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Abstract

In the present paper we study the mapping properties of the nonlinear Boltzmann collision operator on a scale of weighted Sobolev spaces.

Introduction

We consider the classical Boltzmann equation for a simple, dilute gas of particles [2]

$$f_t + (v, \mathrm{grad}_x f) = Q(f,f) \tag{1}$$

which describes the time evolution of the particle density $f(t,x,v)$

$$f \ : \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3 \to \mathbb{R}_+ .$$

Here $\mathbb{R}_+$ denotes the set of non-negative real numbers and $\Omega \subset \mathbb{R}^3$ is a domain in physical space. The right-hand side of the equation (1), known as the collision integral or the collision term, is of the form

$$Q(f,f)(v) = \int\limits_{\mathbb{R}^3} \int\limits_{S^2} B(v,w,e) \Big(f(v')f(w') - f(v)f(w) \Big) \, de \, dw \, . \tag{2}$$

Note that $Q(f,f)$ depends on t and x only as parameters, so we have omitted this dependence in (2) for conciseness. The following notations have been used in (2): $v, w \in \mathbb{R}^3$ are the pre-collision velocities, $e \in S^2 \subset \mathbb{R}^3$ is a unit vector, $v', w' \in \mathbb{R}^3$ are the post-collision velocities and $B(v,w,e)$ is the collision kernel. The operator $Q(f,f)$ represents the change of the distribution function $f(t,x,v)$ due to the binary collisions between particles. A single collision results in a change of the velocities of the colliding partners $v, w \ \to v', w'$ with

$$v' = \frac{1}{2}\Big(v + w + |u| \, e \Big), \ \ w' = \frac{1}{2}\Big(v + w - |u| \, e \Big) \, .$$

where $u = v - w$ denotes the relative velocity. The Boltzmann equation (1) is subjected to an initial condition

$$f(0,x,v) = f_0(x,v), \ \ x \in \Omega, \ \ v \in \mathbb{R}^3$$

and to the boundary conditions on $\Gamma = \partial\Omega$. The kernel $B(v,w,e)$ can be written as

$$B(v,w,e) = B(|u|,\mu) = |u| \sigma(|u|,\mu), \ \ \mu = \cos(\theta) = \frac{(u,e)}{|u|} \, .$$

The function $\sigma : \mathbb{R}_+ \times [-1,1] \to \mathbb{R}_+$ is the differential cross-section and θ is the scattering angle. Some special models for the kernel are as follows:

1. The **hard spheres model** is described by the kernel

$$B(|u|,\mu) = \frac{d^2}{4}|u|\,,$$

   where d denotes the diameter of the particles.

2. The kernel

$$B(|u|,\mu) = |u|^{1-4/m} g_m(\mu)\,,\ m > 1\,. \tag{3}$$

   corresponds to the **inverse power cut-off potential** [5] of the interaction. m denotes the order of the potential and $g_m \in \mathbb{L}_1([-1,1])$ is a given function of the scattering angle only.

3. The special case of $m = 4$ in (3) corresponds to the **Maxwell pseudo-molecules** with

$$B(|u|,\mu) = g_4(\mu)\,.$$

   The collision kernel $B(|u|,\mu)$ here does not depend on the relative speed $|u|$.

4. The **Variable Hard Sphere model** [1](VHS) has an isotropic kernel

$$B(|u|,\mu) = C_\lambda |u|^\lambda\,,\ -3 < \lambda \leq 1. \tag{4}$$

   The model includes, as particular cases the hard spheres model for $\lambda = 1$ and a the Maxwell pseudo–molecules with $\lambda = 0$.

The collision integral (2) decomposes into the natural gain and the loss parts

$$Q(f,f)(v) = Q_+(f,f)(v) - Q_-(f,f)(v)\,,$$

where the bilinear operators $Q_+(\cdot,\cdot)\,,\ Q_-(\cdot,\cdot)$ are

$$Q_+(f,g)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu) f(v')g(w')\,de\,dw \tag{5}$$

and

$$Q_-(f,g)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu) f(v)g(w)\,de\,dw\,. \tag{6}$$

We will also consider the linear operators $Q_+(f)[\cdot]$ and $Q_-(f)[\cdot]$ acting on g for a fixed function f. Before we begin the study of the mapping properties of the operators $Q_+(f)$ and $Q_-(f)$, we discuss the results known from the literature.
T. Gustafson [6] considered the weighted spaces

$$\mathbb{L}_p^{\langle\nu\rangle} = \mathbb{L}_p^{\langle\nu\rangle}(\mathbb{R}^3) = \left\{g : \mathbb{R}^3 \to \mathbb{C}, \langle\cdot\rangle^\nu g \in \mathbb{L}_p(\mathbb{R}^3)\right\}, \quad \langle v\rangle^\nu := (1+|v|^2)^{\frac{\nu}{2}}$$

and the following kernels

$$B(|u|,\mu) = |u|^\lambda g(\mu)\,,\ 0<\lambda\le 1\,,\ g\in\mathbb{L}_1([-1,1])\,. \tag{7}$$

He proved the boundednesses

$$Q_+ \ :\ \left(\mathbb{L}_1^{\langle\nu+\lambda\rangle}\bigcap\mathbb{L}_p^{\langle\nu+\lambda\rangle}\right)\times\left(\mathbb{L}_1^{\langle\nu+\lambda\rangle}\bigcap\mathbb{L}_p^{\langle\nu+\lambda\rangle}\right)\to\mathbb{L}_p^{\langle\nu\rangle} \tag{8}$$

for the weighted $\mathbb{L}_p$ spaces with $1\le p<\infty$ and $0\le\nu<\infty$. As we see, T. Gustafson proved that Q_+ is an operator of the order 0.
P.L. Lions [8] proved the estimates

$$\begin{aligned}\left\|Q_+(f,g)\,|\,\mathbb{W}^1\right\| &\le C\,\|f\,|\,\mathbb{L}_1\|\;\|g\,|\,\mathbb{L}_2\|\,,\\ \left\|Q_+(f,g)\,|\,\mathbb{W}^1\right\| &\le C\,\|f\,|\,\mathbb{L}_2\|\;\|g\,|\,\mathbb{L}_1\|\,,\end{aligned} \tag{9}$$

provided the collision kernel $B(|u|,\mu)$ satisfies

$$B(|u|,\mu)\in\mathbb{C}_0^\infty(\mathbb{R}_+\times[-1,1])\,; \tag{10}$$

i.e. kernels are infinitely smooth with respect to both variables $|u|$ and μ and have compact supports with respect to the variable u.
It is easy to ascertain that the conditions in (10) are too restrictive to cover the models of interaction described above.
The estimates (9) can be written in an equivalent and compact form as the continuity of the mapping

$$Q_+ \ :\ \mathbb{L}_1\times\mathbb{L}_2\to\mathbb{W}^1\,, \tag{11}$$

where $\mathbb{W}^1 := \mathbb{W}^1(\mathbb{R}^3)$ is the Sobolev space (see § 1). If $f\in\mathbb{L}_1$ is fixed, the boundedness (9) shows that $Q_+(f)$ is an operator of the order -1.
B. Wennberg [13] proved that the boundedness property of the operator $Q_+(f)$ similar to (11) for the collision kernel (7)

$$Q_+ \ :\ \left(\mathbb{L}_1^{\langle\nu+1\rangle}\bigcap\mathbb{L}_p^{\langle\nu+1\rangle}\right)\times\left(\mathbb{L}_1^{\langle\nu+1\rangle}\bigcap\mathbb{L}_p^{\langle\nu+1\rangle}\right)\to\mathbb{W}^{1,\langle\nu\rangle}$$

but under the restrictions

$$\frac{1}{2} < \lambda \leq 1\,,\; p > \frac{6}{2\lambda - 1}\,.$$

J. Struckmeier [9] proved the following boundedness property of the gain term of the Boltzmann collision operator in the case $B(|u|, \mu) = const$ which corresponds to the Maxwell molecules

$$Q_+ \;:\; \left(\mathbb{L}_\infty \bigcap \mathbb{L}_1\right) \times \left(\mathbb{L}_\infty \bigcap \mathbb{L}_1\right) \to \mathbb{L}_\infty\,.$$

In the present paper we prove the following boundedness property of the operator $Q_+(f)$ in the scale of weighted Bessel potential spaces

$$Q_+ \;:\; \left(\mathbb{H}_1^{\theta,\langle\nu+\lambda\rangle} \bigcap \mathbb{H}_p^{\theta}\right) \times \mathbb{H}^{s,\langle\nu+\lambda\rangle} \to \mathbb{H}^{s+1,\langle\nu\rangle} \tag{12}$$

(see § 1 for the definitions of spaces) under the minor restrictions on the collision kernel (7)

$$0 \leq \lambda \leq 1\,, \quad p > \frac{3}{2+\lambda}\,, \quad \theta \geq 0\,, \quad |s| \leq \theta\,,\; \nu\, s \geq 0$$

and $\nu \geq 0$ if $s = 0$ (see Theorem 13).A similar result for the operator $Q_-(f)$ reads:

$$Q_- \;:\; \mathbb{H}_p^{s,\langle\nu\rangle} \times \mathbb{H}_q^{\theta,\langle\mu\rangle} \to \mathbb{H}_p^{s,\langle\nu-\lambda\rangle}$$

under the following restriction (see Corollary 18)

$$0 < |\lambda| + 3 - \frac{3}{q} < \mu\,, \quad 1 \leq p,\; q \leq \infty\,, \quad \theta \geq 0\,, \quad s \leq \theta\,,\; \nu \in \mathbb{R}\,.$$

The paper is organised as follows. In Section 1 we introduce the function spaces, the three-dimensional Fourier transform and the pseudodifferential operators. Furthermore we formulate the mapping properties of the pseudodifferential operators using the asymptotic behaviour of their symbols. Then, in Section 2, we deal with the gain part of the collision operator, construct its adjoint and prove that it is a pseudodifferential operator with a given symbol. Then we estimate the symbol and finally prove the main boundedness result formulated in (12). In Section 3 we consider the loss part of the collision operator.

1 Preliminaries

Let $g: \mathbb{R}^3 \to \mathbb{C}$ be a complex-valued function, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ a multi-index of nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We use $\partial^\alpha g$ to denote a mixed partial derivative of g of the order $|\alpha|$

$$\partial^\alpha g = \frac{\partial^{|\alpha|} g}{\partial^{\alpha_1} v_1 \partial^{\alpha_2} v_2 \partial^{\alpha_3} v_3}\,.$$

We will use the inequality $\alpha \preceq \beta$ for two multi-indices in the following sense

$$\alpha \preceq \beta \;\Leftrightarrow\; \alpha_j \le \beta_j\,, \quad j = 1, 2, 3\,.$$

Later we will need the Leibnitz formula for the multidimensional derivative of the product of two functions f and g

$$\partial^\alpha (f \cdot g) = \sum_{\beta \preceq \alpha} \binom{\alpha}{\beta} (\partial^\beta f) \cdot (\partial^{\alpha-\beta} g) \tag{13}$$

with the binomial coefficients

$$\binom{\alpha}{\beta} = \frac{\alpha_1!\, \alpha_2!\, \alpha_3!}{\beta_1!\, \beta_2!\, \beta_3!\, (\alpha_1 - \beta_1)!\, (\alpha_2 - \beta_2)!\, (\alpha_3 - \beta_3)!}\,.$$

1.1 Function spaces

Let $1 \le p < \infty$. The **classical** $\mathbb{L}_p = \mathbb{L}_p(\mathbb{R}^3)$ **spaces** consists of functions g having the property that the following Lebesgue integral is finite

$$\mathbb{L}_p = \mathbb{L}_p(\mathbb{R}^3) = \Big\{ g:\ \mathbb{R}^3 \to \mathbb{C},\ \int_{\mathbb{R}^3} |g(v)|^p\, dv < \infty \Big\}.$$

The $\mathbb{L}_p$ norm of the function g is defined by

$$\|g \,|\, \mathbb{L}_p\| = \left(\int_{\mathbb{R}^3} |g(v)|^p\, dv \right)^{1/p}. \tag{14}$$

For $p = \infty$, we use the canonical generalisation of (14)

$$\|g \,|\, \mathbb{L}_\infty\| = \operatorname{ess}\sup_{v \in \mathbb{R}^3} |g(v)|\,.$$

The following abbreviation will often be used

$$\langle v \rangle^\nu = (1 + |v|^2)^{\nu/2}\,, \quad v \in \mathbb{R}^3\,, \quad \nu \in \mathbb{R}\,.$$

The symbol $|v| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$ denotes here the length of the three-dimensional vector v. The **weighted $\mathbb{L}_p^{\langle\nu\rangle}$ spaces** are defined for $1 \le p \le \infty$ as follows

$$\mathbb{L}_p^{\langle\nu\rangle} = \mathbb{L}_p^{\langle\nu\rangle}(\mathbb{R}^3) = \left\{ g: \ \mathbb{R}^3 \to \mathbb{C}, \ \langle\cdot\rangle^\nu g \in \mathbb{L}_p \right\}.$$

The corresponding norm is

$$\left\| g \,|\, \mathbb{L}_p^{\langle\nu\rangle} \right\| = \| \langle\cdot\rangle^\nu g \,|\, \mathbb{L}_p \| \ .$$

The **classical Sobolev space** $\mathbb{W}_p^m = \mathbb{W}_p^m(\mathbb{R}^3)$ consists of functions with the following property

$$\mathbb{W}_p^m = \mathbb{W}_p^m(\mathbb{R}^3) = \left\{ g: \ \mathbb{R}^3 \to \mathbb{C}, \ \partial^\alpha g \in \mathbb{L}_p \ , \ \forall \alpha : \ |\alpha| \le m \right\}. \tag{15}$$

The norm in the Sobolev space $\mathbb{W}_p^m$ is defined as follows

$$\left\| g \,|\, \mathbb{W}_p^m \right\| = \left(\sum_{|\alpha| \le m} \| \partial^\alpha g \,|\, \mathbb{L}_p \|^p \right)^{1/p} . \tag{16}$$

The corresponding **weighted Sobolev space** $\mathbb{W}_p^{m,\langle\nu\rangle} = \mathbb{W}_p^{m,\langle\nu\rangle}(\mathbb{R}^3)$ is defined via

$$\mathbb{W}_p^{m,\langle\nu\rangle} = \mathbb{W}_p^{m,\langle\nu\rangle}(\mathbb{R}^3) = \left\{ g: \ \mathbb{R}^3 \to \mathbb{C}, \ \partial^\alpha(\langle\cdot\rangle^\nu g) \in \mathbb{L}_p \ , \ \forall \alpha : \ |\alpha| \le m \right\} \tag{17}$$

and has the norm

$$\left\| g \,|\, \mathbb{W}_p^{m,\langle\nu\rangle} \right\| = \left(\sum_{|\alpha| \le m} \| \partial^\alpha(\langle\cdot\rangle^\nu g) \,|\, \mathbb{L}_p \langle\nu\rangle \|^p \right)^{1/p} . \tag{18}$$

For the Sobolev spaces $\mathbb{W}_2^m$ and $\mathbb{W}_2^{m,\langle\nu\rangle}$ the notations $\mathbb{W}^m$ and $\mathbb{W}^{m,\langle\nu\rangle}$ are used respectively.

Remark 1 *The following norm in $\mathbb{W}_p^{m,\langle\nu\rangle}$ is equivalent to the norm (18):*

$$\left\| g \,|\, \mathbb{W}_p^{m,\langle\nu\rangle} \right\| = \left(\sum_{|\alpha| \le m} \| \langle\cdot\rangle^\nu \partial^\alpha g \,|\, \mathbb{L}_p \langle\nu\rangle \|^p \right)^{1/p} .$$

The **Schwartz space** $\mathbb{S} = \mathbb{S}(\mathbb{R}^3)$ of rapidly decreasing smooth test functions is defined as follows:

$$\mathbb{S} = \mathbb{S}(\mathbb{R}^3) = \Big\{ g \in \mathbb{C}^\infty(\mathbb{R}^3) : \ |\langle v\rangle^m \partial^\beta g(v)| \le C_{m,\beta} \Big\} ,$$

with arbitrary $m \in \mathbb{N}_0 \, , \beta \in \mathbb{N}_0^3 \, , \ v \in \mathbb{R}^3$ and with some positive constants $C_{m,\beta}$.

A sequence $\{g_n\}\, , \ n \in \mathbb{N}$ of functions from $\mathbb{S}$ is said to converge to zero ($g_n \to 0$) in $\mathbb{S}$ if for each compact set $\Omega \subset \mathbb{R}^3$, and for all $m \in \mathbb{N}_0 \, , \beta \in \mathbb{N}_0^3$ the sequences $\{\langle v\rangle^m \partial^\beta g_n\}\, , \ n \in \mathbb{N}$ converge to zero uniformly in Ω. The adjoint space $\mathbb{S}' = \mathbb{S}'(\mathbb{R}^3)$ is called the **space of tempered distributions**. If, for example, $\varphi \in \mathbb{C}(\mathbb{R}^3)$ is a continuous function with the property

$$\varphi(v) = \mathcal{O}(\langle v\rangle^a) \, , \quad |v| \to \infty$$

for some $a \in \mathbb{R}$, then φ defines a regular distribution over $\mathbb{S}$ as follows:

$$\langle g, \varphi\rangle = \int\limits_{\mathbb{R}^3} g(v)\, \overline{\varphi(v)}\, dv \, , \quad \forall g \in \mathbb{S} \, .$$

We will reserve the same notation even for a non–regular distribution $\varphi \in \mathbb{S}'$, bearing in mind the duality between the space of test functions and the space of distributions under the integral.

The **space** $\mathbb{C}_0^\infty = \mathbb{C}_0^\infty(\mathbb{R}^3)$ of smooth test functions with compact supports is a proper subset of $\mathbb{S}$ and its dual space of distributions $\mathbb{D} = \mathbb{D}(\mathbb{R}^3)$ contains the space of tempered distributions as a proper subset

$$\mathbb{C}_0^\infty(\mathbb{R}^3) \subset \mathbb{S}(\mathbb{R}^3) \subset \mathbb{S}'(\mathbb{R}^3) \subset \mathbb{D}'(\mathbb{R}^3) \, .$$

Let $\mu = m + \nu$, where $m = 0, 1, \dots$ is an integer and $0 \le \nu < 1$. The space of Hölder functions $\mathbb{C}^\mu$ is defined as follows

$$\mathbb{C}^\mu = \mathbb{C}^\mu(\mathbb{R}^3) = \Big\{ g \in \mathbb{C}(\mathbb{R}^3) \ : \ \|g \,|\, \mathbb{C}^\mu\| < \infty \Big\}$$

and is endowed with the norm

$$\|g \,|\, \mathbb{C}^\mu\| := \sum_{|\alpha| \le m} \sup_{x \in \mathbb{R}^3} |\partial^\alpha g(x)| + \sum_{|\alpha| = m} \|\partial^\alpha g \,|\, \mathbb{C}^\nu\| \, ,$$

$$\|\varphi \,|\, \mathbb{C}^\nu\| := \sup_{x \in \mathbb{R}^3} |\varphi(x)| + \sup_{\substack{x, h \in \mathbb{R}^3 \\ h \ne 0}} \frac{|\varphi(x+h) - \varphi(x)|}{|h|^\nu} \, .$$

1.2 The Fourier transform and further spaces

The three-dimensional Fourier transform of the function g is defined as

$$\hat{g}(\xi) = \mathcal{F}_{v\to\xi}[g(v)](\xi) = \int\limits_{\mathbb{R}^3} g(v) e^{\imath(v,\xi)}\, dv\,,$$

where (v, ξ) denotes the three-dimensional scalar product. The corresponding inverse Fourier transform is then

$$g(v) = \mathcal{F}^{-1}_{\xi\to v}[\hat{g}(\xi)](v) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \hat{g}(\xi) e^{-\imath(\xi,v)}\, d\xi\,.$$

The Fourier transform $\hat{g}$ exists, at least, for $g \in \mathbb{L}_1$. It is well known that the Schwartz space $\mathbb{S}$ is invariant under the Fourier transform $\mathcal{F}$ and under its inverse $\mathcal{F}^{-1}$

$$\mathcal{F}^{\pm 1}:\ \mathbb{S} \to \mathbb{S}\,. \tag{19}$$

The further properties of the Fourier transform are

$$\begin{aligned} \mathcal{F}_{v\to\xi}[\partial^\alpha g(v)](\xi) &= (-\imath\,\xi)^\alpha \mathcal{F}_{v\to\xi}[g](\xi)\,, \\ \partial^\alpha \mathcal{F}_{v\to\xi}[g(v)](\xi) &= (\imath)^{|\alpha|} \mathcal{F}_{v\to\xi}[v^\alpha g(v)](\xi)\,, \end{aligned} \tag{20}$$

which hold for the arbitrary test function $g \in \mathbb{S}$.
From the celebrated Plancherel equality

$$\langle f, g\rangle_{\mathbb{L}_2} = \int\limits_{\mathbb{R}^3} f(v)\, \overline{g(v)}\, dv = (2\pi)^{-3} \langle \hat{f}, \hat{g}\rangle_{\mathbb{L}_2} \tag{21}$$

which holds for every $f, g \in \mathbb{L}_2$ we obtain the well known Parseval identity

$$\| f \,|\, \mathbb{L}_2 \| = (2\pi)^{-3/2} \left\| \hat{f} \,|\, \mathbb{L}_2 \right\|\,. \tag{22}$$

Thus, the mappings $(2\pi)^{-3/2}\mathcal{F}$ and $(2\pi)^{3/2}\mathcal{F}^{-1}$ are isometrical isomorphisms in $\mathbb{L}_2$. The Fourier transform of a tempered distribution $\varphi \in \mathbb{S}'$ is given by the following definition

$$\langle g, \hat{\varphi}\rangle = \langle \hat{g}, \varphi\rangle\,,\ \forall g \in \mathbb{S}$$

and has the property

$$\mathcal{F}^{\pm 1}:\ \mathbb{S}' \to \mathbb{S}'\,.$$

With the help of the Fourier transform we define the **Bessel potential** space $\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbb{R}^3)$, $s \in \mathbb{R}$, $1 < p < \infty$ of the tempered distributions by

$$\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbb{R}^3) = \left\{ \varphi \in \mathbb{S}' \;:\; \left\|\varphi \,|\, \mathbb{H}^s_p\right\| < \infty \right\},$$

where the norm in $\mathbb{H}^s_p$ is defined as follows:

$$\left\|\varphi \,|\, \mathbb{H}^s_p\right\| := \left(\int\limits_{\mathbb{R}^3} \left|\mathcal{F}^{-1}_{\xi\to y} \left[\langle\xi\rangle^s \hat{\varphi}(\xi)\right](y)\right|^p \, dy \right)^{\frac{1}{p}} .$$

In a particular case $p = 2$, due to (22), the norm in the space $\mathbb{H}^s = \mathbb{H}^s_2(\mathbb{R}^3)$ acquires a simpler form

$$\|\varphi \,|\, \mathbb{H}^s\| = \left(\int\limits_{\mathbb{R}^3} \langle\xi\rangle^{2s} |\hat{\varphi}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} .$$

Finally, for all $s, \nu \in \mathbb{R}$ we define the **weighted Bessel potential** space $\mathbb{H}^{s,\langle\nu\rangle}_p = \mathbb{H}^{s,\langle\nu\rangle}_p(\mathbb{R}^3)$ (or $\mathbb{H}^{s,\langle\nu\rangle} = \mathbb{H}^{s,\langle\nu\rangle}_2(\mathbb{R}^3)$ when $p = 2$) via

$$\mathbb{H}^{s,\langle\nu\rangle}_p = \mathbb{H}^{s,\langle\nu\rangle}_p(\mathbb{R}^3) := \left\{\varphi \in \mathbb{S}' \;:\; \left\|\varphi \,|\, \mathbb{H}^{s,\langle\nu\rangle}_p\right\| := \left\|\langle\cdot\rangle^\nu \varphi \,|\, \mathbb{H}^s_p\right\| < \infty \right\}. \quad (23)$$

For an integer $s = m \in \mathbb{N}$ the Bessel potential spaces $\mathbb{H}^m_p$ and $\mathbb{H}^{s,\langle\nu\rangle}_p$ become the classical Sobolev spaces $\mathbb{W}^m_p$ and $\mathbb{H}^{s,\langle\nu\rangle}_p$ (see (15) and (17) respectively) with the equivalent norms (16) and (18) (see [12, § 2.5.6]).
The following embedding property of the weighted Bessel potential spaces is almost trivial:

$$\mathbb{H}^{s,\langle\nu\rangle}_p \subseteq \mathbb{H}^s_p \subseteq \mathbb{H}^{s,\langle\mu\rangle}_p, \quad \forall s \in \mathbb{R}, \quad \forall \nu \geq 0, \quad \forall \mu \leq 0, \quad \forall p \in [1, \infty),$$

while the next one is less trivial and is known as the Sobolev lemma (see [11, § 2.7.1]):

$$\mathbb{H}^{s+\frac{3}{p}}_p(\mathbb{R}^3) \subset \mathbb{C}^s(\mathbb{R}^3), \quad \forall s > 0, \quad \forall p \in [1, \infty). \quad (24)$$

Note that if $\mathbb{X}^*$ denotes the dual (adjoint) space to a Banach space $\mathbb{X}$, then

$$\left(\mathbb{H}^{s,\langle\nu\rangle}_p\right)^* = \mathbb{H}^{-s,\langle-\nu\rangle}_{p'}, \quad p' := \frac{p}{p-1} \quad (25)$$

and, in particular, $\left(\mathbb{H}^{s,\langle\nu\rangle}_2\right)^* = \mathbb{H}^{-s,\langle-\nu\rangle}_2$.

We will also apply the interpolation properties of the spaces $\mathbb{H}_p^{s,\langle\nu\rangle}$, which can be found in [11, § 2.4.2] for the case $\nu = 0$; the case of the weighted spaces $\nu \neq 0$ can easily be reduced to the case $\nu = 0$. To formulate a very short version of the interpolation property, we will use the notation $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ for the set of all linear bounded operators $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$. Then, in particular,

$$\mathcal{A} \in \mathcal{L}(\mathbb{H}_p^{s_0,\langle\nu\rangle}, \mathbb{H}_p^{s_0+r,\langle\nu\rangle}) \bigcap \mathcal{L}(\mathbb{H}_p^{s_1,\langle\nu\rangle}, \mathbb{H}_p^{s_1+r,\langle\nu\rangle}), \ r, s_0, s_1 \in \mathbb{R}, \ 1 < p < \infty$$

yields

$$\mathcal{A} \in \mathcal{L}(\mathbb{H}_p^{s,\langle\nu\rangle}, \mathbb{H}_p^{s+r,\langle\nu\rangle}), \quad \forall s \in [s_0, s_1] \tag{26}$$

(see [11, § 2.4.2] for this and much more general interpolation theorems).

1.3 Pseudodifferential operators

In this subsection we give some basic definitions and properties of pseudodifferential operators for our subsequent applications. For more details we refer the reader to [3],[7],[10].

For a symbol $a \in \mathbb{C}(\mathbb{R}^3, \mathbb{S}')$ the pseudodifferential operator

$$\mathcal{A} = \mathcal{A}(v, D) : \ \mathbb{S} \to \mathbb{S}'$$

is defined as follows

$$\mathcal{A}(v, D)[g](v) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} a(v, \xi) \mathcal{F}_{z\to\xi}[g(z)](\xi) e^{-\imath(\xi, v)} \, d\xi, \ \forall g \in \mathbb{S}. \tag{27}$$

The definition (27) is correct because of the following facts. The function $\hat{g} = \mathcal{F}[g] \in \mathbb{S}$ corresponding to (19) and $a(w, \cdot)\hat{g} \in \mathbb{S}'$ for all $w \in \mathbb{R}^3$ since the dependence of the symbol a on w is continuous. The inverse Fourier transform $b(w, v) = \mathcal{F}^{-1}[a(w, \cdot)\hat{g}] \in \mathbb{S}'$ is therefore a well defined distribution for all values of the parameter w and depends on w continuously in the topology of the Freshet space $\mathbb{S}'$. Therefore,

$$\chi(v) b(v, v) \in \mathbb{S}', \quad \forall \chi \in \mathbb{C}_0^\infty$$

i.e. $b(v, v) \in \mathbb{S}'_{loc} \subset \mathbb{D}'$ and therefore the operator $\mathcal{A}$ is continuous $\mathcal{A}(v, D) : \mathbb{S} \to \mathbb{D}'$. As an immediate consequence of the isomorphism (21) we get: a pseudodifferential operator

$$\mathcal{A}(v, D) : \ \mathbb{L}_2 \to \mathbb{L}_2$$

is bounded, provided the symbol a is uniformly bounded

$$|a(v, \xi)| \leq C < \infty, \quad \forall v, \xi \in \mathbb{R}^3.$$

The main result we will need later is

Theorem 2 *Let* $m \in \mathbb{N}_0,\ r, \nu \in \mathbb{R}$ *and assume*

$$|\partial_v^\alpha a(v,\xi)| \le C_\alpha \langle v\rangle^{-\nu}\langle \xi\rangle^r\,, \quad \forall \alpha \in \mathbb{N}_0^3\,, \quad |\alpha| \le m\,. \tag{28}$$

Then the pseudodifferential operators

$$\begin{aligned} \mathcal{A} &: \quad \mathbb{H}^s \to \mathbb{H}^{s-r,\langle\nu\rangle}\,, \\ \mathcal{A} &: \quad \mathbb{H}^{-s+r,\langle-\nu\rangle} \to \mathbb{H}^{-s} \end{aligned} \tag{29}$$

are bounded for all $0 \le s \le m + r$.

Proof. According to the definition (23) of the weighted Bessel potential space $\mathbb{H}^{s,\langle\nu\rangle}$ the first boundedness in (29) is equivalent to the boundedness

$$\mathcal{A}^{\langle\nu\rangle} :\ \mathbb{H}^s \quad\to\quad \mathbb{H}^{s-r}\,,$$

where the symbol of the operator $\mathcal{A}^{\langle\nu\rangle}$ is $\langle v\rangle^\nu a(v,\xi)$. This symbol obviously satisfies the estimate (28) with $\nu = 0$

$$|\partial_v^\alpha(\langle v\rangle^\nu a(v,\xi))| \le C_\alpha\langle\xi\rangle^r\,, \quad \forall\alpha \in \mathbb{N}_0^3\,, \quad |\alpha| \le m\,.$$

Thus, we can suppose $\nu = 0$. First, we take $s = k \in \mathbb{N}_0\,, k \le m$. Using the formula (13) we obtain from the definition (27) and using (20) for $|\alpha| \le k$

$$\partial^\alpha \mathcal{A}(v,D)[g](v) = \sum_{\beta\preceq\alpha} \left(\begin{array}{c}\alpha\\ \beta\end{array}\right) \mathcal{A}^{(\beta)}(v,D)\left[\partial^{\alpha-\beta}g\right](v)\,,$$

where the operator $\mathcal{A}^{(\beta)}(v,D)$ has the symbol $\partial_v^\beta a(v,\xi)$. With the equivalent norm (16) we then obtain the boundedness

$$\mathcal{A}(v,D) :\ \mathbb{H}^{k+r} \to \mathbb{H}^k\,, \quad k = 0,1,\ldots,m\,. \tag{30}$$

From (30), by the interpolation (26), we get the boundedness

$$\mathcal{A}(v,D) :\ \mathbb{H}^s \to \mathbb{H}^{s-r}\,, \quad 0 \le s-r \le m \tag{31}$$

i.e. the first statement in (29). Now we can prove the second statement in (29) by the duality arguments (cf. (25)).
This proof is based on the boundedness of the adjoint operator

$$\mathcal{A}^*(v,D) :\ \mathbb{H}^s \to \mathbb{H}^{s-r,\langle\nu\rangle}\,, \quad 0 \le s-r \le m\,. \tag{32}$$

The definition of the adjoint operator $\mathcal{A}^*(v,D)$ is as follows:

$$\mathcal{A}^*(v,D)[g](v) = \mathcal{F}^{-1}_{\xi\to v}\left[\int_{\mathbb{R}^3} \overline{a(w,\xi)}g(w)e^{\imath(w,\xi)}\,dw\right](v)\,, \quad \forall g \in \mathbb{S}\,. \tag{33}$$

Now the boundedness (32) can be proved similarly to (30),(31) provided $0 \leq -s \leq m$. Since $\mathcal{A}(v,D)$ is, in its turn, adjoint to $\mathcal{A}^*(v,D)$, the operator should be bounded in the dual pair of spaces

$$\mathcal{A}(v,D):\ \mathbb{H}^{-s+r,\langle -\nu\rangle} = \left(\mathbb{H}^{s-r,\langle\nu\rangle}\right)^* \to \mathbb{H}^{-s} = \left(\mathbb{H}^{s}\right)^* .$$

■

Remark 3 *One can define more general pseudodifferential operators which contain both operators (27) and (33) as particular cases (i.e. the algebra of such operators is closed with respect to the duality). For this we take*

$$\mathcal{A}(v,D)[g](v) = \mathcal{F}^{-1}_{\xi\to v}\left[\int_{\mathbb{R}^3} \overline{a(v,w,\xi)}g(w)e^{\imath(w,\xi)}\,dw\right](v)\,, \quad \forall g \in \mathbb{S}\,.$$

If the symbol $a(v,w,\xi)$ is chosen rigorously, than $\mathcal{A}(v,D) = \mathcal{A}_0(v,D) + \mathcal{T}$, where $\mathcal{A}_0(v,D)$ is the pseudodifferential operator defined in (27) with the symbol $a_0(v,\xi) := a(v,v,\xi)$ and $\mathcal{T} : \mathbb{H}^s_p \to \mathbb{S}$ is a smoothing operator of the order $-\infty$ (see [7],[10]).

2 The gain part of the collision integral

Let us first mention here the equality which we apply to prove the boundedness (12) and which is, probably, of interest in its own right.

Lemma 4 *An arbitrary partial derivative of the functions $Q_\pm(f,g)(v)$ can be represented as*

$$\partial^\alpha Q_\pm(f,g)(v) = \sum_{\beta\preceq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} Q_\pm(\partial^\beta f, \partial^{\alpha-\beta} g)(v)\,. \tag{34}$$

Proof. By direct calculation we get

$$\begin{aligned}
&\frac{\partial}{\partial v_j}\left(Q_+(f,g)(v)\right)\\
&\quad= \int_{\mathbb{R}^3}\int_{S^2}\left(\left(\frac{\partial}{\partial v_j}B(|u|,\mu)\right)f(v')g(w') + B(|u|,\mu)\frac{\partial}{\partial v_j}\Big(f(v')g(w')\Big)\right)\,de\,dw\\
&\quad= \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu)\left(\frac{\partial}{\partial v_j} + \frac{\partial}{\partial w_j}\right) f(v')g(w')\,de\,dw\,.
\end{aligned}$$

Here we have used the convolution property of the collision kernel

$$\frac{\partial}{\partial v_j} B(|u|,\mu) = \frac{\partial}{\partial v_j} B\left(|v-w|, \frac{(v-w,e)}{|v-w|}\right) = -\frac{\partial}{\partial w_j} B(|u|,\mu)$$

and the integration by parts which transports the derivative from the kernel $B(|u|,\mu)$ to the product $f(v')g(w')$. Now we compute the sum of the partial derivatives as follows:

$$\left(\frac{\partial}{\partial v_j} + \frac{\partial}{\partial w_j}\right) f(v')g(w') = \left((\operatorname{grad} f)(v'), \frac{\partial v'}{\partial v_j} + \frac{\partial v'}{\partial w_j}\right) g(w')$$
$$+f(v')\left((\operatorname{grad} g)(w'), \frac{\partial w'}{\partial v_j} + \frac{\partial w'}{\partial w_j}\right) = \frac{\partial f}{\partial v_j}(v')g(w') + f(v')\frac{\partial g}{\partial v_j}(w'),$$

where the obvious identity

$$\frac{\partial v'}{\partial v_j} + \frac{\partial v'}{\partial w_j} = \frac{\partial w'}{\partial v_j} + \frac{\partial w'}{\partial w_j} = e_j$$

has been used; e_j denotes here the $j-$th column of the three-dimensional identity matrix. Thus, we have proved (34) for the case $\alpha = e_j$

$$\partial^{e_j} Q_+(f,g)(v) = \begin{pmatrix} e_j \\ e_j \end{pmatrix} Q_+(\partial^{e_j} f, g) + \begin{pmatrix} e_j \\ 0 \end{pmatrix} Q_+(f, \partial^{e_j} g),$$

where

$$\partial^\alpha = \partial^{e_j} = \frac{\partial}{\partial v_j}, \quad j = 1,2,3.$$

The general case can then be concluded by induction with respect to $|\alpha|$. The operator Q_- in (6) can be written as follows:

$$Q_-(f,g)(v) = f(v)\,\mathcal{B}[g](v)\,, \tag{35}$$

where the linear integral operator

$$\mathcal{B}[g](v) = \int_{\mathbb{R}^3} B_{tot}(|u|)\, g(w) dw$$

is of the convolution type and has the kernel

$$B_{tot}(|u|) = \int_{S^2} B(|u|,\mu)\, de\,.$$

Thus, by using the convolution property

$$\frac{\partial}{\partial v_j} B_{tot}(|u|) = -\frac{\partial}{\partial w_j} B_{tot}(|u|) \tag{36}$$

and the integration by parts, we get

$$\begin{aligned} \partial^{e_j} Q_-(f,g)(v) &= (\partial^{e_j} f(v))\, \mathcal{B}[g](v) + f(v)\, (\partial^{e_j} \mathcal{B}[g](v)) \\ &= (\partial^{e_j} f(v))\, \mathcal{B}[g](v) + f(v)\, \mathcal{B}\left[\partial^{e_j} g\right](v) \\ &= \begin{pmatrix} e_j \\ e_j \end{pmatrix} Q_-(\partial^{e_j} f, g) + \begin{pmatrix} e_j \\ 0 \end{pmatrix} Q_-(f, \partial^{e_j} g)\,. \end{aligned}$$

The proof of the lemma can be accomplished as in the foregoing case by induction. ■

Corollary 5 *The following important boundedness properties hold:*

$$Q_+\,,\ Q_-\,,\ Q\ :\ \mathbb{S} \times \mathbb{S} \to \mathbb{S}\,. \tag{37}$$

Proof. Since the proofs for the operators Q_+, Q_- and Q are verbatim the same, we will consider only the operator Q_+.
Due to the definition (18) and to the property (34) we get

$$\begin{aligned} \left\| Q_+(f)[g] \,|\, \mathbb{W}_2^{m+1,\langle\nu\rangle} \right\|^2 &= \sum_{|\alpha|\le m+1} \left\| \partial^\alpha Q_+(f)[g] \,|\, \mathbb{L}_2^{\langle\nu\rangle} \right\|^2 \\ &= \sum_{|\alpha|\le m+1} \left\| \sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Q_+(\partial^\beta f)[\partial^{\alpha-\beta} g] \,\middle|\, \mathbb{L}_2^{\langle\nu\rangle} \right\|^2 \\ &\le \sum_{|\alpha|\le m+1} \left(\sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \left\| Q_+(\partial^\beta f)[\partial^{\alpha-\beta} g] \,|\, \mathbb{L}_2^{\langle\nu\rangle} \right\| \right)^2 . \end{aligned}$$

Now using the boundedness (8) we obtain the estimate

$$\begin{aligned} \left\| Q_+(f)[g] \,|\, \mathbb{W}_2^{m+1,\langle\nu\rangle} \right\|^2 &\le C \sum_{|\alpha|,|\beta|\le m} \left\| \partial^\beta f \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle} \right\|^2 \left\| \partial^\alpha g \,|\, \mathbb{L}_2^{\langle\nu+\lambda\rangle} \right\|^2 \\ &= C \left\| f \,|\, \mathbb{W}_1^{m,\langle\nu+\lambda\rangle} \right\|^2 \left\| g \,|\, \mathbb{W}_2^{m,\langle\nu+\lambda\rangle} \right\|^2 \end{aligned} \tag{38}$$

for some positive constant C. (38) can be interpreted as the following boundedness property

$$Q_+\ :\ \mathbb{W}_1^{m,\langle\nu+\lambda\rangle} \times \mathbb{W}_2^{m,\langle\nu+\lambda\rangle} \longrightarrow \mathbb{W}_2^{m+1,\langle\nu\rangle}\,, \quad \forall m \in \mathbb{N}_0,\,, \forall \nu \in \mathbb{R}_+\,.$$

Due to the Sobolev embedding lemma (see (24))

$$\begin{aligned} \mathbb{W}_2^{m,\langle\nu\rangle} &\subset \mathbb{C}^{\ell,\langle\nu\rangle}, \ \ell = m - \frac{3}{2}, \\ \mathbb{W}_1^{m,\langle\nu\rangle} &\subset \mathbb{C}^{\ell,\langle\nu\rangle}, \ \ell = m - 3, \end{aligned} \tag{39}$$

i.e.

$$\varphi \in \mathbb{W}_2^{m,\langle\nu\rangle} \Longrightarrow \langle v\rangle^\nu \varphi \in \mathbb{C}^{m-\ell,\langle\nu\rangle} \Longleftrightarrow \varphi \in \mathbb{C}^{m-\ell}.$$

Therefore,

$$\bigcap_{m\in\mathbb{N}_0,\,\nu\in\mathbb{R}_+} \mathbb{W}_2^{m,\langle\nu\rangle} = \bigcap_{\ell\in\mathbb{N}_0,\,\nu\in\mathbb{R}_+} \mathbb{C}^{\ell,\langle\nu\rangle} = \mathbb{S}$$

and from (39) we get the required boundedness (37). ∎

Lemma 6 *Let $G \in \mathbb{S}(\mathbb{R}^3) \times \mathbb{S}(\mathbb{R}^3) \times \mathbb{C}(S^2)$ and $h \in \mathbb{S}(\mathbb{R}^3)$ be given test functions. Then the following formula holds:*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} G(v,w,e)\overline{h(v')}\,de\,dw\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} G(v',w',e')\overline{h(v)}\,de\,dw\,dv \tag{40}$$

with

$$v' = \frac{1}{2}\Big(v+w+|v-w|e\Big),\ w' = \frac{1}{2}\Big(v+w-|v-w|e\Big),\ e' = \frac{v-w}{|v-w|}. \tag{41}$$

Proof. Using the change of variables

$$U = \frac{1}{2}(v+w),\ u = v-w,\ dw\,dv = dU\,du$$

we get for the right integral in (40)

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} G\Big(U+\frac{1}{2}|u|e, U-\frac{1}{2}|u|e, e\Big)\,\overline{h\Big(U+\frac{1}{2}u\Big)}\,de\,dU\,du.$$

Switching to the spherical coordinates

$$u = r\,\tilde e,\ r\in[0,\infty),\ \tilde e\in S^2,\ du = r^2dr\,d\tilde e$$

leads to

$$\int_0^\infty r^2 \int_{S^2}\int_{\mathbb{R}^3}\int_{S^2} G\Big(U+\frac{1}{2}re, U-\frac{1}{2}re, e\Big)\,\overline{h\Big(U+\frac{1}{2}r\tilde e\Big)}\,dedUd\tilde edr.$$

Putting r and e together to the new three-dimensional variable $\tilde{u} = re$ we now get

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} G\Big(U + \frac{1}{2}\tilde{u}, U - \frac{1}{2}\tilde{u}, \frac{\tilde{u}}{|\tilde{u}|}\Big)\, \overline{h\Big(U + \frac{1}{2}|\tilde{u}|\tilde{e}\Big)}\, d\tilde{e}\, dU\, d\tilde{u}\,.$$

Substituting $\tilde{u} = v - w\,,\ U = (v+w)/2$ we obtain the identity (40). ∎

Corollary 7 *Using the equation* (40) *it is possible to define the distribution* $h(v') \in (\mathbb{S}(\mathbb{R}^3) \times \mathbb{S}(\mathbb{R}^3) \times \mathbb{C}(S^2))'$ *for an arbitrary distribution* $h \in (\mathbb{S}(\mathbb{R}^3))'$ *by*

$$\langle G(v,w,e)\,,\, h(v')\rangle = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} \langle G(v',w',e')\,,\, h(v)\rangle\, dw\, de$$

with v', w' *and* e' *defined in* (41).

From (40) we immediately derive two known identities for the bilinear operator $Q_+(f,g)$

$$\langle Q_+(f,g)\,,\, h\rangle = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu) f(v)\, g(w)\, \overline{h(v')}\, de\, dw\, dv \tag{42}$$

and

$$\langle Q_+(f,g)\,,\, h\rangle = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu) f(w)\, g(v)\, \overline{h(w')}\, de\, dv\, dw\,. \tag{43}$$

The second formula is obtained by exchanging $v \leftrightarrow w$ and $e \leftrightarrow -e$ in (42). In order to study the mapping properties of the linear operator $Q_+(f)$ we find the explicit form of the adjoint operator $Q_+^*(f)$.

Lemma 8 *The adjoint operator* $Q_+^*(f)$ *to* $Q_+(f)$ *defined in (5) is a pseudodifferential operator*

$$Q_+^*(f)[g](v) = \frac{1}{(2\,\pi)^3}\int\limits_{\mathbb{R}^3} q_+^*(f;\, v,\xi)\mathcal{F}_{z\to\xi}[g(z)](\xi) e^{-\imath(\xi,\, v)}\, d\xi$$

with the symbol

$$q_+^*(f;\, v,\xi) = \int\limits_{\mathbb{R}^3} \overline{f(v-u)} b(u,\xi)\, du\,, \tag{44}$$

$$b(u,\xi) = e^{\imath\frac{1}{2}(u,\,\xi)} \int\limits_{S^2} B(|u|,\mu)\, e^{\imath\frac{1}{2}|u|(e,\,\xi)} de\,,$$

where B *is the collision kernel.*

Proof. Using the identity (43) we obtain

$$\langle Q_+(f)[g], h\rangle_{\mathbb{L}_2} = \langle g, Q_+^*(f)[h]\rangle_{\mathbb{L}_2}$$

with

$$Q_+^*(f)[h](v) = \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu)\overline{f(w)}\, h(w')\, de\, dw\,. \tag{45}$$

Using the Dirac $\delta-$distribution, $\delta \in \mathbb{S}'$, we rewrite (45) as follows

$$Q_+^*(f)[h](v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} B(|u_z|,\mu_z)\overline{f(w)}\, h(w_z')\delta(z-v)\, de\, dw\, dz\,,$$

where the notation

$$v_z' = \frac{1}{2}\Big(z + w + |z-w|\, e\Big)\,,\ w_z' = \frac{1}{2}\Big(z + w - |z-w|\, e\Big)\,,\ u_z = z - w$$

has been used. Using the property (43) in the opposite direction we get

$$Q_+^*(f)[h](v) = \int_{\mathbb{R}^3} h(w)\left[\int_{\mathbb{R}^3}\int_{S^2} B(|u_z|,\mu_z)\overline{f(w_z')}\,\delta(v_z'-v)\, de\, dz\right] dw$$

and therefore with (21)

$$Q_+^*(f)[h](v) = \langle h, q(v,\cdot)\rangle_{\mathbb{L}_2} = (2\pi)^{-3}\langle \hat{h}, \hat{q}(v,\cdot)\rangle_{\mathbb{L}_2}\,. \tag{46}$$

The function $\hat{q}$ in (46) can be evaluated in the following way

$$\begin{aligned}
\hat{q}(v,\xi) &= \mathcal{F}_{w\to\xi}[q(v,w)](\xi) \\
&= \int_{\mathbb{R}^3} e^{\imath(w,\,\xi)}\left[\int_{\mathbb{R}^3}\int_{S^2} B(|u_z|,\mu_z) f(w_z')\,\delta(v_z'-v)\, de\, dz\right] dw \\
&= \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu) f(w)\, e^{\imath(w',\,\xi)}\, de\, dw \\
&= e^{\imath(v,\,\xi)}\int_{\mathbb{R}^3} f(v-u)e^{-\imath\frac{1}{2}(u,\,\xi)}\int_{S^2} B(|u|,\mu)\, e^{-\imath\frac{1}{2}|u|(e,\,\xi)}\, de\, dw\,.
\end{aligned}$$

In the concluding step we have used the substitution $w = v - u\,,\ dw = du$ and the identity (43) for the third time before removing the $\delta-$distribution. The final result reads as follows

$$\hat{q}(v,\xi) = e^{\imath(v,\,\xi)}\overline{q_+^*(v,\xi)}\,,$$

where the symbol $q_+^*(v,\xi)$ is defined in (44). The operator $Q_+^*(f)$ is then

$$\begin{aligned} Q_+^*(f)[h](v) &= (2\pi)^{-3}\langle \hat{h}, \hat{q}(v,\cdot)\rangle_{\mathbb{L}_2} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{h}(\xi)\overline{\hat{q}(v,\xi)}\, d\xi \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} q_+^*(f;v,\xi)\,\hat{h}(\xi)\, e^{-\imath(v,\xi)} d\xi \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} q_+^*(f;v,\xi)\mathcal{F}_{w\to\xi}[h(w)](\xi) e^{-\imath(v,\xi)} d\xi \qquad (47) \end{aligned}$$

and, therefore, Q_+ is a pseudodifferential operator (cf. (27)) with the symbol (44). The proof is herewith accomplished. ∎

Remark 9 *The property* (34) *is also valid for the operator* $Q_+^*(f)$. *This can be checked directly using the representation* (47).

$$\begin{aligned} &\partial^\alpha Q_+^*(f)[h](v) \\ &= \frac{1}{(2\pi)^3}\sum_{\beta\preceq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^3}\partial^\beta\left(q_+^*(f;v,\xi)\right)\mathcal{F}_{w\to\xi}[h(w)](\xi)\partial^{\alpha-\beta}\left(e^{-\imath(v,\xi)}\right) d\xi \\ &= \frac{1}{(2\pi)^3}\sum_{\beta\preceq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^3} q_+^*(\partial^\beta f;v,\xi)(-\imath\,\xi)^{\alpha-\beta}\mathcal{F}_{w\to\xi}[h(w)](\xi) e^{-\imath(v,\xi)} d\xi \\ &\sum_{\beta\preceq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} Q_+^*(\partial^\beta f)[\partial^{\alpha-\beta}h](v) \qquad (48) \end{aligned}$$

We have used the formulae (44) *and* (20).

Remark 10 *For the VHS model (4) the symbol* $q_+^*(f;v,\xi)$ *can be written more explicitly*

$$q_+^*(f;v,\xi) = 4\pi\, C_\lambda \int_{\mathbb{R}^3} \overline{f(v-u)}|u|^\lambda e^{\imath\frac{1}{2}(u,\xi)}\mathrm{sinc}\left(\frac{1}{2}|u||\xi|\right) du\,. \qquad (49)$$

In fact, the integral over the unit sphere in (44) can be now evaluated analytically

$$\int_{S^2} e^{\imath\frac{1}{2}|u|(e,\xi)} B(|u|,\mu)\, de = 4\pi C_\lambda |u|^\lambda \mathrm{sinc}\left(\frac{1}{2}|u||\xi|\right),$$

where the notation

$$\mathrm{sinc}(y) = \frac{\sin(y)}{y} \qquad (50)$$

has been used.

The main result for the VHS model is the following.

Theorem 11 *Let*

$$0 \le \lambda \le 1\,,\ p > \frac{3}{2+\lambda} \ \textit{and}\ f \in \mathbb{L}_p \cap \mathbb{L}_1^{\langle\lambda\rangle}\,. \tag{51}$$

Then, for the VHS model, the operators

$$\begin{aligned} Q_+^*(f) &: \quad \mathbb{H}^{-1} \to \mathbb{L}_2^{\langle -\lambda\rangle}\,, \\ Q_+(f) &: \quad \mathbb{L}_2^{\langle\lambda\rangle} \to \mathbb{H}^1 \end{aligned}$$

are bounded.

Proof. Due to Theorem 2 it is sufficient to have the estimate

$$|q_+^*(f;v,\xi)| \le C\left(\left\|f\,|\,\mathbb{L}_1^{\langle\lambda\rangle}\right\| + \|f\,|\,\mathbb{L}_p)\|\right)\langle v\rangle^{\lambda}\langle\xi\rangle^{-1}, \tag{52}$$

uniform with respect to $v,\xi \in \mathbb{R}^3$. For $|\xi| \le 1$ we apply the inequalities

$$1 \le \frac{\sqrt{2}}{(1+|\xi|^2)^{1/2}} = \sqrt{2}\,\langle\xi\rangle^{-1}\,, \quad |\mathrm{sinc}(y)| \le 1\,, \quad \forall y \in \mathbb{R} \tag{53}$$

to (49) and obtain the estimate

$$\begin{aligned} |q_+^*(f;v,\xi)| &\le 2^{5/2}\pi\, C_\lambda \left(\int_{\mathbb{R}^3} |f(v-u)||u|^{\lambda}\,du\right)\langle\xi\rangle^{-1} \\ &= 2^{5/2}\pi\, C_\lambda \left(\int_{\mathbb{R}^3} |f(w)||v-w|^{\lambda}\,dw\right)\langle\xi\rangle^{-1}\,. \end{aligned} \tag{54}$$

With the rough estimate $|v-w|^2 \le 2\langle v\rangle^2\langle w\rangle^2$ we obtain for $\lambda \ge 0$

$$|v-w|^{\lambda} \le 2^{\lambda/2}\langle v\rangle^{\lambda}\langle w\rangle^{\lambda} \tag{55}$$

and, therefore,

$$\begin{aligned} |q_+^*(f;v,\xi)| &\le 2^{(5+\lambda)/2}\pi\, C_\lambda \left(\int_{\mathbb{R}^3} |f(w)|\langle w\rangle^{\lambda}\,dw\right)\langle v\rangle^{\lambda}\langle\xi\rangle^{-1} \\ &= C_1\left\|f\,|\,\mathbb{L}_1^{\langle\lambda\rangle}\right\|\langle v\rangle^{\lambda}\langle\xi\rangle^{-1} \le C_1\left(\|f\,|\,\mathbb{L}_p\| + \left\|f\,|\,\mathbb{L}_1^{\langle\lambda\rangle}\right\|\right)\langle v\rangle^{\lambda}\langle\xi\rangle^{-1}\,. \end{aligned}$$

Now let $|\xi| \geq 1$. Then $|\xi|^{-1} \leq \sqrt{2}\,\langle\xi\rangle^{-1}$ and we get the following (cf. (49), (50)):

$$\left|q_+^*(f;v,\xi)\right| = \left|8\pi\, C_\lambda |\xi|^{-1} \int\limits_{\mathbb{R}^3} \overline{f(v-u)}|u|^{\lambda-1} e^{\imath\frac{1}{2}(u,\xi)} \sin\left(\frac{1}{2}|u||\xi|\right) du\right|$$

$$\leq 2^{7/2}\pi\, C_\lambda \left(\int\limits_{u\in\mathbb{R}^3} |f(v-u)|\,|u|^{\lambda-1}\, du\right) \langle\xi\rangle^{-1}. \tag{56}$$

Since $\lambda - 1 \leq 0$ the integrand in (56) has a singularity at $u = 0$. Therefore we decompose this integral into two parts

$$\int\limits_{|u|\leq 1} |f(v-u)|\,|u|^{\lambda-1}\, du + \int\limits_{|u|\geq 1} |f(v-u)|\,|u|^{\lambda-1}\, du\,. \tag{57}$$

and estimate each of them separately. Using the Hölder inequality we obtain the estimate for the first integral in (57)

$$\int\limits_{|u|\leq 1} |f(v-u)||u|^{\lambda-1} du \leq \left(\int\limits_{|u|\leq 1} |f(v-u)|^p du\right)^{1/p} \left(\int\limits_{|u|\leq 1} |u|^{p'(\lambda-1)} du\right)^{1/p'}$$

$$= C_{p,\lambda}\, \|f \mid \mathbb{L}_p(B_1(v))\| \leq C_{p,\lambda}\, \left\|f \mid \mathbb{L}_p(\mathbb{R}^3)\right\|\,. \tag{58}$$

The constant $C_{p,\lambda}$ in (58) is well-defined due to condition $p'(\lambda - 1) > -3$ (cf. (51)):

$$C_{p,\lambda} = \left(\int\limits_{|u|\leq 1} |u|^{p'(\lambda-1)}\, du\right)^{1/p'} = \left(\frac{4\pi}{p'(\lambda-1)+3}\right)^{1/p'}.$$

Using the inequality $1 \leq \langle v\rangle^\lambda$, which obviously holds for $\lambda \geq 0$, we obtain the estimate for the first integral in (57)

$$\int\limits_{|u|\leq 1} |f(v-u)||u|^{\lambda-1}\, du \leq C_{p,\lambda}\, \|f \mid \mathbb{L}_p\|\, \langle v\rangle^\lambda\,.$$

For the estimation of the second integral in (57) we apply (55) and proceed as follows:

$$\int\limits_{|u|\geq 1} |f(v-u)||u|^{\lambda-1}\, du \leq \int\limits_{|u|\geq 1} |f(v-u)||u|^{\lambda}\, du$$

$$\leq \int_{\mathbb{R}^3} |f(w)||v-w|^\lambda \, dw \leq 2^{\lambda/2} \left\| f \mid \mathbb{L}_1^{\langle\lambda\rangle} \right\| \langle v\rangle^\lambda .$$

Thus, for $|\xi| \geq 1$ we get

$$|q_+^*(f;v,\xi)| \leq C_2 \left(\|f \mid \mathbb{L}_p\| + \left\| f \mid \mathbb{L}_1^{\langle\lambda\rangle} \right\| \right) \langle v\rangle^\lambda \langle\xi\rangle^{-1}$$

with $C_2 = 2^{7/2}\pi C_\lambda \max(C_{p,\lambda}, 2^{\lambda/2})$ and the estimate (52) holds for all $\xi \in \mathbb{R}^3$ with $C = \max(C_1, C_2)$. ∎

Corollary 12 *The estimate of the symbol obtained in the foregoing lemma can be extended to the collision kernels of the inverse power potential typ (cf. (3))*

$$B(|u|,\mu) = |u|^\lambda g_\lambda(\mu) \, , \;\; \lambda = 1 - \frac{4}{m} \tag{59}$$

having the additional property

$$g_\lambda \in \mathbb{H}_1^a([-1,1]) \, , \;\; a > 1 \, . \tag{60}$$

Proof. Due to the proof of the above lemma it suffices to show the estimates (54) for $|\xi| \leq 1$ and (56) for $|\xi| \geq 1$. The Sobolev lemma reads in the one-dimensional case as follows (cf. (24))

$$\mathbb{H}_p^{s+\frac{1}{p}}(\mathbb{R}) \subset \mathbb{C}^s(\mathbb{R}) \, , \;\; s > 0 \, , \;\; p \in [1,\infty) \, .$$

Thus the condition (60) means that the function g_λ is continuous on $[-1,1]$ and therefore $|g_\lambda(\mu)| \leq C_0$. Assuming $|\xi| \leq 1$ and using (53) from the definition of the symbol q_+^* in (49) we immediately obtain the estimate

$$|q_+^*(f;v,\xi)| \leq 2^{5/2}\pi \, C_0 \left(\int_{\mathbb{R}^3} |f(w)||v-w|^\lambda \, dw \right) \langle\xi\rangle^{-1} .$$

The estimate for $|\xi| \geq 1$ is a little more delicate. The main problem is to evaluate the integral over the unit sphere

$$\int_{S^2} g_\lambda(\mu) e^{\imath\frac{1}{2}|u|(e,\xi)} de \, , \;\; \mu = \frac{(u,e)}{|u|} \, . \tag{61}$$

We use the following parametrisation of the unit sphere in (61)

$$e = Q \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix} , \;\; 0 \leq \varphi < 2\pi \, , \;\; 0 \leq \theta \leq \pi \, , \;\; de = \sin\theta \, d\varphi \, d\theta \, .$$

where the orthogonal matrix

$$Q = \left(\frac{(\xi \times u) \times \xi}{|\xi \times u|\,|\xi|} \,\vdots\, \frac{\xi \times u}{|\xi \times u|} \,\vdots\, \frac{\xi}{|\xi|} \right)$$

is written columnwise. Thus we get

$$Q^T \xi = |\xi| \Big(0, 0, 1\Big)^T, \; Q^T u = \frac{1}{|\xi|} \Big(|\xi \times u|, 0, (\xi, u) \Big)^T$$

and

$$\mu = \mu(\varphi, \theta) = \frac{(u, e)}{|u|} = \frac{1}{|\xi|\,|u|} \Big(|\xi \times u| \cos\varphi \, \sin\theta + (\xi, u) \cos\theta \Big) .$$

The integral (61) transforms into

$$\int\limits_0^{2\pi} \int\limits_0^{\pi} g_\lambda \Big(\mu(\varphi, \theta) \Big) e^{\imath \frac{1}{2} |u|\,|\xi| \cos\theta} \sin\theta \, d\varphi \, d\theta .$$

Integrating by parts with respect to θ leads to

$$\int\limits_0^{\pi} g_\lambda \Big(\mu(\varphi, \theta) \Big) e^{\imath \frac{1}{2} |u|\,|\xi| \cos\theta} \sin\theta \, d\theta = \frac{2\imath}{|u|\,|\xi|} \left[g_\lambda(\mu) e^{\imath \frac{1}{2} |u|\,|\xi| \cos\theta} \right]_0^{\pi}$$

$$- \frac{2\imath}{|u|\,|\xi|} \int\limits_0^{\pi} g_\lambda' \Big(\mu(\varphi, \theta) \Big) \frac{\partial \mu(\varphi, \theta)}{\partial \theta} e^{\imath \frac{1}{2} |u|\,|\xi| \cos\theta} \, d\theta .$$

Using the inequalities (cf. (60))

$$\Big| g_\lambda \Big(\mu(\varphi, \theta) \Big) \Big| \,|\, \leq C_0 \, , \quad \int_0^{\pi} \Big| g_\lambda' \Big(\mu(\varphi, \theta) \Big) \Big| \, d\theta \leq C_1 \, , \quad \left| \frac{\partial \mu(\varphi, \theta)}{\partial \theta} \right| \leq 2$$

we get the estimate

$$\left| \int\limits_{S^2} g_\lambda(\mu) e^{\imath \frac{1}{2} |u| (e, \xi)} de \right| \leq \frac{C}{|u|\,|\xi|} \, , \; C = 2\pi(4C_0 + 4\pi C_1)$$

and therefore (cf. (56))

$$|q_+^*(f; v, \xi)| \leq \sqrt{2}\, C \left(\int\limits_{\mathbb{R}^3} |f(w)| |v - w|^{\lambda - 1} \, dw \right) \langle \xi \rangle^{-1} .$$

The proof is completed. ∎

Now we can prove the main result of the paper.

Theorem 13 *Let the collision kernel be of the form (4) or (59) and*

$$0 \le \lambda \le 1\,, \quad p > \frac{3}{2+\lambda}\,, \quad \theta \ge 0\,, \quad |s| \le \theta\,, \quad \nu\, s \ge 0$$

and $\nu \ge 0$ *if s=0. Then the operators*

$$Q_+ \; : \; \left(\mathbb{H}_1^{\theta,\langle\nu+\lambda\rangle} \bigcap \mathbb{H}_p^{\theta}\right) \times \mathbb{H}^{s,\langle\nu-\lambda\rangle} \; \to \mathbb{H}^{s+1,\langle\nu\rangle}\,, \tag{62}$$

$$Q_+^* \; : \; \left(\mathbb{H}_1^{\theta,\langle\nu+\lambda\rangle} \bigcap \mathbb{H}_p^{-\theta}\right) \times \mathbb{H}^{-1-s,\langle-\nu\rangle} \to \mathbb{H}^{-s,\langle-\nu+\lambda\rangle} \tag{63}$$

are bounded.

Proof. Since the spaces $\mathbb{L}_2^{\langle\mu\rangle}$ and $\mathbb{L}_2^{\langle-\mu\rangle}$, $\mathbb{H}^{s,\langle\mu\rangle}$ and $\mathbb{H}^{-s\langle-\mu\rangle}$ with $s,\mu \in \mathbb{R}$ are dual (i.e. are conjugate to each other), the boundedness properties (62),(63) are equivalent, provided $Q_+(f,g) = Q_+(f)[g]$ and $Q_+^*(f,g) = Q_+^*(f)[g]$ are considered as operators operating on g for a fixed f. Thus, on each step it is sufficient to prove one of the inequalities (62) or (63).
Now let $\theta = m = 0$, which yields $s = 0\,,\ \nu \ge 0$. Then the proposed boundedness property (62) writes as

$$Q_+ \; : \; \left(\mathbb{L}_1^{\langle\nu+\lambda\rangle} \bigcap \mathbb{L}_p\right) \times \mathbb{L}_2^{\langle\nu-\lambda\rangle} \to \mathbb{H}^{1,\langle\nu\rangle}\,. \tag{64}$$

We can rewrite (64) as follows

$$Q_+^{\langle\nu\rangle} \; : \; \left(\mathbb{L}_1^{\langle\nu+\lambda\rangle} \bigcap \mathbb{L}_p\right) \times \mathbb{L}_2^{\langle-\lambda\rangle} \to \mathbb{H}^{1}\,, \tag{65}$$

where the operator $Q_+^{\langle\nu\rangle}$ has the form (cf. [13, Corollary 4.3])

$$Q_+^{\langle\nu\rangle}(f,g) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B^{\langle\nu\rangle}(v,w,e) f(v')g(w')\, de\, dw$$

with

$$B^{\langle\nu\rangle}(v,w,e) = \frac{\langle v\rangle^\nu}{\langle v'\rangle^\nu \langle w'\rangle^\nu} B(|u|,\mu)\,.$$

Since

$$\frac{\langle v\rangle^\nu}{\langle v'\rangle^\nu \langle w'\rangle^\nu} = \left(\frac{1+|v|^2}{1+|v|^2+|w|^2+|v'|^2|w'|^2}\right)^{\nu/2} \le 1$$

the boundedness (65) follows as the boundedness in the Theorem 11.
Recalling now the definition of the Sobolev norm (16) in $\mathbb{W}_2^{m,\langle\mu\rangle}$, using the property (48) and applying the already proved properties (62) and (63) for $s=0$ we easily obtain the boundedness (62) and (63) for all integer $s=k=1,2,\dots,m$.
By the interpolation (see [12, § 2.4.2] and (26), (26)) we obtain the boundedness (62) and (63) for all real $0\le s\le m=\theta\,,\ \nu\ge 0$.
Next we fix $g\in\mathbb{H}^{s,\langle\nu-\lambda\rangle}$ and prove the boundedness (62) and (63) by interpolation for arbitrary $\theta\ge 0$.
To complete the proof, we recall the duality arguments described at the beginning of this proof and extend the boundedness (62) and (63) to negative $-\theta\le s\le 0$. ∎

Remark 14 *Since $\lambda\ge 0$ the space $\mathbb{H}^{s,\langle\nu-\lambda\rangle}$ in (62) ($\mathbb{H}^{-s,\langle-\nu+\lambda\rangle}$ in (63)), perhaps, can not be replaced by $\mathbb{H}^{s,\langle\nu\rangle}$ ($\mathbb{H}^{-s,\langle-\nu\rangle}$).*

To justify the asserted proposition let us show that if the following physical conditions hold for some $w_0\in\mathbb{R}^3$ and $\varepsilon>0$

$$f:\ \mathbb{R}^3\to\mathbb{R}_+\,,\ f(w)\ge\varepsilon\,,\qquad \forall w\,:\,|w-w_0|\le\delta$$

then the symbol $q_+^*(f;,v,\xi)$ allows the following estimate from below

$$|q_+^*(f;,v,\xi)|\ge c\,\langle v\rangle^\lambda\,,\quad \forall v\in\mathbb{R}^3,\quad |\xi|\le\varepsilon' \tag{66}$$

for some $\varepsilon'>0$.
In fact, it suffices to prove (66) for $\xi=0$ and then apply continuity of $q_+^*(f;v,\xi)$ with respect to ξ to get (66) for small $|\xi|$.
We have $q_+^*(f;v,0)\in\mathbb{R}$ and using (49) obtain

$$\begin{aligned} q_+^*(f;,v,0) &= 4\pi\,C_\lambda\int\limits_{\mathbb{R}^3} f(v-u)|u|^\lambda\,du = 4\pi\,C_\lambda\int\limits_{\mathbb{R}^3} f(w)|v-w|^\lambda\,du \qquad (67)\\ &\ge 4\pi\,C_\lambda\int\limits_{|w-w_0|\le\delta} f(w)|v-w|^\lambda\,dw\ge 4\pi\,C_\lambda\varepsilon\int\limits_{|w-w_0|\le\delta}|v-w|^\lambda\,dw\ge c\langle v\rangle^\lambda\,. \end{aligned}$$

Thus if we choose a function $g\in\mathbb{L}_2$ with $\mathrm{supp}(\hat g)$ inside the ball $|\xi|\le\varepsilon$, we can easily obtain the following estimate

$$\left\|Q_+^*(f)[g]\,|\,\mathbb{L}_2^{\langle\lambda\rangle}\right\|\ge c\,\|g\,|\,\mathbb{L}_2\|$$

for sufficiently small $c>0$.

3 The loss part of the collision integral

The bilinear operator Q_-, which corresponds to the loss part of the collision integral defined in (6), can be written in the following form (cf. (35))

$$Q_-(f,g)(v) = \int\limits_{\mathbb{R}^3} B_{tot}(|v-w|)f(v)g(w)\,dw = f(v)\,\mathcal{B}[g](v)\,,$$

where the linear integral operator $\mathcal{B}$

$$\mathcal{B}[g](v) = \int\limits_{\mathbb{R}^3} B_{tot}(|v-w|)g(w)\,dw \tag{68}$$

is of the convolution type. For the study of the mapping properties of the operator (68) we need to investigate the kernel. For the inverse power potential model (cf. (3)) the kernel B_{tot} is

$$\begin{aligned} B_{tot} &= |u|^{1-4/m} \int\limits_{S^2} g_m\left(\frac{(u,e)}{|u|}\right) de = g_{m,tot}|u|^{1-4/m}\,, \qquad (69)\\ g_{m,tot} &= 2\pi \int\limits_{-1}^{1} g_m(\mu)\,d\mu \end{aligned}$$

and with $\lambda = 1 - 4/m$ the operator $\mathcal{B}$ takes the following form

$$\mathcal{B}[g](v) = g_{m,tot} \int\limits_{\mathbb{R}^3} |v-w|^\lambda g(w)\,dw\,, \quad -3 < \lambda \le 1\,.$$

In the special case of the Maxwell pseudo-molecules the integral operator (68) degenerates into the functional

$$\mathcal{B}[g](v) = g_{4,tot} \int\limits_{\mathbb{R}^3} g(w)\,dw = \varrho\, g_{4,tot}\,,$$

where ϱ denotes the "density" which corresponds to the function g.
The mapping properties of the operator $\mathcal{B}$ can now be formulated as follows.

Lemma 15 *Assume*

$$\mu > 3 - \frac{3}{q} + |\lambda| = \frac{3}{q'} + |\lambda| \ge \frac{3}{q'} + \lambda > 0 \tag{70}$$

with $q' = \dfrac{q}{q-1}$ *and* $1 \le q \le \infty$. *Then*

$$\mathcal{B} \,:\; \mathbb{L}_q^{\langle\mu\rangle} \to \mathbb{L}_\infty^{\langle-\lambda\rangle}$$

is continuous and the inequality

$$\left\|\mathcal{B}[g] \,|\, \mathbb{L}_\infty^{\langle-\lambda\rangle}\right\| \le C_{1,\lambda,\mu,q} \left\|g \,|\, \mathbb{L}_q^{\langle\mu\rangle}\right\|$$

holds for all $g(v) \in \mathbb{L}_q^{\langle\mu\rangle}$.

Proof. We suppose $1 \le q < \infty$. For $q = \infty$ the proof is essentially the same with obvious modifications concerning the supremum norm

$$\left\|g \,|\, \mathbb{L}_\infty^{\langle\mu\rangle}\right\| = \sup_{v\in\mathbb{R}^3} |\langle v\rangle^\mu g(v)| \,.$$

We proceed with the Hölder inequality as follows:

$$\begin{aligned}
\left|\langle v\rangle^{-\lambda}\mathcal{B}[g](v)\right| &= g_{m,tot} \int\limits_{\mathbb{R}^3} \langle v\rangle^{-\lambda}|v-w|^\lambda \langle w\rangle^{-\mu}\langle w\rangle^\mu g(w)\,dw \\
&\le g_{m,tot} \left(\int\limits_{\mathbb{R}^3} \left(\frac{|v-w|^\lambda}{\langle v\rangle^\lambda \langle w\rangle^\mu} \right)^{q'} dw \right)^{\frac{1}{q'}} \left\|g \,|\, \mathbb{L}_q^{\langle\mu\rangle}\right\| .
\end{aligned}$$

For $\lambda > 0$ we use the substitution $\tilde{w} = v - w\,,\; d\tilde{w} = dw$ in the last integral. Removing the tilde sign it results with (cf. (55))

$$|w|^s \le 2^{s/2}\langle v-w\rangle^s\langle v\rangle^s\,,\;\; s>0\,,$$

the integral

$$\begin{aligned}
\int\limits_{\mathbb{R}^3} \left(\frac{|v-w|^\lambda}{\langle v\rangle^\lambda \langle w\rangle^\mu} \right)^{q'} dw &= \int\limits_{\mathbb{R}^3} \langle v\rangle^{-q'\lambda}|w|^{q'\lambda}\langle v-w\rangle^{-q'\mu} dw \\
&\le 2^{\frac{q'\lambda}{2}} \int\limits_{\mathbb{R}^3} \langle v-w\rangle^{(\lambda-\mu)q'} dw < \infty
\end{aligned}$$

is finite because of the assumption of the lemma $(\mu-\lambda)q' > 3$.
For $\lambda < 0$ we similarly find with

$$|v|^{-s} \le 2^{-s/2}\langle v-w\rangle^{-s}\langle v\rangle^{-s}\,,\;\; s<0$$

and using the substitution $\tilde{w} = v - w$, $d\tilde{w} = dw$ again

$$\int\limits_{\mathbb{R}^3} \left(\frac{|v-w|^{\lambda}}{\langle v \rangle^{\lambda} \langle w \rangle^{\mu}} \right)^{q'} dw \le \int\limits_{\mathbb{R}^3} \frac{\langle v-w \rangle^{-q'\lambda}}{|v-w|^{-q'\lambda} \langle w \rangle^{q'(\lambda+\mu)}} \, dw < \infty \, .$$

The last integral converges for $w \to v$ because of the assumption $-q'\lambda < 3$ and for $w \to \infty$ because $q'(\lambda + \mu) > 3$ (see (70)).
The remark that for $\lambda = 0$ the function $\mathcal{B}[g](v)$ is constant (see (70)) completes the proof with the final estimate

$$\left| \langle v \rangle^{-\lambda} \mathcal{B}[g](v) \right| \le C_{1,\lambda,\mu,q} \left\| g \mid \mathbb{L}_q^{\langle \mu \rangle} \right\| \, , \ -3 < \lambda \le 1 \, .$$

■

Remark 16 *The condition* (70) *is not restrictive for the solution of the Boltzmann equation* $f(t,x,v) \ge 0$ *which represents the distribution of particles in the phase space* $\Omega \times \mathbb{R}^3$ *and, therefore,* $f(t,x,v)$ *maintains a finite kinetic energy*

$$\int\limits_{\Omega} \int\limits_{\mathbb{R}^3} |v|^2 f(t,x,v) \, dv \, dx < \infty \, .$$

Corollary 17 *If the condition* (70) *holds, the operator*

$$\mathcal{B} \ : \ \mathbb{H}_q^{s,\langle \nu \rangle} \to \mathbb{H}_\infty^{s,\langle -\lambda \rangle}$$

is bounded for all $s \ge 0$.

Proof. For an integer $s = m \in \mathbb{N}_0$ the proof is a direct consequence of the foregoing lemma because

$$\partial^\alpha \mathcal{B}[g](v) = \mathcal{B}[\partial^\alpha g](v) \, , \ \alpha \in \mathbb{N}_0^3 \, , \ |\alpha| \le m$$

(see Corollary 5 and Theorem 13). For arbitrary $s \ge 0$ the proof follows then by the interpolation (see (26) and (26)). ■

Corollary 18 *Let* (70) *hold and* $s, \theta \ge 0$, $1 \le p \le \infty$, $\nu \in \mathbb{R}$. *Then the bilinear operator*

$$Q_- \ : \ \mathbb{H}_p^{s,\langle \mu \rangle} \times \mathbb{H}_q^{\theta,\langle \nu \rangle} \to \mathbb{H}_p^{\ell,\langle \nu - \lambda \rangle} \tag{71}$$

is bounded for $\ell = \min(s, \theta)$.
In particular, the loss term (6) of the Boltzmann equation (2) has the following boundedness property

$$Q_- \ : \ \mathbb{H}_p^{s,\langle \nu \rangle} \times \mathbb{H}_p^{s,\langle \nu \rangle} \to \mathbb{H}_p^{s,\langle \nu - \lambda \rangle} \, ,$$

provided the conditions (70) *hold with* $q = p$ *and* $\mu = \nu$.

Proof. First let us prove the following assertion:

$$a \in \mathbb{H}_\infty^{\theta,\langle\gamma\rangle},\ \varphi \in \mathbb{H}_p^{\theta,\langle\nu\rangle} \quad \text{yield} \quad a\varphi \in \mathbb{H}_p^{\ell,\langle\nu+\gamma\rangle}\,. \tag{72}$$

The assertion can easily be verified for integers $\theta = m,\ s = n \in \mathbb{N}_0$. Let, for the sake of definity, $s \le \theta$. We fix $a \in \mathbb{W}_\infty^{m,\langle\mu\rangle}$, interpret (72) as a boundedness of the multiplication operator aI, and extend the boundedness property to an arbitrary $0 \le s \le m$ by the interpolation (26) and (26). After this we fix $\varphi \in \mathbb{H}_\infty^{s,\langle\nu\rangle}$, and extend similarly the boundedness property for arbitrary $\theta \ge s$. This completes the proof of (72).
For integers $s = n,\ \theta = m \in \mathbb{N}_0$ the proof of the asserted boundedness (71) is a direct consequence of the property

$$\partial^\alpha Q_-(f,g)(v) = \sum_{\beta \preceq \alpha} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) Q_-(\partial^\beta f, \partial^{\alpha-\beta} g)(v)$$

(see (34) and Corollary 5) and the property (72).
For arbitrary $0 \le s \le \theta$ the proof then follows by the interpolation, applied twice as in the proof of the assertion (72). ∎

Remark 19 *It can be proved that the operator*

$$\mathcal{B}\ :\ \mathbb{H}_{q,com}^s \to \mathbb{H}_{q,loc}^{s+3+\lambda}$$

is bounded for arbitrary $s \in \mathbb{R}$. In fact, the symbol $a(\xi)$ of the operator of the convolution type $\mathcal{B}$

$$\mathcal{B}[g](v) = \int_{\mathbb{R}^3} B_{tot}(|v-w|) f(w)\, dw$$

can be computed as the Fourier transform of its kernel

$$a(\xi) = \mathcal{F}_{u\to\xi}[B_{tot}(|u|)](\xi)$$

(cf. (68),(69)). Thus the symbol of the operator $\mathcal{B}_0$ can be written as

$$a(\xi) = g_{m,tot} \int_{\mathbb{R}^3} |u|^\lambda e^{\imath(u,\xi)}\, du\,.$$

The result is (see e.g. [4])

$$a(\xi) = \begin{cases} -(2\pi)^2 g_{4,tot} \dfrac{\delta'(|\xi|)}{|\xi|} & ,\quad \text{for } \lambda = 0\,, \\ \\ -4\pi\,(\lambda+1)\Gamma(\lambda+1)\sin\left(\dfrac{\lambda\pi}{2}\right) g_{m,tot} \dfrac{1}{|\xi|^{\lambda+3}} & ,\quad \text{for } \lambda \neq 0\,. \end{cases}$$

In the case of the hard spheres model ($\lambda = 1$) we get

$$a(\xi) = -\frac{8\pi^2 d^2}{|\xi|^4}.$$

Thus the symbol $a(\xi)$ always has singularity at $\xi = 0$. By cutting out the neighborhood of 0, with the help of a cut-off function with a compact support we decompose the operator $\mathcal{B}$ in a sum

$$\mathcal{B} = \mathcal{B}^{(1)} + \mathcal{B}^{(2)},$$

where $\mathcal{B}^{(1)}$ has no more singularity at 0 and, having order $-3-\lambda$, maps

$$\mathcal{B}^{(1)} : \mathbb{H}^s_{q,com} \to \mathbb{H}^{s+3+\lambda}_{q,loc}.$$

The operator $\mathcal{B}^{(2)}_0$ is smoothing

$$\mathcal{B}^{(2)} : \mathbb{H}^s_{q,com} \to \mathbb{C}^\infty \subset \mathbb{H}^{s+3+\lambda}_{q,loc}$$

because the symbol has a compact support, but functions $\mathcal{B}^{(2)}[f](v)$ might have problems with integration at infinity.

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