Gamma,\rho)$ by the same reason.

For a closed contour $\partial_t S_\Gamma\varphi = S_\Gamma\partial_t\varphi$ and we get

$$\begin{aligned} \partial_t W_{\Gamma,0}\varphi &:= \frac{1}{4}\left(\partial_t S_\Gamma + \mathscr{V}\frac{dt}{d\overline{t}}\partial_t S_\Gamma\mathscr{V}\right)\varphi \\ &= \frac{1}{4}\left(S_\Gamma + \mathscr{V} h^2 S_\Gamma h^{-2}\mathscr{V}\right)\partial_t\varphi = W^{(2)}_{\Gamma,0}\partial_t\varphi \end{aligned} \qquad (2.28)$$

(cf. (1.17)–(1.21), (1.26)); therefore $W_{\Gamma,0}$ is bounded in $W^1_p(\Gamma,\rho)$ and in $H^0_{1+\mu}(\Gamma,\rho)$. By interpolation (see [Tr1]) we get boundedness of $W_{\Gamma,0}$ in $W^s_p(\Gamma,\rho)$ for $0 \le s \le 1$.

Since the operator $W^*_{\Gamma,0}$ is adjoint to $W_{\Gamma,0}$, it is automatically bounded in adjoint space $W^s_p(\Gamma,\rho)$ (see, e.g., [Tr1]) to $W^{-s}_{p'}(\Gamma,\rho)$ for $-1 \le s \le 0$ and $p' := p/(p-1)$.

Let us prove the last claim.

V_Γ has a weak singular kernel and, therefore,

$$\|V_\Gamma\varphi\big|L_p(\Gamma,\rho)\| \le C_1\|\varphi\big|L_p(\Gamma,\rho)\|\,;$$

on the other hand, due to (1.19),

$$\|\partial_t V_\Gamma\varphi\big|L_p(\Gamma,\rho)\| = \|(S_\Gamma + \mathscr{V}S_\Gamma\mathscr{V})\varphi\big|L_p(\Gamma,\rho)\| \le C_2\|\varphi\big|L_p(\Gamma,\rho)\|\,.$$

and we get the final estimate

$$\|V_\Gamma\varphi\big|W^1_p(\Gamma,\rho)\| = \|V_\Gamma\varphi\big|L_p(\Gamma,\rho)\| + \|\partial_t V_\Gamma\varphi\big|L_p(\Gamma,\rho)\|\,.$$

Similarly for the HÖLDER spaces $H^0_\mu(\Gamma,\rho) \to H^0_{1+\mu}(\Gamma,\rho)$. ∎

Proof of Theorem 1.6. It suffices to show that $W^{(k)}_{\Gamma,0}$ are bounded in $PC(\Gamma,\rho)$ for even $k = 0, 2, \ldots$ and $W^{(0)}_{\Gamma,0} = W_{\Gamma,0}$ is bounded in $C(\Gamma,\rho)$. In fact, $h^\pm I$ are bounded in $PC(\Gamma,\rho)$ and boundedness of $W^*_{\Gamma,0}$ in $PC(\Gamma,\rho)$ follows since

$$W^*_{\Gamma,0} = -hW^{(2)}_{\Gamma,0}\overline{h}I$$

(cf. (1.18), (2.27)). By virtue of (1.22) we get

$$\begin{aligned}\|W_{\Gamma,0}\varphi\big|PC^1(\Gamma,\rho)\| &= \|W_{\Gamma,0}\varphi\big|C(\Gamma,\rho)\| + \|\partial_t W_{\Gamma,0}\varphi\big|PC(\Gamma,\rho)\|\,,\\ &= \|W_{\Gamma,0}\varphi\big|C(\Gamma,\rho)\| + \|W^{(2)}_{\Gamma,0}\partial_t\varphi\big|PC(\Gamma,\rho)\|\,.\end{aligned}$$

which means boundedness of $W_{\Gamma,0}$ in $PC^1(\Gamma,\rho)$.

Integral operator K with a weak singular kernel

$$|k(t,\tau)| \le C|t-\tau|^{\nu-1}\,, \quad 0 < \nu \le 1\,, \quad t,\tau \in \Gamma, \tag{2.29}$$

is bounded (moreover, is compact) in spaces $C(\Gamma,\rho)$ and in $PC(\Gamma,\rho)$.

In fact, this is easy to ascertain for $\rho(t) \equiv 1$. For $\rho(t) \not\equiv 1$ we have to prove that $K_1 := \rho K\rho^{-1}I - K$ is compact in $C(\Gamma)$ and in $PC(\Gamma)$.

The kernel $k_1(t,\tau)$ of K_1 has the following estimate

$$|k_1(t,\tau)| = |\rho(t)-\rho(\tau)|\frac{k(t,\tau)}{\rho(\tau)} \le C\frac{g_\rho(t,\tau)}{\rho(\tau)}|t-\tau|^{\nu-1}\,;$$

here $g_\rho(t,\tau) = |t-\tau|^{\delta_j}$ when both t and τ are close to the knot t_j, $j = 1,\ldots,n$ and $g_\rho(t,\tau) = |t-\tau|$ otherwise. Thus, $k_1(t,\tau)$ is weak singular and compactness (in $C(\Gamma)$ and in $PC(\Gamma)$) follows.

Let Γ' be another LJAPUNOV contour and $\omega : \Gamma \to \Gamma'$ be a diffeomorphism with analytic continuation in some neighbourhood of cuspidal wedge $U_j \subset \Omega^+$ (outward peak of Ω^+) of cusps c_j with $\gamma_j = 0$. Then the operator

$$K_\omega := \omega_*^{-1} S_\Gamma \omega_* - S_\Gamma \,, \quad \omega_* \varphi(t) = \varphi(\omega(t)) \,, \quad t \in \Gamma \,, \tag{2.30}$$

where $\omega^{-1} : \Gamma \to \Gamma'$ is the inverse diffeomorphism, has a weak singular kernel (2.29) (see [DLS1, § 3.5] and [Kh1, GK1]).

Due to representations (1.17)–(1.19), (2.27) and to boundedness of operator K_ω in $C(\Gamma, \rho)$ and in $PC(\Gamma, \rho)$ (see (2.29) and further) the contour Γ can be replaced by another one Γ' for which we can find a diffeomorphism $\omega : \Gamma \to \Gamma'$ with local analytic continuation in the vicinity of cusps.

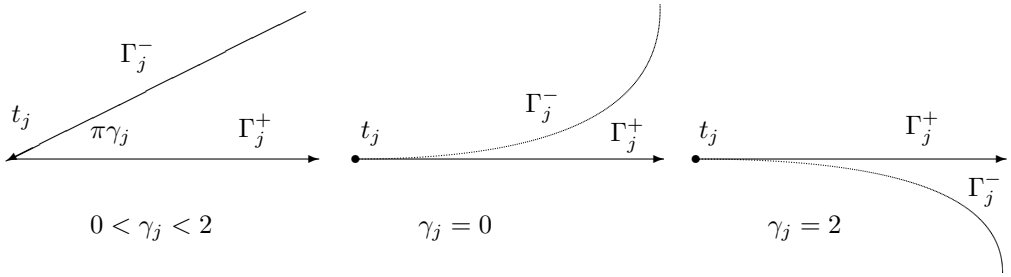

Fig. 2

Due to this we can suppose Γ_j has rectilinear parts Γ_j^+ and Γ_{j+1}^- in some neighbourhoods of the endpoints t_j and t_{j+1} except cusps; for a cusp $\gamma_j = 0, 2$ the right neighbourhood $\Gamma_j^+ \subset \Gamma_j$ is rectilinear in the vicinity of t_j, while the left neighbourhood $\Gamma_j^- \subset \Gamma_{j-1}$ is not (we remind, that $\{t_j\} = \Gamma_{j-1} \bigcup \Gamma_j$; see Fig. 2). Let

$$\Gamma_j^0 = \Gamma_j^- \bigcup \Gamma_j^+ \,, \quad \Gamma^0 = \bigcup_{j=1}^{n} \Gamma_j^0 \,, \quad \Gamma_0 = \Gamma \setminus \Gamma^0 \,. \tag{2.31}$$

Let $v_0 \in C^1(\Gamma)$ be a cut-off function with $\operatorname{supp} v_0 \subset \Gamma^0$ and $v_0(t) = 1$ in some neighbourhoods of all knots $t_1, \ldots, t_n$. Then

$$W_{\Gamma,0}^{(k)} = (1 - v_0) W_{\Gamma,0}^{(k)} + v_0 W_{\Gamma_0,0}^{(k)} + v_0 W_{\Gamma^0,0}^{(k)} \,. \tag{2.32}$$

Γ_0 is free of knots $t_1, \ldots, t_n$ and operators $(1 - v_0) W_{\Gamma,0}^{(k)}$, $v_0 W_{\Gamma_0,0}^{(k)}$ have weak singular kernels. In fact, kernels of these operators read

$$k_2(t, \tau) = [1 - v_0(t)] k_0(t, \tau) \,,$$

$$k_3(t, \tau) = v_0(t) \chi_0(t) k_0(t, \tau) \,, \quad t, \tau \in \Gamma \,,$$

where $\chi_0(t)$ is the characteristic function of Γ_0 and

$$k_0(t,\tau) = \frac{1}{\pi i}\left[\frac{1}{\tau - t} - \frac{\overline{h^k(t)}}{\overline{h^k(\tau)}}\frac{1}{\overline{\tau} - \overline{t}}\frac{d\overline{\tau}}{d\tau}\right]$$

$$= \frac{1}{\pi i}\left[\frac{1}{\tau - t} - \frac{1}{\overline{\tau} - \overline{t}}\frac{d\overline{\tau}}{d\tau}\right] - \frac{\overline{h^k(t)} - \overline{h^k(\tau)}}{\overline{h^k(\tau)}(\overline{\tau} - \overline{t})}\frac{d\overline{\tau}}{d\tau}\,. \qquad (2.33)$$

$k_0(t,\tau) = k_1(t,\tau) = 0$ if $t, \tau \notin \Gamma_0$; therefore we can suppose $t, \tau \in \Gamma_0$ because otherwise $k_2(t,\tau)$ and $k_2(t,\tau)$ are bounded. Γ_0 consists of n disjoint smooth arcs and $k_0(t,\tau)$ is the kernel of $W^{(k)}_{\Gamma,0} = S_\Gamma - \mathscr{V} h^k S_\Gamma h^{-k} \mathscr{V}$; therefore we can apply a diffeomorphism $\omega : \Gamma_0 \to \Gamma_{\mathbb{R}} \subset \mathbb{R}$ which transforms Γ_0 to the finite union of intervals on the real axes. Since $\omega_*^{-1} W^{(k)}_{\Gamma_0,0} \omega_*$ differs from $W^{(k)}_{\Gamma_{\mathbb{R}},0}$ by a compact operator with weak singular kernel, we can consider $W^{(k)}_{\Gamma_{\mathbb{R}},0}$. But the first summand in representation (2.33) of the kernel of operator $W^{(k)}_{\Gamma_{\mathbb{R}},0}$ vanishes

$$\frac{1}{\tau - t} - \frac{1}{\overline{\tau} - \overline{t}}\frac{d\overline{\tau}}{d\tau} = 0\,, \quad t, \tau \in \Gamma_{\mathbb{R}} \subset \mathbb{R}\,,$$

while the second summand is weak singular, because the function $h^k(\omega^{-1}(t))$ is $C^{1+\nu}$-continuous.

Thus, we have to consider only operator $v_0 W^{(k)}_{\Gamma^0,0}$ in (2.32). This can be simplified further and we need to treat only operators $W^{(k)}_{\Gamma^0_j,0}$, because the difference

$$T_0 = v_0\left[W^{(k)}_{\Gamma^0,0} - \sum_{j=1}^{n} W^{(k)}_{\Gamma^0_j,0}\right]$$

is compact (has a bounded kernel).

Let $0 < \gamma_j < 2$. Without loss of generality we can suppose that

$$\Gamma^0_j = \Gamma^-_j \bigcup \Gamma^+_j\,, \quad \Gamma^+_j = (0,1]\,, \quad \Gamma^-_j = \left\{e^{i\gamma_j} x \,:\, 0 \le x \le 1\right\}\,.$$

Consider the transformation

$$\mathscr{Z}_{\gamma_j,\delta_j}\varphi(x) := \left[\begin{array}{c} e^{-\delta_j x}\varphi(e^{-x}) \\ e^{-\delta_j x i}\varphi(e^{\pi\gamma_j - x}) \end{array}\right], \quad x \in \mathbb{R}^+, \qquad (2.34)$$

and its inverse

$$\mathscr{Z}^{-1}_{\gamma_j,\delta_j}\left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right](t) = \chi^0_+(t) t^{-\delta_j}\psi_1(-\log t)$$

$$+\chi^0_-(t) e^{\pi\gamma_j\delta_j i} t^{-\delta_j}\psi_2(\pi\gamma_j - \log t)\,, \quad t \in \Gamma^0_j\,,$$

where χ_+^0 and χ_-^0 are the characteristic functions of Γ_j^+ and Γ_j^-, respectively.

$\mathscr{Z}_{\gamma_j,\delta_j}$ arranges the isomorphism of the space $PC(\Gamma_j^0, t^{\delta_j}) = PC(\Gamma_j^0, \rho)$ with the vector-space $[C(\dot{\mathbb{R}}^+)]^2 = C(\dot{\mathbb{R}}^+) \times C(\dot{\mathbb{R}}^+)$ (see § 1.1). The transformed operator acquires the form

$$\mathscr{Z}_{\gamma_j,\delta_j} W^{(k)}_{\Gamma_j^0,0} \mathscr{Z}^{-1}_{\gamma_j,\delta_j} = \begin{bmatrix} 0 & W_{a^+_{\gamma_j,\delta_j,k}} \\ W_{a^-_{\gamma_j,\delta_j,k}} & 0 \end{bmatrix},$$

where

$$W_{a^\pm_{\gamma_j,\delta_j,k}} \varphi(x) = \int_0^\infty k^\pm_{\gamma_j,\delta_j,k}(x-y)\varphi(y)dy\,,$$

$$a^\pm_{\gamma_j,\delta_j,k}(\lambda) := \mathscr{F}_{t\to\lambda}\left[k^\pm_{\gamma_j,\delta_j,k}(t)\right], \quad \lambda, x \in \mathbb{R}\,,$$

$$k^\pm_{\gamma_j,\delta_j,k}(t) = -\frac{e^{\pm\pi\gamma_j k - \delta_j t}\sin\pi\gamma_j + e^{-(\delta_j+1)t}\sin\pi\gamma_j}{2\pi(1 - 2e^{-t}\cos\pi\gamma_j + e^{-2t})}$$

$$:= \frac{e^{-\delta_j t}}{4\pi i}\left[\frac{1}{1-e^{-\pi\gamma_j i - t}} - \frac{e^{\pm\pi\gamma_j k i}}{1-e^{\pi\gamma_j i - t}}\right].$$

Obviously,

$$k^\pm_{\gamma_j,\delta_j,k} \in L_1(\mathbb{R}) \quad \text{iff} \quad 0 < \delta_j < 1 \tag{2.35}$$

and, due to Lemma 2.2, the transformed operator $\mathscr{Z}_{\gamma_j,\delta_j} W^{(k)}_{\Gamma_j^0,0} \mathscr{Z}^{-1}_{\gamma_j,\delta_j}$ is bounded in $[C(\dot{\mathbb{R}}^+)]^2$ because $0 < \gamma_j < 2\,,\ 0 < \delta_j < 1$.

Now let $\gamma_j = 0, 2$. We can suppose without loss of generality that $t_k = 0$ and

$$\Gamma_j^+ = \mathscr{J} = (0,1] \subset \mathbb{R}^+\,,$$

$$\Gamma_j^- = \{z_j(x) = x + ig_j(x) \ :\ 0 \le x \le 1\}\,,$$

$$g_j \in C^{1+\nu}(\mathscr{J})\,, \quad g_j(0) = g_j'(0) = 0\,, \quad g_j(x) \ge 0\,,$$

$$h(z_j(x)) = 1 + ig_j(x)\,, \quad h(x) = 1\,, \quad x \in \mathscr{J} \quad (\text{see } (2.21))\,.$$

The transformation

$$\mathscr{B}_j\varphi(x) = \begin{bmatrix} \varphi(x) \\ \varphi(z_j(x)) \end{bmatrix}, \quad z_j(x) = x + ig_j(x)\,, \quad x \in \mathscr{J}\,, \tag{2.36}$$

arranges the isomorphism

$$\mathscr{B} \ :\ PC(\Gamma_j^0, |t|^{\delta_j}) \longrightarrow [C(\mathscr{J}, |t|^{\delta_j})]^2$$

and

$$\mathscr{B}_j W^{(k)}_{\Gamma_j,0}\mathscr{B}_j^{-1} = \begin{bmatrix} 0 & -\widetilde{V}_{ig_j} \\ V_{ig_j} & 0 \end{bmatrix} + \begin{bmatrix} 0 & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

where

$$T_{12} = \widetilde{N}_{-ig_j}[(1-ig_j')^{-k}-1]\,, \qquad T_{12} = [(1-ig_j')^k - 1]N_{-ig_j}\,,$$

$$T_{21} = -K_{z_j} - (1-ig_j')^k K_{z_j}(1-ig_j')^{-k} I + (1-ig_j')^k S_{\mathscr{J}}(1-ig_j')^{-k} I - S_{\mathscr{J}}\,,$$

$$V_{ig_j}N_{ig_j} - N_{-ig_j}\,, \qquad \widetilde{V}_{ig_j} = \widetilde{N}_{ig_j}(1+ig_j')I - \widetilde{N}_{-ig_j}(1-ig_j')I\,,$$

$$N_{\pm ig_j}\frac{1}{\pi i}\int\limits_0^1 \frac{\varphi(y)dy}{y-x\pm ig_j(x)}\,, \qquad \widetilde{N}_{\pm ig_j} = \frac{1}{\pi i}\int\limits_0^1 \frac{\varphi(y)dy}{y-x\pm ig_j(y)}$$

and K_{z_j} is defined in (2.30) ($z_j(x)$ see in (2.36)). Operators T_{12}, T_{21}, T_{22} all have weak singular kernels and there is left to prove boundedness of operators V_{ig_j} and $\widetilde{V}_{ig_j}$ only.

It is easy to ascertain that

$$v(x) := V_{ig_j}1(x) = -\frac{2g_j(x)}{\pi}\int\limits_0^1 \frac{dy}{(y-x)^2+g_j^2(x)}$$

$$= \frac{1}{\pi i}\log\frac{[x+ig_j(x)][1-x+ig_j(x)]}{[x-ig_j(x)][1-x-ig_j(x)]}\,,$$

$$\widetilde{v}(x) := \widetilde{V}_{ig_j}1(x) = \frac{1}{\pi i}\frac{1-x+ig_j(x)}{1-x-ig_j(x)}\,, \qquad x\in\mathscr{J}\,,$$

and $v, \widetilde{v}\in C(\mathscr{J})$. Functions

$$V_{ig_j}\varphi(x) = V_{ig_j}[\varphi(y)-\varphi(x)] + \varphi(x)v(x)\,,$$

$$\widetilde{V}_{ig_j}\varphi(x) = \widetilde{V}_{ig_j}[\varphi(y)-\varphi(x)] + \varphi(x)\widetilde{v}(x)$$

are continuous provided $\varphi\in C^1(\mathscr{J})$. On the other hand we get

$$\big|V_{ig_j}\varphi(x)\big| = \frac{2g_j(x)}{\pi}\left|\int\limits_0^1 \frac{\varphi(y)dy}{(y-x)^2+g_j^2(x)}\right|$$

$$\le \frac{\|\varphi|C(\mathscr{J})\|}{\pi}\left|\int\limits_0^1\left[\frac{1}{y-x+ig_j(x)} - \frac{1}{y-x-ig_j(x)}\right]dy\right|$$

$$= \frac{\|\varphi|C(\mathscr{J})\|}{\pi}|v(x)|\,,$$

$$\left|\widetilde{V}_{ig_j}\varphi(x)\right| = \frac{2}{\pi}\left|\int_0^1 \frac{(y-x)g_j'(y)-g_j(y)}{(y-x)^2+g_j^2(y)}\varphi(y)dy\right|$$

$$\le \frac{2}{\pi}\|\varphi|C(\mathscr{J})\|\left\{\left|\int_0^1 \frac{[(y-x)g_j'(y)-g_j(y)][g_j^2(x)-g_j^2(y)]}{[(y-x)^2+g_j^2(y)][(y-x)^2+g_j^2(x)]}dy\right.\right.$$

$$\left.\left.+g_j^2(x)\int_0^1 \frac{(y-x)g_j'(y)-g_j(y)}{(y-x)^2+g_j^2(x)}dy\right|\right\}$$

$$\le \|\varphi|C(\mathscr{J})\|\big[(1+\|g_j'|C(\mathscr{J})\|)^2\|g_j'|C(\mathscr{J})\|$$

$$+\|g_j|C(\mathscr{J})\|(\|g_j'|C(\mathscr{J})\|+\|g_j'|C(\mathscr{J})\|)\big]\left|\frac{1}{\pi}\int_0^1 \frac{2ig_j(x)dy}{(y-x)^2+g_j^2(x)}\right|$$

$$= C_{g_j}\|\varphi|C(\mathscr{J})\||v(x)|\,,$$

$$C_{g_j} = \big[(1+\|g_j'|C(\mathscr{J})\|)^2\|g_j'|C(\mathscr{J})\|$$

$$+\|g_j|C(\mathscr{J})\|(\|g_j'|C(\mathscr{J})\|+\|g_j'|C(\mathscr{J})\|)\big]\,.$$

Obtained inequalities prove that V_{ig_j} and $\widetilde{V}_{ig_j}$ can be extended as continuous operators from $C^1(\mathscr{J}) \longrightarrow C(\mathscr{J})$ to $C(\mathscr{J}) \longrightarrow C(\mathscr{J})$. ∎

3 Equations with non-elliptic symbols

3.1 Convolutions on $\mathbb{R}^+$

Let α, γ and p be as in (2.7) and the symbol $a \in PC_p(\mathbb{R})$ be non-elliptic (vanishing at 0):

$$a(\lambda) = \frac{\lambda}{\lambda - i}a^{(-)}(\lambda)\,, \quad a^{(-)} \in PC_p(\mathbb{R})\,, \qquad \inf_{\lambda\in\mathbb{R}} |\det a^{(-)}(\lambda)| > 0\,. \quad (3.1)$$

Then equation (2.19) is not Fredholm in $L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ due to Theorem 2.6. Namely the image of the operator Im W_a is not closed in $L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ (see [Du4, § 4]).

In the present section, similarly to [Pr1, § 5.2], we define the spaces $\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ and $\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ such that the operators

$$W_a\ :\ L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\,, \tag{3.2}$$

$$W_a\ : \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \tag{3.3}$$

would be FREDHOLM.

Let

$$\mathscr{U}_0\varphi(x) := \int\limits_0^x \varphi(y)dy\,, \qquad \mathscr{V}_\infty\varphi(x) := \int\limits_x^\infty \varphi(y)dy\,, \tag{3.4}$$

where we have to deal with an usual improper integral

$$\int\limits_x^\infty \varphi(y)dy := \lim_{t\to\infty} \int\limits_x^t \varphi(y)dy\,. \tag{3.5}$$

The operator $\mathscr{V}_\infty$ in (3.5), $\widetilde{\mathscr{V}}_{\zeta_j}$ in (1.90) and $\widetilde{\mathscr{V}}_{t_j}$ in (1.75) are equivalent modulo isomorphism of the corresponding spaces (see Lemmata 1.27 and 3.8).

Let

$$\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) := \left\{\varphi\ :\ \varphi, \mathscr{V}_\infty\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\right\}\,, \tag{3.6}$$

$$\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) := \left\{\psi + \mathscr{U}_0\varphi\ :\ \varphi, \psi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\right\}\,.$$

On defining the norms

$$\|\varphi\big|\,\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\| := \|\varphi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|$$

$$+\|\mathscr{V}_\infty\varphi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|\,,$$

$$\|\varphi + \mathscr{U}_0\psi\big|\,\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\| := \|\varphi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|$$

$$+\|\psi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|$$

we make $\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ and $\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ into BANACH spaces.

The embedding

$$C_0^\infty(\mathbb{R}^+) \subset \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \subset L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$$

$$\subset \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \tag{3.7}$$

are dense and follow from definitions.

Let

$$\mathscr{G}_\pm := W_{g_\pm}\,, \qquad g_\pm := \frac{\lambda}{\lambda \pm i}\,. \tag{3.8}$$

Then

$$\mathscr{G}_-\varphi(x) = \varphi(x) - \int\limits_x^\infty e^{x-y}\varphi(y)dy\,, \quad \mathscr{G}_+\varphi(x) = \varphi(x) - \int\limits_0^x e^{y-x}\varphi(y)dy \tag{3.9}$$

and we can give equivalent description of spaces (3.6) in form of the following lemma (see [Pr1, § 5.2] for a similar assertion).

Lemma 3.1 *The following definitions of spaces are equivalent:*

$$\vec{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) := \left\{\mathscr{G}_-\varphi \;:\; \varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\right\}$$

$$= \operatorname{Im}{}_{L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})}\mathscr{G}_-\,,$$

$$\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) := \left\{\varphi \;:\; \mathscr{G}_+\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\right\}$$

$$= \operatorname{Im}{}_{L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})}\mathscr{G}_+^{-1}\,.$$

Proof. It suffices to prove that

$$\|\psi\big|\,\vec{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|^0 := \|\mathscr{G}_-^{-1}\psi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|\,, \tag{3.10}$$

$$\|\varphi\big|\,\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\|^0 := \|\mathscr{G}_+\varphi\big|L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\| \tag{3.11}$$

define equivalent norms. To check this let us prove that the operators

$$W_{g_-^{-1}}\psi = \mathscr{G}_-^{-1}\psi = \psi + \mathscr{V}_\infty\psi\,, \qquad W_{g_+^{-1}}\psi = \mathscr{G}_+^{-1}\psi = \psi + \mathscr{U}_0\psi \tag{3.12}$$

represent inverses to $\mathscr{G}_-$ and to $\mathscr{G}_+$, respectively. Let us check $\mathscr{G}_-(I + \mathscr{V}_\infty)\varphi = \varphi$, because all other cases are similar.

Due to the density of embedding (3.7) we have to check the claimed equality only for $\varphi \in C_0^\infty(\mathbb{R}^+)$. Then

$$\mathscr{V}_\infty\varphi(x) = \int\limits_x^\infty \varphi(y)dy$$

and integrating by parts we find

$$\mathscr{G}_-(I + \mathscr{V}_\infty)\varphi(x) = \varphi(x) + \int\limits_x^\infty \varphi(s)ds - e^{-x}\int\limits_x^\infty e^{-y}\varphi(y)dy$$

$$-e^{-x}\int\limits_x^\infty e^{-y}dy\int\limits_0^\infty \varphi(s)ds = \varphi(x)\,.$$

By the definition of $\varphi \in \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ we get $\mathscr{G}_-^{-1}\varphi = \varphi + \mathscr{V}_\infty\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$; therefore the mappings

$$\mathscr{G}_- \;:\; L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\,,$$

$$\mathscr{G}_-^{-1} \;:\; \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$$

are one-to-one and continuous. Equivalence of the norm in $\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ and of the norm in (3.10) follows from the BANACH theorem.

As we already know

$$\mathscr{G}_+(\varphi + \mathscr{U}_0\varphi) = \mathscr{G}_+\mathscr{G}_+^{-1}\varphi = \varphi\,;$$

on the other hand $\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ implies $\mathscr{G}_+\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ (see (3.8), (3.9)) and therefore $\mathscr{G}_+\mathscr{U}_0\varphi = \mathscr{G}_+(\varphi + \mathscr{U}_0\varphi) - \mathscr{G}_+\varphi = \varphi - \mathscr{G}_+\varphi \in L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$. Thus, the mappings

$$\mathscr{G}_+ \;:\; \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\,,$$

$$\mathscr{G}_+^{-1} \;:\; L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$$

are one-to-one and continuous. Equivalence of the norms in (3.11) and of this in $\overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ follows from the BANACH theorem. ∎

Corollary 3.2 *The spaces* $\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ *and* $\overleftarrow{L}_{p'}\,(\mathbb{R}^+, x^{-\alpha}(1+x)^{-\gamma+\alpha})$, *where* $p' = \dfrac{p}{p-1}$, *are dual.*

Proof. The operators

$$\begin{aligned} W_{g_-} = \mathscr{G}_- \;&:\; L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})\,, \\ W_{g_+} = \mathscr{G}_+ \;&:\; \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \end{aligned} \tag{3.13}$$

define isomorphisms (see Lemma 3.1) and they are dual (conjugate) $W_{g_\pm}^* = W_{g_\mp}$. The claimed result follows since the spaces $L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha})$ and $L_{p'}(\mathbb{R}^+, x^{-\alpha}(1+x)^{-\gamma+\alpha})$ are dual as well. ∎

Lemma 3.3 *The embedding*

$$\begin{aligned} L_p(\mathbb{R}^+, x^\alpha(1+x)^{1+\gamma-\alpha}) \subset \overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \subset L_p(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \\ \subset \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) \subset L_p(\mathbb{R}^+, x^\alpha(1+x)^{-1+\gamma-\alpha}) \end{aligned} \tag{3.14}$$

are continuous and dense.

Proof (see [Pr1, Ch. 5, Theorem 2.3]). We have to prove only the first and the last embedding (see (3.7)).

Density of embedding follow from the density of $C_0^\infty(\mathbb{R}^+)$ in all these spaces.

First we check the embedding in (3.14). Obviously,

$$L_p(\mathbb{R}^+, x^\alpha(1+x)^{1+\gamma-\alpha}) = L_p([0,1], x^\alpha) \dot{+} L_p([1,\infty), (1+x)^{1+\gamma}),$$

$$\overrightarrow{L}_p\,(\mathbb{R}^+, x^\alpha(1+x)^{\gamma-\alpha}) = L_p([0,1], x^\alpha) \dot{+} L_p([1,\infty), (1+x)^{\gamma})$$

and it suffices to prove the embedding

$$L_p([1,\infty), (1+x)^{1+\gamma}) \subset \overrightarrow{L}_p\,([1,\infty), (1+x)^{\gamma}). \tag{3.15}$$

If we prove the inequality

$$\|\mathscr{V}_\infty\varphi\big|L_p([1,\infty), (1+x)^{\gamma})\| \le c_1\|\varphi\big|L_p([1,\infty), (1+x)^{1+\gamma}), \tag{3.16}$$

due to the norm definition in $\overrightarrow{L}_p\,([1,\infty), (1+x)^{\gamma})$ (see (3.6)) there will follow the embedding (3.15).

Invoking the HÖLDER inequality we proceed as follows

$$|\mathscr{V}_\infty\varphi(x)| = \left|\int_x^\infty \varphi(y)dy\right| \le \left(\int_x^\infty y^{-(1+\gamma)p'}dy\right)^{\frac{1}{p'}} \left(\int_x^\infty \left|y^{1+\gamma}\varphi(y)\right|^p dy\right)^{\frac{1}{p}}$$

$$\le \frac{1}{(1+\gamma)p'-1}\|\varphi\big|L_p([1,\infty), (1+x)^{1+\gamma})\|,$$

since $-p'(1+\gamma) < -1$ (see (2.7)). Thus, $\mathscr{V}_\infty\varphi(x)$ exists as an ordinary LEBESGUE integral for arbitrary $\varphi \in L_p([1,\infty), (1+x)^{1+\gamma})$.

For the function

$$f(s,t) := t\,|\varphi(st)|, \qquad s,t \in [1,\infty),$$

we have

$$\int_1^\infty f(s,t)ds = \int_t^\infty |\varphi(y)dy| \ge |\mathscr{V}_\infty\varphi(x)|,$$

$$\left\{\int_1^\infty [t^\gamma f(s,t)]^p dt\right\}^{\frac{1}{p}} = t^{-\frac{1}{p}-\gamma-1}\left\{\int_t^\infty \left|y^{1+\gamma}\varphi(y)\right|^p\right\}^{\frac{1}{p}} dy.$$

The latter equalities, inserted in the following well-known inequality

$$\left\{\int_1^\infty \left[\int_1^\infty t^\gamma f(s,t)ds\right]^p dt\right\}^{\frac{1}{p}} \le \int_1^\infty \left\{\int_1^\infty [t^\gamma f(s,t)]^p\,dt\right\}^{\frac{1}{p}} ds$$

(see [HLP1, Theorem 202]) yield

$$\|\mathscr{V}_\infty\varphi|L_p([1,\infty),(1+x)^\gamma)\| \le 2^\gamma \int_1^\infty s^{-\frac{1}{p}-\gamma-1}\left\{\int_1^\infty \left|y^{1+\gamma}\varphi(y)\right|^p dy\right\}^{\frac{1}{p}} dt$$
$$\le \frac{2^\gamma}{\frac{1}{p}+\gamma}\|\varphi|L_p([1,\infty),(1+x)^{1+\gamma})\|$$

since $-\frac{1}{p}-\gamma<0$ (see (2.7)).

Thus, (3.16) is proved and implies continuity of the first embedding in (3.14).

The second embedding in (3.14) follows by density. In fact, as we already proved the embedding

$$L_{p'}(\mathbb{R}^+,x^{-\alpha}(1+x)^{1-\gamma+\alpha}) \subset \overrightarrow{L}_{p'}\,(\mathbb{R}^+,x^{-\alpha}(1+x)^{-\gamma+\alpha})$$

is continuous and dense. The spaces are reflexive and the embedding of the dual spaces

$$\overleftarrow{L}_p\,(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha}) \subset L_p(\mathbb{R}^+,x^\alpha(1+x)^{-1+\gamma-\alpha})$$

are continuous and dense as well. ∎

Corollary 3.4 *Let $a\in C(\dot{\mathbb{R}}^+)$; then*

$$aI\in\mathscr{L}\left(\overrightarrow{L}_p\,(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha})\right),\quad aI\in\mathscr{L}\left(\overleftarrow{L}_p\,(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha})\right),$$

provided

$$|a(x)-a(\infty)|\le M(1+x))^{-1},\quad x\in\mathbb{R}^+\quad M<\infty.$$

Proof. It suffices to represent

$$a\varphi=[a-a(\infty)]\varphi+a(\infty)\varphi$$

and apply Lemma 3.3 to the first summand, because the second summand, multiplication by a constant, is obviously continuous operator. ∎

Theorem 3.5 *Let $a(\lambda)$ be given by* (3.1) *and* (1.4) *hold. Then operators* (3.2) *and* (3.3) *are continuous.*

Operators (3.2) *and* (3.3) *are* FREDHOLM *or are invertible if and only if the corresponding operators*

$$W_{a^{(-)}}\ :\ L_p(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha})\longrightarrow L_p(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha}),\qquad(3.17)$$

$$W_{a^{(+)}}\ :\ L_p(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha})\longrightarrow L_p(\mathbb{R}^+,x^\alpha(1+x)^{\gamma-\alpha}),\qquad(3.18)$$

$$a^{(+)}(\lambda):=\frac{\lambda+i}{\lambda}a(\lambda)$$

are FREDHOLM *or are invertible, respectively.*

The pairs of operators (3.2) *and* (3.17), (3.3) *and* (3.18) *have the kernels and cokernels of equal dimension and equal indices.*

Proof (see [Pr1, § 5.2.3] for a similar proof). Let $b, d \in V_1(\mathbb{R})$ and either $b(\lambda)$ has a bounded analytic continuation $b(\lambda - i\sigma)$ in the lower half-plane $\sigma > 0$ or $d(\lambda)$ has a bounded analytic continuation $d(\lambda + i\sigma)$ in the upper half-plane $\sigma > 0$; then

$$W_{bd} = W_b W_d \tag{3.19}$$

(see[Du1, GF1]). Since

$$a(\lambda) = \frac{\lambda}{\lambda - i} a^{(-)}(\lambda) = \frac{\lambda}{\lambda + i} a^{(+)}(\lambda)$$

(see (3.1), (3.17) and (3.18)), we get

$$W_a = \mathscr{G}_- W_{a^{(-)}} = \mathscr{G}_+ W_{a^{(+)}} \tag{3.20}$$

(see (3.8) and (3.19)).

All claimed assertions follow from (3.20) since the operators

$$\mathscr{G}_- \ : \ L_p(\mathbb{R}^+, x^\alpha (1+x)^{\gamma-\alpha}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R}^+, (1+x)^{\gamma - al}),$$

$$\mathscr{G}_+ \ : \ \overleftarrow{L}_p\,(\mathbb{R}^+, x^\alpha (1+x)^{\gamma-\alpha}) \longrightarrow L_p(\mathbb{R}^+, (1+x)^{\gamma - al})$$

establish isomorphism (see Lemma (3.1)). ∎

Remark 3.6 *Defining the spaces*

$$\overrightarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta) := \left\{ \varphi \ : \ \varphi, \mathscr{V}_\infty \varphi \in C(\dot{\mathbb{R}}^+, (1+x)^\beta) \right\},$$

$$\overleftarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta) := \left\{ \psi + \mathscr{U}_0 \varphi \ : \ \varphi, \psi \in C(\dot{\mathbb{R}}^+, (1+x)^\beta) \right\},$$

for $\beta \in \mathbb{R}$ *and taking in* (3.1) $a, a^{(-)} \in W_\beta(\mathbb{R})$, *full analogies of Lemma* 3.1 *and of Theorem* 3.5 *can be proved for the convolution operators*

$$W_a \ : \ C(\dot{\mathbb{R}}^+, (1+x)^\beta) \longrightarrow \overrightarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta),$$

$$W_a \ : \overleftarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta) \longrightarrow C(\dot{\mathbb{R}}^+, (1+x)^\beta).$$

As for the analogies of Lemma 3.3 *and Corollary* 3.4 *we easily find that continuity*

$$[g - g(\infty)]I \ : \ C(\dot{\mathbb{R}}^+, (1+x)^\beta) \longrightarrow \overrightarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta),$$

$$[g - g(\infty)]I \ : \overleftarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta) \longrightarrow C(\dot{\mathbb{R}}^+, (1+x)^\beta),$$

$$gI \in \mathscr{L}\left(\overrightarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta)\right), \quad gI \in \mathscr{L}\left(\overleftarrow{C}\,(\dot{\mathbb{R}}^+, (1+x)^\beta)\right)$$

follows if

$$g \in C(\dot{\mathbb{R}}), \quad |g(x) - g(\infty)| \le M(1+x))^{-1-\varepsilon}, \quad x \in \mathbb{R}^+, \ \ M < \infty .$$

3.2 Convolutions on $\mathbb{R}$

Let $a \in PC_p(\mathbb{R})$ be non-elliptic, namely, as in (3.1). Then operator W_a^0 is not FREDHOLM in $L_p(\mathbb{R})$ and, moreover, has non-closed image Im W_a^0 (see [Du4, § 4]).

Let us consider the operators

$$\mathscr{V}_\infty\varphi(x) := \int_x^\infty \varphi(y)dy, \qquad dst\mathscr{V}_{-\infty}\varphi(t) := \int_{-\infty}^t \varphi(y)dy, \qquad (3.21)$$

$$\mathscr{F}_0\varphi := \int_{-\infty}^\infty \varphi(\tau)d\tau, \quad t \in \mathbb{R},$$

where the integrals are understood all as improper (see (3.5)). We define the space

$$\begin{aligned} \overrightarrow{L}_p(\mathbb{R}) &:= \{\varphi \in L_p(\mathbb{R}) \ : \ \mathscr{V}_\infty \in L_p(\mathbb{R})\} \\ &= \{\varphi \in L_p(\mathbb{R}) \ : \ \mathscr{V}_{\pm\infty} \in L_p(\mathbb{R}), \ \ \mathscr{F}_0\varphi = 0\} \end{aligned} \qquad (3.22)$$

and endow it with the norm

$$\|\varphi \mid \overrightarrow{L}_p(\mathbb{R})\| := \|\varphi \mid L_p(\mathbb{R})\| + \|\mathscr{V}_\infty\varphi \mid L_p(\mathbb{R})\| .$$

To justify the second definition in (3.22) let us prove that the conditions

$$\mathscr{V}_{-\infty}\varphi \in L_p(\mathbb{R}), \quad \mathscr{F}_0\varphi = 0$$

follow from the principal condition $\mathscr{V}_\infty\varphi \in L_p(\mathbb{R})$. In fact, the inclusion $\mathscr{V}_{-\infty}\varphi \in L_p(\mathbb{R})$ follows from the principal condition and from $\mathscr{F}_0\varphi = 0$, since

$$\mathscr{V}_{-\infty}\varphi(t) = \mathscr{F}_0\varphi - \mathscr{V}_\infty\varphi(t) = -\mathscr{V}_\infty\varphi(t) .$$

Thus, we have to prove only $\mathscr{F}_0\varphi = 0$. Since

$$\mathscr{F}_0\varphi = \lim_{t\to-\infty} \mathscr{V}_\infty\varphi(t),$$

$\mathscr{V}_\infty\varphi \in L_p(\mathbb{R})$ is absolutely continuous with derivative $(\mathscr{V}_\infty\varphi) = \varphi \in L_p(\mathbb{R})$, we get the result.

The embedding

$$\{\varphi \in C_0^\infty(\mathbb{R}) \ : \ \mathscr{F}_0\varphi = 0\} \subset \overrightarrow{L}_p(\mathbb{R})$$

is dense.

Let us prove that the convolution operator

$$\mathscr{G}_- := W^0_{g_-} \ : \ L_p(\mathbb{R}) \longrightarrow \overrightarrow{L}_p\,(\mathbb{R})\,, \quad g_-(\lambda) := \frac{\lambda}{\lambda - i}\,. \tag{3.23}$$

with vanishing symbol is bounded and, moreover, defines an isomorphism with the inverse operator written as follows

$$(W^0_{g_-})^{-1} = W_{g_-^{-1}} = I + \mathscr{V}_\infty \ : \overrightarrow{L}_p\,(\mathbb{R}) \longrightarrow L_p(\mathbb{R}) \tag{3.24}$$

(cf. Lemma 3.1). In fact, by the definition (3.22) the operator (3.24) is bounded and, due to obvious equality $W^0_g W^0_h = W^0_{gh}$ (see (2.1)) $W_{g_-^{-1}}$ is the inverse from the right to $W^0_{g_-}$:

$$W^0_{g_-} W_{g_-^{-1}} = W^0_{g_- g_-^{-1}} = I\,.$$

Let us prove that the inverse operator (3.24) is bounded. According to the definition (3.22) it suffices to prove that

$$\mathscr{V}_\infty W^0_{g_-}\,\varphi \in L_p(\mathbb{R}) \quad \text{provided} \quad \varphi \in L_p(\mathbb{R})\,.$$

Since

$$W^0_{g_-}\,\varphi(t) = \varphi(t) - \int\limits_x^\infty e^{t-\tau}\varphi(\tau)d\tau\,,$$

we proceed as follows

$$\begin{aligned}
\mathscr{V}_\infty W^0_{g_-}\,\varphi(t) &= \mathscr{V}_\infty\varphi(t) - \int\limits_t^\infty d\tau \int\limits_\tau^\infty e^{\tau-y}\varphi(y)dy \\
&= \mathscr{V}_\infty\varphi(t) - \int\limits_t^\infty \varphi(y)dy \int\limits_t^y e^{\tau-y}dy = \mathscr{V}_\infty\varphi(t) - \int\limits_t^\infty (1-e^{\tau-y})\varphi(y)dy \\
&= \int\limits_t^\infty e^{\tau-y}\varphi(y)dy = \varphi(t) - W^0_{g_-}\,\varphi(t)
\end{aligned} \tag{3.25}$$

and get the inclusion $\mathscr{V}_\infty W^0_{g_-}\,\varphi \in L_p(\mathbb{R})$ because φ, $W^0_{g_-}\,\varphi \in L_p(\mathbb{R})$. Moreover, (3.25) can also be written as follows

$$(I + V_\infty)W^0_{g_-}\,\varphi = \varphi\,,$$

which means, due to (3.24), $W^0_{g_-^{-1}} W^0_{g_-} = I$ and $W^0_{g_-}$ is invertible from the left.

Similarly to (3.14) is proved that the embeddings

$$\{\varphi \in L_p(\mathbb{R}, 1+|x|) \ : \ \mathscr{F}_0\varphi = 0\} \subset \overrightarrow{L}_p(\mathbb{R}) \subset L_p(\mathbb{R}, (1+|x|)^{-1})$$

are continuous and dense.

If $g \in L_\infty(\dot{\mathbb{R}})$ has the estimate

$$|g(x) - g(\infty)| \le M(1+|x|))^{-1}, \quad x \in \mathbb{R}, \ \ M < \infty, \tag{3.26}$$

the following multiplication operators are bounded

$$[g - g(\infty)]I \ : \ L_p(\mathbb{R}) \longrightarrow \overrightarrow{L}_p(\mathbb{R}), \qquad gI \ : \overrightarrow{L}_p(\mathbb{R}) \longrightarrow \overrightarrow{L}_p(\mathbb{R}). \tag{3.27}$$

Theorem 3.7 *Let $a(\lambda)$ be as in* (3.1). *Operator $W_a^0 \ : \ L_p(\mathbb{R}) \longrightarrow \overrightarrow{L}_p(\mathbb{R})$ is* Fredholm *if and only if $W_{a^{(-)}}^0$ is* Fredholm *in the space $L_p(\mathbb{R})$, which reads*

$$\inf_{\lambda\in\mathbb{R}} |a^{(-)}(\lambda)| > 0 \tag{3.28}$$

(*see* (2.5)). *If* (3.28) *holds, $(a^{(-)})^{-1} \in PC_p^{N\times N}(\dot{\mathbb{R}})$ and the inverse is*

$$(W_a^0)^{-1} := W^0_{(a^{(-)})^{-1}}(I + \mathscr{V}_\infty) \ : \overrightarrow{L}_p(\mathbb{R}) \longrightarrow L_p(\mathbb{R}). \tag{3.29}$$

Proof. Due to (3.23), (3.24) the proof can immediately be reduced to the investigation of the operator $W_{a^{(-)}}^0$ in the space $L_p(\mathbb{R})$. In this case the Fredholm criteria is known (see (2.5)). ∎

3.3 Operators $\mathscr{V}_{\pm\infty}$, $\mathscr{V}_{t_j}$

We remind that $\Gamma_1 := \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$ is the unit circumference, $\Xi := \{\zeta_1, \dots, \zeta_n\} \subset \Gamma_1$ is the conformal image of all knots of Γ and Ξ_{ow} is the subset of Ξ (conformal image of all outward peaks of Γ; see (1.74)–(1.91)); $\Gamma_{1\zeta_j} = \Gamma^-_{1\zeta_j} \cup \Gamma^+_{1\zeta_j}$ is a fixed neighbourhood of ζ_j (see (1.74)–(1.91)). We use $L_p(\Gamma_1, \{\zeta_j\})$ for the space $L_p(\Gamma_1, \Xi_{ow})$ when $\Xi_{ow} = \{\zeta_j\}$ consists of a single knot.

For a Banach space $\mathbb{X}$ by $\mathbb{X}^n$ we denote the spaces of vector–elements $\Psi = (\psi_1, \dots, \psi_n)$ with components $\psi_j \in \mathbb{X}$. Let

$$L_p^2(\mathbb{R}, \{\infty\}) := \left\{\Phi = (\varphi_1, \varphi_2) \in L_p^2(\mathbb{R}) \ : \ \widetilde{\mathscr{V}}_\infty\Phi \in L_p(\mathbb{R}^+)\right\}, \tag{3.30}$$

$$\widetilde{\mathscr{V}}_\infty\Phi := \begin{bmatrix} e^{\frac{2\pi}{p}i}\mathscr{V}_\infty & -\mathscr{V}_\infty \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \mathscr{V}_\infty[e^{-\frac{\pi}{p}i}\varphi_1 - \varphi_2]$$

(see (3.4) for $\mathscr{V}_\infty$) denote the subset of $L_p^2(\mathbb{R})$ with the appropriate norm

$$\|\Phi\big|L_p^2(\mathbb{R}, \{\infty\})\| := \|\Phi\big|L_p(\mathbb{R})\| + \|\widetilde{\mathscr{V}}_\infty\Phi\big|L_p(\mathbb{R}^+)\|.$$

Lemma 3.8 *There exists an isomorphism of spaces*

$$\begin{aligned} \mathscr{Z}_{p\zeta_j} &: L_p(\Gamma_1) \longrightarrow L_p^2(\mathbb{R})\,, \\ \mathscr{Z}_{p\zeta_j} &: L_p(\Gamma_1,\{\zeta_j\}) \longrightarrow L_p^2(\mathbb{R},\{\infty\})\,, \end{aligned} \tag{3.31}$$

such that the operators $\widetilde{\mathscr{V}}_{\zeta_j}$ in (1.90) *and $\widetilde{\mathscr{V}}_{\infty}$ in* (3.4) *are equivalent*

$$\mathscr{Z}_{p\zeta_j}\widetilde{\mathscr{V}}_{\zeta_j}\mathscr{Z}_{p\zeta_j}^{-1} = g_j\widetilde{\mathscr{V}}_{\infty}h_j I = g_j\widetilde{\mathscr{V}}_{\infty} + R_j\,, \tag{3.32}$$

where the functions $g_j^{\pm 1}$, $h_j^{\pm 1} \in C^{\infty}(\dot{\mathbb{R}})$ are non-vanishing

$$g_j(x) := \left(\frac{1-ie^{-x}}{1+e^{-2x}}\right)^{\frac{1}{p}}\,, \quad h_j(x) := \frac{(1+e^{-2x})^{\frac{1}{p}}}{(1-ie^{-x})^{1+\frac{1}{p}}}$$

and the operator $R_j : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is bounded.

Proof. The transformations

$$\begin{aligned} \mathscr{Z}_{\zeta_j}\varphi(x) &:= |\varkappa'_{\zeta_j}(x)|^{\frac{1}{p}}\varphi(\varkappa_{\zeta_j}(x)) = 2^{\frac{1}{p}}(x^2+1)^{\frac{1}{p}}\varphi\left(-\zeta_j\frac{x-i}{x+i}\right)\,, \\ \mathscr{Z}_p\psi(\lambda) &:= \left(e^{-\frac{\lambda}{p}}\psi(e^{-\lambda}), e^{-\frac{\lambda}{p}}\psi(-e^{-\lambda})\right)\,, \quad x,\lambda\in\mathbb{R}, \end{aligned} \tag{3.33}$$

where the first one is based on the Kelly transformation

$$\varkappa_{\zeta_j}(x) := -\zeta_j\frac{x-i}{x+i} \;:\; \mathbb{R} \longrightarrow \Gamma_1\,, \quad \varkappa_{\zeta_j}(0) = \zeta_j\,,$$

establish isometric isomorphisms

$$\begin{aligned} \mathscr{Z}_{\zeta_j} &: L_p(\Gamma_1) \to L_p(\mathbb{R})\,, \quad \|\mathscr{Z}_{\zeta_j}\varphi\big|L_p(\mathbb{R})\| = \|\varphi\big|L_p(\Gamma_1)\|\,, \\ \mathscr{Z}_p &: L_p(\mathbb{R}) \to L_p^2(\mathbb{R})\,, \quad \|\mathscr{Z}_p\psi\big|L_p^2(\mathbb{R})\| = \|\psi\big|L_p(\mathbb{R})\| \end{aligned} \tag{3.34}$$

and have the following inverses

$$\mathscr{Z}_{\zeta_j}^{-1}\psi(\zeta) := |(\varkappa_{\zeta_j}^{-1})'(x)|^{\frac{1}{p}}\psi(\varkappa_{\zeta_j}^{-1}(x)) = |\zeta+\zeta_j|^{-\frac{2}{p}}\psi\left(-i\frac{\zeta-\zeta_j}{\zeta+\zeta_j}\right)\,, \tag{3.35}$$

$$\mathscr{Z}_p^{-1}\Phi(x) := \chi_-(x)(-x)^{-\frac{1}{p}}\varphi_2(-\log(-x)) + \chi_+(x)x^{-\frac{1}{p}}\varphi_1(-\log x)\,,$$

where $\Phi = (\varphi_1,\varphi_2)^{\top}$ and $\chi_{\pm}(x)$ are the characteristic functions of $\mathbb{R}^{\pm} \subset \mathbb{R}$. The transformation

$$\mathscr{Z}_{p\zeta_j} := \mathscr{Z}_p\mathscr{Z}_{\zeta_j} \tag{3.36}$$

establishes the first of the claimed isomorphisms in (3.31).

To prove that $\mathscr{Z}_{p\zeta_j}$ arranges the second isomorphisms in (3.31) as well, let us consider the following intermediate space

$$L_p^2(\mathbb{R},\{0\}) := \left\{\psi\in L_p(\mathbb{R}) \;:\; \widetilde{\mathscr{V}}_0\psi \in L_p(\mathbb{R}^+)\right\}\,, \tag{3.37}$$

$$\widetilde{\mathscr{V}}_0\psi(x) := \mathscr{V}_0[e^{-\frac{\pi}{p}i}\psi(x) - \psi(-x)]\,,$$

where the operator $\mathscr{V}_0$ is defined as follows (cf. (3.4)):

$$\mathscr{V}_0\psi(x) := \int\limits_0^x \Big(\frac{y}{x}\Big)^{\frac{1}{p}} \psi(y)\frac{dy}{y}\,. \tag{3.38}$$

It is easy to verify directly the following connection

$$\mathscr{Z}_p\widetilde{\mathscr{V}}_0\mathscr{Z}_p^{-1} = \widetilde{\mathscr{V}}_\infty \tag{3.39}$$

(see (3.34)). Moreover, $\mathscr{Z}_p$ establishes isometric isomorphisms

$$\begin{gathered}\mathscr{Z}_p \;:\; L_p(\mathbb{R},\{0\}) \to L_p^2(\mathbb{R},\{\infty\})\,,\\ \|\mathscr{Z}_p\psi\big|L_p^2(\mathbb{R},\{\infty\})\| = \|\psi\big|L_p(\mathbb{R},\{0\})\|\,.\end{gathered} \tag{3.40}$$

Therefore, to justify the second isomorphism in (3.31) we just need to verify

$$\mathscr{Z}_{\zeta_j}\mathscr{V}_{\zeta_j}\mathscr{Z}_{\zeta_j}^{-1} = g_0\mathscr{V}_0h_0I\,, \tag{3.41}$$

where $g_0^{\pm1}, h_0^{\pm1} \in C^\infty(\mathbb{R})$ are non-vanishing functions

$$g_0(x) := \left(\frac{1-ix}{1+x^2}\right)^{\frac{1}{p}}\,, \quad h_0(x) := \frac{(1+x^2)^{\frac{1}{p}}}{(1-ix)^{1+\frac{1}{p}}}$$

because applying equivalence (3.39) to equality (3.41) we get (3.32) immeadiately.

To prove the second isomorphism in (3.31) we proceed as follows:

$$\begin{aligned}(\mathscr{Z}_{\zeta_j}\chi_{\zeta_j}\mathscr{V}_{\zeta_j}\mathscr{Z}_{\zeta_j}^{-1}\psi)(x) = \Big|\varkappa'_{\zeta_j}(x)\Big|^{\frac{1}{p}} \int\limits_{\zeta_j\varkappa_{\zeta_j}(x)} \left[\frac{\tau-\zeta_j}{\varkappa_{\zeta_j}(x)-\zeta_j}\right]^{\frac{1}{p}}\\ \times\Big|(\varkappa_{\zeta_j}^{-1})'(\tau)\Big|^{\frac{1}{p}}\psi(\varkappa_{\zeta_j}^{-1}(\tau))\frac{d\tau}{\tau-\zeta_j}\,;\end{aligned} \tag{3.42}$$

inserting $\tau = \varkappa_{\zeta_j}(y)$, $d\tau = \varkappa'_{\zeta_j}(y)dy$ and taking into account the equalities

$$\begin{gathered}(\varkappa_{\zeta_j}^{-1})'(\varkappa_{\zeta_j}(y)) = [\varkappa'_{\zeta_j}(y)]^{-1}\,, \quad \frac{\varkappa_{\zeta_j}(x)-\zeta_j}{\varkappa_{\zeta_j}(x)+\zeta_j} = ix\,,\\ \varkappa_{\zeta_j}(x)-\zeta_j = \frac{-2\zeta_jx}{x+i}\,, \quad \varkappa'_{\zeta_j}(x) = \frac{-2i\zeta_j}{(x+i)^2}\,,\end{gathered} \tag{3.43}$$

we continue as follows

$$(\mathscr{Z}_{\zeta_j}\chi_{\zeta_j}\mathscr{V}_{\zeta_j}\mathscr{Z}_{\zeta_j}^{-1}\psi)(x) = \int\limits_z^x \left[\frac{\varkappa_{\zeta_j}(y)-\zeta_j}{\varkappa_{\zeta_j}(x)-\zeta_j}\right]^{\frac{1}{p}}\left|\frac{\varkappa'_{\zeta_j}(x)}{\varkappa'_{\zeta_j}(y)}\right|^{\frac{1}{p}}\frac{\varkappa'_{\zeta_j}(y)\psi(y)dy}{\varkappa_{\zeta_j}(y)-\zeta_j}$$

$$= g_0(x) \int\limits_0^x \left(\frac{y}{x}\right)^{\frac{1}{p}} h_0(y)\psi(y)\frac{dy}{y} g_0 \mathscr{V}_0 h_0 \psi(x)\,,$$

$$g_0(x) := \left(\frac{1-ix}{1+x^2}\right)^{\frac{1}{p}}\,, \quad h_0(x) := \frac{(1+x^2)^{\frac{1}{p}}}{(1-ix)^{1+\frac{1}{p}}} \tag{3.44}$$

and we get (3.41).

Boundedness of

$$R_j := g_j \widetilde{\mathscr{V}}_\infty [h_j - 1]I \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

follows since $h_j(x) - 1 = h_j(x) - h_j(+\infty) = O\left(e^{-\frac{x}{p}}\right)$ as $x \to +\infty$ which yields the boundedness $[h_j - 1]I \;:\; L^2(\mathbb{R}) \to \vec{L}^{\,2}(\mathbb{R})$ (see (3.27)). ∎

Proof of Lemma 1.25. Let us apply the isomorphism $\mathscr{Z}_{p\zeta_j}$, defined in (3.31), (3.33). Then $\varphi, \log(\zeta - \zeta_j)\varphi \in L_p(\Gamma_{1\zeta_j})$ for all $\zeta_j \in \Xi_{ow}$ imply

$$\mathscr{Z}_{p\zeta_j}\varphi \in L_p^2(\mathbb{R}^+),\;\; (\mathscr{Z}_{p\zeta_j} \log(\zeta - \zeta_j)\varphi)(x) = \log\left(-\zeta_j \frac{e^{-x}}{e^{-x}+i}\right)(\mathscr{Z}_{p\zeta_j}\varphi)(x)$$
$$= \left(-x + \log\frac{-\zeta_j}{e^{-x}+i}\right)(\mathscr{Z}_{p\zeta_j}\varphi)(x) \in L_p^2(\mathbb{R}^+)$$

(see (3.33)); due to Lemma 3.3 $\mathscr{Z}_{p\zeta_j}\varphi \in \vec{L}^{\,2}_p(\mathbb{R}^+)$. Applying the inverse isomorphism $\mathscr{Z}^{-1}_{p\zeta_j}$ (see (3.31), (3.35)) we find $\varphi = \mathscr{Z}^{-1}_{p\zeta_j}\mathscr{Z}_{p\zeta_j}\varphi \in \vec{L}^{\,2}_p(\Gamma_{1\zeta_j}, \{\zeta_j\})$.

The remainder claims of the Lemma (see (1.98)) follow from the proved part as Corollary 3.4 from Lemma 3.3. ∎

Remark 3.9 *Due to the above established isomorphism* (3.40) *and to Corollary* 3.4 *if a function* $g(x)$ *has the property*

$$g \in C(\mathscr{J}), \quad g(x) - g(0) = O((1 - \log x)^{-1}), \quad \mathscr{J} := (-c, c) \subset \mathbb{R} \tag{3.45}$$

the following multiplication operators (see (3.37)*)*

$$gI \;:\; L_p(\mathscr{J}, \{0\}) \longrightarrow L_p(\mathscr{J}, \{0\}),$$
$$[g - g(0)]I \;:\; L_p(\mathscr{J}) \longrightarrow L_p(\mathscr{J}, \{0\})$$

are bounded.

3.4 Equations on the circumference (example)

Let Γ_1, $\Xi := \{\zeta_1, \ldots, \zeta_n\} \subset \Gamma_1$, $\Xi_{ow} \subset \Xi$, $\Gamma_{1\zeta_j}$ and $\Gamma^{\pm}_{1\zeta_j}$ be the same as in § 3.3.

We us consider, as an example, the following operator with fixed singularities at Ξ in the kernel

$$A_\Xi\varphi(\zeta) = \varphi(\zeta) + \sum_{j=1}^{n} \chi^+_{\zeta_j}(\zeta)\tfrac{\mu_j\zeta_j}{\pi} \int\limits_{\Gamma^+_{1\zeta_j}} \left(\tfrac{\zeta-\zeta_j}{\tau-\zeta_j}\right)^{\gamma_j} \tfrac{\varphi(\tau)d\tau}{\zeta_j^2-\tau t}\,, \quad \zeta\in\Gamma_1\,, \tag{3.46}$$

where $\chi^+_{\zeta_j}(t)$ is the characteristic function of the arc $\Gamma^+_{1\zeta_j} \subset \Gamma_{1\zeta_j} \subset \Gamma_1$ and

$$\mu_j = \begin{cases} \sin\pi\left(\dfrac{1}{p}+\gamma_j\right) & \text{for} \quad \xi\in\Xi_{ow}\,, \\ \\ \sin\pi\left(\dfrac{1}{p}+\gamma'_j\right) & \text{for} \quad \xi\notin\Xi_{ow}\,, \end{cases} \tag{3.47}$$

$$-\frac{1}{p} < \gamma_j < 1-\frac{1}{p}\,, \quad \xi\in\Xi_{ow}\,,$$

$$-\frac{1}{p} < \gamma'_k \neq \gamma_k < 1-\frac{1}{p}\,, \quad \xi\notin\Xi_{ow}\,. \tag{3.48}$$

Theorem 3.10 *Let conditions* (3.47) *and* (3.48) *hold. Then the operator*

$$A_\Xi \,:\, L_p(\Gamma_1) \to L_p(\Gamma_1, \Xi_{ow})\,, \quad 1<p<\infty \tag{3.49}$$

is FREDHOLM *provided*

$$\frac{1}{p}+\gamma_j \neq \frac{1}{2} \quad \text{for all} \quad j=1,\dots,n \tag{3.50}$$

and then

$$\dim \text{Ker } \mathrm{A}_\Xi = \sum_{\sigma_\mathrm{j}>0} \sigma_\mathrm{j}\,, \quad \dim \text{Coker } \mathrm{A}_\Xi = -\sum_{\sigma_\mathrm{j}<0} \sigma_\mathrm{j}\,, \tag{3.51}$$

where

$$\sigma_j = \begin{cases} 0 & \text{for} \quad \dfrac{1}{2} < \dfrac{1}{p}+\gamma_j < 1\,, & \zeta_j\in\Xi_{ow}\,, \\ 0 & \text{for} \quad \gamma_j\in(\gamma'_j, 1-\gamma'_j)\,, & \zeta_j\notin\Xi_{ow}\,, \\ -1 & \text{for} \quad 0 < \dfrac{1}{p}+\gamma_j < \dfrac{1}{2}\,, & \zeta_j\in\Xi_{ow}\,, \\ 1 & \text{for} \quad \gamma_j > \max\{\gamma'_j, 1-\gamma'_j\}\,, & \zeta_j\notin\Xi_{ow}\,, \\ -1 & \text{for} \quad \gamma_j < \min\{\gamma'_j, 1-\gamma'_j\}\,, & j=m+1,\dots,n\,. \end{cases} \tag{3.52}$$

In particular, A_Ξ in (3.49) *is invertible provided*

$$\frac{1}{2} < \frac{1}{p}+\gamma_j < 1 \quad \text{for all} \quad \xi\in\Xi_{ow}$$

$$\text{and} \quad \gamma_j\in(\gamma'_j, 1-\gamma'_j) \quad \text{for all} \quad \xi\notin\Xi_{ow}\,. \tag{3.53}$$

Proof. Note that since $\Gamma^+_{1\zeta_k} \cap \Gamma^+_{1\zeta_j} = \emptyset$ for $k \neq j$, we have

$$A_\Xi = \prod_{j=1}^{n} A_{\zeta_j}\,,$$
$$A_{\zeta_j} := \varphi(\zeta) + \chi^+_{\zeta_j}(\zeta)\frac{\mu_j\zeta_j}{\pi}\int\limits_{\Gamma^+_{1\zeta_j}} \left(\frac{\zeta-\zeta_j}{\tau-\zeta_j}\right)^{\gamma_j} \frac{\varphi(\tau)d\tau}{\zeta_j^2-\tau t}\,, \quad \zeta\in\Gamma_1\,. \tag{3.54}$$

Therefore it suffices to prove the Theorem for a single knot $\Xi = \{\zeta_j\}$.

We will apply the isomorphisms of spaces

$$\begin{aligned} \mathscr{Z}_{\zeta_j} &: L_p(\Gamma^+_{1\zeta_j}) \longrightarrow L_p(\mathscr{I})\,, \\ \widetilde{\mathscr{Z}_p} &: L_p(\mathscr{I}) \longrightarrow L_p(\mathbb{R}^+)\,, \end{aligned} \tag{3.55}$$

where $\mathscr{I} = [0,1]$ and $\mathscr{Z}_{\zeta_j}$ is defined in (3.33), while

$$\widetilde{\mathscr{Z}_p}\varphi(x) := e^{-\frac{x}{p}}\varphi(e^{-x})\,. \tag{3.56}$$

We have assumed, without loss of generality, that

$$\Gamma^+_{1\zeta_j} = \left\{e^{i\vartheta}\zeta_j \;:\; 0<\vartheta<\pi\right\}$$

is the half-circumference; otherwise we will use another Kelly transformation

$$\varkappa_{\zeta_j}(x) := -\zeta_j\frac{x-i\cot\frac{\vartheta_j}{2}}{x+i\cot\frac{\vartheta_j}{2}} \;:\; \mathscr{I}=[0,1] \longrightarrow \Gamma^+_{1\zeta_j} = \zeta_j(\zeta_j e^{i\vartheta_j}) \subset \Gamma_1$$

while defining the isomorphism $\mathscr{Z}_{\zeta_j}$ in (3.33). The operators $\mathscr{Z}_{\zeta_j}$ and $\widetilde{\mathscr{Z}_p}$, besides (3.55) and similarly to (3.31), (3.40), establish the following isomorphisms

$$\begin{aligned} \mathscr{Z}_{\zeta_j} &: L_p(\Gamma^+_{1\zeta_j}, \{\zeta_j\}) \longrightarrow L_p(\mathscr{I}, \{0\})\,, \\ \widetilde{\mathscr{Z}_p} &: L_p(\mathscr{I}, \{0\}) \longrightarrow \vec{L}_{\,p}(\mathbb{R}^+)\,. \end{aligned} \tag{3.57}$$

Lifting the operator (3.54) to the equivalent operator first by the isomorphism $\mathscr{Z}_{\zeta_j}$, we get, by applying (3.43),

$$\widetilde{B}_{\zeta_j}\psi(x) := (\mathscr{Z}_{\zeta_j}A_{\zeta_j}\mathscr{Z}_{\zeta_j}^{-1}\psi)(x) = \psi(x) + \frac{\mu_j\zeta_j\left|\varkappa'_{\zeta_j}(x)\right|^{\frac{1}{p}}}{\pi}\int\limits_{\Gamma_{1\zeta_j}}\left(\frac{\varkappa_{\zeta_j}(x)-\zeta_j}{\tau-\zeta_j}\right)^{\gamma_j}$$
$$\times\left|(\varkappa_{\zeta_j}^{-1})'(\tau)\right|^{\frac{1}{p}}\psi(\varkappa_{\zeta_j}^{-1}(\tau))\frac{d\tau}{\zeta_j^2-\tau\varkappa_{\zeta_j}(x)}$$
$$= \psi(x) + \frac{\mu_j\zeta_j}{\pi}\int_0^1\left|\frac{\varkappa'_{\zeta_j}(x)}{\varkappa'_{\zeta_j}(y)}\right|^{\frac{1}{p}}\left(\frac{\varkappa_{\zeta_j}(x)-\zeta_j}{\varkappa_{\zeta_j}(y)-\zeta_j}\right)^{\gamma_j}\frac{\varkappa'_{\zeta_j}(y)\psi(y)dy}{\zeta_j^2-\varkappa_{\zeta_j}(y)\varkappa_{\zeta_j}(x)}$$

$$= \psi(x) - \frac{\mu_j}{\pi} \int\limits_0^1 \left[\frac{x}{y}\right]^{\gamma_j} \left[\frac{x+i}{y+i}\right]^{1-\gamma_j} \left[\frac{x^2+1}{y^2+1}\right]^{\frac{1}{p}} \frac{\psi(y)dy}{y+x} = g_j B_{\zeta_j} g_j^{-1} \psi(x)$$

for $x \in \mathscr{I}$, where

$$g_j(x) := \frac{(x+i)^{1-\gamma_j}}{(1+x^2)^{\frac{1}{p}}}\,, \quad B_{\zeta_j}\psi(x) := \psi(x) - \frac{\mu_j}{\pi} \int\limits_0^1 \left[\frac{x}{y}\right]^{\gamma_j} \frac{\psi(y)dy}{y+x} \tag{3.58}$$

and $g_j^{\pm} \in C^\infty(\mathscr{I})$ satisfy condition (3.45). Therefore we can detach invertible operators $g_j^{\pm 1} I$ and study the equivalent operators

$$\begin{array}{c} B_{\zeta_j} \ : \ L_p(\mathscr{I}) \longrightarrow L_p(\mathscr{I}, \{0\}) \quad \text{for} \quad \zeta_j \in \Xi_{ow}\,, \\ B_{\zeta_j} \ : \ L_p(\mathscr{I}) \longrightarrow L_p(\mathscr{I}) \quad \text{for} \quad \zeta_j \notin \Xi_{ow}\,. \end{array}$$

The operator B_{ζ_j} can be lifted further, now by $\mathscr{Z}_p$, to the following equivalent operator

$$\begin{array}{l} W_{\mathscr{B}_{\zeta_j}} = \mathscr{Z}_p B_{\zeta_j} \mathscr{Z}_p^{-1} \ : \ L_p(\mathbb{R}^+) \longrightarrow \vec{L}_p(\mathbb{R}^+) \quad \text{for} \quad \zeta_j \in \Xi_{ow}\,, \\ W_{\mathscr{B}_{\zeta_j}} = \mathscr{Z}_1 B_{\zeta_j} \mathscr{Z}_1^{-1} \ : \ L_p(\mathbb{R}^+) \longrightarrow L_p(\mathbb{R}^+) \quad \text{for} \quad \zeta_j \notin \Xi_{ow} \end{array} \tag{3.59}$$

(see (3.56)), which turn out to be convolutions. In fact,

$$(\mathscr{Z}_1 B_{\zeta_j} \mathscr{Z}_1^{-1}\varphi)(x) = \varphi(x) - \frac{\mu_j}{\pi} \int\limits_0^1 e^{-\frac{x}{p}} \left[\frac{e^{-x}}{y}\right]^{\gamma_j} \frac{z^{-\frac{1}{p}}\varphi(-\log z)dz}{z+e^{-x}}$$

$$= \varphi(x) - \frac{\mu_j}{\pi} \int\limits_0^\infty e^{(y-x)\left(\frac{1}{p}+\gamma_j\right)} \frac{\varphi(y)dy}{1+e^{y-x}} = W_{\mathscr{B}_{\zeta_j}}\varphi(x)\,,$$

where

$$\mathscr{B}_{\zeta_j}(\lambda) := 1 - \frac{\mu_j}{\pi} \mathscr{F}_{t\to\lambda}\left[\frac{e^{-\left(\frac{1}{p}+\gamma_j\right)t}}{1+e^{-t}}\right] = 1 - \frac{\mu_j i}{\sinh \pi\left[\left(\frac{1}{p}+\gamma_j\right)i + \lambda\right]}$$

$$= 1 - \frac{\mu_j}{\sin \pi\left(\frac{1}{p}+\gamma_j - i\lambda\right)}\,, \quad \lambda \in \mathbb{R}\,, \ \ j = 0, \dots, n \tag{3.60}$$

(see [Du1, Ch. II, § 1]).

First let $\zeta_j \notin \Xi_{ow}$; then (see (3.47))

$$\mathscr{B}_{\zeta_j}(\lambda) := 1 - \frac{\sin \pi\left(\frac{1}{p}+\gamma_j'\right)}{\sin \pi\left(\frac{1}{p}+\gamma_j - i\lambda\right)}\,, \quad \lambda \in \mathbb{R}\,, \quad \gamma_j' \neq \gamma_j\,.$$

From the property $\overline{\mathscr{B}_{\zeta_j}(\lambda)} = \mathscr{B}_{\zeta_j}(-\lambda)$ we easily conclude that $\mathscr{B}_{\zeta_j}(\lambda) = 0$ implies $\lambda = 0$ and, due to conditions (3.48),

$$\inf_{\lambda\in\mathbb{R}} |\mathscr{B}_{\zeta_j}(\lambda)| > 0 \quad \text{for} \quad \zeta_j \notin \Xi_{ow}\,.$$

Since $\mathscr{B}_{\zeta_j}(\lambda)$ depends continuously on the parameter $\beta_j := \frac{1}{p} + \gamma_j$, $0 < \beta_j < 1$, the index $\operatorname{ind}\mathscr{B}_{\zeta_j}$ might have at most 3 different values. For $\gamma_j \in (\gamma_j', 1-\gamma_j')$ we apply the homotopy

$$\mathscr{B}_{j,\mu}(\lambda) := 1 - \mu\frac{\sin\pi\left(\frac{1}{p}+\gamma_j'\right)}{\sin\pi\left(\frac{1}{p}+\gamma_j - i\lambda\right)} \neq 0 \quad \text{for} \quad \lambda\in\dot{\mathbb{R}}\,, \quad 0\le\mu\le 1\,,$$

since $\mathscr{B}_{j,\mu}(\lambda) \neq 0$ for all $\lambda \in \dot{\mathbb{R}}$ and $\mu \in [0,1]$ we conclude $\operatorname{ind}\mathscr{B}_{\zeta_j} = \operatorname{ind}\mathscr{B}_{j,1} = \operatorname{ind}\mathscr{B}_{j,0} = 0$.

For $\gamma_j < \min\{\gamma_j', 1-\gamma_j'\}$ and for $\gamma_j > \max\{\gamma_j', 1-\gamma_j'\}$ it is sufficient to calculate the index only for one value of parameters in each case. The images of the test functions on the complex plane are plotted on Fig. 5 in the Appendix with the arrows showing the orientation of the image when the argument λ ranges through $\mathbb{R}$ from $-\infty$ to ∞.

Finally we get

$$\operatorname{ind}\mathscr{B}_{\zeta_j} = \begin{cases} 1 & \text{if} \quad \gamma_j < \min\{\gamma_j', 1-\gamma_j'\}\,, \\ 0 & \text{if} \quad \gamma_j \in (\gamma_j', 1-\gamma_j')\,, \\ -1 & \text{if} \quad \gamma_j > \max\{\gamma_j', 1-\gamma_j'\}\,, \end{cases} \tag{3.61}$$

for $\zeta_j \notin \Xi_{ow}$ (cf. [Du1, Du3]).

Next let $\zeta_j \in \Xi_{ow}$. Then (see (3.47))

$$\mathscr{B}_{\zeta_j}(\lambda) := 1 - \frac{\sin\pi\left(\frac{1}{p}+\gamma_j\right)}{\sin\pi\left(\frac{1}{p}+\gamma_j - i\lambda\right)} = \frac{\lambda}{\lambda - i}\mathscr{B}_j^0(\lambda)$$

and

$$\mathscr{B}_j^0(0) = \lim_{\lambda\to 0}\frac{\lambda - i}{\lambda}\mathscr{B}_j^0(\lambda) = -i\mathscr{B}_j'(0)$$
$$= -\pi\cot\pi\left(\frac{1}{p}+\gamma_j\right) \neq 0 \quad \text{iff} \quad \frac{1}{p}+\gamma_j \neq \frac{1}{2}\,.$$

Therefore,

$$\inf_{\lambda\in\mathbb{R}} |\mathscr{B}_j^0(\lambda)| > 0 \quad \text{iff} \quad \frac{1}{p}+\gamma_j \neq \frac{1}{2}\,, \quad \zeta_j \in \Xi_{ow}\,. \tag{3.62}$$

Further we find easily that $\operatorname{ind}\mathscr{B}_j^0$ might have at most two different values, $\mathscr{B}_j^0(\pm\infty) = 1$, $\pm\mathscr{B}_j^0(0) > 0$ for $\pm\left(\frac{1}{p}+\gamma_j-\frac{1}{2}\right) > 0$ and $\pm\operatorname{Im}\mathscr{B}_j^0(\lambda) > 0$ for

$\pm\lambda > 0$. The images of the test functions on the complex plane are plotted on Fig. 6 in Appendix with the arrows showing the orientation of the image when the argument λ ranges through $\mathbb{R}$ from $-\infty$ to ∞. These tests show that

$$\operatorname{ind} \mathscr{B}_j^0 = \begin{cases} 1 & \text{if} \quad 0 < \frac{1}{p} + \gamma_j < \frac{1}{2}\,, \\ 0 & \text{if} \quad \frac{1}{2} < \frac{1}{p} + \gamma_j < 1 \quad \text{for } \zeta_j \in \Xi_{ow}\,. \end{cases}$$

According to Theorems 2.5 and 3.5 we get: the operator $W_{\mathscr{B}_{\zeta_j}}$ in (3.59) is FREDHOLM iff conditions (3.48) and (3.50) hold (see (3.51) and (3.62)) and $\operatorname{Ind} W_{\mathscr{B}_{\zeta_j}} = -\operatorname{ind} \mathscr{B}_j^0 = \sigma_j$ for $\zeta_j \in \Xi_{ow}$ (see (3.61)), $\operatorname{Ind} W_{\mathscr{B}_{\zeta_j}} = -\operatorname{ind} \mathscr{B}_{\zeta_j} = \sigma_j$ for $\zeta_j \notin \Xi_{ow}$ (see (3.61)), where σ_j is defined in (3.52). ∎

4 Elliptic boundary integral equations

Let Γ be as in § 1.1, the weight function $\rho(t)$ be defined in (1.2).

For our purposes we need to define the order of cusp: $\sigma_j > 0$ is called the order of a cusp $t_j \in \Gamma$ if there exists $q_j \neq 0$ such that

$$\arg \frac{\tau_-(t_j, r) - t_j}{\tau_+(t_j, r) - t_j} = q_j r^{\sigma_j} + o\left(r^{\sigma_j}\right) \quad \text{as} \quad r \to 0,$$

where $\tau_-(t_j, r) \in \Gamma_{j-1}$ and $\tau_+(t_j, r) \in \Gamma_j$ are equidistant points $|\tau_\pm(t_j, r) - t_j| = r$ (see Fig. 3).

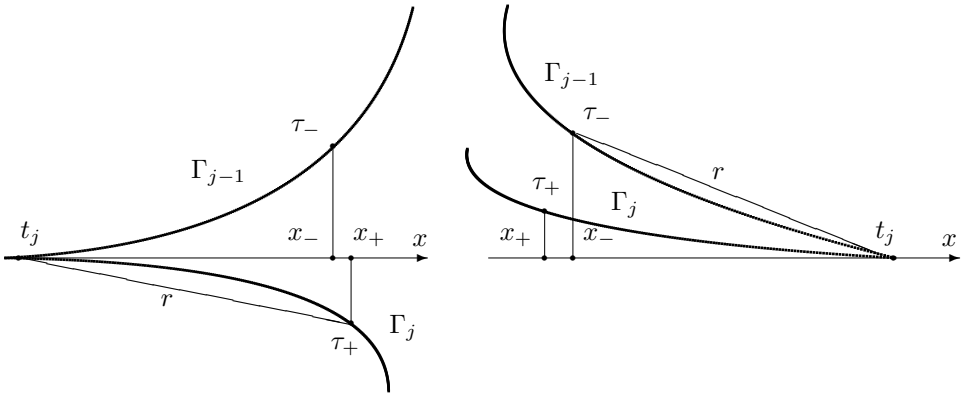

Fig. 3

The obvious equivalent condition is

$$\tau_-(t_j, r) - \tau_+(t_j, r) = q_j r^{1+\sigma_j} + o\left(r^{1+\sigma_j}\right) \quad \text{as} \quad r \to 0\,.$$

Further equivalent definitions of the order can be found in [DLS1].

Throughout this section we assume the orders of cusps are all equal 1

$$\text{if} \quad \gamma_j = 0 \quad \text{or} \quad \gamma_j = 2\,, \quad \text{then} \quad \sigma_j = \sigma(t_j) = 1 \qquad (4.1)$$

$$\text{for all} \quad j = 1, \dots, n$$

(see (3.2)) and will investigate the following integral equations:

$$A_0\varphi = a_0\varphi + a_1 S_\Gamma + a_2 W_{\Gamma,0}\varphi + a_3 W^*_{\Gamma,0}\varphi + a_4 \partial_t V_\Gamma \varphi = f \qquad (4.2)$$

with $N \times N$ matrix coefficients a_0, a_1, a_2, a_3, $a_4 \in PC^{N\times N}(\Gamma)$ (a_0, a_1, a_2, a_3, $a_4 \in PH^{N\times N}_\mu(\Gamma)$) in the vector space $L^N_p(\Gamma,\rho)$ (in the vector space $(H^0_\mu)^N(\Gamma,\rho)$, respectively, provided Γ has no cusps $0 < \gamma_j < 2$, $j = 1, \dots, n$)

$$A_1\varphi = a_0\varphi + a_1 W_{\Gamma,0}\varphi + a_2 W^*_{\Gamma,0}\varphi = f\,, \quad a_0, a_1, a_2 \in PC^{N\times N}(\Gamma) \qquad (4.3)$$

in the vector spaces $L^N_p(\Gamma,\rho)$ and $PC^N(\Gamma,\rho)$,

$$B_0\varphi = b_0\varphi + b_1 W_{\Gamma,0}\varphi = g\,, \qquad b_0, b_1 \in (PC^1)^{N\times N}(\Gamma) \subset C^{N\times N}(\Gamma) \qquad (4.4)$$

in the vector spaces $(W^1)^N_p(\Gamma,\rho)$, $C^N(\Gamma,\rho)$, $(PC^1)^N(\Gamma,\rho)$ and in $(H^0)^N_{\mu+1}(\Gamma,\rho)$ (in the latter case cusps are absent and coefficients belong to $PH^{N\times N}_\mu(\Gamma)$).

Due to Theorems 1.5 and 1.6 respective conditions in (1.4) ensure boundedness of operators A_0, A_1, B_0 in spaces listed above.

4.1 Equation (4.2) in the spaces $L^N_p(\Gamma,\rho)$ and $H^0_\mu(\Gamma,\rho)$

Let $\mathbb{X}(\Gamma)$ denote the space $L^N_p(\Gamma,\rho)$ or, if cusps are absent, the space $H^0_\mu(\Gamma,\rho)$ and appropriate condition in (1.4) hold. Symbol of equation (4.2) in the space $\mathbb{X}(\Gamma)$ reads as follows

$$(\mathscr{A}_0)_{\mathbb{X}(\Gamma)} := \widetilde{a}_0 + \widetilde{a}_1 S_{\mathbb{X}(\Gamma)} + \widetilde{a}_2 W_{\mathbb{X}(\Gamma)} + \widetilde{a}_3 W^*_{\mathbb{X}(\Gamma)} + \widetilde{a}_4 (\partial_t V)_{\mathbb{X}(\Gamma)}\,, \qquad (4.5)$$

where

$$\widetilde{a} := \begin{bmatrix} a(t+0) & 0 \\ 0 & a(t-0) \end{bmatrix}, \qquad a \in PC^{N\times N}(\Gamma)\,, \quad t \in \Gamma\,,$$

$$W_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) := \frac{1}{4}\left[S_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) + \overline{S_{\mathbb{X}(\Gamma)}(t,-\lambda,-\xi)}\right], \quad \lambda,\xi \in \mathbb{R}\,,$$

$$(\partial_t V)_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) := \frac{i}{4}\left[S_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) - \overline{S_{\mathbb{X}(\Gamma)}(t,-\lambda,-\xi)}\right],$$

$$W^*_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) := -\frac{1}{4}\Big[\widetilde{h}^{-1}(t) S_{\mathbb{X}(\Gamma)}(t,\lambda,\xi)\widetilde{h}(t) + \widetilde{h}(t)\overline{S_{\mathbb{X}(\Gamma)}(t,-\lambda,-\xi)}\widetilde{h}^{-1}(t)\Big]$$

$$\widetilde{h}(t) := \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if} \quad t \neq t_1, \dots, t_n\,, \\ \begin{bmatrix} 1 & 0 \\ 0 & e^{\pi(\gamma_j - 1)i} \end{bmatrix} & \text{if} \quad t = t_j,\ \ j = 1 \dots, n\,, \end{cases}$$

$$S_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) := \begin{cases} \begin{bmatrix} \coth \pi(i\beta_t + \lambda) & -\dfrac{e^{\pi(\gamma_t - 1)(i\beta_t + \lambda)}}{\sinh \pi(i\beta_t + \lambda)} \\ \dfrac{e^{\pi(1-\gamma_t)(i\beta_t + \lambda)}}{\sinh \pi(i\beta_t + \lambda)} & -\coth \pi(i\beta_t + \lambda) \end{bmatrix} \\ \qquad \text{if} \quad 0 < \gamma_t < 2\,, \\ (\gamma_j - 1) \begin{bmatrix} -\operatorname{sign} \lambda & 2\chi_-(\lambda) e^{2\lambda} \\ 2\chi_+(\lambda) e^{-2\lambda} & \operatorname{sign} \lambda \end{bmatrix} \\ \qquad \text{if} \quad t = t_j,\ \ \gamma_t = \gamma_j = 0, 2\,, \quad \lambda \neq 0\,, \\ (\gamma_j - 1) \begin{bmatrix} -\coth \pi(i\beta_t + \xi) & 1 + \coth \pi(i\beta_t + \xi) \\ 1 - \coth \pi(i\beta_t + \xi) & -\coth \pi(i\beta_t + \xi) \end{bmatrix} \\ \qquad \text{if} \quad t = t_j,\ \ \gamma_t = \gamma_j = 0, 2, \quad \lambda = 0\,, \end{cases} \tag{4.6}$$

$$\beta_t := \begin{cases} \frac{1}{p} & \text{if } t \neq t_1, \dots, t_n\,, & \mathbb{X}(\Gamma) = L_p(\Gamma, \rho)\,, \\ \frac{1}{2} & \text{if } t \neq t_1, \dots, t_n\,, & \mathbb{X}(\Gamma) = H^0_\mu(\Gamma, \rho)\,, \\ \frac{1}{p} + \alpha_t & \text{if } t = t_j\,, & \mathbb{X}(\Gamma) = L_p(\Gamma, \rho)\,, \\ \alpha_j - \mu & \text{if } t = t_j\,, & \mathbb{X}(\Gamma) = H^0_\mu(\Gamma, \rho)\,, \end{cases}$$

$$\gamma_t := \begin{cases} 1 & \text{if} \quad t \neq t_1, \dots, t_n\,, \\ \gamma_j & \text{if} \quad t = t_j\,, \end{cases} \qquad \chi_\pm(\lambda) := \frac{1}{2}(1 + \operatorname{sign} \lambda)\,.$$

Due to constraints (1.4) $0 < \beta_t < 1$ for all $t \in \Gamma$ and the symbol $(\mathscr{A}_0)_{\mathbb{X}(\Gamma)}(t, \lambda, \xi)$ represents piecewise-continuous uniformly bounded function of all variables.

Although $h(t_j - 0) = h(t_j + 0) e^{\pi(\gamma_j - 1)i}$ (see (1.20)), we have dropped the factor $h(t_j + 0)$ for $t = t_j$ and the factor $h(t)$ for $t \neq t_1, \dots, t_n$ in the definition of the symbol matrix $\widetilde{h}(t)$ above since it cancels out in the combined symbol $(\mathscr{A}_0)_{\mathbb{X}(\Gamma)}(t, \lambda, \xi)$. In fact, $\widetilde{h}(t)$ and $\widetilde{h}^{-1}(t)$ enter the symbol $(\mathscr{A}_0)_{\mathbb{X}(\Gamma)}(t, \lambda, \xi)$ only as the combination $\widetilde{h}^{-1}(t) S_{\mathbb{X}(\Gamma)} \widetilde{h}(t)$ and the constant factors $h^{-1}(t_j + 0)$, $h(t_j + 0)$ cancel out.

Theorem 4.1 *Let* $\mathbb{X}(\Gamma) = L_p^N(\Gamma,\rho)$ *or, if* Γ *has no cusps,* $\mathbb{X}(\Gamma) = (H_\mu^0)^N(\Gamma,\rho)$*). Equation* (4.2) *is* FREDHOLM *in the space* $\mathbb{X}(\Gamma)$ *if and only if*

$$\inf_{t\in\Gamma,\ \lambda,\xi\in\mathbb{R}} \left|\det\left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t,\lambda,\xi)\right| > 0\,. \tag{4.7}$$

If condition (4.7) *holds, then*

$$\begin{aligned}\operatorname{Ind} A_0 = &-\frac{1}{2\pi}\left[\arg\det\left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t,+\infty,0)\right]_\Gamma\\ &-\sum_{j=1}^{n}\frac{1}{2\pi}\Big\{\left[\arg\det\left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t_j,\lambda,0)\right]_{\mathbb{R}\setminus\{0\}}\\ &+\left[\arg\det\left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t_j,0,\xi)\right]_{\mathbb{R}}\Big\}\,.\end{aligned} \tag{4.8}$$

Proof. Due to Lemma 3.1

$$A_0 = a_0 I + a_1 S_\Gamma + \frac{a_2}{4}(S_\Gamma + \mathscr{V}S_\Gamma\mathscr{V}) + \frac{a_3}{4}(S_\Gamma^* + \mathscr{V}S_\Gamma^*\mathscr{V}) + \frac{a_4}{4}(S_\Gamma - \mathscr{V}S_\Gamma\mathscr{V})$$

and the claimed result follows from [DLS1, Theorem 1.1] for the case $\mathbb{X}(\Gamma) = L_p^N(\Gamma,\rho)$ and from [Du6, Du7] for the case $\mathbb{X}(\Gamma) = (H_\mu^0)^N(\Gamma,\rho)$ (when cusps are absent) if we take into account the following:

I. The symbol of operator A_0 defined in [DLS1] and in [Du8] (see also [Du3, Du5]) has a block-diagonal form

$$\begin{bmatrix} \left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t,\lambda,\xi) & 0 \\ 0 & \overline{\left(\mathscr{A}_0\right)_{\mathbb{X}(\Gamma)}(t,-\lambda,-\xi)} \end{bmatrix}$$

and it suffices to consider only the first block as a symbol of A_0. Due to this change we should multiply the index formula by factor $\dfrac{1}{2}$.

Let us note that symbol would be a full matrix-function if the corresponding operator contains terms $\mathscr{V}S_\Gamma$, $\mathscr{V}aI$, $a\mathscr{V}$ or $S_\Gamma\mathscr{V}$.

II. The dual operator $W_{\Gamma,0}^*$ to $W_{\Gamma,0}$ is defined in (3.9) and the symbol for it is composed according to the usual rule (see (4.5)) with $\widetilde{h}(t)$ denoting the symbol of hI (see (3.7) for $h(t)$).

III. If $\mathscr{B}(t,\lambda,\xi)$ is the symbol of B, the symbol of $B^0 = \mathscr{V}B\mathscr{V}$ reads as follows

$$\mathscr{B}^0(t,\lambda,\xi) = \overline{\mathscr{B}(t,-\lambda,-\xi)} \tag{4.9}$$

(see [DLS1, § 1]). ∎

Corollary 4.2 *For the operator*

$$A_0 = a_0 I + a_1 S_\Gamma = (a_o + a_1)(P_+ + GP_-), \quad P_\pm := \frac{1}{2}(I \pm S_\Gamma), \quad G := \frac{a_o - a_1}{a_o + a_1},$$

following conditions are equivalent to (4.7)*:*

(i) $\inf\limits_{t\in\Gamma} |a_0(t) \pm a_1(t)| > 0$*;*

(ii′) $-2\pi\beta_{t_j} < \arg\dfrac{G(t_j - 0)}{G(t_j + 0)} < 2\pi(1 - \beta_{t_j})$, $j = 1, \dots, n$, *where* β_{t_j} *is defined in* (4.6)*;*

(ii″) (equivalent to ii') $G(t)$ has the representation

$$G(t) = G_0(t) \prod_{j=1}^{n} (t - z_0)_{t_j}^{\nu_j}, \quad G_o \in C(\Gamma_1),$$

$$z_0 \in \Omega^+, \quad -\beta_{t_j} < \nu_j < 1 - \beta_{t_j}, \quad j = 1, \dots, n$$

and $(t - z_0)_{t_j}^{\nu_j}$ *has the jump only at the point* $t_j \in \Gamma$*.*

If conditions (i) and (ii') (or (i) and (ii")) hold,

$$\operatorname{Ind} A = \operatorname{ind} G_0.$$

4.2 Equation (4.3) in the spaces $L_p^N(\Gamma,\rho)$ and $PC^N(\Gamma,\rho)$

Although equation (4.3) is a particular case of equation (4.2), in this case we can define substantially simpler symbol and consider equations also in the space $PC^N(\Gamma,\rho)$.

Let $\mathbb{X}(\Gamma)$ denote either $L_p^N(\Gamma,\rho)$ or $PC^N(\Gamma,\rho)$ and (1.4) hold.

Symbol of equation (4.3) in the space $\mathbb{X}(\Gamma)$ reads as follows

$$(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t,\lambda) := \begin{bmatrix} a_0(t+0) & \mathscr{A}_+(t,\lambda) \\ \mathscr{A}_-(t,\lambda) & a_0(t-0) \end{bmatrix}, \tag{4.10}$$

where

$$\mathscr{A}_\pm(t,\lambda) := a_1(t \pm 0) w_{\mathbb{X}(\Gamma)}(t,\lambda) + a_2(t \pm 0) w^*_{\mathbb{X}(\Gamma)}(t,\lambda), \quad t \in \Gamma,\ \lambda \in \mathbb{R},$$

$$w_{\mathbb{X}(\Gamma)}(t,\lambda) = \begin{cases} 0 & \text{if} \quad t \neq t_1, \dots, t_n, \\ \dfrac{\sinh \pi(1 - \gamma_j)(i\beta_j + \lambda)}{\sinh \pi(i\beta_j + \lambda)} & \text{if} \quad t = t_j,\ \ 0 < \gamma_j < 2, \\ \dfrac{\gamma_j - 1}{2} e^{-|\lambda|} & \text{if} \quad t = t_j,\ \ \gamma_j = 0, 2, \end{cases}$$

$$w^*_{\mathbb{X}(\Gamma)}(t,\lambda) = \begin{cases} 0 & \text{if} \quad t \neq t_1, \dots, t_n\,, \\ \dfrac{\sinh \pi(1-\gamma_j)[(\beta_j+1)i+\lambda]}{\sinh \pi(i\beta_j+\lambda)} & \text{if} \quad t = t_j,\ \ 0<\gamma_j<2\,, \\ \dfrac{\gamma_j - 1}{2} e^{-|\lambda|} & \text{if} \quad t = t_j,\ \ \gamma_j = 0, 2\,, \end{cases}$$

$$\beta_j := \begin{cases} \dfrac{1}{p} + \alpha_j & \text{if} \quad \mathbb{X}(\Gamma) = L^N_p(\Gamma,\rho)\,, \\ \alpha_j & \text{if} \quad \mathbb{X}(\Gamma) = PC^N(\Gamma,\rho)\,. \end{cases}$$

Since $0 < \beta_j < 1,\ j = 1, \dots, n$ (see (1.4)) the symbol $(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t,\lambda)$ is a correctly defined $2N \times 2N$ matrix-function, is continuous and

$$(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j, -\infty) = (\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j, +\infty) = \operatorname{diag}\{\mathrm{a_0(t_j - 0), a_0(t_j + 0)}\}\,.$$

Theorem 4.3 *Let* $\mathbb{X}(\Gamma)$ *denote either* $L^N_p(\Gamma,\rho)$ *or* $PC^N(\Gamma,\rho)$ *and* (1.4) *hold.*

Equation (4.3) *is* Fredholm *in* $\mathbb{X}(\Gamma)$ *if and only if*

$$\inf_{t\in\Gamma,\ \lambda\in\mathbb{R}} \left|\det (\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t,\lambda)\right| > 0\,. \tag{4.11}$$

If condition (4.11) *holds, then*

$$\operatorname{Ind} A_1 = \sum_{j=1}^{n} \frac{1}{2\pi} \left[\arg \det\ (\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\lambda)\right]_{\mathbb{R}}\,. \tag{4.12}$$

Remark 4.4 *It is easy to ascertain that condition* (4.11) *for a cusp* t_j *(with* $\gamma_j = 0, 2$*) reads as follows*

$$a_0(t_j-0)a_0(t_j+0) - [a_1(t_j-0)+a_2(t_j-0)][a_1(t_j+0)+a_2(t_j+0)]e^{-\lambda} \neq 0,\ \ \lambda\in\mathbb{R},$$

or, equivalently,

$$\left[\frac{a_0(t_j-0)a_0(t_j+0)}{[a_1(t_j-0)+a_2(t_j-0)][a_1(t_j+0)+a_2(t_j+0)]}\right] > 1\,.$$

Proof of Theorem 4.3. For $\mathbb{X}(\Gamma) = L^N_p(\Gamma,\rho)$ the proof can be derived from Theorem 4.1 (see (4.36) how to get symbol (4.10) from (4.6)). We expose independent proof to cover the case $\mathbb{X}(\Gamma) = PC^N(\Gamma,\rho)$ which is not covered by Theorem 4.1.

We suppose, as in the proof of Theorem 1.6 in § 2.3, that Γ has rectilinear parts $\Gamma_j^-,\ \Gamma_j^+$ in some neighbourhood of all knots $t_1, \dots, t_n$ except cusps; for a cusp $\gamma_j = 0, 2$ the right neighbourhood Γ_j^+ is rectilinear, while the

left one Γ_j^- is not (cf. (2.31) and Fig. 2). Such changes of the contour Γ cause a compact perturbation of equation (4.3) and does not influence the FREDHOLM properties as well as the index of equation (see [DLS1]).

Next we notice that operators $W_{\Gamma_0,0}$ and $W^*_{\Gamma_0,0}$ are compact due to Corollary (1.6) since Γ_0 has no angular points and cusps.

Applying the "macro localization", described in [DLS1, Theorem 1.1, § 3.2], we find that A_1 is FREDHOLM in $\mathbb{X}(\Gamma)$ iff $\det a_0(t) \neq 0$ for $t \in \Gamma \setminus \{t_1, \dots, t_n\}$ and operators

$$A_{1,\Gamma_j^0} = a_{0,j} I + a_{1,j} W_{\Gamma_j^0,0} + a_{2,j} W^*_{\Gamma_j^0,0}\,, \quad \Gamma_j^0 = \Gamma_j^- \cup \Gamma_j^+\,, \tag{4.13}$$

$$a_{k,j}(t) := \begin{cases} a_k(t_j - 0) & \text{if} \quad t \in \Gamma_j^-\,, \\ \\ a_k(t_j + 0) & \text{if} \quad t \in \Gamma_j^+\,, \quad k = 0, 1, 2 \end{cases}$$

are FREDHOLM in $X(\Gamma_j^0)$ for all $j = 1, \dots, n$; for the index we have

$$\text{Ind } A_1 = \sum_{j=1}^{n} \text{Ind } A_{1,\Gamma_j^0}\,. \tag{4.14}$$

First let us consider the space $\mathbb{X}(\Gamma) = L_p^N(\Gamma, \rho)$ and $0 < \gamma_j < 2$; without loss of generality $t_j = 0$.

The transformation $\mathscr{Z}_{\gamma_j,\beta_j}$ with $\beta_j := \frac{1}{p} + \alpha_j$ has the inverse $\mathscr{Z}^{-1}_{\gamma_j,\beta_j}$ (see (2.34)) and arranges an isomorphism

$$\mathscr{Z}_{\gamma_j,\beta_j} \;:\; L_p^N(\Gamma_j^0, |t|^{\alpha_j}) \longrightarrow L_p^{2N}(\mathbb{R}^+)\,. \tag{4.15}$$

Obviously,

$$\begin{aligned} \mathscr{Z}_{\gamma_j,\beta_j} A_{1,\Gamma_j^0} \mathscr{Z}^{-1}_{\gamma_j,\beta_j} &= \begin{bmatrix} a_0(t_j + 0) & 0 \\ 0 & a_0(t_j - 0) \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} a_1(t_j + 0) & 0 \\ 0 & a_1(t_j - 0) \end{bmatrix} \begin{bmatrix} 0 & N^0_{\gamma_j} - N^0_{-\gamma_j} \\ N^0_{\gamma_j} - N^0_{-\gamma_j} & 0 \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} a_2(t_j + 0) & 0 \\ 0 & a_2(t_j - 0) \end{bmatrix} \\ &\times \begin{bmatrix} 0 & e^{-\pi(\gamma_j - 1)i} N^0_{\gamma_j} - e^{\pi(\gamma_j - 1)i} N^0_{-\gamma_j} \\ e^{-\pi(\gamma_j - 1)i} N^0_{\gamma_j} - e^{\pi(\gamma_j - 1)i} N^0_{-\gamma_j} & 0 \end{bmatrix}, \end{aligned}$$

where

$$N^0_{\pm\gamma_j}\varphi(x) := \frac{1}{2\pi i}\int\limits_0^\infty \frac{e^{-(x-y)\beta_j}\varphi(y)dy}{1-e^{-(x-y)\pm\pi\gamma_j i}} \tag{4.16}$$

(see (4.6) where the symbols of hI, S_Γ and of $\mathscr{V}S_\Gamma\mathscr{V}$ is possible to pick up).

Thus, we get a convolution operator

$$\mathscr{Z}_{\gamma_j,\beta_j}A_{1,\Gamma_j^0}\mathscr{Z}^{-1}_{\gamma_j,\beta_j} = W_{(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\cdot)} \,:\, L_p^{2N}(\mathbb{R}^+) \longrightarrow L_p^{2N}(\mathbb{R}^+) \tag{4.17}$$

(cf. (2.6)) with the symbol $(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\lambda)$ defined in (4.10). In fact, $N^0_{\pm\gamma_j}$ in (4.16) are convolutions with the symbols

$$\mathscr{N}^0_{\pm\gamma_j}(\lambda) := \frac{1}{2\pi i}\int\limits_{-\infty}^\infty \frac{e^{i\lambda y-\beta_j y}dy}{1-e^{-y\pm\pi\gamma_j i}} = \frac{1}{2\pi i}\int\limits_0^\infty \frac{t^{\beta_j-i\lambda-1}dt}{1-e^{\pm\pi\gamma_j i}t} \tag{4.18}$$

$$= \frac{e^{\pm\pi(1-\gamma_j)(\beta_j-i\lambda)}}{\sinh\pi(i\beta_j+\lambda)} = \frac{e^{\pm\pi(1-\gamma_j)\Lambda_{t_j}i}}{\sinh\pi(i\beta_j+\lambda)}\,, \quad \beta_j = \frac{1}{p}+\alpha_j\,, \quad \lambda\in\mathbb{R}\,,$$

since $-\pi < \pi-\pi\gamma_j < \pi$ (see [GR1, 3.194.4]). Thus, $N^0_{\pm\gamma_j} = W_{\mathscr{N}^0_{\pm\gamma_j}}$ and from (4.18) we get (4.17).

From (4.17) and from Theorem 2.6 follows: A_{1,Γ_j} is FREDHOLM iff

$$\inf\big|\det(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\lambda)\big| > 0\,, \quad \lambda\in\mathbb{R} \tag{4.19}$$

and, for $0<\gamma_j<2$

$$\text{Ind } A_{1,\Gamma_j} = -\,\text{ind }\det(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\cdot)\,. \tag{4.20}$$

Now let $\gamma_j = 0$ or $\gamma_j = 2$. Then $\Gamma_j^+ = [0,1]$ and, due to condition (4.1) Γ_j^- can be taken as the quarter part of the circumference centered at $z_0 = \frac{1-\gamma_j}{2}i$, starting at $z_1 = \frac{i}{2}+\frac{1-\gamma_j}{2}i$ and terminating at $z_2 = 0$ (see Fig. 4).

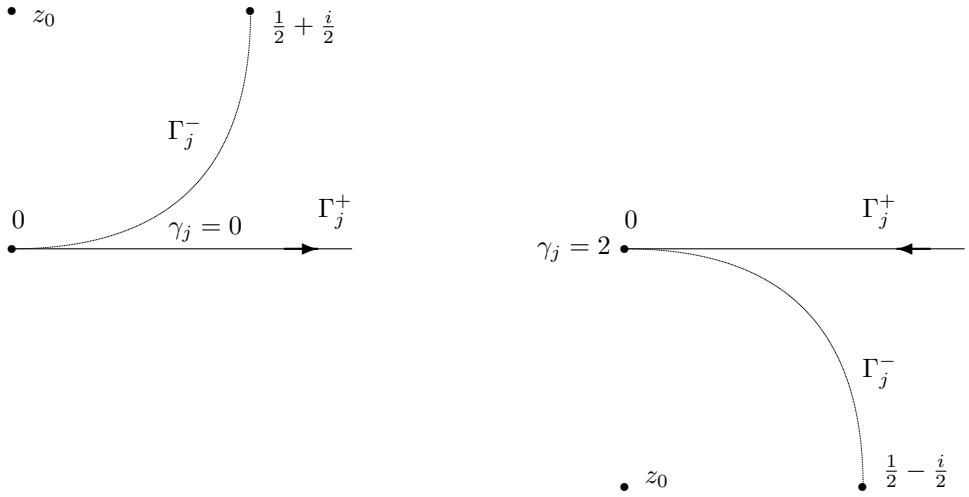

Fig. 4

The transformations

$$\mathscr{Z}_0\varphi(x) := \begin{bmatrix} \dfrac{1}{x+1}\varphi\left(\dfrac{1}{x+1}\right) \\ \dfrac{1}{x-i+1}\varphi\left(\dfrac{1}{x-i+1}\right) \end{bmatrix} \quad \text{if} \quad \gamma_j = 0\,,$$

$$\mathscr{Z}_2\varphi(x) := \begin{bmatrix} \dfrac{1}{x+1}\varphi\left(\dfrac{1}{x+1}\right) \\ \dfrac{1}{x+i+1}\varphi\left(\dfrac{1}{x+i+1}\right) \end{bmatrix} \quad \text{if} \quad \gamma_j = 2\,, \quad x \in \mathbb{R}^+, \quad (4.21)$$

define isomorphisms

$$\mathscr{Z}_{\gamma_j} \;:\; L_p^N(\Gamma_j^0, |t|^{\alpha_j}) \longrightarrow L_p^{2N}(\mathbb{R}^+, (1+x)^{\widetilde{\alpha}_j})\,, \quad \widetilde{\alpha}_j := p - \alpha_j - 2 \quad (4.22)$$

and their inverses read

$$\mathscr{Z}_0^{-1}\begin{bmatrix} \psi_1 \\ \\ \psi_2 \end{bmatrix}(t) = \chi_+^0(t)\frac{1}{t}\psi_1(t-1) + \chi_-^0(t)\frac{1}{t}\psi_2(\frac{1}{t}+i-1)\,,$$

$$\mathscr{Z}_2^{-1}\begin{bmatrix} \psi_1 \\ \\ \psi_2 \end{bmatrix}(t) = \chi_+^0(t)\frac{1}{t}\psi_1(t-1) + \chi_-^0(t)\frac{1}{t}\psi_2(\frac{1}{t}-i-1)\,,$$

where χ_+^0 and χ_-^0 are the characteristic functions of Γ_j^+ and Γ_j^-, respectively.

Obviously $1 < \widetilde{\alpha}_j < p-1$ and

$$\mathscr{Z}_{\gamma_j} A_{1,\Gamma_j^0} \mathscr{Z}_{\gamma_j}^{-1} = \begin{bmatrix} a_0(t_j+0) & 0 \\ 0 & a_0(t_j-0) \end{bmatrix}$$

$$+\frac{\gamma_j - 1}{2} \begin{bmatrix} a_1(t_j+0)+a_2(t_j+0) & 0 \\ 0 & a_1(t_j-0)+a_2(t_j-0) \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & N_i - N_{-i} \\ N_i - N_{-i} & 0 \end{bmatrix},$$

where

$$N_{\pm i}\varphi(x) := \frac{1}{2\pi i} \int_0^\infty \frac{\varphi(y)dy}{y - x \pm i} = W_{\mathscr{N}_{\pm i}}\varphi(x) \tag{4.23}$$

are convolutions with the symbols

$$\mathscr{N}_{\pm i}(\lambda) := \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{e^{i\lambda y}dy}{\pm i - y} = \mp \chi_\pm(\lambda) e^{\mp\lambda}, \tag{4.24}$$

$$\chi_\pm(\lambda) := \frac{1}{2}(1 \pm \operatorname{sign}\lambda), \quad \lambda \in \mathbb{R}.$$

Therefore,

$$\mathscr{Z}_{\gamma_j} A_{1,\Gamma_j} \mathscr{Z}_{\gamma_j}^{-1} = \begin{bmatrix} a_0(t_j+0) & 0 \\ 0 & a_0(t_j-0) \end{bmatrix}$$

$$+\frac{\gamma_j - 1}{2} \begin{bmatrix} a_1(t_j+0)+a_2(t_j+0) & 0 \\ 0 & a_1(t_j-0)+a_2(t_j-0) \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & W_{\mathscr{N}_i - \mathscr{N}_{-i}} \\ W_{\mathscr{N}_i - \mathscr{N}_{-i}} & 0 \end{bmatrix} = W_{(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\cdot)}$$

and, due to Theorem 2.6, A_{1,γ_j} is FREDHOLM iff (4.19) holds; the index formula (4.20) remains valid for $\gamma_j = 0, 2$.

Now let $\mathbb{X}(\Gamma) = PC^N(\Gamma, \rho)$.

For $0<\gamma_j<2$ we consider the transformation $\mathscr{Z}_{\gamma_j,\delta_j}$, defined in (3.20). Similarly to (4.15)–(4.18) we find that

$$\mathscr{Z}_{\gamma_j,\delta_j} \;:\; C^N(\Gamma_j^0,|t|^{\delta_j}) \longrightarrow C^{2N}(\dot{\mathbb{R}}^+)$$

defines an isomorphism and

$$\mathscr{Z}_{\gamma_j,\delta_j} A_{1,\Gamma_j^0} \mathscr{Z}^{-1}_{\gamma_j,\delta_j} = W_{(\mathscr{A}_1)_{\mathbb{X}(\Gamma)}(t_j,\cdot)}$$

is a FREDHOLM operator in the space $C^{2N}(\dot{\mathbb{R}}^+)$ iff

$$\inf \left| \det (\mathscr{A}_1)_{PC(\Gamma,\rho)}(t_j,\lambda)\right| > 0\,, \quad \lambda\in\mathbb{R} \tag{4.25}$$

and

$$\operatorname{Ind} A_{1,\Gamma_j^0} = -\left[\arg\det (\mathscr{A}_1)_{PC(\Gamma,\rho)}(t_j,\lambda)\right]_{\mathbb{R}}\,, \tag{4.26}$$

provided $0<\gamma_j<2$.

For $\gamma_j=0$ and $\gamma_j=2$ (see Fig. 4) the transformation

$$\mathscr{Z}_{\gamma_j} \;:\; PC^N(\Gamma_j^0,|t|^{\delta_j}) \longrightarrow C^{2N}(\dot{\mathbb{R}}^+,(1+x)^{-\delta_j+1})\,, \quad \Gamma_j^0=\Gamma_j^-\cup\Gamma_j^+\,,$$

defined in (4.21), arranges an isomorphism and

$$\mathscr{Z}_{\gamma_j} A_{1,\Gamma_j^0} \mathscr{Z}^{-1}_{\gamma_j} = W_{(\mathscr{A}_1)_{PC(\Gamma,\rho)}(t_j,\cdot)}$$

is FREDHOLM in the space $PC^{2N}(\dot{\mathbb{R}}^+,(1+x)^{-\delta_j+1})$ iff condition (4.25) holds (see Theorem 2.6); again the index is defined by (4.26). ∎

Remark 4.5 *If $S_{\mathbb{X}}(\Gamma)(t,\lambda)$ is the symbol of S_Γ (see* (4.5), (4.6), (4.10)*), the symbol of $\mathscr{V}S_\Gamma\mathscr{V}$ is $\overline{S_{\mathbb{X}(\Gamma)}(t,-\lambda)}$. We know the symbol of aI for $a\in PC^{N\times N}(\Gamma)$ ($\mathbb{X}(\Gamma)=L_p^N(\Gamma,\rho)$ or $\mathbb{X}(\Gamma)=PC^N(\Gamma,\rho)$). Therefore we can compose the symbol of equation*

$$a_0\varphi + a_1 W_{\Gamma,0}\varphi + a_2 W^*_{\Gamma,0}\varphi + \sum_{k=1}^{M} a_{2+k} W^{(k)}_{\Gamma,0}\varphi = f\,, \tag{4.27}$$

$$a_0,\dots,a_{2+M}\in PC^{N\times N}(\Gamma)$$

and prove Theorem 4.3 *for equation* (4.27).

4.3 Equation (4.4) in the spaces $(W_p^1)^N(\Gamma,\rho)$, $(H^0_{\mu+1})^N(\Gamma,\rho)$, $C^N(\Gamma,\rho)$ and $(PC^1)^N(\Gamma,\rho)$

Let $\mathbb{X}(\Gamma)$ denote one of the spaces mentioned in the headline.

To equation (4.4) in the space $\mathbb{X}(\Gamma)$ with smooth matrix coefficients we assign the symbol

$$(\mathscr{B}_0)_{\mathbb{X}(\Gamma)}(t,\lambda) := \begin{bmatrix} b_0(t) & b_1(t)e^{-\pi(1-\gamma_j)i}w_{\mathbb{X}(\Gamma)}(t,\lambda) \\ b_1(t)e^{\pi(1-\gamma_j)i}w_{\mathbb{X}(\Gamma)}(t,\lambda) & b_0(t) \end{bmatrix}, \quad (4.28)$$

where

$$w_{\mathbb{X}(\Gamma)}(t,\lambda) = \begin{cases} 0 & \text{if} \quad t \neq t_1,\dots,t_n\,, \\ \dfrac{\gamma_j - 1}{2}e^{-|\lambda|} & \text{if} \quad t = t_j,\ \ \gamma_j = 0,2 \end{cases}$$

and $w_{\mathbb{X}(\Gamma)}(t_j,\lambda)$ has following values for the different spaces $\mathbb{X}(\Gamma)$:

$$w_{W^1_p(\Gamma,\rho)}(t_j,\lambda) = \frac{\sinh\pi(1-\gamma_j)\left(\frac{i}{p}+\alpha_j i - i + \lambda\right)}{2\sinh\pi\left(\frac{i}{p}+\alpha_j i+\lambda\right)}\,,$$

$$w_{H^0_{\mu+1}(\Gamma,\rho)}(t_j,\lambda) = \frac{\sinh\pi(1-\gamma_j)\left(\alpha_j i - \mu i - i + \lambda\right)}{2\sinh\pi\left(\frac{i}{p}+\alpha_j i+\lambda\right)}\,,$$

$$w_{PC^1(\Gamma,\rho)}(t_j,\lambda) = \frac{\sinh\pi(1-\gamma_j)(\alpha_j i - i + \lambda)}{2\sinh\pi(\alpha_j i+\lambda)}\,,$$

$$w_{C(\Gamma,\rho)}(t_j,\lambda) = \frac{\sinh\pi(1-\gamma_j)(\alpha_j i + \lambda)}{2\sinh\pi(\alpha_j i+\lambda)}\,.$$

Due to conditions (1.4) the symbol $(\mathscr{B}_0)_{\mathbb{X}(\Gamma)}(t,\lambda)$ is correctly defined, i.e., is a piecewise-continuous and uniformly bounded function of all variables.

Theorem 4.6 *Let* $\mathbb{X}(\Gamma)$ *denote one of the following spaces* $(W^1_p)^N(\Gamma,\rho)$, $(H^0_{\mu+1})^N(\Gamma,\rho)$ *(if cusps are absent),* $(PC^1)^N(\Gamma,\rho)$ *or* $C^N(\Gamma,\rho)$ *and conditions* (1.4) *hold.*

Equation (4.4) *is* FREDHOLM *in the space* $\mathbb{X}(\Gamma)$ *if and only if* [9)]

$$\inf_{t\in\Gamma,\ \lambda\in\mathbb{R}} \left|\det(\mathscr{B}_0)_{\mathbb{X}(\Gamma)}(t,\lambda)\right| > 0\,. \quad (4.29)$$

If condition (4.29) *holds, then (cf.* (4.12)*)*

$$\operatorname{Ind} B_0 = -\sum_{j=1}^{n}\frac{1}{2\pi}\left[\arg\det(\mathscr{B}_0)_{\mathbb{X}(\Gamma)}(t_j,\lambda)\right]_{\mathbb{R}}\,. \quad (4.30)$$

[9)] An equivalent condition for a cusp see in Remark 4.4.

Proof. For the space $\mathbb{X}(\Gamma) = C^N(\Gamma,\rho)$ the proof is verbatim the case $\mathbb{X}(\Gamma) = PC^N(\Gamma,\rho)$, exposed in Theorem 4.3.

Let

$$g(s) \,:\, [0,\ell] \longrightarrow \Gamma\,, \quad r(t) := g^{-1}(t) \,:\, \Gamma \longrightarrow [0,\ell]\,, \quad g(r(t)) \equiv t$$

be some parametrisation of Γ and the inverse to the parametrisation. The operator

$$\Lambda^1_\Gamma\varphi(t) := \partial_t\varphi(t) + r'(t)\frac{2\pi i}{\ell}[\varphi(t) - \varphi(t_n)] + \varphi(t_n)e^{-\frac{2\pi i}{\ell}r(t)}$$

$$= \partial_s\varphi_0(s) + \frac{2\pi i}{\ell}[\varphi_0(s) - \varphi_0(0)] + \varphi_0(0)e^{-\frac{2\pi i}{\ell}s}\,, \tag{4.31}$$

$$s = r(t)\,, \quad \varphi_0(s) = \varphi(g(s))\,, \quad 0 \le s \le \ell\,, \quad t \in \Gamma$$

(see [Du3, § 2.2]) defines an isomorphism of spaces

$$\Lambda^1_\Gamma \,:\, (W^1_p)^N(\Gamma,\rho) \longrightarrow L^N_p(\Gamma,\rho)\,. \tag{4.32}$$

and the inverse operator reads

$$\Lambda^{-1}_\Gamma\psi(t) := e^{-\frac{2\pi i}{\ell}r(t)} \int\limits_{t_n t} e^{\frac{2\pi i}{\ell}r(\tau)}\psi(\tau)d\tau$$

$$+\frac{1}{\ell}\left[1 - r(t)e^{-\frac{2\pi i}{\ell}r(t)}\right] \int\limits_{\Gamma} e^{\frac{2\pi i}{\ell}r(\tau)}\psi(\tau)d\tau\,. \tag{4.33}$$

Namely,

$$\Lambda^{-1}_\Gamma\Lambda^1_\Gamma\psi = \psi\,, \quad \psi \in L^N_p(\Gamma,\rho)\,, \quad \Lambda^1_\Gamma\Lambda^{-1}_\Gamma\varphi = \varphi\,, \quad \varphi \in (W^1_p)^N(\Gamma,\rho)$$

and

$$\Lambda^1_\Gamma = \partial_t + \mathscr{R}\,, \quad \partial_t, \mathscr{R} \,:\, (W^1_p)^N(\Gamma,\rho) \longrightarrow L^N_p(\Gamma,\rho)\,,$$

where $\mathscr{R}$ is a compact operator.

Then the equation

$$B_1\psi := \Lambda^1_\Gamma B_0\Lambda^{-1}_\Gamma\psi = u\,, \tag{4.34}$$

$$u, \psi \in L^N_p(\Gamma,\rho)\,, \quad \psi := \Lambda^1_\Gamma\varphi\,, \quad u = \Lambda^1_\Gamma f$$

is equivalent to (4.3). Since

$$\partial_t\Lambda^{-1}_\Gamma = I + K\,, \quad K \,:\, L^N_p(\Gamma,\rho) \longrightarrow L^N_p(\Gamma,\rho)$$

where K is a compact operator, applying (2.27). we get

$$\begin{aligned} B_1 &= (\partial_t + \mathscr{R})(a_0 I + a_1 W_{\Gamma,0})\Lambda_\Gamma^{-1} = a_0 I + a_1 W_{\Gamma,0}^{(2)} + T \\ &= a_0 I + a_1[S_\Gamma + h^{-2}\mathscr{V} S_\Gamma \mathscr{V} h^2 I] + T\,, \qquad (4.35) \\ T &= (a_0' I + a_1' W_{\Gamma,0})\Lambda_\Gamma^{-1} + \mathscr{R}(a_0 I + a_1 W_{\Gamma,0})\Lambda_\Gamma^{-1} \\ &\quad + (a_0 I + a_1 W_{\Gamma,0}^{(2)})K \;:\; L_p^N(\Gamma,\rho) \longrightarrow L_p^N(\Gamma,\rho)\,. \end{aligned}$$

T is a compact operator because Λ_Γ^{-1}, $\mathscr{R}$ and K are compact in $L_p^N(\Gamma,\rho)$.

Symbol of the operator B_1 in $L_p^N(\Gamma,\rho)$, according to (4.6) and to Remark 4.5, reads

$$(\mathscr{B}_1)_{L_p(\Gamma,\rho)}(t,\lambda) := \begin{bmatrix} b_0(t) & 0 \\ 0 & b_0(t) \end{bmatrix} \qquad \text{if} \quad t \neq t_1, \dots, t_n\,,$$

while for the knots $t = t_j$ we get

$$\begin{aligned} (\mathscr{B}_1)_{L_p(\Gamma,\rho)}(t_j,\lambda) &= \begin{bmatrix} b_0(t_j) & 0 \\ 0 & b_0(t_j) \end{bmatrix} + \begin{bmatrix} b_1(t_j) & 0 \\ 0 & b_1(t_j) \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} \coth\pi(i\beta_j+\lambda) & -\dfrac{e^{-\pi(1-\gamma_j)(i\beta_j+\lambda)}}{\sinh\pi(i\beta_j+\lambda)} \\ \dfrac{e^{\pi(1-\gamma_j)(i\beta_j+\lambda)}}{\sinh\pi(i\beta_j+\lambda)} & -\coth\pi(i\beta_j+\lambda) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi(1-\gamma_j)i} \end{bmatrix} \right. \\ &\times \left. \begin{bmatrix} -\coth\pi(i\beta_j+\lambda) & \dfrac{e^{\pi(1-\gamma_j)(i\beta_j+\lambda)}}{\sinh\pi(i\beta_j+\lambda)} \\ -\dfrac{e^{-\pi(1-\gamma_j)(i\beta_j+\lambda)}}{\sinh\pi(i\beta_j+\lambda)} & \coth\pi(i\beta_j+\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi(1-\gamma_j)i} \end{bmatrix} \right\} \qquad (4.36) \\ &= \begin{bmatrix} b_0(t) & b_1(t)e^{-\pi(1-\gamma_j)i} w_{W_p^1(\Gamma,\rho)}(t_j,\lambda) \\ b_1(t)e^{\pi(1-\gamma_j)i} w_{W_p^1(\Gamma,\rho)}(t_j,\lambda) & b_0(t) \end{bmatrix}, \\ w_{W_p^1(\Gamma,\rho)}(t_j,\lambda) &:= \frac{\sinh\pi(1-\gamma_j)[i(\beta_j-1)+\lambda)}{\sinh\pi(i\beta_j+\lambda)}\,, \end{aligned}$$

where β_j is defined in (1.79). Thus, we get the symbol defined in (4.28).

As proved above, the operator B_1 (see (4.35)) in the space $L_p^N(\Gamma,\rho)$ is equivalent (as a Fredholm operator) with B_0 (see (4.4)) in the space $(W_p^1)^N(\Gamma,\rho)$ and their indices are equal $\operatorname{Ind} B_0 = \operatorname{Ind} B_1$ (see (4.34)). Thus, the symbol $(\mathscr{B}_0)_{L_p(\Gamma,\rho)}(t_j,\lambda) := (\mathscr{B}_1)_{L_p(\Gamma,\rho)}(t,\lambda)$ defined in (4.36) is responsible for the Fredholm properties and the index of B_0 in the space $(W_p^1)^N(\Gamma,\rho)$. Now the assertion follows from Theorem 4.1 (and from Theorem 4.3).

In the cases $\mathbb{X}(\Gamma) = (H^0_{\mu+1})^N(\Gamma,\rho)$ and $\mathbb{X}(\Gamma) = (PC^1)^N(\Gamma,\rho)$ the proofs follow verbatim the above exposed case $\mathbb{X}(\Gamma) = (W^1_p)^N(\Gamma,\rho)$. ∎

5 Conformal mapping and BVPs

Through this section we use the notation from § 1.1: for domains $\Omega^{\pm}$, for their boundary $\Gamma = \partial\Omega^{\pm}$, for the weight function $\rho(t)$ (see (1.2), (1.4)), for the unit disk $\mathscr{D}_1$ and the unit circumference $\Gamma_1 = \partial\mathscr{D}_1$.

5.1 The Cisotti formula and its applications

In the present subsection we prove the Cisotti formula (5.5). It was published in 1921 (see [LS1, Ch. III, § 1, n^o. 44, Example 5]) and was rediscovered in [PK1] for piecewise-smooth curves by a different method (namely, by reducing the problem to the Riemann–Hilbert BVP for analytic functions). This formula has several interesting applications (see [KKP1]) and we will give some further applications below. Returning to the original method (see [LS1]) we prove the Cisotti formula for arbitrary domain bounded by a rectifiable Jordan curve.

Next Theorem is easy to ascertain if properties of conformal mapping $\omega : \mathscr{D}_1 \longrightarrow \Omega^+$ and of the inverse to it $\omega^{-1} : \Omega^+ \longrightarrow \mathscr{D}_1$ are taken into account: it suffices to change variables in the integrals $\zeta = \omega(z)$, $z = \omega^{-1}(\zeta)$. (see (1.47) and [Ev1, Ch. V, § 1]).

Theorem 5.1 *The derivatives $\omega'(z)$ and $(\omega^{-1})'(\zeta)$ of conformal mapping* (1.46) *and its inverse are both square integrable*

$$\int\limits_{\Omega^+} |(\omega^{-1})'(\zeta)|^2 |d\zeta| = \pi^2\,, \quad \int\limits_{\mathscr{D}_1} |\omega'(z)|^2 |dz| = (\operatorname{mes}\Omega^+)^2\,, \tag{5.1}$$

while restricted to the boundaries they become absolutely integrable

$$\int\limits_{\Gamma} |(\omega^{-1})'(\zeta)||d\zeta| = 2\pi\,, \quad \int\limits_{\Gamma_1} |\omega'(z)||dz| = \operatorname{mes}\Gamma\,. \tag{5.2}$$

Next Theorem is a far non-trivial and subtle consequence of the foregoing theorem and we quote [Go1, p.p. 405–411] (see also [Ko1, Ch. I, II]) for rigorous proofs.

Theorem 5.2 *If $\omega(z)$ in* (5.1) *is a conformal mapping of the unit disk $\mathscr{D}_1$ onto a simply connected domain Ω^+ with the rectifiable* Jordan *boundary, then:*

i. $\omega \in \mathscr{W}^1_1(\overline{\mathscr{D}_1})$ (*see* § 1.1).

ii. $\omega(z)$ *is absolutely continuous on the boundary* Γ_1.

iii. For almost all $t_0 \in [0, 2\pi]$ *there exists an angular (i.e., non-tangential) boundary limit* $\overset{\wedge}{\to}$ *of the function* $\omega'(z)$

$$\lim_{re^{it} \overset{\wedge}{\to} e^{it_0}} \omega'(re^{it}) = -ie^{-it_0} \left. \frac{d\omega(e^{i\tau})}{d\tau} \right|_{\tau=t_0} . \tag{5.3}$$

The limit is denoted again by $\omega'(e^{it_0})$.

Theorem 5.3 *The derivatives* $\omega'(z)$ *of the conformal mapping* $\omega : \mathscr{D}_1 \to \Omega^+$ *has the following representation*

$$\omega'(z) = \omega'(0) \exp\left[\frac{1}{\pi} \int\limits_{|\zeta|=1} \frac{\beta(\zeta) d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|=1} \beta(\zeta) \frac{d\zeta}{\zeta} \right], \quad z \in \mathscr{D}_1 , \tag{5.4}$$

$$\beta(e^{it}) := \alpha(t) - t - \frac{\pi}{2} = \vartheta(t) - t \quad \text{for a.a.} \quad t \in [-\pi, \pi] , \tag{5.5}$$

where $\alpha(t)$ *and* $\vartheta(t) = \vartheta_{\omega(e^{it})} = \arg \vec{\nu}(\omega(e^{it}))$ *denote the inclinations with respect to the abscissa axes of the tangent and the outer unit normal vectors at the point* $\omega(e^{it})$*, respectively (see Fig.* 1).

Proof. Due to (5.3) $\beta_0(t) := \beta(e^{it})$ in (5.5) exists for almost all $t \in (-\pi, \pi]$ and for those t we have

$$\omega'(e^{it}) = -ie^{-it} \frac{d\omega(e^{it})}{dt} = e^{i\beta_0(t)} \left| \frac{d\omega(e^{it})}{dt} \right| .$$

Since $\omega'(e^{it}) \neq 0$ ($\omega(z)$ is a conformal mapping!)

$$\text{Re } [-i \log \omega'(e^{it})] = \text{Im } \log \omega'(e^{it}) = \beta_0(t) = \beta(e^{it}) \quad \text{for a.a. } t \in (-\pi, \pi]$$

and the SCHWARTZ integral recovers the analytic function $-i \log \omega'(z) \in w_1^1(\overline{\mathscr{D}_1})$ by its real part on the boundary

$$-i \log \omega'(z) = iC + \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{e^{i\tau} + z}{e^{i\tau} - z} \beta(e^{i\tau}) d\tau$$

(see [Ko1, Ch. I, II], [LS1, §. 44]); therefore

$$\begin{aligned} \omega'(z) &= \exp(-C) \exp\left[\frac{i}{2\pi} \int\limits_{-\pi}^{\pi} \frac{e^{i\tau} + z}{e^{i\tau} - z} \beta(e^{i\tau}) d\tau \right] \\ &= C_0 \exp\left[-\frac{i}{2\pi} \int\limits_{-\pi}^{\pi} \beta(e^{i\tau}) d\tau + \frac{i}{\pi} \int\limits_{-\pi}^{\pi} \frac{\beta(e^{i\tau}) e^{i\tau} d\tau}{e^{i\tau} - z} \right] \\ &= C_1 \exp\left[\frac{1}{\pi} \int\limits_{|\zeta|=1} \frac{\beta(\zeta) d\zeta}{\zeta - z} \right] \end{aligned}$$

and taking $z = 0$ easily locate the constant C_1:

$$C_1 = \omega'(0) \exp\left[-\frac{1}{\pi} \int\limits_{|\zeta|=1} \beta(\zeta) \frac{d\zeta}{\zeta}\right] . \qquad \blacksquare$$

It is sometimes helpful to have the CISOTTI formula (5.4) in the following equivalent form

$$\omega'(re^{it}) = \omega'(0) \exp\left[i(\mathscr{P}_r\beta_0)(re^{it}) - (\widetilde{\mathscr{P}}_r\beta_0)(re^{it}) - \frac{i}{2\pi} \int\limits_{\pi}^{\pi} \beta_0(\tau) d\tau\right], \quad (5.6)$$
$$0 < r < 1\,, \quad -\pi < t \leq \pi\,,$$

where $\mathscr{P}_r\varphi(z)$ is the POISSON operator and $\widetilde{\mathscr{P}}_r\varphi(z)$ defines the adjoint harmonic function to $\mathscr{P}_r\varphi(z)$ ($|z| < 1$; see [Ko1, Ch. I]):

$$\begin{aligned} \mathscr{P}_r\varphi(re^{it}) &:= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(t-\tau) + r^2} \varphi(\tau) d\tau\,, \\ \widetilde{\mathscr{P}}_r\varphi(re^{it}) &= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{r\sin(t-\tau)}{1 - 2r\cos(t-\tau) + r^2} \varphi(\tau) d\tau\,. \end{aligned} \quad (5.7)$$

In the next theorem we have collected properties of the POISSON operator $\mathscr{P}_r$ and its adjoint $\widetilde{\mathscr{P}}_r$ from [Ko1, Ch.I] and [Ko1, Ch. V, § D.1^o], necessary for further investigations.

Theorem 5.4 *Let $\varphi \in L_p(\Pi)$, $\Pi := [-\pi, \pi]$, $1 \leq p < \infty$. Then*

i. $\mathscr{P}_r\varphi(z)$ is harmonic in $\mathscr{D}_1$ and

$$\|\mathscr{P}_r\varphi \big| L_p(\Pi)\| \leq \|\varphi \big| L_p(\Pi)\|\,, \quad 0 \leq r < 1\,, \quad \lim_{r\to 0} \|\mathscr{P}_r\varphi - \varphi \big| L_p(\Pi)\| = 0$$

ii. If $\varphi(t)$ is continuous at some $t_0 \in \Pi$, then

$$\lim\, \mathscr{P}_r\varphi(z) = \varphi(t_0) \quad \text{as} \quad z = re^{it} \to e^{it_0}\,, \quad r < 1. \qquad (5.8)$$

In particular, if $\varphi \in C(\Pi)$, $\varphi(-\pi) = \varphi(\pi)$, then the convergence in (5.8) is uniform (including convergence across tangent paths) with respect to $t_0 \in \Pi$.

iii. If $\operatorname{Im} g(t) \equiv 0$, $|g(t)| \leq \lambda\frac{\pi}{2}$ for all $t \in \Pi$ and $\lambda < 1$, then

$$\int\limits_{-\pi}^{\pi} \exp\left[\left|\widetilde{\mathscr{P}}_r g(e^{it})\right|\right] \leq \frac{4\pi}{\cos\frac{\pi}{2}\lambda}\,. \qquad (5.9)$$

Remark 5.5 *Easy to check that*

$$\begin{aligned} \mathscr{P}_r\varphi(z) &= \operatorname{Im} \mathrm{C}_\Gamma\varphi(\mathrm{z}) - \frac{1}{2\pi}\int\limits_{\pi}^{\pi}\varphi(\tau)\mathrm{d}\tau\,, \\ \widetilde{\mathscr{P}}_r\varphi(z) &= -\operatorname{Re} \mathrm{C}_\Gamma\varphi(\mathrm{z}) \quad \text{for} \quad \operatorname{Im}\varphi(\mathrm{t}) \equiv 0 \end{aligned} \tag{5.10}$$

(*see* (1.3)). *Therefore for* $\widetilde{\mathscr{P}}_r\varphi$ *we can apply the* Plemelji *formulae and get*

$$\lim_{z\overset{\wedge}{\to}e^{it}} \widetilde{\mathscr{P}}_r\varphi(z) = -\frac{1}{2}\varphi(t) - \frac{1}{2\pi}\int\limits_{\pi}^{\pi}\cot\frac{t-\tau}{2}\varphi(\tau)d\tau\,, \tag{5.11}$$

where the limit is angular (*see* (1.25)).

Corollary 5.6 *If the inclination of the tangent vector to the boundary curve* Γ *is continuous on the entire boundary, derivative* $\omega'(\zeta)$ *of the conformal mapping in* (1.46) *belongs to the* Smirnov–Lebesgue *space* $\omega' \in e_p(\overline{\mathscr{D}_1})$ *for all* $1 < p < \infty$.

Proof. Due to the asserted conditions $\beta_0(t) = \beta(e^{it})$ in (5.5) is a continuous function $\beta_0 \in C(\Pi)$ and $\beta_0(\pi) = \beta_0(-\pi)$; then

$$\beta_0(t) = \beta_1(t) + \beta_2(t)\,, \quad \beta_1 \in C^1(\Pi)\,, \quad \beta_1(\pi) = \beta_1(-\pi)\,,$$

$$|\beta_2(t)| \le \frac{\pi}{4p} = \frac{1}{2p}\frac{\pi}{2} \quad \text{for all} \quad t \in \Pi\,.$$

From (5.6) and (5.9) we get

$$\begin{aligned} \int\limits_{-\pi}^{\pi}|\omega'(re^{i\tau})|^p d\tau &= \int\limits_{-\pi}^{\pi}|\exp\left[\left|p\widetilde{\mathscr{P}}_r\beta_1(e^{i\tau}) + p\widetilde{\mathscr{P}}_r\beta_2(e^{i\tau})\right|\right]d\tau \\ &\le \frac{4\pi C_0}{\cos\frac{\pi}{4}} \quad \text{for all} \quad 0 < r \le 1\,, \end{aligned}$$

where

$$C_0 = \int\limits_{-\pi}^{\pi}\exp\left[\left|p\widetilde{\mathscr{P}}_r\beta_1(e^{i\tau})\right|\right]d\tau < \infty$$

since $\beta_1 \in H_1(\Gamma_1)$ and $\mathscr{P}_r\beta_1(\zeta)$ is uniformly bounded with respect to $0 < r \le 1$ (see (5.8)). ∎

Let us formulate several consequences of the foregoing results. First of them is a weak form of the Lindelöf theorem; in full generality it can be found e.g. [Ko1] and deals with arbitrary domain with Jordan boundary. For a domain with the smooth boundary it is proved e.g in [Go1] by a different method and in [KKP1, p. 141]–as here, by using the Cisotti formula, but for piecewise-smooth curves.

Theorem 5.7 *Let $\Omega^{\pm}$ be a simply connected domain with the rectifiable* JORDAN *boundary Γ and $\omega(z)$ be a conformal mapping of the unit disk $\mathscr{D}_1$ onto the domain Ω^+. If the tangent exists at some point of the boundary $t_0 \in \Gamma$, then the argument $\arg \omega'(z)$ of the derivative of the conformal mapping is continuous at $e^{i\vartheta_0} \in \Gamma_1 = \partial\mathscr{D}_1$, where $t_0 = \omega(e^{i\vartheta_0})$:*

$$\lim \arg \omega'(z) = \arg \omega'(e^{i\vartheta_0}) = e^{i\alpha(\vartheta_0)} \quad \text{as} \quad z \to e^{i\vartheta_0} \quad \text{and} \quad z \in \Omega^+.$$

In particular, if the tangent exists at each point of the boundary Γ, then $\arg \omega'(x)$ is a continuous function on the closed domain $\overline{\Omega^+}$.

Proof. The proof follows from Theorem 5.4.ii and from the equality

$$\arg \omega'(z) = \arg \omega'(0) + \mathscr{P}_r \beta_0(t) - \frac{1}{2\pi} \int_{\pi}^{\pi} \beta_0(\tau) d\tau \,, \tag{5.12}$$

$$z = re^{i\vartheta} \,, \;\; 0 < r < 1 \,, \;\; -\pi < t \le \pi$$

(see (5.6)), where $\beta_0(t) := \beta(e^{it})$ is defined in (5.5). ∎

Let $0 < \mu < \infty$ and $\mathbb{X}$ be a compact sufficiently smooth manifold (we can take $\mathbb{X} = [0,1]$, $\mathbb{X} = \overline{\Omega^+}$ or even $\mathbb{X} = \Gamma$ if the latter is sufficiently smooth). Norm in the ZYGMUND space $\mathbb{Z}^\mu(\mathbb{X})$ is defined as follows

$$||\varphi|\mathbb{Z}^\mu(\mathbb{X})|| = ||f|C^{[\mu]^-}(\mathbb{X})|| + \sum_{|\alpha|=[\mu]^-} \sup_{x, x\pm h \in \mathbb{X}} \frac{|(\partial^\alpha \varphi)(x+h) - 2(\partial^\alpha \varphi)(x) + (\partial^\alpha \varphi)(x-h)|}{|h|^{\{\mu\}^+}},$$

$$\mu = [\mu]^- + \{\mu\}^+ \quad [\mu]^+ \in \mathbb{N}_0, \quad 0 < \{\mu\}^+ \le 1,$$

where

$$\|f|C^m(\mathbb{X})\| = \sum_{|\alpha| \le m} \sup_{x \in \mathbb{X}} |\partial^\alpha f(x)|.$$

For $\mu \in \mathbb{R}^+ \setminus \mathbb{N}$ the space $\mathbb{Z}^\mu(\mathbb{X})$ coincides with the generalized HÖLDER space $H_\mu(\mathbb{X})$ (see [St1]), where (cf. § 1.1)

$$||\varphi\big|H^\mu(\mathbb{X})|| = ||f|C^{[\mu]}(\mathbb{X})|| + \sum_{|\alpha|=[\mu]} \sup_{x,y \in \mathbb{X}, x \ne y} \frac{|(\partial^\alpha \varphi)(y) - (\partial^\alpha \varphi)(x)|}{|y-x|^{\{\mu\}}},$$

$$\mu = [\mu] + \{\mu\}, \quad [\mu] \in \mathbb{N}_0, \qquad 0 < \{\mu\} < 1.$$

$\mathbb{Z}^\mu(\Gamma)$ coincides with the BESOV space $B^\mu_{\infty,\infty}(\Gamma)$ (see [Tr1]) and the next theorem represents very particular case of [Du10, Theorem 3.2] (cf. Theorem 1.8 above). The assertion can readily be derived from the MUSKHELISHVILI–PRIVALOV theorem (the case $\mu < 1$), proved in [Mu1, § 21], for non-integer $\mu \in \mathbb{R}$ and extended to integer values $\mu = 1, 2, \ldots$ by the interpolation of ZYGMUND spaces (see [St1, Tr1] for theorems on interpolation).

Theorem 5.8 *Let $0 < \mu < \infty$ and the boundary $\Gamma = \partial\Omega^{\pm}$ be m-smooth, where $m \in \mathbb{N}_0$, $m \geq \mu$.*

The potential operators

$$\begin{array}{rcl} C_\Gamma & : & \mathbb{Z}^\mu(\Gamma) \longrightarrow \mathbb{Z}^\mu(\overline{\Omega^{\pm}})\,, \\ W_\Gamma & : & \mathbb{Z}^\mu(\Gamma) \longrightarrow \mathbb{Z}^\mu(\overline{\Omega^{\pm}})\,, \\ V_\Gamma & : & \mathbb{Z}^\mu(\Gamma) \longrightarrow \mathbb{Z}^{\mu+1}(\overline{\Omega^{\pm}}) \end{array} \tag{5.13}$$

(*see* (1.3) *and* (1.16)) *are bounded.*

In particular, if Γ is piecewise-smooth, we should restrict $0 < \mu < 1$. ∎

Kellogg proved that if the inclination of the tangent vector is a Hölder continuous function with some exponent $0 < \mu < 1$ (so called Ljapunov boundary), then the derivative $\omega'(x)$ of the conformal mapping $\omega \;:; \mathscr{D}_1 \to \Omega^+$ also is Hölder continuous with the same exponent μ. The simple proof of this assertion is exposed in [KKP1, p. 143] and is based on the Cisotti formula. The next theorem generalizes Kellogg's theorem for $\mu \geq 1$.

Theorem 5.9 *Let Ω be a simply connected domain and the inclination of the tangent to the boundary $\Gamma = \partial\Omega$ with respect to some fixed direction belongs to the* Zygmund *space $\mathbb{Z}^\mu([0,\ell])$ for some $0 < \mu < \infty$.*

If $\omega(z)$ is a conformal mapping of the unit disk $\mathscr{D}_1$ onto the domain Ω, then $\omega \in \mathbb{Z}^{\mu+1}(\overline{\mathscr{D}_1})$.

Proof. Let us consider the natural parametrisation of the curve Γ by the arc length parameter $\zeta(s)\,[0,\ell] \to \Gamma$, $\zeta(0) = \zeta(\ell)$ (cf. (1.21)). The derivative $\zeta'(s)$ coincides with the unit tangent vector to Γ and the condition of the theorem can be written as follows

$$\arg\zeta'(\cdot) \in \mathbb{Z}^\mu([0,\ell])\,, \quad \arg\partial_s^{k+1}\zeta(\ell{-}0) = \arg\partial_s^{k+1}\zeta(0{+}0)\,, \quad k = 0,\ldots,[\mu]\,.$$

By the definition (see (5.5))

$$\beta(e^{it(s)}) = \alpha(t(s)) - t(s) - \frac{\pi}{2}\,,$$

where $t(s) \;:\; [0,\ell] \to [-\pi,\pi]$ is a continuous function of the arc length parameter, defined by the equality $\omega(e^{it(s)}) = \zeta(s)$. Thus, we need to prove the implication

$$t'(\cdot) \in \mathbb{Z}^\mu([0,\ell]) \;\Rightarrow\; t'(s(\omega(\cdot))) \in \mathbb{Z}^\mu(\Gamma_1)\,.$$

From the asserted conditions $\beta \in C(\Gamma_1)$ and from the Corollary 5.6 we get $\omega' \in e_2(\overline{\mathscr{D}_1})$. Then

$$\begin{aligned} |s(\omega(\zeta_2)) - s(\omega(\zeta_1))| &= \left|\int\limits_{\zeta_1}^{\zeta_2} |\omega'(\zeta)||d\zeta|\right| \leq \left(\int\limits_{\zeta_1}^{\zeta_2} |\omega'(\zeta)|^2|d\zeta|\right)^{\frac{1}{2}} \left(\int\limits_{\zeta_1}^{\zeta_2} |d\zeta|\right)^{\frac{1}{2}} \\ &= C_0|\zeta_2 - \zeta_1|^{\frac{1}{2}}\,. \end{aligned} \tag{5.14}$$

Thus, $s(\omega(\cdot)) \in H_{\frac{1}{2}}(\Gamma_1)$ and we find the first crude inclusion $\beta(s(\omega(\cdot))) \in \mathbb{Z}^{\nu_1}(\Gamma_1) = H_{\nu_1}(\Gamma_1)$ with $\nu_1 = \min\left\{\frac{1}{2}, \frac{\mu}{2}\right\}$. Due to Theorem 5.8 and to the CISOTTI formula (5.4) we get another crude result $\omega' \in \mathbb{Z}^{\nu_1}(\overline{\mathscr{D}_1})$. We return to (5.14) and find

$$|s(\omega(\zeta_2)) - s(\omega(\zeta_1))| = \int_{\zeta_1}^{\zeta_2} |\omega'(\zeta)||d\zeta| \le C_1|\zeta_2 - \zeta_1|\,, \quad \zeta_1, \zeta_2 \in \Gamma\,,$$

where $C_1 = \sup_{\zeta\in\Gamma_1} |\omega'(\zeta)|$. The obtained estimate and the inclusion $\beta(\cdot) \in \mathbb{Z}^{\mu}([0,\ell])$ give the second crude inclusion $\beta(s(\omega(\cdot))) \in \mathbb{Z}^{\nu_2}(\Gamma_1)$ with $\nu_2 = \min\{1,\mu\}$. Due to Theorem 5.8 and the formula (5.4) this inclusion yields $\omega' \in \mathbb{Z}^{\nu_2}(\overline{\mathscr{D}_1})$, which is the final result provided $0 < \mu \le 1$.

If $\mu > 1$ we take the derivative in (5.4)

$$\omega''(z) = \omega'(0)\exp\left[C_{\Gamma_1}\beta(z) - B_0\right] C_{\Gamma_1}\beta'(z)\,, \tag{5.15}$$
$$B_0 := \frac{1}{\pi}\int_{|\zeta|=1} \beta(\zeta)\frac{d\zeta}{\zeta}, \quad z \in \mathscr{D}_1\,.$$

On the other hand,

$$(\partial_\zeta\beta)(s(\omega(\zeta))) := \frac{d\beta(s(\omega(\zeta)))}{d\zeta} = \beta'(s(\omega(\zeta)))(\partial_\zeta s)(\omega(\zeta))\,. \tag{5.16}$$

Since —s'(t)—=1, we find that

$$(\partial_\zeta s)(\omega(\zeta))| = |\omega'(\zeta)|$$

and from the inclusion $\omega' \in \mathbb{Z}^1(\Gamma_1) \subset H_1(\Gamma_1)$ we conclude $(\partial_\zeta s)(\omega(\cdot)) \in H_1(\Gamma_1)$. This inclusion, together with $\beta'(\cdot) \in \mathbb{Z}^{\mu-1}([0,\ell])$ yields $\partial_\zeta\beta(s(\omega(\cdot))) \in \mathbb{Z}^{\nu_3}(\Gamma_1)$ (see (5.16)) with $\nu_3 = \min\{1, \mu-1\}$.

Again, we derive $\omega'' \in \mathbb{Z}^{\nu_3}(\overline{\Omega^+}) \Rightarrow \omega \in \mathbb{Z}^{\nu_3+2}(\overline{\Omega^+})$ from (5.15) and from Theorem 5.8. The final result is obtained if $\mu \le 2$ which implies $\nu_3 = \mu$.

If $\mu > 2$ we repeat the foregoing proof, taking further derivatives in (5.15) and accomplish the proof by the mathematical induction. ∎

Corollary 5.10 (*see also* [KKP1]). *The inequality*

$$0 < C_1 \le \left|\frac{\log[\omega(\zeta) - \omega(\zeta_j)]}{\log[\zeta - \zeta_j]}\right| \le C_2 < \infty \tag{5.17}$$

holds for all $|\zeta| = 1$ *provided* $t_j = \omega(\zeta_j)$ *is not a cusp of* Γ, *i.e., if* $0 < \gamma_j < 2$.

Proof. Invoking the LAGRANGE theorem and CISOTTI formula (5.5) with the PLEMELJI formula (the last one in (1.25)) we get

$$\log[\omega(\zeta)-\omega(\zeta_j)]=\log(\zeta-\zeta_j)+\log\omega'(\zeta')$$
$$=C_0+\log(\zeta-\zeta_j)+\beta(\zeta')+\frac{1}{\pi}\int\limits_{|\tau|=1}\frac{\beta(\tau)d\tau}{\tau-\zeta'}\,,$$

where $\zeta'=\zeta'(\zeta,\zeta_j)\in\overset{\smile}{\zeta_j\zeta}$ and

$$C_0:=\log\omega'(0)-\frac{1}{\pi}\int\limits_{|\tau|=1}\beta(\tau)\frac{d\tau}{\tau}=\text{ const }.$$

The density $\beta(\tau)$ in the CAUCHY integral is piecewise-HÖLDER continuous $\beta\in H_\nu(\Gamma_{1\zeta_j}\setminus\{\zeta_j\})$ by condition and has the following jump at $\zeta_j\in\Xi_{ow}$

$$\frac{\beta(\zeta_j+0)-\beta(\zeta_j-0)}{\pi}=1-\gamma_j\,.$$

Applying the estimates

$$\frac{1}{\pi}\int\limits_{|\tau|=1}\frac{\beta(\tau)d\tau}{\tau-\zeta'}=-\frac{\beta(\zeta_j+0)-\beta(\zeta_j-0)}{\pi}\log(\zeta_j-\zeta')+\beta_1(\zeta')$$

$$=(\gamma_j-1)\log(\zeta'-\zeta_j)+\beta_1(\zeta')=(\gamma_j-1)\log(\zeta-\zeta_j)+\beta_2(\zeta')$$

as $\zeta\to\zeta_j$, $|\zeta'-\zeta_j|/|\zeta-\zeta_j|\le 1$, where $\beta_1,\beta_2\in H_\nu(\Gamma_{1\zeta_j}\setminus\{\zeta_j\})$ (see [Mu1, § 26]) we find

$$\log[\omega(\zeta)-\omega(\zeta_j)]=\gamma_j\log(\zeta-\zeta')+\beta_3(\zeta_j,\zeta)$$

with uniformly bounded $\beta_3(\zeta_j,\cdot)\in H_\nu(\Gamma_{1\zeta_j}\setminus\{\zeta_j\})$ when $\zeta\to\zeta_j$ and (5.17) follows. ∎

5.2 Proof of Lemma 1.11

Repeating verbatim the arguments exposed in the proof of Theorem 1.16 (see (1.51)–(1.56)) we find easily that the Riemann–Hilbert problem (1.35) in the space $\Psi\in\mathscr{E}_p(\overline{\Omega^\pm},\rho)$, $g\in L_p(\Gamma,\rho)$ is equivalent to the singular integral equation (1.50) in the space $L_p(\Gamma)$.

Let, for definiteness, consider the domain Ω^+. The case of outer domain differs only by angles: we should replace all γ_j by $2-\gamma_j$ (i.e., by the measure of the complementary angle).

First let us prove that $G\in PC(\Gamma_1)$; namely,

$$\frac{G(\zeta_j-0)}{G(\zeta_j+0)}=\exp\left[-\frac{2\pi}{p}i+2\pi\left(\frac{1}{p}+\alpha_j\right)\gamma_j i\right]\,,\quad j=1,\dots,n\,.\qquad(5.18)$$

In fact, in the vicinity of $t_j \in \Gamma$ we get

$$\rho_0(\omega(\zeta)) = \rho_j(\zeta)\,[\omega(\zeta) - \omega(\zeta_j)]^{\alpha_j} = \rho_j(\zeta)\left[\omega'(\zeta_j')\right]^{\alpha_j}(\zeta - \zeta_j)^{\alpha_j}\,, \quad \zeta \to \zeta_j\,,$$

$$\zeta_j' := \lambda_j\zeta_j + (1-\lambda_j)\zeta\,, \quad 0 < \lambda_j < 1\,, \quad \rho_j(\zeta) = \prod_{k\neq j}[\omega(\zeta) - \omega(\zeta_k)]^{\alpha_k}$$

(see (1.46), (1.48)) and $\rho_j(t)$ is continuous at t_j: $\rho_j(t_j - 0) = \rho_j(t_j + 0)$. Therefore,

$$\frac{G(\zeta_j - 0)}{G(\zeta_j + 0)} = \frac{\rho_0(\omega(\zeta_j - 0))}{\overline{\rho_0(\omega(\zeta_j - 0))}}\,\frac{\overline{\rho_0(\omega(\zeta_j + 0))}}{\rho_0(\omega(\zeta_j + 0))}\left[\frac{\omega'(\zeta_j - 0)}{\overline{\omega'(\zeta_j - 0)}}\,\frac{\overline{\omega'(\zeta_j + 0)}}{\omega'(\zeta_j + 0)}\right]^{\frac{1}{p}}$$

$$= \frac{(\zeta_j - 0 - \zeta_j)^{\alpha_j}}{\overline{(\zeta_j - 0 - \zeta_j)^{\alpha_j}}}\,\frac{\overline{(\zeta_j + 0 - \zeta_j)^{\alpha_j}}}{(\zeta_j + 0 - \zeta_j)^{\alpha_j}}\left[\frac{\omega'(\zeta_j - 0)}{\overline{\omega'(\zeta_j - 0)}}\,\frac{\overline{\omega'(\zeta_j + 0)}}{\omega'(\zeta_j + 0)}\right]^{\frac{1}{p}+\alpha_j}$$

$$= \exp\left\{2\pi\alpha_j i + 2\left(\frac{1}{p} + \alpha_j\right)[\arg\,\omega'(\zeta_j - 0) - \arg\,\omega'(\zeta_j + 0)]i\right\}\,.$$

We proceed with the help of (1.66) (see also (5.6) and (5.8))

$$\frac{G(\zeta_j - 0)}{G(\zeta_j + 0)} = \exp\left\{2\pi\alpha_j i + 2\left(\frac{1}{p} + \alpha_j\right)[\arg\,\beta(\zeta_j - 0) - \arg\,\beta(\zeta_j + 0)]i\right\}$$

$$= \exp\left[2\pi\alpha_j i - 2\pi\left(\frac{1}{p} + \alpha_j\right)(1 - \gamma_j)i\right]$$

$$= \exp\left[-\frac{2\pi}{p}i + 2\pi\left(\frac{1}{p} + \alpha_j\right)\gamma_j i\right]\,.$$

The function $\zeta_{\zeta_j}^{-\nu_j}$ with

$$\nu_j := -\frac{1}{p} + \left(\frac{1}{p} + \alpha_j\right)\gamma_j\,, \quad j = 1, \ldots, n, \tag{5.19}$$

has discontinuity on the unit circumference if $\nu_j \neq 0, \pm 1, \ldots$ and this discontinuity we fix at the point $\zeta_j \in \Gamma_1$; then

$$\frac{(\zeta_j - 0)_{\zeta_j}^{-\nu_j}}{(\zeta_j + 0)_{\zeta_j}^{-\nu_j}} = \exp(-2\pi\nu_j i) = \exp\left[\frac{2\pi}{p}i - 2\pi\left(\frac{1}{p} + \alpha_j\right)\gamma_j i\right]$$

and consider the function

$$G_0(\zeta) := G(\zeta)\prod_{j=1}^{n}\zeta_{\zeta_j}^{-\nu_j}\,, \quad \zeta \in \Gamma_1\,. \tag{5.20}$$

Let us prove that

$$G_0 \in C(\Gamma_1)\,, \quad |G_0(\zeta)| = 1 \quad \text{for all} \quad |\zeta| = 1 \quad \text{and} \quad \text{ind } \mathrm{G}_0 = 0\,. \quad (5.21)$$

Continuity on Γ_1 follows from (5.18) $G_0(\zeta_j - 0) = G_0(\zeta_j + 0)$, $j = 1, \dots, n$, while from (1.51), (5.20) we find immediately that the function is unimodular $|G_0(\zeta)| = 1$.

To prove the last claim $\text{ind } \mathrm{G}_0 = 0$ we rewrite (5.20) as follows

$$G_0(\zeta) = G(\zeta) \prod_{j=1}^{n} \left(-\frac{\zeta - \zeta_j}{\zeta_j \left(\overline{\zeta} - \overline{\zeta_j}\right)} \right)_{\zeta_j}^{-\nu_j} = G(\zeta) \prod_{j=1}^{n} (-\zeta_j)^{\nu_j} \left(\frac{\zeta - \zeta_j}{\overline{\zeta} - \overline{\zeta_j}} \right)^{-\nu_j}$$

$$= c_0 \frac{\rho_0(\omega(\zeta))}{\overline{\rho_0(\omega(\zeta))}} \left[\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}} \right]^{\frac{1}{p}} \left(\frac{\zeta - \zeta_j}{\overline{\zeta} - \overline{\zeta_j}} \right)^{-\nu_j}, \qquad \zeta \in \Gamma_1\,, \quad c_0 := \prod_{j=1}^{n} (-\zeta_j)^{\nu_j}\,.$$

Thus, $G_0(\zeta)$ has a continuous extension inside the unit disk

$$G_0 \in C(\overline{\mathscr{D}_1})\,, \quad G_0(z)| \neq 0 \ \text{ for all } z \in \overline{\mathscr{D}_1}$$

and the homotopy

$$G_{0,r}(\zeta) := G_0(r\zeta)\,, \quad |\zeta| = 1\,, \quad 0 \leq r \leq 1$$

is continuous, non-vanishing and connects the function $G_0 = G_{0,1}$ with the constant $G_{0,0} = G_0(0)$, confirming $\text{ind } \mathrm{G}_0 = 0$.

Let us rewrite (5.20) in the form

$$G(\zeta) := G_0(\zeta) \prod_{j=1}^{n} \zeta_{\zeta_j}^{\nu_j}\,, \quad \zeta \in \Gamma_1\,. \quad (5.22)$$

From (5.22), (5.21) and Corollary 4.2 we find that conditions (1.36) (1.32) are necessary and sufficient the singular integral equation (1.50) to have a solution, because under these conditions A is Fredholm in $L_p(\Gamma_1)$ and has the following index

$$\text{Ind } \mathrm{A} = \sum_{\nu_j > 1} 1\,,$$

since $\text{ind } \zeta_{\zeta_j}^{\nu_j} = 0$ when $\nu_j < 1$ and $\text{ind } \zeta_{\zeta_j}^{\nu_j} = 1$ when $\nu_j > 1$.

In conclusion it is worth mentioning that the problem has alwayes non-negative index $\text{Ind}_{L_p(\Gamma_1)} A \geq 0$, i.e., is surjective if it is Fredholm. ∎

5.3 Proof of Theorem 1.26

As in the proof of Lemma 1.11 in § 5.2 we treat, for definiteness, the domain Ω^+. In the case of outer domain we have just to replace all γ_j by $2 - \gamma_j$.

First we suppose $\Xi_{ow} = \emptyset$. Then

$$G(\zeta) := \widetilde{G}_0(\zeta) \prod_{j=1}^{n} \zeta_{\zeta_j}^{\widetilde{\nu}_j}\,, \quad \zeta \in \Gamma_1\,, \tag{5.23}$$

$$\widetilde{\nu}_j := \begin{cases} \nu_j & \text{for} \quad \frac{1}{p} \le \nu_j\,, \\ \nu_j - 1 & \text{for} \quad \frac{1}{p} > \nu_j\,, \end{cases} \qquad \widetilde{G}_0(\zeta) := G_0(\zeta)\zeta^{\sigma}\,, \quad \sigma := \sum_{\substack{\zeta_j \notin \Xi_{ow} \\ \nu_j > 1}} 1$$

(see (5.22) and (1.93)–(1.95)). Due to Corollary 4.2 equation (1.50) is Fredholm in $L_p(\Gamma_1)$ if and only if conditions (1.94) hold and then

$$\text{Ind } A = \text{ ind } \widetilde{G}_0 = \sigma = \sum_{\substack{\zeta_j \notin \Xi_{ow} \\ \nu_j > 1}} 1$$

(see (5.23)). Proposition (1.95) follows because the equivalent Riemann–Hilbert BVP (1.55) has non-negative index $\sigma \ge 0$ and has the trivial kernel dim Ker $A = 0$ (if the index is positive, BVP (1.55) would have the trivial cokernel dim Coker $A = 0$; cf. [GK1, Kh1, Mu1]).

Now let $\Xi_{ow} \ne \emptyset$ and consider equation (1.50) for $g_0 \in L_p(\Gamma_1, \Xi_{ow})$, $\varphi \in L_p(\Gamma_1)$ or, what is equivalent, consider operator (1.93). We should start by proving boundedness of (1.93). First note that due to Lemma 1.25 the operator

$$\frac{G-1}{2} K \ : \ L_p(\Gamma) \longrightarrow PC(\Gamma) \subset L_p(\Gamma, \Xi_{ow})$$

is bounded and since is one-dimensional influences neither the Fredholm property nor the index of the operator

$$A = P_{\Gamma_1}^{+} + G(\zeta) P_{\Gamma_1}^{-} + \frac{G(\zeta) - 1}{2} K\,.$$

Therefore, in what follows, we ignore this summand in the operator A and put

$$A = P_{\Gamma_1}^{+} + G(\zeta) P_{\Gamma_1}^{-}\,.$$

Let $\Gamma_{1j}^{\pm} := \{\zeta \in \Gamma_1 \ : \ \pm \operatorname{Im}(\zeta/\zeta_j) > 0\}$ be the semi-circles having $\pm\zeta_j$ as endpoints and $\chi_{\zeta_j}^{\pm}(\zeta)$ be the corresponding characteristic functions ($\zeta \in \Gamma_1$).

Boundedness of the operator in (1.93) follows from the boundedness of the restrictions

$$A_{\zeta_j} := (1 - \chi_{\zeta_j}) I + g_1 \chi_{\zeta_j} A \chi_{\zeta_j} g_1^{-1} I \ : \ L_p(\Gamma_{1\zeta_j}) \longrightarrow L_p(\Gamma_{1\zeta_j}, \{\zeta_j\})\,, \tag{5.24}$$

$$g_1(\zeta) := \frac{|\zeta + \zeta_j|^{\frac{2}{p}}}{\zeta + \zeta_j}$$

for all $\zeta_j \in \Xi_{ow}$. Easy to ascertain that if

$$G_{\zeta_j}(\zeta) := G(\zeta_j - 0)\chi^-_{\zeta_j}(\zeta) + G(\zeta_j + 0)\chi^+_{\zeta_j}(\zeta)\,,$$

then

$$G(\zeta) - G_{\zeta_j}(\zeta) = O(|\zeta - \zeta_j|) \quad \text{as} \quad \zeta \to \zeta_j \in \Xi_{ow}\,. \tag{5.25}$$

Due to Lemma 1.22 the operator

$$A_{\zeta_j} - A^0_{\zeta_j} = g_1[G(\zeta) - G_{\zeta_j}]P^-_{\Gamma_1}g_1^{-1}I \;:\; L_p(\Gamma_{1\zeta_j}) \longrightarrow L_p(\Gamma_1, \{\zeta_j\})\,,$$

$$A^0_{\zeta_j} := g_1[P^+_{\Gamma_1} + G_{\zeta_j}P^-_{\Gamma_1}]g_1^{-1}I \tag{5.26}$$

is bounded. Moreover, if $\varepsilon > 0$ and $\chi_{\zeta_j,\varepsilon}$ is the characteristic function of the neighbourhood $\Gamma_{1\zeta_j,\varepsilon} \subset \Gamma_{1\zeta_j}$, contracting to $\{\zeta_j\}$ as $\varepsilon \to 0$, then

$$\|\chi_{\zeta_j,\varepsilon}(A_{\zeta_j} - A^0_{\zeta_j})\big|\mathscr{L}(L_p(\Gamma_1), L_p(\Gamma_1, \{\zeta_j\}))\|$$

$$\le M_0\|\chi_{\zeta_j,\varepsilon}(G - G_{\zeta_j})\big|L_\infty(\Gamma_1)\|^{1-\delta}\,,$$

which yields

$$\lim_{\varepsilon\to 0}\|\chi_{\zeta_j,\varepsilon}(A_{\zeta_j} - A^0_{\zeta_j})\big|\mathscr{L}(L_p(\Gamma_1), L_p(\Gamma_1, \{\zeta_j\}))\| = 0 \quad \text{as} \quad \varepsilon \to 0$$

since $\delta > 0$ is arbitrary. Thus, boundedness of operator (1.93) follows from the boundedness of the operator

$$A^0_{\zeta_j} \;:\; L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\})\,.$$

The boundedness of A^0, in its turn, follows from the estimates

$$\|\widetilde{\mathscr{V}}_{\zeta_j}A^0_{\zeta_j}\varphi\big|L_p(\Gamma_1^+)\| \le M_j\|\varphi\big|L_p(\Gamma_1)\|\,, \quad M_j < \infty \quad \text{for all} \quad \zeta_j \in \Xi_{ow}$$

(see (1.90), (1.91)).

We can suppose, that

$$G_{\zeta_j}(\zeta) = \begin{cases} e^{\frac{2\pi}{p}i} & \text{for} \quad \zeta \in \Gamma_1^+\,, \\ 1 & \text{for} \quad \zeta \in \Gamma_1^- \end{cases} \tag{5.27}$$

In fact, the operator

$$B_j := P^+_{\Gamma_1} + G^{-1}(\zeta_j - 0)P^-_{\Gamma_1} \tag{5.28}$$

has constant coefficients $G(\zeta_j - 0)$ =const$\neq 0$ and due to the following well-known properties of the singular projections

$$(P^\pm_{\Gamma_1})^2 = P^\pm_{\Gamma_1}\,, \quad P^+_{\Gamma_1}P^-_{\Gamma_1} = P^-_{\Gamma_1}P^+_{\Gamma_1} = 0\,, \quad P^-_{\Gamma_1} + P^+_{\Gamma_1} = I \tag{5.29}$$

is invertible $B_j^{-1} = P_{\Gamma_1}^+ + G(\zeta_j - 0)P_{\Gamma_1}^-$, $B_j^{-1}B_j = B_jB_j^{-1} = I$. Therefore it suffices to prove boundedness of the operator

$$AB_j = P_{\Gamma_1}^+ + G^{-1}(\zeta_j - 0)GP_{\Gamma_1}^- \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{ow}) \tag{5.30}$$

instead of (1.93). The coefficient $G^0(\zeta) := G^{-1}(\zeta_j - 0)G(\zeta)$ of the operator (5.30) has limits $G^0(\zeta_j - 0) = 1$ and $G^0(\zeta_j + 0) = e^{\frac{2\pi}{p}i}$ and corresponding local representative $G^0_{\zeta_j}(\zeta)$ has the form (5.27).

Let us apply the isomorphisms $\mathscr{Z}_{p\zeta_j} = \mathscr{Z}_p\mathscr{Z}_{\zeta_j}$ defined in (3.31)–(3.36). Since

$$\begin{aligned} A_{0,\zeta_j} := \mathscr{Z}_{p\zeta_j}A^0_{\zeta_j}\mathscr{Z}^{-1}_{p\zeta_j} &= \frac{1}{2}(I + \mathscr{Z}_{p\zeta_j}g_1S_{\Gamma_1}g_1^{-1}\mathscr{Z}^{-1}_{p\zeta_j}) \\ &+\frac{1}{2}\begin{bmatrix} e^{\frac{2\pi}{p}i} & 0 \\ 0 & 1 \end{bmatrix}(I - \mathscr{Z}_{p\zeta_j}g_1S_{\Gamma_1}g_1^{-1}\mathscr{Z}^{-1}_{p\zeta_j}) \end{aligned} \tag{5.31}$$

it suffices to find $\mathscr{Z}_{p\zeta_j}g_1S_{\Gamma_{1\zeta_j}}g_1^{-1}\mathscr{Z}^{-1}_{p\zeta_j}$. Applying (3.43) we proceed as follows

$$\begin{aligned} \mathscr{Z}_{\zeta_j}g_1S_{\Gamma_1}g_1^{-1}\mathscr{Z}^{-1}_{\zeta_j}\psi(x) &= \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\left|\frac{\varkappa'_{\zeta_j}(x)}{\varkappa'_{\zeta_j}(y)}\right|^{\frac{1}{p}}\frac{\varkappa_{\zeta_j}(y)}{\varkappa_{\zeta_j}(x)}\frac{\varkappa'_{\zeta_j}(y)\varphi(y)dy}{\varkappa_{\zeta_j}(y) - \varkappa_{\zeta_j}(x)} \\ &= \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\varphi(y)dy}{y - x} = S_{\mathbb{R}}\varphi(x) \end{aligned}$$

and further

$$\mathscr{Z}_{p\zeta_j}S_{\Gamma_1}\mathscr{Z}^{-1}_{p\zeta_j} = \mathscr{Z}_p(\mathscr{Z}_{\zeta_j}S_{\Gamma_1}\mathscr{Z}^{-1}_{\zeta_j})\mathscr{Z}_p^{-1} = \begin{bmatrix} S_p & -N_p \\ N_p & -S_p \end{bmatrix},$$

where

$$\begin{aligned} &S_p\varphi(x) := \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{e^{-\frac{x-y}{p}}\varphi(y)dy}{1 - e^{-(x-y)}} = W^0_{s_p}, \\ &s_p(\lambda) := \coth\pi\left(\frac{i}{p} + \lambda\right), \quad \lambda, x \in \mathbb{R}, \\ &N_p\varphi(x) := \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{e^{-\frac{x-y}{p}}\varphi(y)dy}{1 + e^{-(x-y)}} = W^0_{n_p}, \\ &n_p(\lambda) := \frac{1}{\sinh\pi\left(\frac{i}{p} + \lambda\right)}. \end{aligned} \tag{5.32}$$

Easy to ascertain that

$$A_{0,\zeta_j} := \mathscr{Z}_{p\zeta_j} A^0_{\zeta_j} \mathscr{Z}^{-1}_{p\zeta_j} = \begin{bmatrix} \frac{1}{2}(I+S_p) + e^{\frac{2\pi}{p}i}\frac{1}{2}(I-S_p) & \frac{1}{2}(e^{\frac{2\pi}{p}i}-1)N_p \\ 0 & I \end{bmatrix}$$

$$= W_{\mathscr{A}_0(\zeta_j,\cdot)} \tag{5.33}$$

(see (5.31)–(5.32)), where $\mathscr{A}_0(\zeta_j,\cdot)$ is the symbol. Since

$$\cosh z \sinh w - \sinh z \cosh w = \sinh(w-z)\,, \quad z,w \in \mathbb{C}\,,$$

we find the symbol

$$\mathscr{A}_0(\zeta_j,\lambda) := \begin{bmatrix} e^{\frac{\pi}{p}i}\left[\cosh\frac{\pi}{p}i - \sinh\frac{\pi}{p}i\coth\pi\left(\frac{i}{p}+\lambda\right)\right] & e^{\frac{\pi}{p}i}\frac{\sinh\frac{\pi}{p}i}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\frac{\pi}{p}i}\frac{\sinh\pi\lambda}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} & e^{\frac{\pi}{p}i}\frac{\sinh\frac{\pi}{p}i}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} \\ 0 & 1 \end{bmatrix} \tag{5.34}$$

Applying (3.30), (3.32) we get

$$\mathscr{Z}_{p\zeta_j}\widetilde{\mathscr{V}}_{\zeta_j}\mathscr{Z}^{-1}_{p\zeta_j} = g_j\widetilde{\mathscr{V}}_\infty I = g_j\begin{bmatrix} e^{-\frac{\pi}{p}i}\mathscr{V}_\infty & -\mathscr{V}_\infty \\ 0 & 0 \end{bmatrix} + R_j = g_j W_{\widetilde{v}_0} + R_j \tag{5.35}$$

$$\widetilde{v}_0 := \begin{bmatrix} e^{-\frac{\pi}{p}i}\left[1-g_-^{-1}(\lambda)\right] & g_-^{-1}(\lambda)-1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{-\frac{\pi}{p}i}\frac{i}{\lambda} & -\frac{i}{\lambda} \\ 0 & 0 \end{bmatrix}$$

(see (3.8)). From (5.33) and (5.35) we have

$$\mathscr{Z}_{p\zeta_j}\widetilde{\mathscr{V}}_{\zeta_j}A^0_{\zeta_j}\varphi = g_j\widetilde{\mathscr{V}}_\infty A_{0,\zeta_j}(\psi_1,\psi_2)^\top + R_j A_{0,\zeta_j}(\psi_1,\psi_2)^\top = (\mathscr{V}_\infty W^0_{a_{1j}}\psi_1$$

$$+\mathscr{V}_\infty W^0_{a_{2j}}\psi_2, 0)^\top = (W^0_{b_{1j}}\psi_1 + W_{b_{2j}}\psi_2, 0)^\top + R_j A_{0,\zeta_j}(\psi_1,\psi_2)^\top\,, \tag{5.36}$$

where $(\psi_1,\psi_2)^\top := \mathscr{Z}_{p\zeta_j}\varphi$ and

$$a_{1j}(\lambda) := \frac{\sinh\pi\lambda}{\sinh\pi\left(\frac{i}{p}+\lambda\right)}\,, \quad b_{1j}(\lambda) := \frac{-i\sinh\pi\lambda}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)}\,, \tag{5.37}$$

$$a_{2j}(\lambda) := \frac{\sinh\frac{\pi}{p}i}{\sinh\pi\left(\frac{i}{p}+\lambda\right)}\,, \quad b_{2j}(\lambda) := \frac{i\left[\sinh\frac{\pi}{p}i - \sinh\pi\left(\frac{i}{p}+\lambda\right)\right]}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)}\,,$$

because $\mathscr{V}_\infty = W^0_{g_-^{-1}} - I = W^0_{g_-^{-1}-1}$ (see (3.23)) and $g_-^{-1}(\lambda) - 1 = -i/\lambda$. The functions $b_{kj}(\lambda)$ satisfy conditions (3.4) and, therefore, $b_{kj} \in PC_p(\dot{\mathbb{R}})$.

This yields the estimate

$$\|\widetilde{\mathscr{V}}_{\zeta_j} A^0_{\zeta_j}\varphi\big|L_p(\Gamma^+_{1\zeta_j})\| \le \|\mathscr{Z}^{-1}_{p\zeta_j}\|\|\mathscr{Z}_{p\zeta_j}\widetilde{\mathscr{V}}_{\zeta_j} A^0_{\zeta_j}\varphi\big|L_p(\mathbb{R}^+)\| \tag{5.38}$$

$$= \|\mathscr{Z}^{-1}_{p\zeta_j}\|\|g_j\widetilde{\mathscr{V}}_\infty A_{0,\zeta_j}(\psi_1,\psi_2)^\top\big|L^2_p(\mathbb{R})\| + \|R_j A_{0,\zeta_j}(\psi_1,\psi_2)^\top\big|L^2_p(\mathbb{R})\|$$

$$\le \|\mathscr{Z}^{-1}_{p\zeta_j}\|\left[\sum_{k=1,2}\|g_j W^0_{b_{kj}}\psi_k\big|L_p(\mathbb{R})\| + \|R_j A_{0,\zeta_j}(\psi_1,\psi_2)^\top\big|L^2_p(\mathbb{R})\|\right]$$

$$\le M^0_j\|(\psi_1,\psi_2)^\top\big|L^2_p(\mathbb{R})\| = M^0_j\|\mathscr{Z}_{p\zeta_j}\varphi\big|L^2_p(\mathbb{R})\| \le M_j\|\varphi\big|L_p(\Gamma_{1\zeta_j})\|\,.$$

Estimates (5.30) follow and imply the boundedness in (1.93).

To prove the Fredholm criteria (1.94) we apply the localization method, due to I.Gohberg and N.Krupnik (see [GK1, RS1]) modified for operators between two different spaces (see [Du9, § 3]). We skip over exposing details of the method because they are well-known and even modified version is operating with similar objects–localization classes, local equivalence, local representatives, local invertibility etc.

We choose a standard covering system of localizing classes $\{M_\zeta\}_{\zeta\in\Gamma_1}$, where M_ζ consists of all multiplication operators vI by smooth functions $v\in C^\infty(\Gamma_1)$, $|v(t)|\le 1$ $(t\in\Gamma_1)$ which are equal 1 in some neighbourhood of ζ. Boundedness of operators $vI\in M_\zeta$ in the space $L_p(\Gamma_1)$ is trivial, while in $L_p(\Gamma_1.\Xi_{ow})$ follows from Lemma 1.22. Another essential property–compactness of commutators

$$[vI,A] = vA - AvI\ :\ L_p(\Gamma_1)\longrightarrow L_p(\Gamma_1.\Xi_{ow})\,,$$

which is a bounded operator already, follows from the well-known criteria of compactness in $L_p(\Gamma_1)$ space modified with the help of Lemma 1.22

$$\int\limits_{\Gamma_1}\left[\int\limits_{\Gamma_1}|\log(\zeta-\zeta_j)k(\zeta,\tau)|^{p'}|d\tau|\right]^{\frac{p}{p'}}|d\zeta|<\infty\,,$$

since the kernel $k(\zeta,\tau)$ of the commutator $[vI,A]$ is a uniformly bounded function.

As a local representative of A at a regular point $\zeta_0\ne\zeta_1,\dots\zeta_n$ we choose the following operator

$$A\overset{M_{\zeta_0}}{\sim}A_{\zeta_0} := P^+_{\Gamma_1} + G(\zeta_0)P^-_{\Gamma_1}\,,\quad A_{\zeta_0}\ :\ L_p(\Gamma_1)\longrightarrow L_p(\Gamma_1) \tag{5.39}$$

with the constant ("frozen" at ζ_0) coefficient. This operator is invertible $A^{-1}_{\zeta_0} := P^+_{\Gamma_1} + G^{-1}(\zeta_0)P^-_{\Gamma_1}$ (see (5.28), (5.29)).

Before localizing at the point ζ_j, where the coefficient has discontinuity $G(\zeta_j+0)\ne G_j(\zeta_j-0)\ne 0$ let us simplify the operator by taking composition

with the invertible operator B_j in (5.28). The composition AB_j has the same image $\operatorname{Im} A_j = \operatorname{Im} A$ and due to invertibility of B_j we can consider the composition

$$A_j := P^+_{\Gamma_1} + G^{-1}(\zeta_j - 0)GP^-_{\Gamma_1} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{ow}) \tag{5.40}$$

instead of (1.93). The local representative of the operator (5.40) at the point $\zeta_j \in \Gamma_1$ is chosen as follows

$$A_j \overset{M_{\zeta_j}}{\sim} A^0_{\gamma_j,\zeta_j} := g_1[P^+_{\Gamma_1} + G_{\zeta_j}P^-_{\Gamma_1}]g_1^{-1}I\,, \quad G_{\zeta_j}(t) := e^{2\pi\nu_j i}\chi^+_j + \chi^-_j\,, \tag{5.41}$$

$$\nu_j = \frac{1}{p} - \left(\frac{1}{p} + \alpha_j\right)\gamma_j\,,$$

since $G^{-1}(\zeta_j - 0)G(\zeta_j + 0) = e^{2\pi\nu_j i}$ (see (5.18) and note that in (5.24)–(5.27) we have taken the outward peak which means $\gamma_j = 0$); $\chi^\pm_j$ in (5.41) are the characteristic functions of the semi-circumference $\pm\operatorname{Im}(\zeta/\zeta_{\mathrm{j}}) \geq 0$.

The localized operator $A^0_{\gamma_j,\zeta_j}$ should be considered in the appropriate local spaces:

$$\begin{aligned} &A^0_{\gamma_j,\zeta_j} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1) && \text{if} \quad 0 < \gamma_j \leq 2\,, \\ &A^0_{0,\zeta_j} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\}) && \text{if} \quad \gamma_j = 0, \ \ (\text{i.e., } \ \zeta_j \in \Xi_{ow})\,. \end{aligned} \tag{5.42}$$

The lifted operators (cf. (5.33))

$$\begin{aligned} &A_{\gamma_j,\zeta_j} := \mathscr{Z}_{p\zeta_j} A^0_{\gamma_j,\zeta_j} \mathscr{Z}^{-1}_{p\zeta_j} =: L^2_p(\mathbb{R}) \longrightarrow L^2_p(\mathbb{R}) && \text{if } \ 0 < \gamma_j \leq 2\,, \\ &A_{0,\zeta_j} := \mathscr{Z}_{p\zeta_j} A^0_{0,\zeta_j} \mathscr{Z}^{-1}_{p\zeta_j} =: L^2_p(\mathbb{R}) \longrightarrow L^2_p(\mathbb{R}, \{\infty\}) && \text{if } \ \gamma_j = 0 \end{aligned} \tag{5.43}$$

are convolutions

$$A_{\gamma_j,\zeta_j} = W^0_{\mathscr{A}_{\gamma_j}(\zeta_j,\cdot)} \tag{5.44}$$

(cf. (5.33)) with the symbols

$$\begin{aligned} \mathscr{A}_{\gamma_j}(\zeta_j, \lambda) &:= \begin{bmatrix} \frac{1}{2}(I + s_p(\lambda)) + e^{2\pi\nu_j i}\frac{1}{2}(I - s_p(\lambda)) & \frac{1}{2}(e^{2\pi\nu_j i} - 1)n_p(\lambda) \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} e^{\pi\nu_j i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sinh\pi\left(\frac{i}{p} - \nu_j i + \lambda\right)}{\sinh\pi\left(\frac{i}{p} + \lambda\right)} & \frac{\sinh\pi\nu_j i}{\sinh\pi\left(\frac{i}{p} + \lambda\right)} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\pi\nu_j i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sinh\pi\left(\lambda - \left(\frac{1}{p} + \alpha_j\right)\gamma_j i\right)}{\sinh\pi\left(\frac{i}{p} + \lambda\right)} & \frac{\sinh\pi\left[\frac{i}{p} - \left(\frac{1}{p} + \alpha_j\right)\gamma_j\right]}{\sinh\pi\left(\frac{i}{p} + \lambda\right)} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{5.45}$$

The operator $A_{\gamma_j,\zeta_j} = W^0_{\mathscr{A}_{\gamma_j}(\zeta_j,\cdot)}$ for $\gamma_j \neq 0$ is invertible in $L^2_p(\mathbb{R})$ iff

$$\mathscr{A}_0(\zeta_j,\lambda) = e^{\pi\nu_j i}\frac{\sinh\pi\left(\lambda-\left(\frac{1}{p}+\alpha_j\right)\gamma_j i\right)}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} \neq 0 \Longrightarrow \left(\frac{1}{p}+\alpha_j\right)\gamma_j \neq 1, \quad (5.46)$$

as it follows from (5.45) and (2.5). Condition (1.94) is justified.

Now let $\gamma_j = 0$; then $\nu_j = \frac{1}{p}$ and (see (5.45))

$$\mathscr{A}_0(\zeta_j,\lambda) := \begin{bmatrix} e^{\frac{\pi}{p}i}\frac{\sinh\pi\lambda}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} & e^{\frac{\pi}{p}i}\frac{\sinh\frac{\pi}{p}i}{\sinh\pi\left(\frac{i}{p}+\lambda\right)} \\ 0 & 1 \end{bmatrix} \quad (5.47)$$

(cf. (5.36), (5.37)). The operator

$$\widetilde{\mathscr{V}}_\infty := \begin{bmatrix} I+\mathscr{V}_\infty & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} e^{-\frac{\pi}{p}i}I & -I \\ 0 & I \end{bmatrix} = \begin{bmatrix} e^{-\frac{\pi}{p}i}(I+\mathscr{V}_\infty) & -(I+\mathscr{V}_\infty) \\ 0 & I \end{bmatrix}$$

$$= W^0_{v_0} \,:\, L^2_p(\mathbb{R},\{\infty\}) \longrightarrow L^2_p(\mathbb{R})\,, \quad v_0(\lambda) := \begin{bmatrix} e^{-\frac{\pi}{p}i}\frac{\lambda-i}{\lambda} & -\frac{\lambda-i}{\lambda} \\ 0 & 1 \end{bmatrix} \quad (5.48)$$

arranges an isomorphism (see (1.91) and (5.35)). Therefore, the operator A_{0,ζ_j} in (5.43), (5.44) (the case $\gamma_j = 0$) is equivalent to the operator

$$\widetilde{\mathscr{V}}_\infty W^0_{A_0(\zeta_j,\cdot)} = W^0_{v_0} W^0_{\mathscr{A}_0(\zeta_j,\cdot)} = W^0_{\mathscr{A}^1_0(\zeta_j,\cdot)} \,:\, L^2_p(\mathbb{R}) \longrightarrow L^2_p(\mathbb{R})\,, \quad (5.49)$$

where

$$\mathscr{A}^1_0(\zeta_j,\lambda) := v_0(\lambda)\mathscr{A}_0(\zeta_j,\lambda) := \begin{bmatrix} \frac{(\lambda-i)\sinh\pi\lambda}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)} & \frac{(\lambda-i)\left[\sinh\frac{\pi}{p}i-\sinh\pi\left(\frac{i}{p}+\lambda\right)\right]}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)} \\ 0 & 1 \end{bmatrix}$$

Obviously, $\mathscr{A}^1_0(\zeta_j,\cdot) \in PC^{2\times 2}_p(\mathbb{R})$ (see (3.4)) and

$$\det\mathscr{A}^1_0(\zeta_j,\lambda) = \frac{(\lambda-i)\sinh\pi\lambda}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)} \neq 0 \quad \text{for all} \quad \lambda\in\dot{\mathbb{R}}\,.$$

$W_{\mathscr{A}^1_0(\zeta_j,\cdot)}$ is invertible in $L^2_p(\mathbb{R})$ and yields invertibility of the local representatives A_{ζ_j} in (5.42) for all $\zeta_j \in \Xi_{ow}$.

Thus, under conditions (1.94), all local representatives of the operator (1.93) are invertible, which implies that (1.93) is Fredholm.

To prove the index formula (1.95) we recall the representation (5.23) and arrange a homotopy sending the function $G(\zeta)$ to

$$g(\zeta) := g_0(\zeta)\prod_{\zeta_j\in\Xi_{ow}} g_j(\zeta)\,, \quad \zeta\in\Gamma_1\,, \quad (5.50)$$

where the functions $g_0(\zeta)$ and $g_j(\zeta)$ have the same images (accept the same values) as $\widetilde{G}_0(\zeta)$ and $\zeta_j^{-1}\zeta_{\zeta_j}^{\frac{1}{p}}$, respectively, when ζ ranges over Γ_1 (we remind that $\widetilde{\nu}_j = \nu_j = \frac{1}{p}$ as soon as $\gamma_j = 0$). More of this, supports of $g_0 - 1$ and of $g_j - 1$ are "squeezed" and belong to Γ_{10} and Γ_{ζ_j}, respectively. Therefore,

$$\begin{gathered}\operatorname{supp}(g_k - 1) \cap (\zeta) \operatorname{supp}(g_j - 1) := \emptyset \quad \text{for all} \quad k \neq j \quad , \\ g_0 \in C^1(\Gamma_1) \,, \quad \operatorname{ind} g_0 = \operatorname{ind} \widetilde{G}_0 = \sigma \,. \\ g_j \in C^1(\Gamma_1 \setminus \{\zeta_j\}) \,, \quad \frac{g_j}{\zeta_{\zeta_j}^{\frac{1}{p}}} \in C^1(\Gamma_1) \,, \quad \operatorname{ind} \frac{g_j}{\zeta_{\zeta_j}^{\frac{1}{p}}} = 0 \,.\end{gathered} \tag{5.51}$$

To arrange such homotopy we just define

$$G_\vartheta(\zeta) := \widetilde{G}_0(\zeta) \left[\frac{g_0(\zeta)}{\widetilde{G}_0(\zeta)}\right]^\vartheta \prod_{\zeta_j \in \Xi_{ow}} \left[\frac{g_j(\zeta)}{\zeta_{\zeta_j}^{\frac{1}{p}}}\right]^\vartheta \zeta_{\zeta_j}^{\frac{1}{p}} \prod_{\zeta_j \notin \Xi_{ow}} \zeta_{\zeta_j}^{(1-\vartheta)\nu_j} \tag{5.52}$$

for $0 \leq \vartheta \leq 1$. Since the functions $[g_0(\zeta)/\widetilde{G}_0(\zeta)]^\vartheta$ and $[g_j(\zeta)/\zeta_{\zeta_j}^{\frac{1}{p}}]^\vartheta$ are continuous for all $0 \leq \vartheta \leq 1$ (see (5.51)) and the exponents $(1-\vartheta)\widetilde{\nu}_j$ continue to satisfy conditions (1.94) when $\zeta_j \notin \Xi_{ow}$, we get the operators

$$A_\vartheta := P_{\Gamma_1}^+ + G_\vartheta P_{\Gamma_1}^- \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \Xi_{ow})$$

which are FREDHOLM for all $0 \leq \vartheta \leq 1$. Then these operators maintain the index

$$\operatorname{Ind} A = \operatorname{Ind} A_0 = \operatorname{Ind} A_1 = \operatorname{Ind} (P_{\Gamma_1}^+ + G_1 P_{\Gamma_1}^-) \,. \tag{5.53}$$

Due to the disjoint supports of $g_j - 1$ (see (5.51)) we get

$$A_1 = P_{\Gamma_1}^+ + G_1 P_{\Gamma_1}^- = D_0 \prod_{\zeta_j \in \Xi_{ow}} D_j \,,$$

$$D_0 := P_{\Gamma_1}^+ + g_0 P_{\Gamma_1}^- \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1) \,, \tag{5.54}$$

$$D_j := P_{\Gamma_1}^+ + g_j P_{\Gamma_1}^- \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\})$$

and the operators commute $D_j D_k = D_k D_j$. Therefore

$$\operatorname{Ind} A_1 = \operatorname{Ind} D_0 + \sum_{\zeta_j \in \Xi_{ow}} \operatorname{Ind} D_j = \operatorname{ind} \widetilde{G}_0 + \sum_{\zeta_j \in \Xi_{ow}} \operatorname{Ind} D_j \tag{5.55}$$

and to justify the index formula (1.95) we just have to show that

$$\operatorname{Ind} D_j = 0 \quad \text{for all} \quad \zeta_j \in \Xi_{ow} \,. \tag{5.56}$$

By the condition the image of $g_j(\zeta)$ coincides with the image of $\zeta_j^{-1}\zeta_{\zeta_j}^{\frac{1}{p}}$ which means that

$$|g_j(\zeta)| = 1 \,, \quad g_j(\zeta_j - 0) = e^{\frac{2\pi}{p} i} \,, \quad g_j(\zeta_j + 0) = +1 \,. \tag{5.57}$$

Let us consider the operator

$$H_{\zeta_j} = I + \mathscr{Z}_{p\zeta_j}^{-1} W_{\mathscr{H}}^0 \mathscr{Z}_{p\zeta_j} = \mathscr{Z}_{p\zeta_j}^{-1} W_{1+\mathscr{H}}^0 \mathscr{Z}_{p\zeta_j} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\}) \,,$$

$$1 + \mathscr{H}(\lambda) = \begin{bmatrix} e^{\frac{\pi}{p} i} \frac{\lambda}{\lambda - i} & e^{\frac{\pi}{p} i} \\ 0 & 1 \end{bmatrix} . \tag{5.58}$$

The lifted operator (see (3.37)–(3.40))

$$\mathscr{Z}_{p\zeta_j} H_{\zeta_j} \mathscr{Z}_{p\zeta_j}^{-1} = W_{1+\mathscr{H}}^0 \ : \ L_p^2(\mathbb{R}) \longrightarrow L_p^2(\mathbb{R}, \{\infty\}) \tag{5.59}$$

is invertible. In fact,

$$[1 + \mathscr{H}(\lambda)]^{-1} = \begin{bmatrix} e^{-\frac{\pi}{p} i} \frac{\lambda - i}{\lambda} & -\frac{\lambda - i}{\lambda} \\ 0 & 1 \end{bmatrix} = v_0(\lambda)$$

(cf. (5.48)) and therefore $\widetilde{\mathscr{V}}_\infty$ in (5.48) is the inverse operator to (5.58)

$$\widetilde{\mathscr{V}}_\infty W_{1+\mathscr{H}}^0 = W_{v_0(1+\mathscr{H})}^0 = I \tag{5.60}$$

(see (3.23)).

For the parameter-dependent operator

$$R_\vartheta := (1 - \vartheta) D_j - \vartheta e^{\mu i} H_{\zeta_j} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\}) \,, \quad 0 \le \vartheta \le 1 \,, \tag{5.61}$$

where $\frac{2\pi}{p} \le \mu \le 2\pi$ will be chosen later, the local representatives for $\zeta_0 \notin \Xi_{ow}$ read

$$\begin{aligned} R_\vartheta \overset{M_{\zeta_0}}{\sim} R_{\vartheta,\zeta_0} &= g_1^{-1}[(1-\vartheta) P_{\Gamma_1}^+ + (1-\vartheta) g_j(\zeta_0) P_{\Gamma_1}^-] g_1 I - \vartheta e^{\mu i} I \\ &=] g_1^{-1}[(1 - \vartheta - \vartheta e^{\mu i}) P_{\Gamma_1}^+ + [(1-\vartheta) g_j(\zeta_0)]] g_1 - \vartheta e^{\mu i}] P_{\Gamma_1}^- \\ &\qquad : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1) \,, \end{aligned} \tag{5.62}$$

while for $\zeta_j \in \Xi_{ow}$ we get

$$\begin{aligned} R_\vartheta \overset{M_{\zeta_j}}{\sim} R_{\vartheta,\zeta_j} &=] g_1^{-1}[(1-\vartheta) P_{\Gamma_1}^+ + (1-\vartheta) G_{\zeta_j}(\zeta_0) P_{\Gamma_1}^-]] g_1 I \\ &\quad - \vartheta e^{\mu i} \mathscr{Z}_{p\zeta_j}^{-1} W_{1+\mathscr{H}} \mathscr{Z}_{p\zeta_j} \\ &= \mathscr{Z}_{p\zeta_j}^{-1} W_{\mathscr{R}_\vartheta(\zeta_j, \cdot)} \mathscr{Z}_{p\zeta_j} \ : \ L_p(\Gamma_1) \longrightarrow L_p(\Gamma_1, \{\zeta_j\}) \end{aligned} \tag{5.63}$$

(cf. (5.39), (5.41), (5.42)–(5.47)), where $G_{\zeta_j}(\zeta) = +1$ for $\operatorname{Im}(\zeta \zeta_j^{-1}) > 0$ and $G_{\zeta_j}(\zeta) = e^{\frac{2\pi}{p} i}$ for $\operatorname{Im}(\zeta \zeta_j^{-1}) < 0$ (cf. (5.27), (5.41)) and

$$\mathscr{R}_\vartheta(\zeta_j, \lambda) = (1 - \vartheta) \mathscr{A}(\zeta_j, \lambda) - \vartheta e^{\mu i} [1 + \mathscr{H}(\lambda)] \,. \tag{5.64}$$

The operators R_{ϑ,ζ_0} in (5.62) are invertible having constant non-vanishing coefficients

$$1-\vartheta(1+e^{\mu i})\neq 0\,,\ (1-\vartheta)g_j(\zeta_0)-\vartheta e^{\mu i}\neq 0\quad\text{for all}\quad 0\leq\vartheta\leq 1\,,\quad \zeta_0\neq\zeta_j$$

provided $\mu>\pi$ (we remind that $g_j(\zeta_0)=e^{\mu i}$ with $\frac{2\pi}{p}\leq\mu\leq 2\pi$ is impossible since $\zeta_0\neq\zeta_j$). The inverse operator is written as in (5.28)–(5.30).

The operators R_{ϑ,ζ_j} in (5.63) are also invertible because the lifted operators

$$W^0_{\mathscr{R}_\vartheta(\zeta_j,\cdot)}=\mathscr{Z}_{p\zeta_j}R_{\vartheta,\zeta_j}\mathscr{Z}^{-1}_{p\zeta_j}\ :\ L^2_p(\mathbb{R})\longrightarrow L^2_p(\mathbb{R},\{\infty\}) \tag{5.65}$$

are invertible. To verify this we should apply the isomorphism $\widetilde{\mathscr{V}}_\infty$ from (5.48)

$$\widetilde{\mathscr{V}}_\infty W^0_{\mathscr{R}_\vartheta(\zeta_j,\cdot)}=W^0_{v_0\mathscr{R}_\vartheta(\zeta_j,\cdot)}\ :\ L^2_p(\mathbb{R})\longrightarrow L^2_p(\mathbb{R}) \tag{5.66}$$

(see (3.23)), where

$$\begin{aligned}v_0(\lambda)\mathscr{R}_\vartheta(\zeta_j,\lambda)&=(1-\vartheta)\mathscr{A}^1_0(\zeta_j,\lambda)-\vartheta e^{\mu i}v_0(\lambda)[1+\mathscr{H}(\lambda)]\\&=(1-\vartheta)\mathscr{A}^1_0(\zeta_j,\lambda)-\vartheta e^{\mu i}I\\&=\begin{bmatrix}(1-\vartheta)\frac{(\lambda-i)\sinh\pi\lambda}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)}-\vartheta e^{\mu i} & (1-\vartheta)(\lambda-i)\frac{\sinh\frac{\pi}{p}i-\sinh\pi\left(\frac{i}{p}+\lambda\right)}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)}\\ 0 & 1-\vartheta(1+e^{\mu i})\end{bmatrix}\end{aligned}$$

(see (5.49), (5.60)). The image of the function

$$h_p(\lambda):=\frac{(\lambda-i)\sinh\pi\lambda}{\lambda\sinh\pi\left(\frac{i}{p}+\lambda\right)}=\frac{\sinh\pi\lambda}{\lambda}\left[\frac{\lambda\sinh\pi\lambda\,\cos\frac{\pi}{p}-\cosh\pi\lambda\,\sin\frac{\pi}{p}}{\left|\sinh\pi\left(\frac{i}{p}+\lambda\right)\right|^2}\right.$$

$$\left.-i\,\frac{\sinh\pi\lambda\,\cos\frac{\pi}{p}+\lambda\cosh\pi\lambda\,\sin\frac{\pi}{p}}{\left|\sinh\pi\left(\frac{i}{p}+\lambda\right)\right|^2}\right]\,,\qquad \overline{h_p(\lambda)}=h_p(-\lambda)$$

on the complex plane $\mathbb{C}$ when λ ranges through $\mathbb{R}$ is a continuous curve connecting points $h_p(\pm\infty)=e^{\pm\frac{\pi}{p}i}$ on the unit circumference and passing through $h_p(0)=-\frac{\pi}{\sin\frac{\pi}{p}}<0$ on the negative semi-axes. Easy to ascertain, that $\frac{\pi}{p}\leq\arg h_p(\lambda)\leq 2\pi-\frac{\pi}{p}$ and the constraints

$$\max\left\{\pi,2\pi-\frac{\pi}{p},\frac{2\pi}{p}\right\}<\mu\leq 2\pi$$

on the parameter μ ensure the ellipticity

$$\det v_0(\lambda)\mathscr{R}_\vartheta(\zeta_j,\lambda)=[(1-\vartheta)h_p(\lambda)-\vartheta e^{\mu i}][1-\vartheta(1+e^{\mu i})]\neq 0$$

$$\text{for all}\quad 0\leq\vartheta\leq 1\,,\quad \lambda\in\dot{\mathbb{R}},$$

which yields invertibility of the operator in (5.66) (see (2.5)).

Thus, the operator R_ϑ in (5.61) depends on the parameter $\vartheta \in [0,1]$ continuously and connects the operator B_j with the invertible one $-e^{\mu i} H_{\zeta_j}$ in the group of FREDHOLM operators, which yields equality of indices

$$\text{Ind } D_j = \text{Ind } R_0 = \text{Ind } R_1 = \text{Ind } H_{\zeta_j} = 0\,.$$

■

5.4 Proof of Theorem 1.23

First suppose Γ has no peaks $\mathscr{T}_{pk} = \emptyset$.

Let us write the symbols of equations (1.39) and (1.40) in the spaces $\mathbb{X}^m(\Gamma,\rho) = W_p^m(\Gamma,\rho), H^0_{\mu+m}(\Gamma,\rho), C(\Gamma,\rho), PC^m(\Gamma,\rho)$ according to (4.6), (4.10) and (4.28)

$$(\mathscr{A}_\pm)_{\mathbb{X}^m(\Gamma,\rho)}(t,\lambda) := \begin{cases} \dfrac{1}{p}\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} & \text{if} \quad t \neq t_1, \dots, t_n\,, \\ \dfrac{1}{p}\begin{bmatrix} \pm 1 & \mathscr{H}_{m,j}(\lambda) \\ \mathscr{H}_{m,j}(\lambda) & \pm 1 \end{bmatrix} & \text{if} \quad t = t_j\,, \end{cases}$$

where $m = 0, 1$, β_j is defined in (1.79) and

$$\mathscr{H}_{m,j}(\lambda) := \frac{\sinh \pi(1-\gamma_j)(i\beta_j - mi + \lambda)}{\sinh \pi(i\beta_j + \lambda)}\,. \tag{5.67}$$

According to Theorems 4.1, 4.3 and 4.6 equations (1.39) and (1.40) are FREDHOLM in $\mathbb{X}^m(\Gamma,\rho)$ if and only if

$$\inf_{\lambda\in\mathbb{R}} \left|\det (\mathscr{A}_\pm)_{\mathbb{X}^m(\Gamma,\rho)}(t,\lambda)\right| = \frac{1}{4} \inf_{\lambda\in\mathbb{R}} \left|1 - \mathscr{H}^2_{m,j}(\lambda)\right| \neq 0\,. \tag{5.68}$$

Invoking the formulae

$$\sinh^2 a - \sinh^2 b = \sinh(a-b)\sinh(a+b)\,,$$

$$\sinh(a + 2\pi k) = \sinh\, a\,, \quad a, b \in \mathbb{C}\,, \;\; k = 0, \pm 1, \dots\,.$$

we find easily

$$\begin{aligned} \frac{1}{4}(1 - \mathscr{H}^2_{m,j}(\lambda)) &= \frac{\sinh^2 \pi(i\beta_j + \lambda) - \sinh^2 \pi(1-\gamma_j)(i\beta_j - mi + \lambda)}{4\sinh^2 \pi(i\beta_j + \lambda)} \\ &= -\frac{\sinh \pi[(2-\gamma_j)(i\beta_j + \lambda) - mi + 2\gamma_j i]\sinh \pi\gamma_j(i\beta_j + \lambda - mi)}{\sinh^2 \pi(i\beta_j + \lambda)} \\ &= -\frac{\sinh \pi(2-\gamma_j)(i\beta_j + \lambda - mi)\sinh \pi\gamma_j(i\beta_j + \lambda - mi)}{\sinh^2 \pi(i\beta_j + \lambda)}\,. \end{aligned} \tag{5.69}$$

Due to (5.69) condition (5.68) holds if and only if

$$(2-\gamma_j)(i\beta_j+\lambda-mi)\neq 0,\pm i,\dots\,,\quad \gamma_j(i\beta_j+\lambda-mi)\neq 0,\pm i,\dots\,.$$

Since $0<\beta_j<1$, $m=0,1$ the latter conditions can be written as follows

$$\beta_j\neq\begin{cases}\gamma_j^0 & \text{if}\quad m=0\,,\\ 1-\gamma_j^0 & \text{if}\quad m=1\,.\end{cases}\tag{5.70}$$

and the condition of the theorem is justified.

On the other hand due to (5.70) the group of non-degenerate symbols (5.68) is divided in four homotopy groups (two for each $m=0,1$); the symbols inside each group have equal indices and it suffices to find the value for one representative of the group. Since

$$\det\,(\mathscr{A}_\pm)_{\mathbb{X}^m(\Gamma,\rho)}(t,\lambda)=\frac{1}{4}[1-\mathscr{H}^2_{m,j}(\lambda)]=\frac{1}{4}[1-\mathscr{H}_{m,j}(\lambda)][1+\mathscr{H}_{m,j}(\lambda)]$$

it is sufficient to investigate simpler functions $1\pm\mathscr{H}_{m,j}(\lambda)$. Images on the complex plane of representatives

$$\gamma_j=\frac{1}{p}\,,\quad \beta_j=\frac{1}{4},\frac{3}{4}\,,\quad m=0,1$$

are plotted on Fig. 7–Fig.10 in Appendix. The result can be summarized as follows:

$$\text{ind }\det\,(\mathscr{A}_\pm)_{\mathbb{X}^m(\Gamma,\rho)}(t_j,\cdot)=\begin{cases}0 & \text{for} & \beta_j<1-\gamma_j^0 & \text{and } m=0\,,\\ -1 & \text{for} & \beta_j>1-\gamma_j^0 & \text{and } m=0\,,\\ 1 & \text{for} & \beta_j<\gamma_j^0 & \text{and } m=-1\,,\\ 0 & \text{for} & \beta_j>\gamma_j^0 & \text{and } m=-1\,.\end{cases}$$

From Theorems 4.1, 4.3 and 4.6 we get the index formula (we remind that $\mathscr{T}_{pk}=\emptyset$)

$$\text{Ind }A_\pm=-\sum_{j=1}^n\text{ind }\det\,(\mathscr{A}_\pm)_{\mathbb{X}^m(\Gamma,\rho)}(t_j,\cdot)=\begin{cases}\displaystyle\sum_{\substack{t_j\notin\mathscr{T}_{pk}\\ \beta_j>\gamma_j^0}}1 & \text{for }\ m=0,\\ \displaystyle-\sum_{\substack{t_j\in\mathscr{T}\\ \beta_j<1-\gamma_j^0}}1 & \text{for }\ m=1\end{cases}$$

(see (1.81)).

Now we need information about the kernels dim Ker $A_\pm$ to derive the remainder equalities in (1.81).

Solvability results follow from from (1.81) provided (1.82) or (1.83) hold.

First of all note, that due to Lemma 1.21 it suffices to establish values of dim $\mathrm{Ker}_{\mathbb{X}} A_{\pm}$ and dim $\mathrm{Coker}_{\mathbb{X}} A_{\pm}$ only for one space among those where operators $A_{\pm}$ have equal indices.

Equalities dim Ker $A_{\pm} = \varepsilon_{\pm}$, dim Coker $A_{-} = \varepsilon_{\pm}$ under condition (1.82) and, in general, equalities in (1.81) can be derived from the equivalence of BVPs and our BIEs stated in Theorem 1.12 by invoking Remark 1.10, Lemma 1.15 and equivalence of BVPs with the Riemann–Hilbert problem, stated in Theorem 1.16, because either the kernel or the cokernel of the Riemann–Hilbert problem (and of characteristic singular integral equation) are trivial (see [Du1, GK1, Kh1]).

If one of conditions of the theorem is missing we can apply above mentioned equivalence with the Riemann–Hilbert problem to find that our BIEs are not Fredholm. Moreover, since in all cases the kernels and cokernels are finite dimensional dim Ker $\mathrm{A}_{\pm} \leq \mathrm{n}+1$ and dim Coker $\mathrm{A}^{*}_{\pm} \leq \mathrm{n}+1$, the images Im $\mathrm{A}_{\pm}$ can not be closed.

Now suppose Γ has peaks $\mathscr{T}_{pk} \neq \emptyset$.

Localization method applied in §,5.3, can be applied in the present situation as well. Due to Corollary 1.7 local representatives of operators $A_{\pm}$ in (1.39) at $t_0 \notin \mathscr{T}_{pk}$ are

$$A_{\pm} \overset{M_{t_0}}{\sim} \pm\frac{1}{2} I$$

and are invertible in $L_p(\Gamma)$.

At the inward peak $t_l \notin \mathscr{T}_{iw}$ we should localize the operator $A_{\pm}$ to the same one, but replace the curve Γ by a new one $\mathscr{L}_j$ which coincides with Γ in the vicinity of t_j and has t_j as a single outward peak. Therefore we can suppose, without restricting generality, that Γ has a single knot $\mathscr{T} = \{t_1\}$, which is either an angular point or an outward peak.

WARNING! While changing from the inward peak to outward, we change the orientation of the curve. Then operators $A_{\pm}$ and $B_{\pm}$ are replaced by $\mp A_{\mp}$ and $\mp B_{\mp}$, respectively. We should also interchange one-side neighbourhoods $\Gamma^{-}_{t_j}$ and $\Gamma^{+}_{t_j}$ which leads, due to non-equal rights of these neighbourhoods in the definition of the space $L_p(\Gamma, \rho, \mathscr{T}_{iw})$ (see (1.76)) to differences, which should be taken into consideration.

Due to Lemma 1.13 the Riemann–Hilbert problem is surjective and we can enjoy equivalent reduction of (1.39) and of (1.44) to the corresponding BVPs (1.6)–(1.8) for the domain Ω^{+} justified in Theorems 1.12 and 1.14. Due to equivalence established in Theorems 1.16 and 1.17 we find that equation (1.39) is equivalent to (1.50) while (1.44)–to (1.60). By applying Theorem 1.26 and 1.29 we accomplish the proof of Fredholm properties.

The same equivalence can be used to prove the index formulae for the case of one knot. In case of multiple knots we can use exactly the same approac as in (5.51)–(5.54) and reduce the proof to the case of one knot.

For equations (1.40) and (1.45) we make conclusions as for dual equations to (1.39) and to (1.44), respectively.

As for dim Ker $A_\pm$ and dim Ker $B_\pm$ in (1.39)–(1.40) and in (1.44)–(1.45), the formulae can be derived from the index formulae and above mentioned results on kernels in $L_p(\Gamma)$ spaces (see Remark 1.10). ∎

Remark 5.11 *Due to Lemma* 1.21 *any integrable solutions* $\varphi_\pm \in L_p(\Gamma, \rho)$ *of integral equations* (1.39) *and* (1.40) *are continuous (are* HÖLDER *continuous with the exponent* $0 < \mu < 1$ *or even belong to the* ZYGMUND *space* $\mathbb{Z}^\nu(\Gamma)$ *for* $0 < \nu < \infty$*) provided the right-hand sides are continuous (belong to* $H^\mu(\Gamma)$ *or to* $\mathbb{Z}^\nu(\Gamma)$*, respectively and, in the latter cases,* Γ *sufficiently smooth).*

Moreover, invoking Theorem 5.8 *we find that the solution* $u(x)$ *to the* DIRICHLET *BVP* (1.6), (1.7) *is continuous on* $\overline{\Omega^\pm}$ *(is* HÖLDER *continuous with the exponent* $0 < \mu < 1$ *or even belongs to the* ZYGMUND *space* $\mathbb{Z}^\nu(\Gamma)$ *for* $0 < \nu < \infty$*) provided the same condition holds for the date* $g(t)$ *on* Γ*.*

Similar assertions for L_p*-spaces and continuous solutions can be found in [Mi2,* § *14] and in [Ma1, Ch. I, Theorems 3 and 5].*

Remark 5.12 *Non-equal rights of curves* $\Gamma_{t_j}^\pm$ *in the definition of the space* $L_p(\Gamma, \rho, \mathscr{T}_{pk})$ *in* (1.76) *originates in the behavior of the convolution operator with* 2×2 *matrix symbol which is a local representative of the boundary integral operator and can easily be traced in the proof of Theorem* 1.26 *in* § *5.3 (see* (5.42)–(5.47)*). Difference of conditions on the function* $\varphi \in L_p(\Gamma, \rho, \mathscr{T}_{pk})$ *at outward and inward peaks in the definition* (1.76) *reflected in* $\varepsilon_j = \pm 1$*, is due to the above-mentioned non-equal rights of curves* $\Gamma_{t_j}^\pm$ *and can be explained by the change of domain* Ω^+ *to some outer domain by localization to make an inward peak outward (see the proof of Theorem* 1.23 *above).*

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Appendix

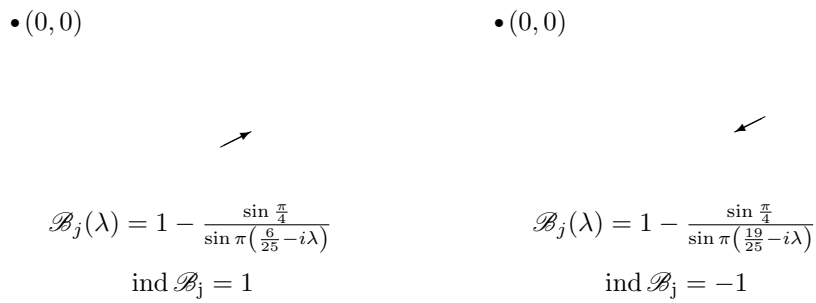


Fig. 5

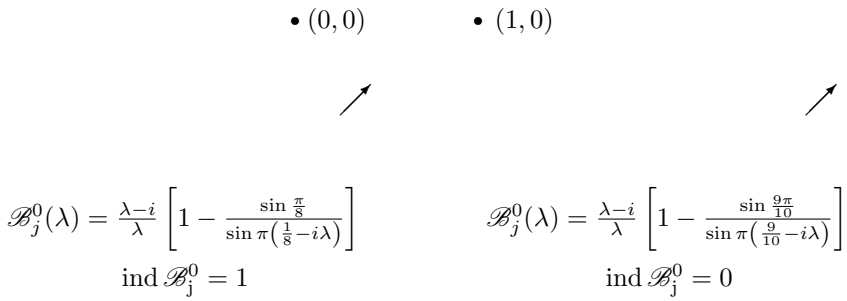


Fig. 6

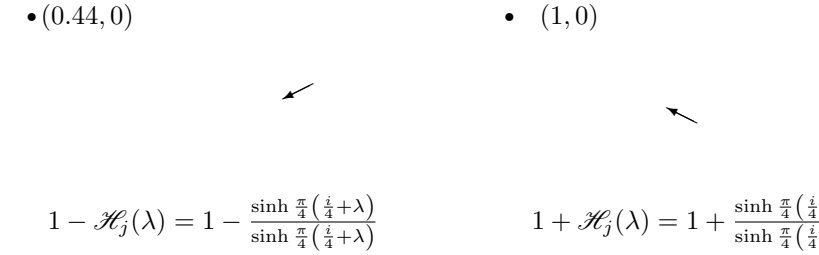
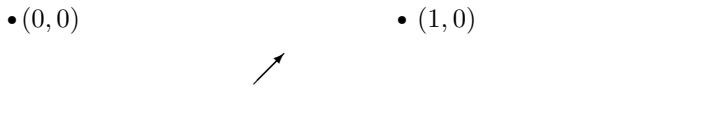


Fig. 7

$$1 - \mathscr{H}_j(\lambda) = 1 - \frac{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)} \qquad 1 + \mathscr{H}_j(\lambda) = 1 + \frac{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)}$$

$$\operatorname{ind}\left(1 - \mathscr{H}_{\mathrm{j}}^2\right) = -1$$

Fig. 8



$$1 - \mathscr{H}_j(\lambda) = 1 - \frac{\sinh \frac{\pi}{4}\left(\frac{3i}{4} - \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{i}{4} + \lambda\right)} \qquad 1 + \mathscr{H}_j(\lambda) = 1 + \frac{\sinh \frac{\pi}{4}\left(\frac{3i}{4} - \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{i}{4} + \lambda\right)}$$

$$\operatorname{ind}\left(1 - \mathscr{H}_{\mathrm{j}}^2\right) = 1$$

Fig. 9

$$1 - \mathscr{H}_j(\lambda) = 1 - \frac{\sinh \frac{\pi}{4}\left(\frac{i}{4} - \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)} \qquad 1 + \mathscr{H}_j(\lambda) 1 + \frac{\sinh \frac{\pi}{4}\left(\frac{i}{4} - \lambda\right)}{\sinh \frac{\pi}{4}\left(\frac{3i}{4} + \lambda\right)}$$

$$\operatorname{ind}\left(1 - \mathscr{H}_{\mathrm{j}}^2\right) = 0$$

Fig. 10

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Contents