Interface cracks in anisotropic composites

R. Duduchava,
A.Razmadze Mathematical Institute, Academy of Sciences of Georgia,
1, M.Alexidze str., TBILISI 93, GEORGIA

A.-M. Sändig, W. L. Wendland
Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57,
70569 STUTTGART, GERMANY

Dedicated to Professor G.Fichera

Abstract

The linear model equations of elasticity give rise to oscillatory solutions in some vicinity of interface crack fronts. In this paper we apply the Wiener–Hopf method which yields the asymptotic behaviour of the elastic fields and, in adition, criteria to prevent oscillatory solutions. The exponents of the asymptotic expansions are found as eigenvalues of the symbol of corresponding boundary pseudodifferential equations. The method works for three–dimensional anisotropic bodies and we demonstrate it for the example of two anisotropic bodies, one of which is bounded and the other one is its exterior complement. The common boundary is a smooth surface. On one part of this surface, called the interface, the bodies are bonded, while on the complementary part there is a crack. By applying the potential method, the problem is reduced to an equivalent system of boundary pseudodifferential equations (BPE) on the interface with the stress vector as unknown. The BPEs are defined via Poincaré–Steklov operators. We

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Contents

Introduction

It is well known that the solutions of elliptic boundary value problems in domains with corners, edges and interfaces have singularities at these geometrical and structural peculiarities regardless the smoothness properties of the given data. Both, mathematicians [?, ?, ?, ?, ?, ?, ?] and mechanists [?, ?, ?, ?, ?, ?] have analysed local asymptotic expansions for the elliptic system of linear elasticity. In this paper we study interface cracks between three–dimensional anisotropic composites. It should be emphasized that G.Fichera with his fundamental work in elasticity [?], eigenvalue problems [?] and his early work on the Zaremba problem [?] has significantly inspired this work. The corresponding two–dimensional case has already been investigated thoroughly by many authors, see [?] and references therein.

In addition to existence, uniqueness and a priori estimates for solutions to crack, punch and similar problems of mathematical physics [?, ?, ?], the explicit asymptotic expansion of the solution near singular submanifolds, such as conical points, edges, crack fronts etc., provides important information for applications in solid mechanics.

The linear elasticity model often gives rise to oscillatory solutions in some vicinity of interface crack fronts. In some investigations the boundary conditions are modified in the vicinity of crack front in order to prevent oscillatory solutions (see [?, ?, ?]). Others formulate conditions for the elastic constants which ensure absence of oscillations (see [?] and references therein). Mostly these investigations are devoted to two–dimensional problems.

In this paper we present a method which yields criteria (i.e.necessary and sufficient conditions) preventing oscillations in three–dimensional anisotropic bodies. It is based on the Wiener–Hopf technique and provides asymptotics of solutions. The corresponding exponents are found via eigenvalues of the symbol of an associated boundary pseudodifferential equation. We demonstrate the method for two anisotropic three–dimensional bodies, one of which is bounded and the other one is its exterior complement. The common boundary is a smooth surface. On one part of this surface, called the interface, the

The local leading edge asymptotics can e.g. be obtained by reduction to the two–dimensional case [**?**, **?**, **?**]. Here we present a different rigorous analysis of the three–dimensional case and find an explicit description of the exponents in the asymptotics, which is based on the Wiener-Hopf method and does not exploit the reduction to the two–dimensional case (see [**?**, §§ 9, 25], [**?**, **?**, **?**, **?**, **?**]).

Although the method is general, in the present paper we apply the method to a boundary–transmission problem for two anisotropic three–dimensional bodies Ω_1 and Ω_2, where Ω_1 is bounded and $\Omega_2 = \mathbb{R}^3 \setminus \overline{\Omega_1}$. The common boundary manifold $\mathscr{S}$ is supposed to be smooth and consists of two parts $\mathscr{S}_1$ and $\mathscr{S}_2$ with $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \Gamma$, where $\Gamma := \partial\mathscr{S}_1 = \partial\mathscr{S}_2$ is a smooth curve. On the part $\mathscr{S}_1$, called the interface, the bodies are bonded, while the part $\mathscr{S}_2 = \mathscr{S} \setminus \mathscr{S}_1$ models a crack.

After formulating the transmission-boundary value problem in Section 1 we recall some properties of potential operators in Subsection 2.1. In Subsection 2.2 the boundary–transmission problem is equivalently reduced to a system of boundary pseudodifferential equations of order -1 on the interface $\mathscr{S}_1$. The principal part of the corresponding operator is given by Poincaré–Steklov operators corresponding to the different materials

$$P_{-1} := \overset{2}{V}_{-1} \left(\frac{1}{2} I - \overset{2}{W^*}_0 \right)^{-1} + \overset{1}{V}_{-1} \left(\frac{1}{2} I + \overset{1}{W^*}_0 \right)^{(-1)} , \qquad (0.1)$$

where the second summand is a pseudoinverse. P_{-1} is a pseudodifferential operator of order -1, is positive definite, however the symbol matrix $\mathscr{P}_{-1}(x, \xi_1, \xi_2)$ does not possess a generalised transmission property

$$\mathscr{P}_{-1,0}(x) := [\mathscr{P}_{-1}(x, 0, +1)]^{-1}\, \mathscr{P}_{-1}(x, 0, -1)) \neq I , \quad x \in \mathscr{S}_1 . \qquad (0.2)$$

It is proved that the eigenvalues $\lambda_1(\sigma)$, $\lambda_2(\sigma)$, $\lambda_3(\sigma)$ of the matrix $\mathscr{P}_{-1,0}(\sigma)$, which we look for on the boundary $\Gamma = \partial\mathscr{S}_1$, have the properties

$$\lambda_1(\sigma) \equiv 1 , \quad \lambda_2(\sigma) = \lambda_3^{-1}(\sigma) = \lambda_0(\sigma) , \quad \operatorname{Im} \lambda_0(\sigma) \equiv 0 , \quad \sigma \in \Gamma . \qquad (0.3)$$

Everywhere in this paper the boldface Greek $\boldsymbol{\rho}^{\mu}$ (or $\boldsymbol{\zeta}^{\mu}$ etc.) is used to denote the diagonal matrix–function

$$\boldsymbol{\rho}^{\mu} := \text{ diag } \{\rho^{\mu_1}, \ldots, \rho^{\mu_1}\} \,, \qquad \mu := (\mu_1, \ldots, \mu_n) \,. \tag{0.4}$$

which is the vector exponent of the scalar variable $\rho \in \mathbb{R}$ (or $\zeta \in \mathbb{C}$).

Then the resulting asymptotic expansion for the traction field on the interface $\mathscr{S}_1$ reads (see Theorem **??**)

$$\mathfrak{t}(\sigma,\rho) = \sum_{k=0}^{M} \mathscr{K}(\sigma)\boldsymbol{\rho}^{-\frac{1}{2}+i\nu(\sigma)+k}\mathscr{K}^{-1}(\sigma)\sum_{l=0}^{k} c_{kl}(\sigma)\log^{l}\rho + \widetilde{\mathfrak{t}}_{M+1}(\sigma,\rho)\,, \tag{0.5}$$

where $\widetilde{\mathfrak{t}}_{M+1} \in C^{M}(\mathscr{S}_1)$ for all sufficiently small $\rho > 0$,

$$\nu(\sigma) = (0, \nu_0(\sigma), -\nu_0(\sigma))\,, \qquad \nu_0(\sigma) := \frac{\log \lambda_0(\sigma)}{2\pi}$$

and arbitrary integer $M \in \mathbb{N}_0$; all 3–vectors c_{kl} and 3×3 matrix–functions $\mathscr{K}(\sigma)$ belong to $C^{\infty}(\partial\mathscr{S}_1)$. The vector $c_{00}(\sigma)$ and the matrix $\mathscr{K}(\sigma)$ are given by the principal symbol matrix of the BPE, while in the definition of vectors $c_{kl}(\sigma)$ for $k > 0$ the full symbol is involved. The leading term for $k = 0$ in (0.4) does not contain $\log\rho$. The logarithmic terms vanish completely provided $\nu_0(\sigma)$ =const:

$$\mathfrak{t}(\sigma,\rho) = \sum_{k=0}^{M} \mathscr{K}(\sigma)\boldsymbol{\rho}^{-\frac{1}{2}+i\nu+k}\mathscr{K}^{-1}(\sigma)c_k(\sigma) + \widetilde{\mathfrak{t}}_{M+1}(\sigma,\rho)\,. \tag{0.6}$$

This is the case when e.g. both materials in Ω_1 and Ω_2 are isotropic.

Moreover, we can present the full spatial asymptotic expansion for displacement and stress fields in $\mathbb{R}^3$ in the vicinity of the crack front $\Gamma = \partial\mathscr{S}$. To this end let us extend the l.t.c.s. to the special local coordinate system (s.l.c.s. in short) into the vicinity of the crack front Γ in $\mathbb{R}^3$; the point $x = (\sigma, \rho, r) \in \Gamma \times \mathbb{R}^2$ with $\sigma \in \Gamma$ and $\rho, r \in \mathbb{R}$ will belong to Ω_k for $(-1)^{k+1}r > 0$ with $|r|$ denoting the distance $|r|$ =dist$(x, \mathscr{S})$, while with $\widetilde{u}_{M+1} \in C^{M+1}(\Gamma \times \mathscr{I}_\varepsilon^2)$, $\mathscr{I}_\varepsilon := (-\varepsilon, \varepsilon)$, smooth 3×3 matrix–functions $d^{mj}_{k,\pm}(\cdot, \pm)$ and 3–vectors $a^{sj}_{kl,\pm}(\vartheta, \cdot) \in C^\infty(\Gamma)$. The scalar complex variables

$$\zeta_{m,-1} = \overline{\zeta}_{m,+1} := \rho + \tau_m r\,, \qquad \mathrm{Im}\,\zeta_m \neq 0\,.$$

are defined with the help of (all different) roots $\{\tau_m\}_{m=1}^{\ell} \subset \tau_m \in C^\infty(\Gamma)$ of a certain polynomial equation, defined by the symbol of original partial differential equation (more details cf. in Theorem **??** in the Appendix; for the definition of the diagonal matrix $\boldsymbol{\zeta}^\mu$ see (0.4)).

With the displacement field available in (0.7), by differentiation and Hooks law, applying Theorem **??**, we also get the full asymptotics of the stress tensor field in the form

$$\mathfrak{T}(\sigma,\rho,r) = \sum_{\vartheta=\pm1}\sum_{m=1}^{\ell}\sum_{j=0}^{p''_m}\sum_{k=0}^{M+1} g^{mj}_{k,\pm}(\sigma,\vartheta)\; r^j \boldsymbol{\zeta}_{m,\mp\vartheta}^{-\frac{1}{2}+i\nu(\sigma)-j+k} \sum_{l=0}^{2k} b^{sj}_{kl,\pm}(\vartheta,\sigma) \log^l\!\rho$$
$$+\widetilde{\mathfrak{T}}_{M+1}(\sigma,\rho,r) \quad \text{for} \quad \pm r > 0 \tag{0.8}$$

with $\widetilde{\mathfrak{T}}_{M+1} \in C^M(\Gamma \times \mathscr{I}_\varepsilon^2)$, with smooth 3×3 matrix–functions $g^{mj}_{k,\pm}(\cdot, \pm)$, $b^{sj}_{kl,\pm}(\vartheta, \cdot) \in C^\infty(\Gamma)$

As we see, again, logarithmic terms do not appear in the leading terms of asymptotics in (0.7) and (0.8) and they vanish provided $\nu_0(\sigma)$ =const.

The main conclusion for oscillating solutions can be formulated in the following theorem.

Theorem 0.1 *The displacement vector field $u(x)$ and the stress tensor field $\mathfrak{T}(x)$ fields of the interface crack problem are non-oscillating solutions of the linearised boundary value problem of anisotropic elasticity if and only if $\mathscr{P}_{-1,0}(\sigma) = I$ in (0.2) on the boundary $\sigma \in \Gamma = \partial\mathscr{S}_1$.*

In Subsection 3.4 we investigate more detailed the interface crack problem for isotropic bodies. As an important consequence of Theorem 0.1 we present a criterion for non–oscillating solutions: if

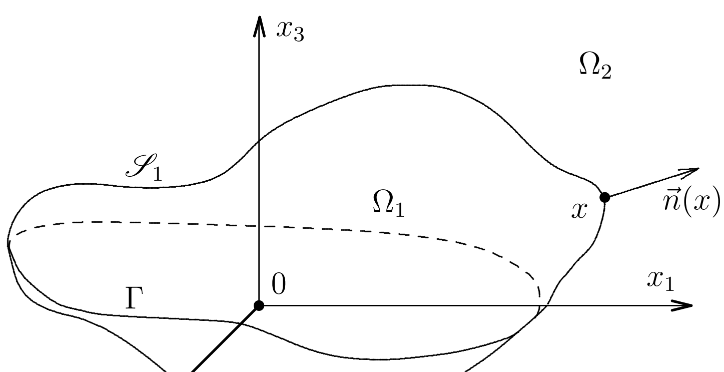
For convenience we collect in the Appendix the definitions and results from the theory of function spaces, pseudodifferential equations on manifolds with smooth boundary and asymptotics of corresponding solutions from [?, ?, ?, ?, ?], on which rely our results in Sections 1–3.

Such an asymptotic expansion of the displacement field can be obtained by a different and well–known method going back to V. Kondratjev in [?] and further developed e.g. in [?, ?, ?, ?]. This method is applicable in cases of non–smooth boundary manifolds. However, the Wiener–Hopf method, presented here, provides more explicit formulae for the exponents and transparent connections between the coefficients of the surface two–dimensional asymptotics in (0.5) (of solutions to boundary pseudodifferential equations) with spatial three–dimensional asymptotics in (0.7) and (0.8) (of solutions to corresponding boundary value problems; see [?, ?, ?, ?, ?] and [?, ?]).

Some preliminary results of this paper have already been announced in [?].

1 The boundary value problem

Let $\Omega_1 \subset \mathbb{R}^3$ be a simply connected bounded domain with $0 \in \Omega_1$ and $\Omega_2 := \mathbb{R}^3 \setminus \overline{\Omega_1}$. Let the surface $\mathscr{S} = \partial\Omega_1 = \partial\Omega_2$, described above, be their common boundary (see Fig.1).



The traces $\overset{k}{\gamma}_{\mathscr{S}_j}\,\varphi$ on $\mathscr{S}_j$ are defined correspondingly $(k, j = 1, 2)$.

If a distribution ψ is defined on the surface $\mathscr{S}$, then we denote by

$$r_{\mathscr{S}_j}\psi := \psi|_{\mathscr{S}_j} \quad \text{for} \quad j = 1, 2 \tag{1.2}$$

the restriction operator.

The domains Ω_1 and Ω_2 are occupied by possibly different elastic, homogeneous, anisotropic materials which are bonded along the interface $\mathscr{S}_1$, while $\mathscr{S}_2$ models a crack.

The displacement vector $u(z) = (u_1(z), u_2(z), u_3(z))$ of elastic media in $\Omega_1 \bigcup \Omega_2$ satisfies the system of linear second order partial differential equations of anisotropic elasticity (see [**?**] and [**?**, Section 1])

$$\mathscr{L}_k(D)u(z) + \overset{k}{X}(z) = 0 \quad \text{for} \quad z \in \Omega_k, \quad \text{and} \quad k = 1, 2 \tag{1.3}$$

with the given volume forces $\overset{k}{X} = (\overset{k}{X}_1, \overset{k}{X}_2, \overset{k}{X}_3)$ and the differential operators

$$\mathscr{L}_k(D) := \left(\sum_{j,l,m,n} \overset{k}{c}_{jlmn}\, D_j D_n \right)_{3\times 3} \quad \text{and} \quad D_j := i\partial_j = i\frac{\partial}{\partial z_j}\,. \tag{1.4}$$

The elastic moduli $\overset{k}{c}_{jlmn}$ are real-valued and satisfy the symmetry relations

$$\overset{k}{c}_{jlnm} = \overset{k}{c}_{jlmn} = \overset{k}{c}_{mnjl}\ . \tag{1.5}$$

The conservation of energy yields positive definiteness of the corresponding quadratic forms; namely, there exist constants $M_{0,i} > 0$ such that

$$\sum_{l,j,m,n} \overset{k}{c}_{ljmn}\, \xi_{lj}\overline{\xi_{mn}} \geq M_0 \sum_{l,j} |\xi_{lj}|^2 \quad \text{for all } \xi_{lj} = \xi_{jl} \in \mathbb{C} \text{ and } k = 1, 2\,. \tag{1.6}$$

To formulate the boundary value problem describing the displacement

By $\mathbb{H}^1{}_{loc}(\mathbb{R}^3_{\mathscr{S}})$ we denote the Frechét–Sobolev space of vector–functions $\varphi(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))^\top$ on $\mathbb{R}^3_{\mathscr{S}} := \mathbb{R}^3 \setminus \mathscr{S} = \Omega_1 \cup \Omega_2$, equipped with the seminorms

$$\|\varphi|\mathbb{H}^1(\Omega_{\mathscr{S}})\| := \left[\int\limits_{\Omega_{\mathscr{S}}} \left(|\varphi(z)|^2 + |\nabla\varphi(z)|^2\right) dz\right]^{\frac{1}{2}} < \infty\,, \tag{1.8}$$

$$\Omega_{\mathscr{S}} = \Omega \setminus \mathscr{S} = \Omega_1 \cup (\Omega_2 \cap \Omega)\,, \quad \nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi),$$

defined for any compact domain $\Omega \Subset \mathbb{R}^3$.

For the definition of all other spaces used $\mathbb{H}^1_{\rm comp}(\mathbb{R}^3_{\mathscr{S}})$, $\mathbb{H}^1(\mathbb{R}^3_{\mathscr{S}})$, $\mathbb{H}^{\pm\frac{1}{2}}(\mathscr{S})$, $\mathbb{H}^{\pm\frac{1}{2}}(\mathscr{S}_l)$, $\widetilde{\mathbb{H}}^{\pm\frac{1}{2}}(\mathscr{S}_l)$ we refer the reader to the Appendix.

In what follows we assume that $\overset{k}{X}= 0$ and consider the homogeneous system

$$\mathscr{L}_k(D)u(z) = 0 \quad \text{for} \quad z \in \Omega_k \quad \text{and} \quad k = 1, 2; \tag{1.9}$$

otherwise we superpose a corresponding particular elastic field.

Problem (the mixed transmission – Neumann problem): *find a solution*

$$u \in \mathbb{H}^1_{\rm loc}(\mathbb{R}^3_{\mathscr{S}}) \quad \text{satisfying} \quad u(z) = o(1) \quad \text{as} \quad |z| \to \infty, \tag{1.10}$$

which fulfils the equations (1.9) *and the following boundary and transmission conditions:*

$$\overset{k}{\gamma}_{\mathscr{S}_2} \overset{k}{\mathscr{T}} u(x) = g_k(x) \quad \textit{for} \quad x \in \mathscr{S}_2 \qquad \textit{and} \qquad k = 1, 2 \tag{1.11}$$

$$\overset{1}{\gamma}_{\mathscr{S}_1} u(x) - \overset{2}{\gamma}_{\mathscr{S}_1} u(x) = f_0(x) \qquad \textit{for} \qquad x \in \mathscr{S}_1, \tag{1.12}$$

$$\overset{1}{\gamma}_{\mathscr{S}_1} \overset{1}{\mathscr{T}} u(x) - \overset{2}{\gamma}_{\mathscr{S}_1} \overset{2}{\mathscr{T}} u(x) = f_1(x) \qquad \textit{for} \qquad x \in \mathscr{S}_1 \tag{1.13}$$

$$\textit{with} \quad \overset{k}{\mathscr{T}} := \overset{k}{\mathscr{T}}(D, \vec{n}(x))\,.$$

Here the functions $g_1, g_2 \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S}_2)$ *and* $f_0, f_1 \in \mathbb{H}^{\frac{1}{2}-j}(\mathscr{S}_1)$ *are given.*

If volume forces are present $\overset{k}{X}\neq 0$ (see equation (1.3)), in (1.11)–(1.13) we should take $g_j = f_j = 0,\ j = 1, 2$. The proposed superposition of some particular elastic fields, which eliminate the body forces in equation (1.9), results in non–homogeneous boundary and transmission conditions $g_j \neq 0,\ f_j \neq 0$.

The traces $\overset{k}{\gamma}_{\mathscr{S}_j} v$ are defined for functions $v \in \mathbb{H}^s_{\rm loc}(\mathbb{R}^3_{\mathscr{S}})$, provided $s > 1/2$; therefore we should justify existence of the trace of $\overset{k}{\mathscr{T}}(D, \vec{n})u \in L^2_{\rm loc}(\mathbb{R}^3_{\mathscr{S}})$ from (1.11), (1.13). To this end we recall that $u(z)$ is a solution to the elliptic system (1.9) and, therefore, the following Green formula is valid:

$$\int\limits_{\Omega_k} [v\mathscr{L}_j u + \mathscr{E}_j(u, v)]\,{\rm d}\,z = (-1)^{k+1}\langle \overset{k}{\gamma}_{\mathscr{S}}\, v, \overset{k}{\gamma}_{\mathscr{S}} \overset{j}{\mathscr{T}}\,(D, \vec{n})u\rangle_{\mathscr{S}}\,, \tag{1.15}$$

$$\mathscr{E}_j(u, v) := \sum_{p,l,m,n} \overset{j}{c}_{plmn}\, \partial_p u_l \partial_m v_n$$

for $j, k = 1, 2$, where $\langle\cdot,\cdot\rangle_{\mathscr{S}}$ denotes the sesquilinear form

$$\langle \varphi, \psi\rangle_{\mathscr{S}} := \int\limits_{\mathscr{S}} \varphi(x)\overline{\psi(x)}\,{\rm d}\,_x\mathscr{S} \quad \text{for} \quad \varphi \in L_2(\mathscr{S}).$$

By duality and density, relation (1.15) can be extended to pairs $\mathbb{H}^{\frac{1}{2}}(\mathscr{S})\times \mathbb{H}^{-\frac{1}{2}}(\mathscr{S})$, $\mathbb{H}^{\frac{1}{2}}(\mathscr{S}_k)\times \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_k)$ and $\mathbb{H}^{-\frac{1}{2}}(\mathscr{S}_k)\times \widetilde{\mathbb{H}}^{\frac{1}{2}}(\mathscr{S}_k)$, respectively.

If u is a solution of (1.9), then the trace $\overset{k}{\gamma}_{\mathscr{S}} \overset{j}{\mathscr{T}}\,(D, \vec{n})u \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S})$ exists as a bounded linear functional: if $v \in \mathbb{H}^1_{\rm comp}(\Omega_2)$, then $\overset{k}{\gamma}_{\mathscr{S}}\, v \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S})$ and (1.15) provides the estimate

$$\langle \overset{k}{\gamma}_{\mathscr{S}}\, v, \overset{k}{\gamma}_{\mathscr{S}} \overset{j}{\mathscr{T}}\,(D, \vec{n})u\rangle_{\mathscr{S}} = \int\limits_{\Omega_k} \mathscr{E}_j(u, v)\,{\rm d}\,z \leq M_1\|u|\mathbb{H}^1_{\rm loc}(\,{\rm supp}\,v)\|\cdot\|v|\mathbb{H}^1(\Omega_k)\|. \tag{1.16}$$

Furthermore, if ${\rm supp}\,\overset{k}{\gamma}_{\mathscr{S}}\, v \subset \mathscr{S}_k$, then $\overset{2}{\gamma}_{\mathscr{S}}\, v \in \widetilde{\mathbb{H}}^{\frac{1}{2}}(\mathscr{S}_k)$ and, by duality,

2 Boundary pseudodifferential equations

2.1 Potential operators of elasticity

Let

$$\mathscr{L}_k(\xi) := \left(\sum_{l,j,m,n} \overset{k}{c}_{ljmn}\, \xi_j \xi_n\right)_{3\times 3} \quad \text{and} \quad \xi_1, \xi_2 \in \mathbb{R}\,, \quad k = 1, 2$$

denote the symbol of the operator $\mathscr{L}_k(D)$ in (1.9). Then

$$G_k(z) = \mathscr{F}^{-1}_{\xi\to x}\mathscr{L}^{-1}_k(z) \quad \text{for} \quad z \in \mathbb{R}^3, \quad k = 1, 2\,, \tag{2.1}$$

where $\mathscr{F}$ is the Fourier transform (see (**??**)) defines the fundamental solution of equation (1.9)

$$\mathscr{L}_k(D)\, G_k(z) = \delta(z)$$

with the Dirac δ-function $\delta(z)$.

The single and the double layer potentials, associated with the operator $\mathscr{L}_k$, are defined for $z \in \mathbb{R}^3_{\mathscr{S}}$ as follows:

$$\overset{k}{V}\varphi(z) := \int_{\mathscr{S}} G_k(z-y)\varphi(y)\,\mathrm{d}_y\mathscr{S}, \tag{2.2}$$

$$\overset{k}{W}\varphi(z) := \int_{\mathscr{S}} \left[\overset{k}{\mathscr{T}}(D_y, \vec{n}(y))G_k(y-z)\right]^\top \varphi(y)\,\mathrm{d}_y\mathscr{S}\,. \tag{2.3}$$

In our investigations we also use the following boundary (pseudodifferential) operators on $\mathscr{S}$ for $k = 1, 2$:

$$\overset{k}{V}_{-1}\,\varphi(x) = \int_{\mathscr{S}} G_k(x-y)\varphi(y)\,\mathrm{d}_y\mathscr{S}, \tag{2.4}$$

$$\overset{k}{W}\varphi(x) = \int_{\mathscr{S}} \left[\overset{k}{\mathscr{T}}(D_y, \vec{n}(y))G_k(y-x)\right]^\top \varphi(y)\,\mathrm{d}_y\mathscr{S} \tag{2.5}$$

$\overset{k}{\mathscr{T}}(D,\vec{n})\overset{k}{V}$ (see (1.7), (2.2)) on $\mathscr{S}$, respectively; the operator $\overset{k}{W}{}_0^*$ is the adjoint to $\overset{k}{W}_0$.

$\overset{k}{V}_{-1}$ and $\overset{k}{W}_0$, $\overset{k}{W}{}_0^*$ are pseudodifferential operators and the subscript indices $0, -1$ indicate their orders (see Appendix, § A.3). Therefore the following boundedness properties are easy to verify (cf. §§ A.2, A.3):

$$\overset{k}{V}_{-1} \;:\; \mathbb{H}^s(\mathscr{S}) \to \mathbb{H}^{s+1}(\mathscr{S})\,, \tag{2.7}$$

$$\overset{k}{W}_0, \overset{k}{W}{}_0^* \;:\; \mathbb{H}^s(\mathscr{S}) \to \mathbb{H}^s(\mathscr{S}) \quad \text{for all} \quad s \in \mathbb{R}\,. \tag{2.8}$$

Moreover, the operators (2.7) are invertible, are positive definite for $s = -\frac{1}{2}$,

$$\langle \overset{k}{V}_{-1}\,\varphi, \varphi\rangle \geq M_2 \|\varphi|\mathbb{H}^{-\frac{1}{2}}(\mathscr{S})\|^2, \quad M_2 > 0, \quad k = 1, 2 \tag{2.9}$$

(see [**?**, Theorem3.9]) and [**?**, Theorem 3] for more general spaces); and they have positive definite symbols

$$\overset{k}{V}_{-1}(x,\xi)\eta \cdot \eta^\top \geq M_3|\xi|^{-1}|\eta|^2, \quad x \in \mathscr{S}\,, \tag{2.10}$$

$$\xi \in \mathbb{R}^2, \quad \eta \in \mathbb{C}^3, \quad M_3 > 0, \quad k = 1, 2$$

(see [**?**, **?**]; for the proof see [**?**, Subsection 3.2]).

The following jump relations are also well-known (see [**?**, **?**]):

$$(\overset{m}{\gamma}_{\mathscr{S}} \overset{k}{V}\varphi)(x) = \overset{k}{V}_{-1}\,\varphi(x), \tag{2.11}$$

$$(\overset{m}{\gamma}_{\mathscr{S}} \overset{k}{W}\varphi)(x) = \frac{(-1)^{m+1}}{2}\varphi(x) + \overset{k}{W}_0\,\varphi(x), \tag{2.12}$$

$$(\overset{m}{\gamma}_{\mathscr{S}} \overset{k}{\mathscr{T}}(D_z, \vec{n}(z))\, \overset{k}{V}\varphi(x) = \frac{(-1)^m}{2}\varphi(x) + \overset{k}{W}{}_0^*\,\varphi(x), \tag{2.13}$$

$\in H^s(\mathscr{S})$, for $s \in \mathbb{R}$ and $k = 1, 2$, $x \in \mathscr{S}$

Theorem 2.1 *Let $s \in \mathbb{R}$. Then the operator*

$$\overset{2}{M}_0 = \frac{1}{2}I - \overset{2}{W}{}^*_0 : \mathbb{H}^s(\mathscr{S}) \to \mathbb{H}^s(\mathscr{S}) \tag{2.15}$$

is invertible. The homogeneous equation

$$\frac{1}{2}v + \overset{1}{W}{}^*_0 \, v = 0 \tag{2.16}$$

and its conjugate homogeneous equation

$$\frac{1}{2}h + \overset{1}{W}_0 \, h = 0 \tag{2.17}$$

have six linearly independent solutions in $\mathbb{H}^s(\mathscr{S})$ each and these solutions have the form

$$v(x) = \left(\overset{1}{V}_{-1}\right)^{-1} h(x) \quad \text{where} \quad h(x) = [\vec{c}_1 \times x] + \vec{c}_2\,, \quad x \in \mathscr{S}\,; \tag{2.18}$$

$h(x)$ is the trace of a rigid motion with arbitrary constant vectors $\vec{c}_1, \vec{c}_2 \in \mathbb{R}^3$ and $[\cdot \times \cdot]$ denotes the vector product.

For $k = 1$ equation $\overset{1}{M}_0 \, \varphi = g$ has a solution φ_0 if and only if

$$\langle g, h \rangle_{\mathscr{S}} = 0 \tag{2.19}$$

for all h in (2.18) *and then the general solution has the form*

$$\varphi = \varphi_0 + \left(\overset{1}{V}_{-1}\right)^{-1} h \tag{2.20}$$

with h from (2.18).

Proof (see [**?**]). We insert $u = v = \overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi$ into (1.15) and recall that $\overset{2}{\mathscr{L}}_2(D)\,\overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi = 0$, and $\overset{2}{\gamma}_{\mathscr{S}}\overset{2}{\mathscr{T}}(D,\vec{n})\,\overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi = -\varphi$ (see (2.13), (2.14)) to obtain the inequality

$$\|\nabla\,\overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi|L_2(\Omega_2)\|^2 \le -\langle\overset{2}{\gamma}_{\mathscr{S}}\overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi,\overset{2}{\gamma}_{\mathscr{S}}\overset{2}{\mathscr{T}}(D,\vec{n})\,\overset{2}{V}\overset{2}{M}{}_0^{-1}\varphi\rangle_{\mathscr{S}} = \langle\overset{2}{P}_{-1}\varphi,\varphi\rangle_{\mathscr{S}}\,.$$

Thus, $\overset{2}{P}_{-1}$ is nonnegative; it is also invertible since both operators $\overset{2}{V}_{-1}$ and $\overset{2}{M}_0$, composing $\overset{2}{P}_{-1}$ (see (2.21)) are invertible as noted in (2.7) and in Theorem 2.1). Therefore $\overset{2}{P}_{-1}$ is positive definite. ■

An operator $B : Y \to X$ is called a **pseudoinverse** to $A : X \to Y$ if $BAB = B$ and $ABA = A$ (cf. e.g. [**?**]). If A is invertible, then $A^{-1} = A^{(-1)}$ is the only pseudoinverse of A. If there exist bounded projections onto the image and onto the kernel of the operator A,

$$\begin{aligned} P_0 : Y &\to \operatorname{Im} A, \quad P_0^2 = P_0\,, \\ Q_1 : X &\to \operatorname{Ker} A, \quad Q_1^2 = Q_1\,, \end{aligned} \tag{2.23}$$

$$X = \operatorname{Ker} A \oplus X_0\,, \qquad Q_0 := I - Q_1 : X \to X_0, \qquad Q_0^2 = Q_0\,,$$

then $A_0 := A|_{X_0} : X_0 \to \operatorname{Im} A$ has the property $\operatorname{Ker} A_0 = \{0\}$ and is invertible. Therefore we can define the pseudoinverse by

$$A^{(-1)} := A_0^{-1}P_0. \tag{2.24}$$

Consequently, every Fredholm operator has a pseudoinverse. This pseudoinverse $A^{(-1)}$ becomes unique as soon as P_0 and Q_0 are specified. In particular, we have:

Corollary 2.3 *The operator*

Here $h_1, \dots, h_6$ *and* $v_1, \dots, v_6$ *are some orthonormal bases of the rigid motions*

$$\mathscr{R}^6 := \{[\vec{c}_1 \times x] + \vec{c}_2 \quad : \quad \vec{c}_1, \vec{c}_2 \in \mathbb{R}^3\} \tag{2.27}$$

and transformed rigid motions $\overset{1}{V}_{-1} \mathscr{R}^6$*, respectively. Then (cf. Theorem 2.1 and* (2.23)*):*

$$\begin{aligned} Y &= \mathbb{H}^s(\mathscr{S}) = \operatorname{Im} \overset{1}{M}_0 \oplus \mathscr{R}^6 \,, \\ X &= \mathbb{H}^s(\mathscr{S}) = X_1 \oplus X_0, \quad X_1 = \overset{1}{V}_{-1} \mathscr{R}^6 \,. \end{aligned} \tag{2.28}$$

Lemma 2.4 *The Poincaré–Steklov operator*

$$\overset{1}{P}_{-1} := \overset{1}{V}_{-1} \overset{1}{M}{}_0^{(-1)} = \overset{1}{V}_{-1} \left(\frac{1}{2} I + \overset{1}{W}{}_0^* \right)^{(-1)} , \tag{2.29}$$

with the pseudoinverse $\overset{1}{M}{}_0^{(-1)}$*, is non–negative:*

$$\langle \overset{1}{P}_{-1} \varphi, \varphi \rangle_{\mathscr{S}} \geq 0 \quad \text{for all} \quad \varphi \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S}) \,. \tag{2.30}$$

The left–hand side vanishes if and only if $\varphi \in \operatorname{Ker} \overset{1}{P}_{-1} = \operatorname{Ker} \overset{1}{M}{}_0^{(-1)} = \mathscr{R}^6$*.*

Proof. We proceed as in Lemma 2.2. Let us insert $u = v = \overset{1}{V} \overset{1}{M}{}_0^{(-1)} \varphi$ into (1.15), $k = j = 1$, and recall that $\mathscr{L}_1(D)u = 0$, $\overset{1}{\gamma}_{\mathscr{S}} \overset{1}{\mathscr{T}} (D, \vec{n}) \overset{1}{V} = \overset{1}{M}_0$ (see (2.13), (2.25)). Due to (1.17), (2.11) we get

$$\|\nabla \overset{1}{V} \overset{1}{M}{}_0^{-1} \varphi | L_2(\Omega_1)\|^2 \leq \langle \overset{1}{\gamma}_{\mathscr{S}} \overset{1}{V} \overset{1}{M}{}_0^{(-1)} \varphi, \overset{1}{\gamma}_{\mathscr{S}} \overset{1}{\mathscr{T}} \overset{1}{V} \overset{1}{M}{}_0^{(-1)} \varphi \rangle_{\mathscr{S}}$$

2.2 Reduction to a boundary pseudodifferential equation

Let $\hat{g}_k \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S})$ be some fixed extension of the boundary datum $g_k \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S}_2)$ in (1.11); then any other extension $\widetilde{g}_k \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S})$ of the same boundary datum g_k is represented by $\widetilde{g}_k = \hat{g}_k + \mathfrak{t}_k$, where $\mathfrak{t}_k \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1)$ $(k = 1, 2)$. Now let us consider the following system, which is the key boundary integral equation for the BVP (1.9)–(1.13):

$$\left.\begin{aligned} & r_{\mathscr{S}_1} P_{-1} \mathfrak{t}_1 + \sum_1^6 c_j h_j^0 = f_2 \,, \\ & \langle \mathfrak{t}_1, h_j \rangle = -\langle \hat{g}_1, h_j \rangle_{\mathscr{S}}, \quad j = 1, \dots, 6 \,, \\ & \mathfrak{t}_2 = \mathfrak{t}_1 - f_1^* \,, \end{aligned}\right\} \tag{2.31}$$

where

$$P_{-1} := \overset{1}{P}_{-1} + \overset{2}{P}_{-1} = \overset{1}{V}_{-1} \overset{1}{M}{}_0^{(-1)} + \overset{2}{V}_{-1} \overset{2}{M}{}_0^{-1} \tag{2.32}$$

(see (2.21) and (2.29) for $\overset{1}{P}_{-1}$, $\overset{2}{P}_{-1}$).

Unknowns in the system (2.31) are the vector–functions $\mathfrak{t}_1, \mathfrak{t}_2 \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1)$ and the constant vector $\vec{c} := (c_1, \dots, c_6)^\top \in \mathbb{R}$, while the functions

$$\begin{aligned} & f_0^* := f_0 - r_{\mathscr{S}_1} \overset{1}{P}_{-1} \hat{g}_1 - r_{\mathscr{S}_1} \overset{2}{P}_{-1} \hat{g}_2 \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S}_1) \,, \\ & f_1^* := f_1 - r_{\mathscr{S}_1} \hat{g}_1 + r_{\mathscr{S}_1} \hat{g}_2 \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1) \,, \\ & f_2 := f_0^* + r_{\mathscr{S}_1} \overset{2}{P}_{-1} f_1^* \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S}_1) \,, \quad h_j^0 := r_{\mathscr{S}_1} h_j \end{aligned} \tag{2.33}$$

are all known boundary data (see (1.11), (1.13) and cf. Remark 1.1. Let us remind, that $h_1, \dots, h_6$ is some fixed basis of rigid motions; see (2.27)).

Theorem 2.5 *The function*

Remark 2.6 *Let us note that the unknown functions* $\mathfrak{t}_k$ *in* (2.31) *do not coincide with the traces of the traction vectors* $\overset{k}{\mathfrak{t}}(x) = \overset{k}{\mathscr{T}}(D,\vec{n}(x))\overset{k}{u}(x)$ *on* $\mathscr{S}_1$, *but carry all singularities of the traction vectors on the interface* $\mathscr{S}_1$ *provided the extended function* $\hat{g}_k$ *is smooth, because* $\mathfrak{t}_k(x) = \overset{k}{\mathfrak{t}}(x) - \hat{g}_k$ *(k=1,2; see* (1.11)*)* .

Proof of Theorem 2.5. The function $\overset{\circ}{u}(z)$ clearly satisfies the requirements (1.9) and (1.10), because it is represented by the single layer potentials.

Recalling the jump relations (2.13) with $k = j$ and invoking Theorem 2.1 and Corollary 2.3 we get the following equalities

$$\begin{aligned}\overset{1}{\gamma}_{\mathscr{S}_2}\overset{1}{\mathscr{T}}\overset{\circ}{u}(x) &= r_{\mathscr{S}_2}\overset{1}{M}_0\left[\sum_{j=1}^{6} c_j\left(\overset{1}{V}_{-1}\right)^{-1} h_j + \overset{1}{M}{}_0^{(-1)}(\hat{g}_1 + \mathfrak{t}_1)\right] \\ &= r_{\mathscr{S}_2}\overset{1}{M}_0\overset{1}{M}{}_0^{(-1)}(\hat{g}_1 + \mathfrak{t}_1) = r_{\mathscr{S}_2}\hat{g}_1 = g_1\,, \\ \overset{2}{\gamma}_{\mathscr{S}_2}\overset{2}{\mathscr{T}}\overset{\circ}{u}(x) &= r_{\mathscr{S}_2}\overset{2}{M}_0\overset{2}{M}{}_0^{-1}(\hat{g}_2 + \mathfrak{t}_2) = r_{\mathscr{S}_2}\hat{g}_2 = g_2\,,\end{aligned}$$

because $\left(\overset{1}{V}_{-1}\right)^{-1} h_j \in \operatorname{Ker}\overset{1}{M}_0$ (see (2.1)), $r_{\mathscr{S}_2}\mathfrak{t}_k = 0$, $k = 1,2$ and $\hat{g}_1 + \mathfrak{t}_1 \in \operatorname{Im}\overset{1}{M}_0$, since $\hat{g}_1 + \mathfrak{t}_1$ satisfies the conditions (2.19) (see the second equation in (2.31)). Thus, the conditions (1.11) are satisfied.

Next we invoke (2.11) and find the following equations,

$$\begin{aligned}\overset{1}{\gamma}_{\mathscr{S}_1}\overset{\circ}{u}(x) - \overset{2}{\gamma}_{\mathscr{S}_1}\overset{\circ}{u}(x) &= \sum_{1}^{6} c_j r_{\mathscr{S}_1}\left(\overset{1}{V}_{-1}\right)^{-1}\overset{1}{V}_{-1} h_j + r_{\mathscr{S}_1}\overset{1}{V}_{-1}\overset{1}{M}{}_0^{(-1)}(\hat{g}_1 + \mathfrak{t}_1) \\ &+ r_{\mathscr{S}_1}\overset{2}{V}_{-1}\overset{2}{M}{}_0^{-1}(\hat{g}_2 + \mathfrak{t}_2) = \sum_{1}^{6} c_j h_j^0 + r_{\mathscr{S}_1}\overset{1}{P}_{-1}(\hat{g}_1 + \mathfrak{t}_1) + \overset{2}{P}_{-1}(\hat{g}_2 + \mathfrak{t}_2)\end{aligned}$$

$$= \sum_1^6 c_j h_j^0 + r_{\mathscr{S}_1} P_{-1}\mathfrak{t}_1 - r_{\mathscr{S}_1} \overset{2}{P}_{-1} f_1^* + r_{\mathscr{S}_1} \overset{1}{P}_{-1}\hat{g}_1 + r_{\mathscr{S}_1} \overset{2}{P}_{-1}\hat{g}_2$$

$$= f_2 - r_{\mathscr{S}_1} \overset{2}{P}_{-1} f_1^* + r_{\mathscr{S}_1} \overset{1}{P}_{-1}\hat{g}_1 + r_{\mathscr{S}_1} \overset{2}{P}_{-1}\hat{g}_2$$

$$= f_0^* + r_{\mathscr{S}_1} \overset{1}{P}_{-1}\hat{g}_1 + r_{\mathscr{S}_1} \overset{2}{P}_{-1}\hat{g}_2 = f_0 \,.$$

Thence, condition (1.12) is satisfied.

Finally, we invoke (2.13) and find the following relations,

$$\overset{1}{\gamma}_{\mathscr{S}_1} \overset{1}{\mathscr{T}} (D, \vec{n}(x)) \overset{\circ}{u} (x) - \overset{2}{\gamma}_{\mathscr{S}_1} \overset{2}{\mathscr{T}} (D, \vec{n}(x)) \overset{\circ}{u} (x) = \sum_1^6 c_j r_{\mathscr{S}_1} \overset{1}{M}_0 \left(\overset{1}{V}_{-1}\right)^{-1} h_j$$

$$+ r_{\mathscr{S}_1} \overset{1}{M}_0 \overset{1}{M}{}_0^{(-1)}(\hat{g}_1 + \mathfrak{t}_1) - r_{\mathscr{S}_1} \overset{2}{M}_0 \overset{2}{M}{}_0^{-1}(\hat{g}_2 + \mathfrak{t}_2) = r_{\mathscr{S}_1}(\hat{g}_1 + \mathfrak{t}_1) - r_{\mathscr{S}_1}(\hat{g}_2 + \mathfrak{t}_2) \,,$$

since $\overset{1}{M}_0 \left(\overset{1}{V}_{-1}\right)^{-1} h_j = 0$ and $\overset{1}{M}_0 \overset{1}{M}{}_0^{(-1)}(\hat{g}_1 + \mathfrak{t}_1) = \hat{g}_1 + \mathfrak{t}_1$ (see Theorem 2.1, (2.31) and Corollary 2.3) and $\overset{2}{M}_0 \overset{2}{M}{}_0^{-1} = I$ (see Theorem 2.1). Further we invoke (2.31), (2.33) and proceed as follows:

$$\overset{1}{\gamma}_{\mathscr{S}_1} \overset{1}{\mathscr{T}} (D, \vec{n}(x)) \overset{\circ}{u} (x) - \overset{2}{\gamma}_{\mathscr{S}_1} \overset{2}{\mathscr{T}} (D, \vec{n}(x)) \overset{\circ}{u} (x) = r_{\mathscr{S}_1}(\hat{g}_1 + \mathfrak{t}_1)$$

$$- r_{\mathscr{S}_1}(\hat{g}_2 + \mathfrak{t}_1 - f_1^*) = r_{\mathscr{S}_1} \hat{g}_1 - r_{\mathscr{S}_1} \hat{g}_2 + f_1^* = f_1 \,.$$

This is the last boundary condition (1.13). ■

The first two blocks of the system (2.31) can be written in short as

$$P_{-1}^0 \mathfrak{t}^0 = f^0 \,, \tag{2.35}$$

where

$$\mathfrak{t}^0 := (\mathfrak{t}_1, \vec{c}) \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1) \dotplus \mathbb{R}^6 \quad f^0 := (f_2, \vec{d}) \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S}^1) \dotplus \mathbb{R}^6 \,,$$

$$\vec{c} := (c_1, \dots, c_6) \in \mathbb{R}^6 \qquad \vec{d} := -\{\langle g_1, h_j \rangle_{\mathscr{S}}\}_1^6 \in \mathbb{R}^6 \,,$$

and L^* is the adjoint operator to L:

$$\langle L\psi, \vec{c}\rangle := \sum_1^6 (L\psi)_j c_j = \sum_1^6 \langle \psi, h_j\rangle_{\mathscr{S}} c_j = \langle \psi, \sum_1^6 c_j h_j\rangle_{\mathscr{S}} = \langle \psi, L^*\vec{c}\rangle_{\mathscr{S}}\,.$$

Lemma 2.7 *The operator* $r_{\mathscr{S}_1} P_{-1} \,:\, \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1) \to \mathbb{H}^{\frac{1}{2}}(\mathscr{S}_1)$ *(see* (2.32)*) is positive definite*

$$\langle r_{\mathscr{S}} P_{-1}\varphi, \varphi\rangle_{\mathscr{S}_1} \geq M_4 \|\varphi|\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1)\|^2, \quad M_4 > 0\,. \tag{2.37}$$

Proof. Let $\widetilde{\varphi} \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{S})$ be the zero-extension of $\varphi \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1)$. Invoking (2.22) and (2.30) we get:

$$\langle r_{\mathscr{S}} P_{-1}\varphi, \varphi\rangle_{\mathscr{S}_1} = \langle P_{-1}\widetilde{\varphi}, \widetilde{\varphi}\rangle_{\mathscr{S}_1} = \langle \overset{1}{P}_{-1}\,\widetilde{\varphi}, \widetilde{\varphi}\rangle_{\mathscr{S}} + \langle \overset{2}{P}_{-1}\,\widetilde{\varphi}, \widetilde{\varphi}\rangle_{\mathscr{S}}$$

$$\geq M_3 \|\widetilde{\varphi}|\mathbb{H}^{-\frac{1}{2}}(\mathscr{S})\|^2 \geq M_3 \|\varphi|\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1)\|^2. \qquad \blacksquare$$

Lemma 2.8 P^0_{-1} *is a selfadjoint operator* $(P^0_{-1})^* = P^0_{-1}$.

The operator

$$r_{\mathscr{S}_1} P_{-1} : \widetilde{\mathbb{H}}^s(\mathscr{S}_1) \to \mathbb{H}^{s+1}(\mathscr{S}_1)\,, \quad s \in \mathbb{R}\,, \tag{2.38}$$

is Fredholm if and only if the operator

$$P^0_{-1} : \widetilde{\mathbb{H}}^s(\mathscr{S}_1) \dot{+} \mathbb{R}^6 \to \mathbb{H}^{s+1}(\mathscr{S}_1) \dot{+} \mathbb{R}^6 \tag{2.39}$$

is Fredholm and, if Fredholm, their indices coinside

$$\text{Ind}\; r_{\mathscr{S}_1} P_{-1} = \text{Ind}\; P^0_{-1}\,.$$

If operator (2.38) *is invertible, then* (2.39) *is also invertible*[1)]

Proof. The first assertion follows from the self–adjointness $P^*_{-1} = P_{-1}$ (moreover, P_{-1} is positive definite (see (2.37)):

be the regulariser (see [?]); then from $RP^0_{-1} = I + T_1$, $P^0_{-1}R = I + T_2$, where $T_1,\ T_2$ are compact, we find immediately

$$R_{11} r_{\mathscr{S}_1} P_{-1} = I - R_{12}L + T_{11}\,, \qquad r_{\mathscr{S}_1} P_{-1} R_{11} = I - L^* R_{21} + T_{21}$$

with certain compact operators T_{11} and T_{21}. Since L and L^* are finite–dimensional, $r_{\mathscr{S}_1} P_{-1}$ has a regulariser and is Fredholm.

Now let (2.38) be Fredholm and R_0 be the regulariser $R_0 r_{\mathscr{S}_1} P_{-1} = I + T_1,\ \ r_{\mathscr{S}_1} P_{-1} R_0 = I + T_2$, where $T_1,\ T_2$ are compact operators. Then

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & 0 \end{pmatrix}$$

is the regulariser for P^0_{-1} and the latter is Fredholm.

Since R is a 6–dimensional extension of the operator R_0, their indices coincide $\operatorname{Ind} R = \operatorname{Ind} R_0$, and we get the equality

$$\operatorname{Ind} P^0_{-1} = -\operatorname{Ind} R = -\operatorname{Ind} R_0 = \operatorname{Ind} r_{\mathscr{S}_1} P_{-1}\,.$$

Furter we have

$$P^0_{-1} = \begin{pmatrix} r_{\mathscr{S}_1} P_{-1} & L^* \\ L & 0 \end{pmatrix} = \begin{pmatrix} r_{\mathscr{S}_1} P_{-1} & 0 \\ L & I \end{pmatrix} \begin{pmatrix} I & r_{\mathscr{S}_1} P_{-1} L^* \\ 0 & -L(r_{\mathscr{S}_1} P_{-1})^{-1} L^* \end{pmatrix}$$

and the first factor is invertible provided (2.38) is invertible:

$$\begin{pmatrix} r_{\mathscr{S}_1} P_{-1} & 0 \\ L & I \end{pmatrix}^{-1} = \begin{pmatrix} (r_{\mathscr{S}_1} P_{-1})^{-1} & 0 \\ -L(r_{\mathscr{S}_1} P_{-1})^{-1} & I \end{pmatrix}.$$

By similar reasoning the second factor is invertible if $L(r_{\mathscr{S}_1} P_{-1})^{-1} L^* : \mathbb{R}^6 \to \mathbb{R}^6$ is invertible. Let $\vec{c} \in \operatorname{Ker} L(r_{\mathscr{S}_1}, P_{-1})^{-1} L^*$ and insert $\varphi = L^*\vec{c}$ into (2.37); then $L^*\vec{c} = 0$. The linear independence of $h_1 \cdots, h_6$ implies $\vec{c} = 0$. this yields the invertibility of the finite–dimensional operator $L(r_{\mathscr{S}_1} P_{-1})^{-1} L^*$.

Thus, invertibility of (2.38) yields the invertibility of (2.39). ■

$\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\mathscr{S}_1) \to \mathbb{H}^{\frac{1}{2}}(\mathscr{S}_1)$. We shall extend this invertibility property to more general spaces (namely, to the anisotropic Bessel potential spaces with weight; see Theorem **??**). For this purpose and for investigations of the asymptotics of solutions we need the symbol of the pseudodifferential operator P_{-1}. Let

$$\mathscr{S}_1 = \bigcup_{j=1}^{N} X_j, \quad \varkappa_j = (\varkappa_{j1}, \varkappa_{j2}, \varkappa_{j3})^\top : Y_j \to X_j, \quad Y_j \subset \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}^+ \tag{3.1}$$

be some C^∞-smooth atlas of the surface $\mathscr{S}_1 \subset \mathbb{R}^3$ and let

$$\widetilde{\varkappa}_j : \widetilde{Y}_j \to \widetilde{X}_j, \quad \widetilde{Y}_j = Y_j \times (-\varepsilon, \varepsilon), \quad \widetilde{X}_j \subset \mathbb{R}^3, \quad \widetilde{X}_j \cap \mathscr{S}_1 = X_j,$$

$$\widetilde{\varkappa}_j(z) := \varkappa_j(z') - z_3 \vec{n}(\varkappa_j(z')) \quad \widetilde{\varkappa}_j\big|_{Y_j} = \widetilde{\varkappa}_j(z', 0) = \varkappa_j(z')\,, \tag{3.2}$$

$$z = (z_1, z_2, z_3) \in \mathbb{R}^3\,, \quad z' = (z_1, z_2) \in \mathbb{R}^2\,, \quad j = 1, 2, \dots, N$$

be extensions of the local diffeomorphisms in (3.1). By $\varkappa'_j(z') = (\partial_k \varkappa_{j\ell}(z'))_{2\times 3}$ and by $\widetilde{\varkappa}'_j(z) = (\partial_k \widetilde{\varkappa}_{j\ell}(z))_{3\times 3}$ for $z' \in Y_j$ and $z \in \widetilde{Y}_j$ we denote the corresponding Jacoby matrices, respectively. $\varkappa'_j(z')$ coincides with $\widetilde{\varkappa}'_j(z', 0)$ for $z' \in Y_j$ if we delete the last column, i. e. the entries $(\partial_3 \widetilde{\varkappa}_{j\ell})(z', 0)$, $\ell = 1, 2, 3$; therefore $\widetilde{\varkappa}'_j(z', 0)(\eta, 0) = \varkappa'_j(z')\eta$ for $z' \in Y_j$, $\eta \in \mathbb{R}^2$.

Let us invoke l.t.c.s. (σ, ρ), introduced in Introduction and use $g(\sigma)$ instead of $g(\sigma, 0, 0)$. Clearly,

$$\widetilde{\varkappa}'_j(\sigma) := \widetilde{\varkappa}'_j(\varkappa_j^{-1}(\sigma), 0) = (\vec{e}_1(\sigma), \vec{e}_2(\sigma), \vec{e}_3(\sigma))\ , \tag{3.3}$$

where the column–vectors $\vec{e}_1(\sigma)$, $\vec{e}_2(\sigma)$ and $\vec{e}_3(\sigma)$ on the boundary $\sigma = \varkappa_j(z_1, 0) \in \Gamma = \partial\mathscr{S}_1$ can be chosen mutually orthogonal. Moreover, $\vec{e}_3(\sigma) = -\vec{n}(\sigma)$ coinsides with the **invard** unit normal vector, while $\vec{e}_1(\sigma), \vec{e}_2(\sigma)$ are tangential to $\mathscr{S}_1$, $\vec{e}_1(\sigma)$ is tangential and $\vec{e}_2(\sigma)$ is cotangential (directed inside $\mathscr{S}_1$) to $\partial\mathscr{S}_1$ at $\sigma \in \partial\mathscr{S}_1$. Therefore $(\vec{e}_1(\sigma), \vec{e}_2(\sigma), \vec{e}_3(\sigma))$ is positively oriented, orthonormal smooth vector–field on Γ. The unit vector–fields $\vec{e}_1(\sigma, \rho)$ and $\vec{e}_2(\sigma, \rho)$ on $\mathscr{S}_1$ are not orthogonal in general in the contrary to the pairs $\vec{e}_1(\sigma, \rho), \vec{e}_3(\sigma, \rho)$ and $\vec{e}_2(\sigma, \rho), \vec{e}_3(\sigma, \rho)$.

Theorem 3.1 *P_{-1} in (2.32) is a pseudodifferential operator with the homogeneous principal symbol*

$$\mathscr{P}_{-1}(\sigma,\rho;\xi) = \overset{1}{\mathscr{V}}_{-1}(\sigma,\rho;\xi)\left[\frac{1}{2}I + i\,\overset{1}{\mathscr{W}}{}_0^{\top}(\sigma,\rho;\xi)\right]^{-1}$$
$$+\overset{2}{\mathscr{V}}_{-1}(\sigma,\rho;\xi)\left[\frac{1}{2}I - i\,\overset{2}{\mathscr{W}}{}_0^{\top}(\sigma,\rho;\xi)\right]^{-1}, \quad \sigma\in\Gamma,\ \rho\in\mathbb{R}^+,\ \xi\in\mathbb{R}^2, \quad (3.6)$$

where the matrix–functions $\overset{k}{\mathscr{V}}_{-1}(\sigma,\rho;\xi)$, $\overset{k}{\mathscr{W}}{}_0^{\top}(\sigma,\rho;\xi)$, are real valued and given by:

$$\overset{k}{\mathscr{V}}_{-1}(\sigma,\rho;\xi) = \frac{\mathscr{G}_{\varkappa_j}(\sigma,\rho)}{2\pi\det\widetilde{\varkappa}'_j(\sigma,\rho)}\int\limits_{-\infty}^{\infty}\mathscr{L}_k^{-1}\left([\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}]^{-1}(\lambda,\xi)\right)\mathrm{d}\,\lambda$$
$$= \frac{\mathscr{G}_{\varkappa_j}(\sigma,\rho)}{2\pi\det\widetilde{\varkappa}'_j(\sigma,\rho)}\int\limits_{\mathscr{C}_\pm}\mathscr{L}_k^{-1}\left([\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}]^{-1}(\zeta,\xi)\right)\mathrm{d}\,\zeta,$$

$$\overset{k}{\mathscr{W}}{}_0^{\top}(\sigma,\rho;\xi) = \frac{\mathscr{G}_{\varkappa_j}(\sigma,\rho)}{2\pi\det\widetilde{\varkappa}'_j(\sigma,\rho)}\int\limits_{\mathscr{C}_\pm}\overset{k}{\mathscr{T}}_0\left(\widetilde{\varkappa}_j(\varkappa_j^{-1}(\sigma,\rho)),[\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}]^{-1}(\zeta,\xi)\right)$$
$$\times\mathscr{L}_k^{-1}\left([\widetilde{\varkappa}'_j(0,x)^{\top}]^{-1}(\zeta,\xi)\right)\mathrm{d}\,\zeta,$$

since the symbol of the stress operator is pure imaginary (cf. (1.7))

$$\overset{k}{\mathscr{T}}(\vartheta,\vec{n}(\sigma,\rho)) = -i\,\overset{k}{\mathscr{T}}_0(\sigma,\rho;\vartheta), \quad \overset{k}{\mathscr{T}}_0(\sigma,\rho;\vartheta) = \left[\sum_{j,l,m,n}\overset{k}{c}_{jlmn}\,n_j(\sigma,\rho)\vartheta_n\right]_{3\times3}$$
$$\text{for}\quad k=1,2,\quad \sigma,\rho\in\mathscr{S}_1^+=\Gamma\times\mathbb{R}^+,\quad \vartheta\in\mathbb{R}^3.$$

The second and the third line–integrations are performed along a smooth

Proof After lifting the operator P_{-1} in (2.32) from the surface $\mathscr{S}_1$ to the half-space $\mathbb{R}^2_x$ by means of the "pull–back" operator

$$\varkappa_{j*}\psi(z') := \begin{cases} \chi^0_j(z')\psi\left(\varkappa_j(z')\right), & \text{if} \quad z' \in Y_j \subset \mathbb{R}^2_+\,, \\ 0, & \text{if} \quad z' \not\in Y_j\,, \end{cases} \qquad \chi^0_j(z') := \chi_j(\varkappa_j(z'))\,,$$

$$\chi_j \in C^\infty(\mathscr{S}_1)\,, \quad \operatorname{supp}\chi_j \subset X_j\,, \quad \sum_{j=1}^N \chi_j(x) \equiv 1\,, \quad x \in \mathscr{S} \tag{3.8}$$

(see (3.1)) and the inverse $\varkappa^{-1}_{j*}$, we can easily find that the difference between the lifted operator $\varkappa_{j*}P_{-1}\varkappa^{-1}_{j*}$ and the pseudodifferential operator $P_{-1}(\varkappa_j(z'), D')$, $\quad D' := (D_1, D_2)$ is an operator of order -2 (for details see in [**?**, **?**]). Therefore $\mathscr{P}_{-1}(z', \xi)$ is the symbol of P_{-1} in accordance with the definition (cf. Section 1.4 in [**?**]). Moreover, if we apply to (2.37) the lifting and "coefficient freezing method" (i.e. apply the quasi–equivalence; see [**?**, Sect. 3.4], [**?**, Sect. 3.2]), we get

$$\langle\chi_0 P_{-1}(z'_0, D')\psi, \psi\rangle_{\mathbb{R}^2} = \langle P_{-1}(z'_0, D')\chi_0\psi, \chi_0\psi\rangle_{\mathbb{R}^2} \geq \frac{M_5}{2}\|\chi_1\psi_0|L_2(\mathbb{R}^2)\|^2\,, \tag{3.9}$$

where $\chi_0 \in C^\infty_0(\mathbb{R}^2)$ is an appropriate cut–off function with $\chi_0(z') = 1$ in some small neighbourhood of the point $z'_0 \in \mathbb{R}^2$. For $\psi \in \mathbb{H}^{-\frac{1}{2}}(\mathbb{R}^2)$ there holds

$$\psi = \lambda^{\frac{1}{2}}(D)\psi_0 \quad \text{with} \quad \psi_0 \in L_2(\mathbb{R}^2) \quad \text{and} \quad \lambda^{\frac{1}{2}}(\xi) := (1+|\xi|^2)^{\frac{1}{4}}$$

(see [**?**, **?**]). The commutators of pseudodifferential operators

$$\chi_0\lambda^{\frac{1}{2}}(D) - \lambda^{\frac{1}{2}}(D)\chi_0 I \text{ and } \chi_0\left[\lambda^{\frac{1}{2}}(D) - \lambda^{\frac{1}{2}}_0(D)\right] \quad \text{with} \quad \lambda^{\frac{1}{2}}_0(\xi) := |\xi|^{\frac{1}{2}}$$

are bounded in $\mathbb{H}^s(\mathbb{R}^2) \to \mathbb{H}^{s-\frac{1}{2}}(\mathbb{R}^2)$ (moreover, they are also bounded in $\mathbb{H}^s(\mathbb{R}^2) \to \mathbb{H}^{s-1}(\mathbb{R}^2)$; see [**?**, **?**] and [**?**, Lemma 1.8]); therefore and, therefore, $P_0(z'_0, D')$ commutes with the dilation operator $R_\varepsilon\varphi(z') := \varphi(\varepsilon z')$. This property together with (??) yields

$$\langle P_0(z'_0, D')\psi_0, \psi_0\rangle_{\mathbb{R}^2} \geq \frac{M_5}{3}\|\psi_0|L_2(\mathbb{R}^2)\|^2. \tag{3.11}$$

From (??) and the Plancherel theorem follows

$$\langle \mathscr{P}_0(y_0, \xi)\, \overset{\wedge}{\psi}_0, \overset{\wedge}{\psi}_0\rangle_{\mathbb{R}^2} \geq \frac{M_5}{3}\|\, \overset{\wedge}{\psi}_0\, |L_2(\mathbb{R}^2)\|^2, \quad \overset{\wedge}{\psi}_0 \in \mathscr{F}\psi_0\,. \tag{3.12}$$

Then (??) is an obvious consequence of (??) (see (??)). ■

Now we are able to investigate system (2.31). To get better regularity results for solutions we will invoke anisotropic Bessel potential spaces with weight $H^{(\mu,s),m}_p(\mathscr{S}_1)$, defined in § A.1. The reader can stay with the usual spaces $H^s(\mathscr{S}_1) = H^{(0,s),0}_2(\mathscr{S}_1)$, but in these spaces one can find only weak solutions.

Theorem 3.2 *The operator*

$$P^0_{-1} : \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathscr{S}_1) \dotplus \mathbb{R}^6 \to \mathbb{H}^{(\mu,s+1),m}_p(\mathscr{S}_1) \dotplus \mathbb{R}^6 \tag{3.13}$$

(cf. (2.35)*) is bounded for every* $1 < p < \infty$, $1 - \mu \leq s \leq \mu$, $m = 0, 1, \dots$ *and is Fredholm if and only if the conditions*

$$\frac{1}{p} - \frac{3}{2} < s < \frac{1}{p} - \frac{1}{2} \tag{3.14}$$

are satisfied. Under these conditions P^0_{-1} *has index zero and the system* (2.35) *has a unique solution* $\mathfrak{t}^0 = (\mathfrak{t}_1, \vec{c}) \in \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathscr{S}_1) \dotplus \mathbb{R}^6$ *for any given* $f^0 \in \mathbb{H}^{(\mu,s+1),m}_p(\mathscr{S}_1) \dotplus \mathbb{R}^6$.

Proof. Since boundedness of operators L and L^* in (2.36) are out of doubt, we only have to check the boundedness of the pseudodifferential operator conditions (**??**) hold (note, that in this particular case the order $\kappa = -1$ and condition (**??**) reads as (**??**)). If Fredholm, (**??**) has a trivial index $\operatorname{Ind} r_{\mathscr{S}_1} P_{-1} = 0$.

Thus, P^0_{-1} in (**??**) is Fredholm if and only if condition (**??**) holds and, if fredholm, $Ind\, P^0_{-1} = 0$ (see Lemma 2.8).

On the other hand, the homogeneous equation $r_{\mathscr{S}_1} P_{-1}\psi = 0$ has only a trivial solution $\psi = 0$ in the space $\mathbb{H}^{\frac{1}{2}}(\mathscr{S}_1)$ due to the positive definiteness (2.37). This yields the invertibility of (**??**) and, consequently of (**??**)), in all spaces which fit into the conditions (**??**) (cf. Theorem **??** and Lemma 2.8).

■

Theorem 3.3 *There is a one–to–one correspondence between the (unique) solutions of the boundary pseudodifferential equation system* (2.31) *and the solutions of the mixed transmission– Neumann problem* (1.9)–(1.13)*, given by formulae* (2.34)*. The inverse correspondence is given by the formula* $\mathfrak{t}_k = r_{\mathscr{S}_1} \overset{k}{\mathscr{T}} (D, \vec{n})u$, $k = 1, 2$.

Proof Theorem **??** follows immediately from Theorems 2.5 and Remark 2.6.

■

Remark 3.4 *The solutions* $\mathfrak{t}_1$, $\mathfrak{t}_2$ *to the system* (2.31) *have the property*

$$\rho^k \mathfrak{t}_j(\sigma, \rho) \in \mathbb{H}^{(\mu,s+k),m}_p(\mathscr{S}_1) \subset \mathbb{H}^{s+k}_p(\mathscr{S}_1) \subset C^{s+k-\frac{1}{p}-\varepsilon_1}(\mathscr{S}_1) \subset C^{k-\frac{1}{2}-\varepsilon_2}(\mathscr{S}_1)$$

for arbitrary $\varepsilon_1 < \varepsilon_2$ *(see Theorem* **??** *and embedding* (**??**)*). Moreover* $\rho^k \mathfrak{t}_j \in C^{k-\frac{1}{2}}(\mathscr{S}_1)$ *and even* $\rho^{\frac{1}{2}} \mathfrak{t}_j \in C^\infty(\mathscr{S}_1)$ *provided* $\nu_0(\sigma)$ *=const (see Theorem* **??***).*

3.2 Asymptotics of solutions to boundary pseudodifferential equation

$$\vec{e}_k(\sigma,\rho) := (e_{k1}(\sigma,\rho), e_{k2}(\sigma,\rho), e_{k3}(\sigma,\rho))^\top\,, \quad k=1,2\,,$$

$$\det \vec{\mathscr{K}}'_j(\sigma) = 1, \quad \mathscr{G}_{\varkappa_j}(\sigma) = 1, \quad \sigma = (\sigma,0) \in \Gamma\,.$$

From (**??**) and (**??**) we find

$$\mathscr{P}_{-1}\,(\sigma;0,\pm 1) = \overset{1}{\mathscr{V}}_{-1}\,(\sigma;0,1)\left[\frac{1}{2}I \pm i\,\overset{1}{\mathscr{W}}{}_0^\top\,(\sigma;0,1)\right]^{-1}$$

$$+\,\overset{2}{\mathscr{V}}_{-1}\,(\sigma;0,1)\left[\frac{1}{2}I \mp \overset{2}{\mathscr{W}}{}_0^\top\,(\sigma;0,1)\right]^{-1}.$$

Due to Lemma **??** the eigenvalues $\lambda_1(\sigma)$, $\lambda_2(\sigma)$, $\lambda_3(\sigma)$ of the matrix

$$\mathscr{P}_{-1,0}(\sigma) := [\mathscr{P}_{-1}(\sigma;0,+1)]^{-1}\,\mathscr{P}_{-1}(\sigma;0,-1) \tag{3.17}$$

are real positive numbers and $\mathscr{P}_{-1,0}(\sigma)$ has no associated eigenvectors, hence is diagonalisable:

$$\mathscr{P}_{-1,0}(\sigma) = \mathscr{K}(\sigma)\Lambda(\sigma)\mathscr{K}^{-1}(\sigma)\,, \qquad \det\mathscr{K}(\sigma) \neq 0\,,$$

$$\Lambda(\sigma) := \operatorname{diag}\{\lambda_1(\sigma),\lambda_2(\sigma),\lambda_3(\sigma)\}\,, \quad \lambda_0, \mathscr{K} \in C^\infty(\partial\mathscr{S}_1)\,. \tag{3.18}$$

Moreover,

$$\lambda_1(\sigma) \equiv 1\,, \quad \lambda_2(\sigma) = \lambda_3^{-1}(\sigma) = \lambda_0(\sigma)\,, \quad \operatorname{Im}\lambda_0(\sigma) \equiv 0 \tag{3.19}$$

(see [**?**] and cf. (**??**) below). In fact, if $\lambda_0(\sigma)$ is an eigenvalue,

$$\det\,[\mathscr{P}_{-1,0}(\sigma) - \lambda_0(\sigma)I] = 0\,,$$

Theorem 3.5 *Let $1 < p < \infty$, $\mu \in \mathbb{R}$ and $m, M \in \mathbb{N}_0$. Then for any $f_2 \in \mathbb{H}_p^{(\mu,\frac{1}{p}+s),m}(\mathscr{S}_1)$, the system of pseudodifferential equations, given by* (2.31)*, has a unique solution $\mathfrak{t}_1, \mathfrak{t}_2 \in \widetilde{\mathbb{H}}_p^{(\mu,\frac{1}{p}-1+s),m}(\mathscr{S}_1)$ if and only if* (**??**) *holds.*

Let $g_{j+1} \in \mathbb{H}_p^{(\infty,\frac{1}{p}+s+M+1),M}(\mathscr{S}_2)$, $f_j \in \mathbb{H}_p^{(\infty,\frac{1}{p}+j+s+M+1),M}(\mathscr{S}_1)$ $(j = 0, 1)$. Then $f_2 \in \mathbb{H}_p^{(\infty,\frac{1}{p}+s+M+2),M}(\mathscr{S}_1)$ in (2.31)*,* (2.33) *and for any $M \in \mathbb{N}_0$ the solutions $\mathfrak{t}_m$ of the system* (2.31) *have the form (cf.* (0.4)*)*

$$\mathfrak{t}_m(\sigma,\rho) = \sum_{k=0}^{M} \mathscr{K}(\sigma)\boldsymbol{\rho}^{-\frac{1}{2}+i\nu(\sigma)+k}\mathscr{K}^{-1}(\sigma)\sum_{j=0}^{k} c_{mkj}(\sigma)\log^j\rho + \widetilde{\mathfrak{t}}_{m,M+1}(\sigma,\rho) \quad (3.20)$$

with $\widetilde{\mathfrak{t}}_{m,M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,\frac{1}{p}+s+M),M}(\mathscr{S}_1)$ for sufficiently small $\rho > 0$, $(m = 1, 2)$. The 3–vector $c_{k00} \in C^\infty(\Gamma)$ is given by the principal symbol of (2.31)*, while the 3–vectors $c_{mkj} \in C^\infty(\Gamma)$ for $k = 1, 2, \ldots$ are given by the full symbol of the equation. Here*

$$\nu(\sigma) := (0, \nu_0(\sigma), -\nu_0(\sigma))^\top \qquad \nu_0(\sigma) := \frac{\log\lambda_0(\sigma)}{2\pi}\,.$$

with $\lambda_0(\sigma)$ given by (**??**) *and we refer* (0.4) *for the definition of the matrix $\boldsymbol{\rho}^{\vartheta+i\nu(\sigma)}$.*

For the displacement vector field $u(z) = u(\sigma,\rho,r)$ and the stress tensor field $\mathfrak{T}(z) = \mathfrak{T}(\sigma,\rho,r)$ we have the expansions (0.7) *and* (0.8)*, respectively.*

Proof The proof follows from Theorems **??**, **??**, **??** in the Appendix if we invoke (**??**), (**??**) and (**??**). ■

3.3 Example: two isotropic bodies

T.C.Ting, using an appropriate ansatz, obtained in [**?**] a criterion for the absence of oscillations of the displacement field describing interface–cracks between two anisotropic bodies. We shall derive this condition for the case

Theorem 3.6 *The matrix* $\mathscr{K}(\sigma)$ *in* (**??**), (**??**) *has the following form*

$$\mathscr{K}(\sigma) = \begin{pmatrix} e_{11}(\sigma) & e_{21}(\sigma) & n_1(\sigma) \\ e_{12}(\sigma) & e_{22}(\sigma) & n_2(\sigma) \\ e_{13}(\sigma) & e_{23}(\sigma) & n_1(\sigma) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & 1 \end{pmatrix}. \qquad (3.22)$$

The parameter $\nu_0 = 0$ *in the asymptotic expansion* (**??**) *vanishes if and only if*

$$\frac{\mu_1}{\mu_2} = \frac{1-2\sigma_1}{1-2\sigma_2} \qquad (3.23)$$

and then the asymptotic expansion (**??**) *simplifies substantially*

$$\mathfrak{t}_m(\sigma,\rho) = \sum_{k=0}^{M} \boldsymbol{\rho}^{-\frac{1}{2}+k} c_k(\sigma) + \widetilde{\mathfrak{t}}_{m,M+1}(\sigma,\rho)\,, \quad m = 1, 2\,. \qquad (3.24)$$

Proof. The fundamental solutions of the Lamé equations (**??**) (see (2.1)) are given by

$$G_k(z) = \frac{1}{16\pi\mu_k(1-\sigma_k)} \left(\frac{3-4\sigma_k}{|z|}\delta_{jm} + \frac{z_j z_m}{|z|^3} \right)_{3\times 3},$$

where $\sigma_k = \lambda_k/2(\lambda_k + \mu_k)$ denotes the corresponding Poisson ratio for $k = 1, 2$. The symbols of the operators $\overset{k}{V}_{-1}$ (see (2.4), (**??**)) were given in [**?**] and [**?**]:

The symbols $\overset{k}{\mathscr{W}_0^{\top}}(\sigma,\rho;\xi)$ can be found in [**?**, Section XIV.6],

$$\overset{k}{\mathscr{W}_0^{\top}}(\sigma,\rho;\xi) = [\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}]^{-1}\,\overset{k}{\mathscr{W}_0^{\top}}(\xi)\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}$$

$$\overset{k}{\mathscr{W}_0^{\top}}(\xi) = \begin{pmatrix} 0 & 0 & \overset{k}{\gamma_3}\,\xi_1|\xi|^{-1} \\ 0 & 0 & \overset{k}{\gamma_3}\,\xi_2|\xi|^{-1} \\ \overset{k}{\gamma_3}\,\xi_1|\xi|^{-1} & \overset{k}{\gamma_3}\,\xi_2|\xi|^{-1} & 0 \end{pmatrix},$$

$$\overset{k}{\gamma_3} = \frac{\mu_k}{\lambda_k + 2\mu_k} = \frac{1-2\sigma_k}{2(1-\sigma_k)}.$$

Now we can write the symbol $\mathscr{P}_{-1}(\sigma,\rho;\xi)$ (see (**??**)) in the explicit form

$$\mathscr{P}_{-1}(\sigma,\rho;0,\pm1) = [\widetilde{\varkappa}'_j(\sigma,\rho)^{\top}]^{-1}\mathscr{P}_{-1}(\pm1)\widetilde{\varkappa}'_j(\sigma,\rho)^{\top},$$

$$\begin{aligned}
\mathscr{P}_{-1}(\pm1) &= \frac{2\,\overset{1}{\gamma_1}}{1-(\overset{1}{\gamma_3})^2}\begin{pmatrix} (1+\overset{1}{\gamma_2}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-(\overset{1}{\gamma_3})^2 & 0 & 0 \\ 0 & 1 & \pm i\,\overset{1}{\gamma_3} \\ 0 & \mp i\,\overset{1}{\gamma_3} & 1 \end{pmatrix} \\
&+ \frac{2\,\overset{2}{\gamma_1}}{1-(\overset{2}{\gamma_3})^2}\begin{pmatrix} (1+\overset{2}{\gamma_2}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-(\overset{2}{\gamma_3})^2 & 0 & 0 \\ 0 & 1 & \mp i\,\overset{2}{\gamma_3} \\ 0 & \pm i\,\overset{2}{\gamma_3} & 1 \end{pmatrix} \\
&= 2\begin{pmatrix} \gamma_4 & 0 & 0 \\ 0 & \gamma_5 & \pm i\gamma_6 \\ 0 & \mp i\gamma_6 & \gamma_5 \end{pmatrix},
\end{aligned}$$

where

$$
= \widetilde{\mathscr{K}}_j'(\sigma) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\gamma_5^2+\gamma_6^2}{\gamma_5^2-\gamma_6^2} & \dfrac{-2i\gamma_5\gamma_6}{\gamma_5^2-\gamma_6^2} \\ 0 & \dfrac{-2i\gamma_5\gamma_6}{\gamma_5^2-\gamma_6^2} & 1 \end{pmatrix} \widetilde{\mathscr{K}}_j'(\sigma)^\top
$$

$$
= \mathscr{K}(\sigma) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\gamma_5-\gamma_6}{\gamma_5+\gamma_6} & 0 \\ 0 & 0 & \dfrac{\gamma_5+\gamma_6}{\gamma_5-\gamma_6} \end{pmatrix} \mathscr{K}^{-1}(\sigma) \quad (3.25)
$$

and $\mathscr{K}(\sigma)$ is defined in (**??**). Comparing the representations (**??**) and (**??**), we find the formulae for $\mathscr{K}(\sigma)$, for the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and for $\delta = i\nu$; namely,

$$
\begin{aligned}
&\lambda_1 = 1, \quad \lambda_2 = \frac{\gamma_5-\gamma_6}{\gamma_5+\gamma_6}, \quad \lambda_3 = \frac{\gamma_5+\gamma_6}{\gamma_5-\gamma_6} = \frac{1}{\lambda_2}, \\
&\nu := (0, -\nu_0, \nu_0), \quad \nu_0 = \frac{1}{2\pi} \log \frac{\gamma_5-\gamma_6}{\gamma_5+\gamma_6}\,.
\end{aligned} \quad (3.26)
$$

As for (**??**), the desired absence of oscillation follows if and only if $\lambda_2 = \lambda_3 = 1$, i.e. if γ_6=0 (see (**??**)); this can be rewritten in the form

$$
\frac{\mu_1}{\mu_2} = \frac{(3-10\sigma_1+8\sigma_1^2)(3-4\sigma_2)}{(3-4\sigma_1)(3-10\sigma_2+8\sigma_1^2)} = \frac{1-2\sigma_1}{1-2\sigma_2}\,. \qquad \blacksquare
$$

A Appendix

In the Appendix we recall some results on pseudodifferential equations in Bessel potential spaces, mostly from [**?**, **?**, **?**, **?**, **?**, **?**, **?**]).

A.1 Spaces

are bounded operators in both spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, the convolution operator

$$a(D)\varphi = W_a^0\varphi := \mathscr{F}^{-1}a\mathscr{F}\varphi \quad \text{with} \quad a \in \mathbb{S}'(\mathbb{R}^n), \quad \varphi \in \mathbb{S}(\mathbb{R}^n) \tag{A.2}$$

is a bounded transformation from $\mathbb{S}(\mathbb{R}^n)$ into $\mathbb{S}'(\mathbb{R}^n)$ (see [**?**, **?**]).

The Bessel potential space $\mathbb{H}_p^s(\mathbb{R}^n)$ is defined as a subset of $\mathbb{S}'(\mathbb{R}^n)$ endowed with the norm ([**?**, **?**])

$$\|u|\mathbb{H}_p^s(\mathbb{R}^n)\| := \|\langle D\rangle^s u|L_p(\mathbb{R}^n)\|, \quad \text{where} \langle\xi\rangle^s := (1+|\xi|^2)^{\frac{s}{2}}. \tag{A.3}$$

For the Hilbert space $\mathbb{H}_2^s(\mathbb{R}^n)$, usually the index 2 is dropped and the notation $\mathbb{H}^s(\mathbb{R}^n)$ is used.

For $\Omega \subset \mathbb{R}^n$, by $C^\sigma(\Omega)$ with $\sigma = m + \mu$, $m = 0, 1, \dots,$ $0 < \mu < 1$ we denote, as usual, the Hölder space of continuous functions, having Hölder continuous (with exponent μ) derivatives of the order m.

The space $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^n_+) \subset \mathbb{H}_p^s(\mathbb{R}^n)$ is defined as the subspace of those functions $\varphi \in \mathbb{H}_p^s(\mathbb{R}^n)$, which are supported in the half space, i.e. $\operatorname{supp}\varphi \subset \overline{\mathbb{R}^n_+}$ whereas $\mathbb{H}_p^s(\mathbb{R}^n_+)$ denotes the quotient space $\mathbb{H}_p^s(\mathbb{R}^n_+) = \mathbb{H}_p^s(\mathbb{R}^n)/\widetilde{\mathbb{H}}_p^s(\mathbb{R}^n_-)$, $\mathbb{R}^n_- := \mathbb{R}^n \setminus \mathbb{R}^n_+$ and can be identified with the space of distributions φ on $\mathbb{R}^n_+$ which admit an extension $\ell\varphi \in \mathbb{H}_p^s(\mathbb{R}^n)$. Therefore $r_{\mathscr{M}}\mathbb{H}_p^s(\mathbb{R}^n) = \mathbb{H}_p^s(\mathbb{R}^n_+)$.

For $\mu, s \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $1 < p < \infty$, the anisotropic Bessel potential spaces with weight $\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)$ consists of functions (of distributions when $\mu < 0$ or $\mu + s < 0$) which have the following finite norm

$$\begin{aligned} &\|u|\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)\| := \sum_{k=0}^{m} \|\langle D'\rangle^\mu \langle D\rangle^{s+k} z_n^k u|L_p(\mathbb{R}^n)\|, \\ &\xi = (\xi', \xi_n), \quad \xi' \in \mathbb{R}^{n-1}, \quad \xi_n \in \mathbb{R}. \end{aligned} \tag{A.4}$$

For integer $\ell, \vartheta = 0, 1, \dots$ we get anisotropic Sobolev spaces with weight, endowed with the norm

$$\|u|\mathbb{H}^{(\ell,\vartheta),m}(\mathbb{R}^n)\| := \sum_{k=0}^{m} \sum \sum \|\partial^\alpha \partial^\beta z_n^k u(x)|L_p(\mathbb{R}^n)\|$$

If $\{X_j\}_{j=1}^{\ell}$ is a sufficiently refined covering of $\mathscr{M}$, the spaces $\mathbb{H}_p^s(\mathscr{M})$, $C^\sigma(\mathscr{M})$, $\widetilde{\mathbb{H}}_p^s(\mathscr{M})$, $\widetilde{C}^\sigma(\mathscr{M})$, $\mathbb{H}_p^{(\mu,s),m}(\mathscr{M})$ and $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathscr{M})$ can be defined by a partition of the unity $\{\psi_j\}_{j=1}^{\ell}$ subordinated to the covering $\{X_j\}_{j=1}^{\ell}$ and local coordinate diffeomorphism

$$\varkappa_j : Y_j \to X_j, \qquad Y_j \subset \mathbb{R}^n_+. \tag{A.5}$$

The space $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathscr{M})$ can also be defined as the subspace of $\mathbb{H}_p^{(\mu,s),m}(\mathscr{S})$ of those functions $\varphi \in \mathbb{H}_p^{(\mu,s),m}(\mathscr{S})$ for which $\operatorname{supp}\varphi \subset \overline{\mathscr{M}}$. The space $\mathbb{H}_p^{(\mu,s),m}(\mathscr{M})$ is the quotient space $\mathbb{H}_p^{(\mu,s),m}(\mathscr{M}) = \mathbb{H}_p^{(\mu,s),m}(\mathscr{S})/\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathscr{S} \setminus \mathscr{M})$ and can be identified with the space of distributions φ on $\mathscr{M}$ which admit an extension $\ell\varphi \in \mathbb{H}_p^{(\mu,s),m}(\mathscr{S})$. Therefore $r_{\mathscr{M}}\mathbb{H}_p^{(\mu,s),m}(\mathscr{S}) = \mathbb{H}_p^{(\mu,s),m}(\mathscr{M})$.

If $\mathscr{B}^*$ denotes the dual space to the space $\mathscr{B}$ and $\partial\mathscr{M} \neq \emptyset$, then the following relations are valid (see [**?**]):

$$\left(\widetilde{\mathbb{H}}_p^s(\mathscr{M})\right)^* = \mathbb{H}_{p'}^{-s}(\mathscr{M}), \qquad \left(\mathbb{H}_p^r(M)\right)^* = \widetilde{\mathbb{H}}_{p'}^{-r}(\mathscr{M})\,, \tag{A.6}$$

provided $s, r \in \mathbb{R}$, $r \geq \dfrac{1}{p}$, $1 < p < \infty$, $p' = \dfrac{p}{p-1}$. If $\mathscr{S}^m \subset \mathbb{R}^n$ is an m-dimensional C^∞–smooth submanifold, where $m < n$, then the trace operator

$$\gamma_{\mathscr{S}^m} : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{B}_{p,p}^{s-\frac{n-m}{p}}(\mathscr{S}^m) \subset \mathbb{H}_p^{s-\frac{n-m}{p}-\varepsilon_1}(\mathscr{S}^m) \subset C^{s-\frac{n-2m}{p}-\varepsilon_2}(\mathscr{S}^m) \tag{A.7}$$

is correctly defined and bounded, provided $1 < p < \infty\,$, $\dfrac{n-m}{p} < s$ and $0 < \varepsilon_1 < \varepsilon_2$. Here $\mathbb{B}_{p,q}^s(\mathscr{S}^m)$ denotes the Besov space (see [**?**]).

A.2 Pseudodifferential equations

If the convolution operator defined in (**??**) has a bounded extension

$$W_a^0 : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n),$$

we get that the operator $W_a^0 : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-\kappa}(\mathbb{R}^n)$ is bounded if and only if $a \in M_p^{(\kappa)}(\mathbb{R}^n)$.

Let $a \in M_p^{(\kappa)}(\mathbb{R}^n)$. Then the operator

$$W_a := r_+ a(D) : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) \to \mathbb{H}_p^{s-\kappa}(\mathbb{R}_+^n) \tag{A.9}$$

is bounded, where $r_+ := r_{\mathbb{R}_+^n}$ is the restriction operator.

If the symbol $a(t;\xi)$ depends on the variable t, then the corresponding convolution operator (see (**??**))

$$a(t,D)\varphi(t) = W_{a(t;\cdot)}^0 \varphi(t) := \left(\mathscr{F}_{\xi\to t}^{-1} a(t;\xi) \mathscr{F}_{y\to\xi}\varphi(y)\right)(t) \tag{A.10}$$

with the symbol $a \in C(\mathbb{R}^n, \mathbb{S}'(\mathbb{R}^n))$ is called a general pseudodifferential operator (PsDO in short) acting on $\varphi \in \mathbb{S}(\mathbb{R}^n)$. Here $C(\Omega, \mathscr{B})$ denotes the set of all continuous functions $a : \Omega \to \mathscr{B}$ with $\mathscr{B}$ any metric space.

Let $M_p^{(\kappa)}(\mathbb{R}^n \times \mathbb{R}^n)$ denote the class of symbols $a(t;\xi)$ for which the operator in (**??**) can be extended to a bounded mapping

$$a(t,D) : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-\kappa}(\mathbb{R}^n) \quad \text{for all} \quad s \in \mathbb{R}\,.$$

Theorem A.1 [**?**, Theorem 5.3] *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. If for a function $a(t;\xi)$, $t \in \Omega$, $\xi \in \mathbb{R}^n$ there exist constants $M_{\alpha,\beta}$ such that*

$$\begin{gathered}\int_\Omega |(\xi')^{\beta'} \partial_t^\alpha \partial_\xi^\beta a(t;\xi)| dt \leq M_{\alpha,\beta} \langle\xi\rangle^{\kappa-\beta_n}, \\ \text{for all } \alpha, \beta = (\beta', \beta_n) \in \mathbb{N}_0^n, \quad |\beta'| \leq \left[\frac{n}{2}\right] + 1\,, \quad \beta' \leq 1\end{gathered} \tag{A.11}$$

and all $\beta_n = 0, 1, .., \ \xi \in \mathbb{R}^n$, then $a \in M_p^{(\kappa)}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $1 < p < \infty$.

Definition A.2 *Let $\mathbb{S}_{cl,\mu}(\Omega, \mathbb{R}^n)$ denote the class of functions $a(t;\xi)$ which satisfy condition* (**??**) *and admit an asymptotic expansion*

$$a(t;\xi) \simeq a_0(t;\xi) + a_1(t;\xi) + \cdots\,, \tag{A.12}$$

where:

Theorem A.3 [**?**, Theorem 1.5] *Let $m \in \mathbb{N}_0, \kappa \in \mathbb{R}$ and $1 < p < \infty$. If $\partial^k_{\xi_n} a \in M_p^{(\kappa-k)}(\mathbb{R}^n, \mathbb{R}^n)$ for every $k = 0, 1, \cdots, m$, then the operator*

$$a(t, D) : \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \to \mathbb{H}_p^{(\mu,s-\kappa),m}(\mathbb{R}^n) \tag{A.13}$$

is bounded for all $\mu, s \in \mathbb{R}$.

*In particular, if $a \in \mathbb{S}_{cl,\kappa}(\mathbb{R}^n, \mathbb{R}^n)$, then $a(t, D)$ in (**??**) is bounded for all $m \in \mathbb{N}_0$ and $\mu, s \in \mathbb{R}$.*

Let $\mathscr{M}$ be an n-dimensional, C^∞–smooth compact manifold with smooth boundary $\Gamma := \partial\mathscr{M} \neq \emptyset$ and $1 < p < \infty, \quad s, \kappa \in \mathbb{R}$.

It is easy to prove that the symbols of the class $\mathbb{S}_{cl,\kappa}(\mathscr{M}, \mathbb{R}^l)$ are invariant with respect to the diffeomorphism $(t; \xi) \mapsto (g_0(t; \xi), g_1(t; \xi))$ with positively homogeneous $g_k \in C^\infty(\mathscr{M}, S^{l-1})$ of order k with respect to ξ ($k = 0, 1$; cf. [**?**, Lemma 1.2]). Therefore the symbol class $\mathbb{S}_{cl,\kappa}(\mathscr{T}^*\mathscr{M})$ is defined correctly on the cotangent manifold $\mathscr{T}^*\mathscr{M}$ (see [**?**, Subsection A.3]).

Moreover, the principal symbol $a_0(t; \xi)$ is defined invariantly and is independent of the particular chart chosen.

Definition A.4 (see [**?**, **?**] etc.). *An operator*

$$A : \widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathscr{M}) \to \mathbb{H}_p^{(\mu,s-\kappa),m}(\mathscr{M}) \tag{A.14}$$

is called a pseudodifferential operator with the symbol $a \in \mathbb{S}_{cl,\kappa}(\mathscr{T}^*\mathscr{M})$, if:
i. $\chi_1 A \chi_2 I : \mathbb{H}_p^{(\mu,s),m}(\mathscr{M}) \to C^\infty(\mathscr{M})$ are continuous for all pairs $\chi_1, \chi_2 \in C^\infty(\mathscr{M})$ with disjoint supports $\operatorname{supp}\chi_1 \bigcap \operatorname{supp}\chi_2 = \emptyset$, i.e. $\chi_1 A \chi_2 I$ has the order $-\infty$;
ii. The "pull–back" operators

$$\varkappa_{j,*} A \varkappa_{j,*}^{-1} u = a^{(j)}(t, D)u, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \qquad j = 1, ..., \ell$$

*(cf. (**??**)) and (**??**)) are pseudodifferential operators on $\mathbb{R}^n_+$ with the symbols*

If $-\infty \le \kappa < 1$ *and* $a = \mathscr{F}k \in \mathbb{S}_{cl,\kappa}(\mathbb{R}^3)$ *is a classical* $N \times N$ *matrix–symbol, where*

$$a(\xi) = a_0(\xi) + a_1(\xi) + \cdots\,, \quad a_k(\lambda\xi) = \lambda^{\kappa-k}a_k(\xi)\,, \quad \xi \in \mathbb{R}^3\,, \lambda > 0\,,$$

then the integral operator

$$\begin{aligned} a_{\mathscr{M}}(x,D)\varphi(x) &= \int\limits_{\mathscr{M}} k(x-\tau)\varphi(\tau)d_\tau\mathscr{M} \quad \text{for} \quad t \in \mathscr{M}\,, \\ a_{\mathscr{M}}(x,D) &: \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathscr{M}) \to \mathbb{H}^{(\mu,s-\kappa-1),m}_p(\mathscr{M}) \end{aligned} \tag{A.15}$$

is a classical pseudodifferential operator

$$a_{\mathscr{M}}(x;\xi') \simeq \sum_{k=0}^{\infty} a_{\mathscr{M},k}(t;\xi')\,, \quad a_{\mathscr{M}} \in \mathbb{S}_{cl,\kappa}(\mathscr{T}^*\mathscr{M})\,, \quad \xi' \in \mathbb{R}^2 \tag{A.16}$$

and the homogeneous principal symbol reads

$$\begin{aligned} a_{\mathscr{M},pr}(x;\xi) &:= a_{\mathscr{M}_1,0}(x;\xi') \\ &= \frac{\mathscr{G}_{\varkappa_j}(x)}{2\pi\det\widetilde{\varkappa}'_j(x)} \int\limits_{-\infty}^{\infty} a_0\left(\left[\widetilde{\varkappa}'_j(x)^\top\right]^{-1}(\xi',\lambda)\right) \mathrm{d}\,\lambda\,, \quad t \in Y_j\,. \end{aligned} \tag{A.17}$$

Here $\mathscr{G}_{\varkappa_j}(x) := \mathscr{G}_{\varkappa_j}(\varkappa_j^{-1}(x))\,,\;\; \widetilde{\varkappa}'_j(x) := \widetilde{\varkappa}'_j(\varkappa_j^{-1}(x))\,,\;\; x \in X_j \subset \mathscr{M}$ *and*

$$\mathscr{G}_{\varkappa_j}(t) := [\det(\partial_k\varkappa_j(t)\,\cdot\,\partial_l\varkappa_j(t))_{2\times 2}]^{\frac{1}{2}}\,, \quad t \in Y_j \subset \mathbb{R}^2$$

with $\partial_k\varkappa_j := (\partial_k\varkappa_{j1}, \partial_k\varkappa_{j2}, \partial_k\varkappa_{j3})^\top$*, denotes the square root of the Gram determinant of the vector–function* $\varkappa_j = (\varkappa_{j1}, \varkappa_{j2}, \varkappa_{j3})^\top$ *for* $j = 1, 2, \dots, N$*.*

A.3 Fredholm property and asymptotics

Theorem A.6 [?, Theorem 2.7], [?, Theorem 1.9]. *Let the symbol* $a_{\mathscr{M}}(x;\xi)$ *in* (**??**) *be elliptic, i. e.*

$$\inf\{|\det a_{\mathscr{M};0}(x;\xi)| : x \in \overline{\mathscr{M}},\ |\xi| = 1\} > 0\,, \tag{A.19}$$

where $a_{\mathscr{M},0}(x;\xi)$ *denotes the principal symbol (see Definition* **??***), and positive definite on the boundary* [2)]

$$\begin{aligned} a_{\mathscr{M},0}(x;\xi)\eta \cdot \eta^{\top} \quad &\geq M|\xi|^{\kappa}|\eta|^2 \\ \text{for all} \quad x \in \partial\mathscr{M}, \qquad &\xi \in \mathbb{R}^n \qquad \text{and} \qquad \eta \in \mathbb{C}^N \end{aligned} \tag{A.20}$$

with some constant $M > 0$.

Then the system of equations (**??**) *is Fredholm if and only if*

$$\frac{1}{p} + \frac{\kappa}{2} - 1 < s < \frac{1}{p} + \frac{\kappa}{2}\,. \tag{A.21}$$

If the symbol is strongly elliptic on $\overline{\mathscr{M}}$*, i.e.*

$$\mathrm{Re}\ (a_{\mathscr{M}}(x,\vartheta)\eta,\eta) \geq M > 0\,, \quad \text{for all} \quad x \in \overline{\mathscr{M}}\,,\ |\vartheta| = |\eta| = 1\,,$$

the Fredholm index of equation (**??**) *vanishes:* $\mathrm{Ind}\, a_{\mathscr{M}}(x,D) = 0$.

If the conditions (**??**) *hold, then* (**??**) *has one and the same kernel in all the spaces* $\widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathscr{M})$, $m \in \mathbb{N}_0$, $\mu \in \mathbb{R}$. *In particular if equation* (**??**) *is uniquelly solvable in one of these spaces, it is uniquelly solvable in all of them and*

$$\psi \in \widetilde{\mathbb{H}}^{(\infty,s),\infty}_p(\mathscr{M}) := \bigcap_{\mu,\,m} \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathscr{M}) \quad \textit{provided} \quad v \in \mathbb{H}^{(\infty,s-\kappa),\infty}_p(\mathscr{M})\,.$$

Note that the Fredholm properties, the index and the kernel $\mathrm{Ker}\, a_{\mathscr{M}}(x,D)$ of (**??**) are independent of the parameters $m \in \mathbb{N}_0$ and $\mu \in \mathbb{R}$.

To formulate results on the asymptotics of the solution $\psi(x)$ to the system

Lemma A.7 *If the matrices* $a_{\mathscr{M}}(\omega,\pm 1)$ *are positive definite, then* $a^0_{\mathscr{M},0}(\omega)$ *in* (**??**) *has only positive eigenvalues* $\lambda'_1(\omega) > 0, \cdots, \lambda'_N(\omega) > 0$ *and has the simple Jordan representation*

$$a^0_{\mathscr{M},0}(\omega) = \mathscr{K}(\omega)\Lambda(\omega)\mathscr{K}^{-1}(\omega) \tag{A.23}$$

with

$$\Lambda(\omega) := \operatorname{diag}\{\lambda_1(\omega), \cdots, \lambda_N(\omega)\}, \quad \mathscr{K} \in C^\infty(\partial\mathscr{M}), \quad \det\mathscr{K}(\omega) \neq 0\,.$$

The numbers

$$\nu_j(\omega) = \frac{\log\lambda_j(\omega)}{2\pi} \tag{A.24}$$

are then real, i.e. $\operatorname{Im}\nu_j(\omega) = 0 \quad \text{for} \quad j = 1, \cdots, N.$

Proof. Since the matrix $a_{\mathscr{M},0}(\omega,+1)$ is positive definite, there exist the square roots $a^{\pm\frac{1}{2}}_{\mathscr{M},0} := [a_{\mathscr{M},0}(\omega,+1)]^{\pm\frac{1}{2}}$ which are positive definite as well. The equivalent matrix

$$a^1_{\mathscr{M},0}(\omega) := a^{\frac{1}{2}}_{\mathscr{M},0} a^0_{\mathscr{M},0}(\omega) a^{-\frac{1}{2}}_{\mathscr{M},0} = [a_{\mathscr{M},0}(\omega,+1)]^{-\frac{1}{2}}\, a^0_{\mathscr{M},0}(\omega)\,\left[a^0_{\mathscr{M},0}(\omega,+1)\right]^{-\frac{1}{2}}$$

has the same eigenvalues, the same eigenvectors and the same Jordan representation as $a^0_{\mathscr{M},0}(\omega)$. Since $a^1_{\mathscr{M},0}(\omega)$ is selfadjoint, it has no associated eigenvectors (i.e. is diagonalisable; see (**??**)) and $\mathscr{K} \in C^\infty(\partial\mathscr{M})$ (see [**?**]). Let $\eta(\omega), \ldots, \eta_N(\omega) \in \mathbb{C}^N$ be eigenvectors corresponding to the eigenvalues $\lambda_1(\omega), \ldots, \lambda_N(\omega)$; then

$$a^0_{\mathscr{M},0}(\omega)\eta_j(\omega) = \lambda_j\eta_j(\omega), \quad j = 1, \ldots, N$$

and we get

$$\lambda_j(\omega) = \frac{\left(a^0_{\mathscr{M},0}(\omega,+1)\eta_j(\omega), \eta_j(\omega)\right)}{\left(a^0_{\mathscr{M},0}(\omega,-1)\eta_j(\omega), \eta_j(\omega)\right)} > 0$$

since $a_{\mathscr{M},0}(\omega,+1)$ is positive definite. □

If $v \in \mathbb{H}_p^{(\infty,s-\mu+M),\infty}(\mathscr{M})$, *then the solution has the following asymptotic expansion* [3)]

$$\psi(\omega,\rho) = \sum_{k=0}^{M} \mathscr{K}(\omega)\boldsymbol{\rho}^{\frac{\kappa}{2}+i\nu(\omega)+k}\mathscr{K}^{-1}(\omega)\sum_{l=0}^{k} c_{kl}(\omega)\log^l\rho + \widetilde{\psi}_{M+1}(\omega,\rho) \quad \text{(A.25)}$$

for all sufficiently small $\rho > 0$, *with* $\widetilde{\psi}_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),M}(\mathscr{M}^+)$. *Here the* N*–vectors* c_{kl} *belong to* $C^\infty(\partial\mathscr{M})$. c_{00} *depends only on the principal symbol of equation* (**??**), *while* $c_{1l}, c_{2l}\ldots$ *depend on the full symbol of the equation. The components of the vector* $\nu := (\nu_1,\cdots,\nu_N)^\top$ *are defined in* (**??**); *the vector exponent of the scalar variable is understood as a diagonal matrix*

$$\boldsymbol{\rho}^{\vartheta+i\nu} := \mathrm{diag}\left\{\rho^{\vartheta+i\nu_1},\cdots,\rho^{\vartheta+i\nu_N}\right\}$$

for arbitrary scalar $\vartheta \in \mathbb{R}$ *(cf* (0.4)*).*

If $\nu_1(\omega) = \mathrm{const}$, *the logarithmic terms in* (**??**) *vanish and:*

$$\psi(\omega,\rho) = \sum_{k=0}^{M} \mathscr{K}(\omega)\boldsymbol{\rho}^{\frac{\kappa}{2}+i\nu(\omega)+k}\mathscr{K}^{-1}(\omega)c_k(\omega) + \widetilde{\psi}_{M+1}(\omega,\rho)\,. \quad \text{(A.26)}$$

Note that the presence of an oscillation $\nu(\omega)$ in the asymptotics (**??**) can be seen as a logarithmic singularity since

$$\boldsymbol{\rho}^{-\frac{\kappa}{2}+i\nu(\omega)+k} = \rho^{-\frac{\kappa}{2}+k}\,\mathrm{diag}\,\{cos[\nu_j(\omega)\log\rho] + i\sin[\nu_j(\omega)\log\rho]\}_{j=1}^N\,.$$

Now let the closed manifold $\mathscr{S}$, which contains $\mathscr{M}$ as a part, be a compact, smooth surface in $\mathbb{R}^n$, $\mathscr{M} \subset \mathscr{S}$ and $\mathscr{S}$ be the common boundary of a compact domain Ω_1 and its outer complement Ω_2 (see Fig. 1 in § 1).

We consider here some homogeneous $N \times N$ system of differential equations

$$A(D_z)u = 0 \qquad \text{in} \qquad \Omega_1 \cup \Omega_2\,, \quad \text{(A.27)}$$

$$A(D_z) := \sum a_\alpha D_z^\alpha$$

is elliptic: $\det A_{pr}(\xi) \neq 0$ for all $|\xi| = 1$, $\xi \in \mathbb{R}^n$. Note, that we consider a homogeneous operator and therefore the principal symbol coincides with the complete symbol. The fundamental solution of the equation (**??**) (see [**?**]) can be written as the following matrix–function

$$H_A(z) = \mathscr{F}^{-1}_{\xi' \to z'} \left[\frac{1}{2\pi} \int_{\mathscr{L}_\pm} A^{-1}(\xi', \tau) e^{-i\tau z_n} d\tau \right] \quad \text{if} \quad \mp z_n > 0 \,, \tag{A.29}$$

where $z = (z', z_n)$, $z' = (z_1, \cdots, z_{n-1})$, $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. The contour $\mathscr{L}_+$ ($\mathscr{L}_-$) is disposed in the upper (lower) complex half–plane $\mathbb{C}^+ := \mathbb{R} \oplus i\mathbb{R}^+$ (in $\mathbb{C}^-$) and is oriented counterclockwise (clockwise) circumventing all roots of the polynomial $\det A(\xi', \tau)$ with respect to τ in the corresponding half–planes $\tau \in \mathbb{C}^\pm$.

For the direct value of the single layer potential

$$Vg(z) = \int_{\mathscr{S}} H_A(z - y) g(y) d_y \mathscr{S}, \qquad z \in \Omega_1 \cup \Omega_2 \tag{A.30}$$

on the surface $\mathscr{S}$ we use the notation

$$V_{1-2m} g(x) = \int\limits_{\mathscr{S}} H_A(x - y) g(y) d_y \mathscr{S}, \qquad x \in \mathscr{S} \,. \tag{A.31}$$

Let B_q be a differential operator with real C^∞–coefficients of order $q = 0, 1, \ldots$ on $\mathbb{R}^n$ with the symbol

$$B_q(z, \xi) = \sum_{|\alpha|=0}^{q} b_\alpha(z) \xi^\alpha \,, \quad B^0_{q-k}(z, \xi) := \sum_{|\alpha|=q-k} b_\alpha(z) \xi^\alpha \,, \quad b_\alpha \in C^\infty(\mathbb{R}^n) \,,$$

with $B^0_q(z, \xi)$ standing for the homogeneous principal symbol ($z, \xi \in \mathbb{R}^n$).

We are interested in the asymptotics of the following potential–type function

A local coordinate diffeomorphism $\varkappa_j : Y_j(\subset \mathbb{R}^2) \to X_j \subset \mathscr{S}$ is extended to a diffeomorphism of "layers"

$$\varkappa_j : \widetilde{Y}_j \to \widetilde{X}_j, \quad \widetilde{Y}_j = Y_j \times (-\varepsilon, \varepsilon), \quad \widetilde{X}_j \cap \mathscr{M} = X_j,$$

and $\mathscr{J}_{\widetilde{\varkappa}_j}(\omega, \rho) := \mathscr{J}_{\widetilde{\varkappa}_j}(\varkappa_j^{-1}(\omega, \rho))$, $\mathscr{J}_{\widetilde{\varkappa}_j}(\omega) := \mathscr{J}_{\widetilde{\varkappa}_j}(\omega, 0)$, denotes the image of the Jacobian matrix $\mathscr{J}_{\widetilde{\varkappa}_j}(z')$ under inverse diffeomorphism $\varkappa_j^{-1}(\omega, \rho)$: $Y_j \to X_j \subset \mathscr{M}$ (cf. § 3.1).

Theorem A.9 *Let the conditions of Theorem* **??** *be valid and let* $\dfrac{\kappa}{2}, \ \dfrac{\kappa}{2} - q, \neq 0, \pm 1, \ldots$; *let* $\psi(\omega, \rho, r)$ *be as in* (**??**)*. The potential–type function* $\mathfrak{T}(\omega, \rho, r)$ *in* (**??**) *then has the following asymptotic expansion:*

$$\mathfrak{T}(\omega, \rho, r) = \sum_{s=1}^{\ell(N)} \sum_{\vartheta=\pm 1} \sum_{k=0}^{M+2m-q} \sum_{j=0}^{n_s(k)} d^{sj}_{k,\pm}(\omega, \vartheta) x^j_n \boldsymbol{\zeta}_{s,\mp\vartheta}^{\frac{\kappa}{2}+i\nu(\omega)-q+2m-j+k} \sum_{l=0}^{2k} a^{sj}_{kl,\pm}(\vartheta, \sigma) \log^l \rho$$

$$+ \widetilde{\mathfrak{T}}_{M+1}(\omega, \rho, r)\,, \quad \widetilde{\mathfrak{T}}_{M+1} \in \mathbb{H}^{s+M+\frac{1}{p}}_{com,p}(\partial\mathscr{M} \times \mathbb{R}^2), \quad \pm r > 0 \qquad \text{(A.33)}$$

for sufficiently small $|\rho| + |r|$ *and arbitrary* $M = 0, 1, \ldots$. *Here* $d^{sj}_{k,\pm}(\cdot, \vartheta), a^{sj}_{kl,\pm}(\cdot, \pm) \in C^\infty(\partial\mathscr{M})$ *and the coefficients of the leading terms* $a^{sj}_{00,\pm}(\omega, \vartheta) = K_j(\omega)$ *are independent of* $s, \vartheta, \pm$.

In (**??**) *by* $\boldsymbol{\zeta}^{\vartheta+i\nu(\omega)}_{s,\mp\vartheta}$ *we denote the diagonal matrix–functions (cf* (0.4)*), which are vector exponents of the scalar variables*

$$\zeta_s = \zeta_{s,+1} := x_{n-1} + x_n \tau_s, \quad \zeta_{s,-1} := \overline{\zeta_{s,+1}}\,, \quad -\pi < \operatorname{Arg} \zeta_s < \pi\,; \qquad \text{(A.34)}$$

$\{\tau_s\}_{s=1}^{\ell(N)} \subset C^\infty(\partial\mathscr{M})$ *are all different roots of the polynomial equation*

$$\det A([\mathscr{J}^{-1}_{\widetilde{\varkappa}_j}(\omega)]^\top (0, +1, \tau)) = 0\,, \quad \operatorname{Im} \tau < 0$$

in the complex lower half–plane and $n_s(0)$ *is the multiplicity of the pole* τ_s *of the matrix–function* $B^0_q(\omega, 0, 0; [\mathscr{J}^{-1}_{\widetilde{\varkappa}_j}(\omega)]^\top (0, 1, \tau)) A^{-1}([\mathscr{J}^{-1}_{\widetilde{\varkappa}_j}(\omega)]^\top (0, 1, \tau))$, *while for* $k = 1, 2, \ldots$ *we have the estimate* $n_s(k) \le k(n_s(0) - 1) + M - k$.

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