ON THE APPROXIMATION OF SINGULAR INTEGRAL EQUATIONS BY EQUATIONS WITH SMOOTH KERNELS

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1. INTRODUCTION

Let $\Gamma \subset \mathbb{C}$ be a finite union of closed or open, compact, oriented, smooth curves without common points. Let $c_1, ..., c_{2m} \in \Gamma$ be the end points of open arcs where $c_2, c_4, ..., c_{2m}$ represent the right end points and $c_1, c_3, ..., c_{2m-1}$ the left ones. Introduce the weight function

$$\varrho(t) = \prod_{j=1}^{n} |t - c_j|^{\alpha_j}\,,\ 1 < p < \infty\,,\ -1 < \alpha_j < p - 1\,, \quad c_1, ..., c_{2m}, c_{2m+1}, ..., c_n \in \Gamma\,. \qquad (1)$$

By $L_p(\Gamma, \varrho)$ we denote the Lebesgue space of functions φ equipped with the norm

$$||\varphi|L_p(\Gamma, \varrho)|| := ||\varrho|\varphi|^p|L_1(\Gamma)||\,.$$

$PC(\Gamma)$ will denote the algebra of piecewise–continuous functions $a(t)$ on Γ which have finite limits $a(t \pm 0)$ at any inner point $t \neq c_1, ..., c_{2m}$ and one–sides limits $a(c_{2j-1}) := a(c_{2j-1} + 0)$, $a(c_{2j}) := a(c_{2j} - 0)$ at the end points c_{2j-1}, c_{2j} $(j = 1, ..., m)$, respectively.

$L_p^N(\Gamma, \varrho)$ and $PC^{N\times N}(\Gamma)$ stand for the space of vector–functions $(\varphi_1, ..., \varphi_N)$, $\varphi_j \in L_p(\Gamma, \varrho)$, and for the algebra of $N \times N$ matrix–functions $a = ||a_{jk}||_{N\times N}$, $a_{jk} \in PC(\Gamma)$, respectively.

Consider the following singular integral equation

$$\begin{aligned} A\varphi &:= a\varphi + bS_\Gamma\varphi + T\varphi = f\,, \\ a, b \in PC^{N\times N}(\Gamma)\,, \quad S_\Gamma\varphi(t) &:= \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t}\,, \quad \varphi, f \in L_p^N(\Gamma, \varrho)\,, \end{aligned} \qquad (2)$$

where T is a compact integral operator in $L_p^N(\Gamma, \varrho)$:

$$T\varphi(t) := \int_\Gamma k(t, \tau)\varphi(\tau)d\tau\,.$$

With (2) we associate the following family of Fredholm integral equations

$$\begin{aligned} A_\varepsilon\psi &:= a\psi + bS_{\Gamma,\varepsilon}\psi + T\psi = f\,, \\ S_{\Gamma,\varepsilon}\psi(t) &:= \frac{1}{\pi i}\int_\Gamma \frac{(\tau - t)\psi(\tau)d\tau}{(\tau - t)^2 - n^2(t)\varepsilon^2}\,, \quad \varepsilon > 0\,, \end{aligned} \qquad (3)$$

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where $n(t) = \cos\omega(t) + i\sin\omega(t) = \exp[i\omega(t)]$, $0 \le \omega(t) < 2\pi$, is a continuous field of unit vectors non–tangential to Γ at t for all $t \in \Gamma$. Thus $\omega(t) - \varphi(t) \neq 0$ for all $t \in \Gamma$ where $\mathcal{T}(t) = \exp[i\varphi(t)]$, $0 \le \varphi(t) < 2\pi$, is the unit tangential vector to Γ at $t \in \Gamma$.

The kernel function of $S_{\Gamma,\varepsilon}$ is continuous and, moreover, belongs to $C^{r-1}(\Gamma \times \Gamma)$ if Γ is r–smooth.

Notice that the harmonic extension, applied in ([5]) for the definition of the index of Toeplitz operators with quasicontinuous symbols, provides a similar method of regularization of singular kernels by C^∞–kernels.

The next three examples show how $n(t)$ can be selected in particular cases:

(a) if $\Gamma = [0, 1]$, we can take $n(t) \equiv i$;
(b) if $\Gamma = \{\zeta \in \mathbb{C}\colon |\zeta| = 1\}$, then $n(t) \equiv t$ has the necessary properties;
(c) if there exists a point $z_o \notin \Gamma$ such that $t - z_o$ is non–tangential to Γ for any $t \in \Gamma$, then $n(t) = |t - z_o|^{-1}(t - z_o)$ can be chosen.

The main purpose of this paper is to solve the following approximation problem.

PROBLEM A. *Let* (2) *be uniquely solvable for any given* $f \in L_p^N(\Gamma, \varrho)$. *Under what conditions does there exist* $\varepsilon_o > 0$ *such that equations* (3) *have unique solutions* φ_ε *for all* $0 < \varepsilon < \varepsilon_o$ *and these solutions converge in* $L_p^N(\Gamma, \varphi)$ *to the solution* φ *of* (1) :

$$\varphi = \lim_{\varepsilon \to 0} \varphi_\varepsilon \ .$$

To formulate the theorem which solves Problem A we consider the following operators associated with (2) and depending on the parameter $t \in \Gamma$:

$$A_{t,\theta}^\varepsilon = a_t I + \delta_t b_t S_{\mathbb{R}_\approx,\theta}^\varepsilon \,, \tag{4}$$

$$S_{\mathbb{R}_\approx,\theta}^\varepsilon \psi(x) := \frac{1}{\pi i} \int_{\mathbb{R}_\approx} \frac{(y-x)\psi(y)dy}{(y-x)^2 - \varepsilon^2 \exp 2\theta(t) i} \,,$$

where $\theta(t) = \omega(t) - \varphi(t)$ denotes the angle between the vector $n(t)$ (see (3)) and the tangent $\mathcal{T}(t)$ to Γ at $t \in \Gamma$, while

$$g_t(x) := \begin{cases} g(t-0) & \text{for} \quad x < 0 \quad \text{and} \quad t \neq c_1, ..., c_{2m} \,, \\ g(t+0) & \text{for} \quad x \ge 0 \quad \text{and} \quad t \neq c_1, ..., c_{2m} \,, \\ g(t) & \text{for} \quad t \in \{c_1, ..., c_{2m}\} \,, \end{cases}$$

$$\mathbb{R}_\approx := \begin{cases} \mathbb{R} = (-\infty, \infty) & \text{for} \quad t \neq c_1, ..., c_{2m} \,, \\ \mathbb{R}^+ = [\nvdash, \infty) & \text{for} \quad t \in \{c_1, ..., c_{2m}\} \,, \end{cases}$$

$$\delta_t := \begin{cases} \ \ 1 & \text{for} \quad t \notin \{c_2, c_4, ..., c_{2m}\} \,, \\ -1 & \text{for} \quad t \in \{c_2, c_4, ..., c_{2m}\} \,. \end{cases}$$

The operators $A_{t,\theta}^\varepsilon$ will be considered in the space $L_p^N(\mathbb{R}_\approx, |\frown|^{\alpha_\approx})$, where

$$\alpha_t := \begin{cases} \alpha_j & \text{for} \quad t = c_j \,, \\ 0 & \text{for} \quad t \neq c_1, ...c_n \,. \end{cases} \tag{5}$$

THEOREM 1. *Problem A has a positive solution for equations* (2) *and* (3) *if and only if the operator* $A^1_{t,\theta}$ *is invertible in the space* $L^N_p(\mathbb{R}_{\approx}, |\frown|^{\alpha_{\approx}})$ *for each* $t \in \Gamma$.

Proof follows from Lemma 5 and Theorem 6 proved below. ∎

Some equivalent reformulations of Theorem 1 with more explicit conditions can be found in Section 3. The next theorem is one of such equivalent reformulations in the particular case $p = 2$, $\varrho(t) \equiv 1$, $N = 1$, which in our opinion represents special interest, since locally strongly elliptic operators play an outstanding role in different approximation methods (see e.g. [2], [19]–[24], [29]). For this we need the following definition.

DEFINITION (see [20, 23, 29]). *An operator*

$$A : L^N_2(\Gamma) \to L^N_2(\Gamma)$$

is said to be locally strongly elliptic if there exist an invertible matrix–function $\theta_o \in PC^{N\times N}(\Gamma)$ *and a compact operator* T_o *such that*

$$A = \theta_o(A_o + T_o)\ ,$$

where A_o *is strongly positive definite*

$$Re(A_o\varphi, \varphi) \geq \delta||\varphi||^2 \quad \textit{for some } \delta > 0 \quad \textit{and any } \varphi \in L^N_2(\Gamma)\,.$$

THEOREM 2. *Let* $\theta(t) \equiv \pi/2$ *(i.e.* $n(t)$ *is the outer normal vector for all* $t \in \Gamma$*). The following assertions are equivalent:*

I. *Problem A has a positive solution for equations* (2), (3) *in the space* $L_2(\Gamma)$ *(i.e. for* $N = 1$, $p = 2$, *and* $\varrho(t) \equiv 1$*).*

II. *The operator A is locally strongly elliptic in* $L_2(\Gamma)$.

III. *There exists* $G_t \in C(\Gamma)$ *such that*

$$d(t \pm 0) \neq 0\,,\ Re\,G_t > 0\,, \quad Re\,G_t c(t \pm 0)d^{-1}(t \pm 0) > 0 \quad \textit{for all } t \in \Gamma\,.$$

IV. *The following conditions are fulfilled:*

$$\inf\{|[a(t \pm 0) + \mu b(t \pm 0)]| : t \in \Gamma\,, \quad \mu \in [-1, 1]\} > 0\,,$$
$$\inf\{|[(1-\mu)c(t-0)d^{-1}(t-0) + (1+\mu)c(t+0)d^{-1}(t+0)]| : t \in \Gamma\,,\ \mu \in [-1, 1]\} > 0\,,$$

where $a(c_{2j} + 0) = a(c_{2j-1} - 0) := 1$, $b(c_{2j} + 0) = b(c_{2j-1} - 0) := 0$ $(j = 1, 2, ..., m)$ *and* $c(t) := a(t) + b(t)$, $d(t) := a(t) - b(t)$.

Proof. The equivalence of conditions II to IV is proved in [20]; the equivalence I $\iff$ IV follows from Theorem 1 and Lemma 8 since $\coth \pi(i/2 + \xi) \equiv \mu \in [-1, 1]$, and $S_{\pi/2}(\xi) = -\text{sgn}\xi \exp(-|\xi|) \equiv \mu \in [-1, 1]$. ∎

REMARK 3. *For the matrix–case* $N > 1$ *see Theorems 12 and 13 below.*

2. STABILITY

DEFINITION (cf. [23]). *The sequence of operators* $\{A_\varepsilon\}_\varepsilon$ *is called stable if:*

I. *it converges strongly to some bounded operator* A*:*

$$\lim_{\varepsilon\to 0} A_\varepsilon\psi = A\psi \qquad \text{for all } \psi \in L_p^N(\Gamma,\varrho)\,;$$

II. *there exists* ε_o *such that* A_ε *is invertible for each* ε*,* $0<\varepsilon<\varepsilon_o$*;*

III. *the inverses are uniformly bounded*

$$\sup_{\varepsilon<\varepsilon_o} ||A_\varepsilon^{-1}|| \le M_A < \infty\,.$$

The next two assertions show the connection between the stability of $\{A_\varepsilon\}_\varepsilon$ and the solution of Problem A for equation (2).

LEMMA 4. *The strong convergence*

$$\lim_{\varepsilon\to 0} S_{\Gamma,\varepsilon}\psi = S_\Gamma\psi \tag{6}$$

holds for all $\psi\in L_p^N(\Gamma,\varrho)$ *(see* (2), (3)*).*

Proof follows immediately since

$$S_{\Gamma,\varepsilon}\psi(t) = \frac{1}{2\pi i}\int_\Gamma \left[\frac{1}{\tau - t - n(t)\varepsilon} + \frac{1}{\tau - t + n(t)\varepsilon}\right]\psi(\tau)d\tau \tag{7}$$

and the Plemelj formulas

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_\Gamma \frac{\psi(\tau)d\tau}{\tau - [t\pm n(t)\varepsilon]} = \pm\frac{1}{2}\psi(t) + S_\Gamma\psi(t) \tag{8}$$

hold if the non–tangential vector $n(t)$ points to the left of the oriented curve Γ (for (8) see e.g. [11]). ∎

LEMMA 5. *Problem A has a positive solution for equations* (2), (3) *if and only if the sequence* $\{A_\varepsilon\}_\varepsilon$ *is stable.*

Proof is well–known (see e.g. [12, 18, 23]) and follows easily from the strong convergence (cf. (6))

$$\lim_{\varepsilon\to 0} A_\varepsilon\psi = A\psi \qquad \text{for all} \qquad \psi\in L_p^N(\Gamma,\varrho)\,. \tag{9}$$

∎

Our main concern is now to get stability conditions for the operator in (3). The first contribution to this topic is given by the following theorem.

THEOREM 6. *The sequence* $\{A_\varepsilon\}_\varepsilon$ *defined in* (3) *is stable if and only if the operator* $A^1_{t,\theta}$ *is invertible in the space* $L_p^N(\mathbb{R}_\approx,\,|x|^{\alpha_t})$ *for each* $t\in\Gamma$*.*

Proof. Sufficiency. In this part we follow the proof of a similar assertion in [18], where the operators $S^\varepsilon_{\Gamma,\theta}$ are defined as follows

$$S^\varepsilon_{\Gamma,\theta}\psi(t) = \frac{1}{\pi i}\int_{\Gamma(t,\varepsilon)}\frac{\varphi(\tau)d\tau}{\tau - t}\,, \qquad \Gamma(t,\varepsilon) = \Gamma\cap\{\zeta\in\mathbb{C} : |\zeta - \approx| \ge \varepsilon\}\,. \tag{10}$$

This proof makes use of the local principle of Gohberg–Krupnik (see [11]) together with some ideas of [15] and [26]. Let $\mathfrak{A}_b(\Gamma) := \mathfrak{A}_b(L_p^N(\Gamma, \varrho))$ denote the Banach algebra of bounded sequences $\{A_\varepsilon\}_{0<\varepsilon\leq 1}$ of operators endowed with the pointwise composition (as multiplication)

$$\{A_\varepsilon\}_\varepsilon \cdot \{B_\varepsilon\}_\varepsilon := \{A_\varepsilon B_\varepsilon\}_\varepsilon$$

and the uniform norm

$$||\{A_\varepsilon\}|| := \sup_\varepsilon ||A_\varepsilon|| \,.$$

Let further $\mathfrak{A}_o(\Gamma) := \mathfrak{A}_o(L_p^N(\Gamma, \varrho))$ denote the ideal in $\mathfrak{A}_b(L_p^N(\Gamma, \varrho))$ consisting of sequences $\{A_\varepsilon\}_\varepsilon$ which converge to 0:

$$\lim_\varepsilon ||A_\varepsilon|| = 0 \,.$$

It is known that the stability of $\{A_\varepsilon\}$ is equivalent to the invertibility of the corresponding quotient classes $\{A_\varepsilon\}_\varepsilon^\wedge$ in the quotient algebra $\mathfrak{A}_b(L_p^N(\Gamma, \varrho))/\mathfrak{A}_o(L_p^N(\Gamma, \varrho))$ (see [15, 23, 26]). This observation makes it possible to apply the local principle to the investigation of stability (see [15, 18, 23, 26]). We stick here to the local principle suggested in [18]. Introduce the notation

$$\begin{aligned}
\mathfrak{A}_c(\Gamma) &= \{\{B_\varepsilon + T\}_\varepsilon : \{B_\varepsilon\}_\varepsilon \in \mathfrak{A}_o(\Gamma)\,,\ T \text{ is compact in } L_p^N(\Gamma, \varrho)\}\,,\\
\mathfrak{A}_s(\Gamma) &= \{\{D_\varepsilon\}_\varepsilon \in \mathfrak{A}_b(\Gamma) \,:\, \lim_{\varepsilon\to 0} D_\varepsilon \varphi = 0 \quad \text{for all } \varphi \in L_p^N(\Gamma, \varrho)\}\,.
\end{aligned}$$

Since

$$\mathfrak{A}_c(\Gamma) \cap \mathfrak{A}_s(\Gamma) = \mathfrak{A}_o(\Gamma)\,,$$

the invertibility in the quotient algebra $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_o(\Gamma)$ is equivalent to the invertibility of the corresponding quotient classes in the quotient algebras $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_c(\Gamma)$ and $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_s(\Gamma)$ (see [18], Lemma 7).

The invertibility of $\{A_\varepsilon\}_\varepsilon^\wedge$ in $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_s(\Gamma)$ is equivalent to the invertibility of the limit operator $A\psi = \lim_{\varepsilon\to 0} A_\varepsilon \psi$ since the strong convergence holds [18]. Thus we have to look only for the invertibility conditions in the quotient algebra $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_c(\Gamma)$.

Let $M_t(\Gamma)$ denote the class of r–smooth cut–off functions on Γ which are equal to 1 in some neighbourhood of $t \in \Gamma$ (r denotes the smoothness of the contour Γ). By $M_t^\wedge(\Gamma)$ we denote the quotient class $\{g_t I\}^\wedge \in \mathfrak{A}_b(\Gamma)/\mathfrak{A}_c(\Gamma)$ of stationary sequences where $g_t \in M_t(\Gamma)$. It can be proved that $\{g_t I\}^\wedge$ and $\{A_\varepsilon\}_\varepsilon^\wedge$ commute (see [18]) and there holds the quasiequivalence (cf. [18, 27, 28])

$$\begin{gathered}
\{A_\varepsilon\}_\varepsilon^\wedge \quad \overbrace{\qquad}^{M_t^\wedge(\Gamma)} \quad \beta_t \quad \overbrace{\qquad}^{M_o^\wedge(\mathbb{R}_\approx)} \quad \{A_{t,\theta}^\varepsilon\}_\varepsilon^\wedge\,, \\
M_o^\wedge(\mathbb{R}_\approx) \subset \mathfrak{A}(\mathbb{R}_\approx)/\mathfrak{A}(\mathbb{R}_\approx)\,,\ \ \{\mathbb{A}_{\approx,\theta}^\varepsilon\}_\varepsilon^\wedge \in \mathfrak{A}(\mathbb{R}_\approx)/\mathfrak{A}(\mathbb{R}_\approx)\,,
\end{gathered} \tag{11}$$

where

$$\mathfrak{A}_b(\mathbb{R}_\approx) := \mathfrak{A}(\mathbb{L}_{|}^{\mathbb{N}}(\mathbb{R}_\approx, |\curvearrowleft|^{\alpha\approx}))\,,\ \ \mathfrak{A}(\mathbb{R}_\approx) := \mathfrak{A}(\mathbb{L}_{|}^{\mathbb{N}}(\mathbb{R}_\approx, |\curvearrowleft|^{\alpha\approx}))$$

and $\beta_t : U_t \to V_o$ denotes a diffeomorphism between the domains $U_t \subset \Gamma$, $t \in U_t$, $V_o \subset \mathbb{R}_\approx$, $0 \in V_o$, $\beta_t(t) = 0$.

If $A^1_{t,\theta}$ is invertible, then

$$A^\varepsilon_{t,\theta} = H_{1/\varepsilon} A^1_{t,\theta} H_\varepsilon \,, \quad H_\varepsilon \psi(t) := \varepsilon^{\frac{1+\alpha_t}{p}} \psi(\varepsilon t) \tag{12}$$

and, therefore, $A^\varepsilon_{t,\theta}$ have uniformly bounded inverses (note that $||H_\varepsilon|L_p(\mathbb{R}_\approx, |\frown|^{\alpha\approx})|| = \nVdash$, $\varepsilon > 0$). Thus, $\{A^\varepsilon_{t,\theta}\}_\varepsilon$ is invertible in $\mathfrak{A}_b(\mathbb{R}_\approx)$ and this implies the invertibility of $\{A^\varepsilon_{t,\theta}\}^\wedge_\varepsilon$ in the quotient algebra $\mathfrak{A}_b(\mathbb{R}_\approx)/\mathfrak{A}(\mathbb{R}_\approx)$.

If $A^1_{t,\theta}$ is invertible for all $t \in \Gamma$, we get due to the local principle (see [11, 27, 28]) that $\{A_\varepsilon\}^\wedge_\varepsilon$ is invertible in $\mathfrak{A}_b(\Gamma)/\mathfrak{A}_c(\Gamma)$.

Necessity. This part of the proof in [18] is given only for the case $\Gamma = \mathbb{R}$, $\mathbb{R}^+$ which simplifies the argumentation. Therefore we display here the detailed proof.

Due to the quasiequivalence (11) and the local principle we have to prove only that the local invertibility of $\{A^\varepsilon_{t,\theta}\}^\wedge_\varepsilon \in \mathfrak{A}_b(\mathbb{R}_\approx)/\mathfrak{A}(\mathbb{R}_\approx)$ at $0 \in \mathbb{R}_\approx$ implies the invertibility of $A^1_{t,\theta}$ in $L^N_p(\mathbb{R}_\approx, |\frown|^{\alpha\approx})$.

Suppose $\{A^\varepsilon_{t,\theta}\}^\wedge_\varepsilon$ is locally invertible. Then there exist $\{L_\varepsilon\}^\wedge_\varepsilon$, $\{R_\varepsilon\}^\wedge_\varepsilon \in \mathfrak{A}_b(\mathbb{R}_\approx)/\mathfrak{A}(\mathbb{R}_\approx)$ and $g_1, g_2 \in M_t(\mathbb{R}_\approx)$ such that

$$\begin{aligned} L_\varepsilon A^\varepsilon_{t,\theta} g_1 I &= g_1 I + B_\varepsilon + T_1 \,, && (13)\\ g_2 A^\varepsilon_{t,\theta} R_\varepsilon &= g_2 I + D_\varepsilon + T_2 \,, && (14)\\ \lim_{\varepsilon\to 0} ||B_\varepsilon|| = \lim_{\varepsilon\to 0} ||D_\varepsilon|| &= 0 \,, \end{aligned}$$

where T_1 and T_2 are compact operators in $L^N_p(\mathbb{R}_\approx, |\frown|^{\alpha\approx})$. Therefore g'_1, g''_1 can be chosen so that $g'_1 g''_1 = g'_1 g_1 = g'_1$ and $||T_1 g''_1 I|| + ||B_\varepsilon|| < 1$ if ε and $\mathrm{supp} g''_1$ are sufficiently small. Thus (see (13))

$$L_\varepsilon A^\varepsilon_{t,\theta} g'_1 I = (I + B_\varepsilon + T_1 g''_1 I) g'_1 I$$

and due to the invertibility of $I + B_\varepsilon + T_1 g''_1 I$ we get

$$L'_\varepsilon A^\varepsilon_{t,\theta} g'_1 I = g'_1 I \,. \tag{15}$$

Similarly from (14) we derive

$$g'_2 A^\varepsilon_{t,\theta} R'_\varepsilon = g'_2 I \,. \tag{16}$$

Due to (12), from (15) and (16) we get

$$\begin{aligned} L''_\varepsilon A^1_{t,\theta} g'_{1,\varepsilon} I &= g'_{1,\varepsilon} I \,, && (17)\\ g'_{2,\varepsilon} A^1_{t,\theta} R''_\varepsilon &= g'_{2,\varepsilon} I \,, && (18) \end{aligned}$$

where

$$\begin{aligned} &g'_{j,\varepsilon}(x) := \varepsilon^{-\frac{1+\alpha_t}{2}} H_\varepsilon g'_j(x) = g'_j(\varepsilon x) \,, \qquad j = 1, 2 \,, \\ &L''_\varepsilon = H_\varepsilon L'_\varepsilon H_{1/\varepsilon} \,, \qquad R''_\varepsilon = H_\varepsilon R'_\varepsilon H_{1/\varepsilon} \,. \end{aligned} \tag{19}$$

From (19) it follows that

$$\lim_{\varepsilon\to 0} g'_{j,\varepsilon} \equiv 1 \,, \quad j = 1, 2 \,, \tag{20}$$

$$\sup_\varepsilon ||L''_\varepsilon|| \le M < \infty \,, \quad \sup_\varepsilon ||R''_\varepsilon|| \le M < \infty \,. \tag{21}$$

Let now $\varphi_o \in \operatorname{Ker} A_{t,1}$. Then (see (20))

$$\lim_{\varepsilon\to 0} g'_{2,\varepsilon}\varphi_o = \varphi_o$$

and from (21) we get

$$\varphi_o = \lim_{\varepsilon\to 0} L''_\varepsilon A^1_{t,\theta} g_{\varepsilon,2}\varphi_o = 0\,.$$

Thus, $\operatorname{Ker} A^1_{t,\theta} = \{0\}$. Similarly, due to (18), we get $\operatorname{Ker} (A^1_{t,\theta})^* = \{0\}$.

Assume now that $A^1_{t,\theta}$ is not normally solvable; then there exists a sequence $\{\varphi_j\}_1^\infty$, $||\varphi_j|| = 1$, such that $\lim_{j\to\infty} A^1_{t,\theta}\varphi_j = 0$. For sufficiently small ε_j we get (see (20), (21))

$$||(1-g'_{1,\varepsilon_j})\varphi_j|| < \frac{1}{4}\min\left\{\frac{1}{M}\,,\ \frac{1}{M||A_{t,1}||}\,,1\right\}$$

and therefore (see (17))

$$\begin{aligned} 1 &= ||\varphi_j|| \le ||g'_{1,\varepsilon_j}\varphi_j|| + ||(1-g'_{1,\varepsilon_j})\varphi_j|| \le \\ &\le ||L''_{\varepsilon_j} A^1_{t,\theta} g'_{1,\varepsilon_j}\varphi_j|| + \frac{1}{4} \le ||L''_{\varepsilon_j} A^1_{t,\theta}\varphi_j|| + \\ &+ ||L''_{\varepsilon_j} A^1_{t,\theta}(1-g'_{1,\varepsilon_j})\varphi_j|| + \frac{1}{4} < \frac{3}{4}\,, \end{aligned} \tag{22}$$

if j is sufficiently large so that

$$||A^1_{t,\theta}\varphi_j|| < \frac{1}{4M}\,.$$

The obtained contradiction in (22) proves that $A_{t,1}$ is normally solvable. This together with $\operatorname{Ker} A^1_{t,\theta} = \{0\}$, $\operatorname{Ker}(A^1_{t,\theta})^* = \{0\}$ yields the invertibility of $A^1_{t,\theta}$. ∎

REMARK 7. *Some sufficient conditions for the stability of sequences* $\{\lambda I + S_{J,\varepsilon}\}$, *where* $\lambda \in \mathbb{C}$, $J = [0,1]$, $n(t) \equiv i$, *in the Lebesgue space* $L_p(J)$ *are announced in* [25].

3. EQUIVALENT CONDITIONS

To reformulate the conditions of stability of the operator sequences $\{A_\varepsilon\}_\varepsilon$, $A = \lim_{\varepsilon\to 0} A_\varepsilon$ (see (2), (3)), i.e. to solve Problem A we shall give invertibility conditions for the operators (see Lemma 5 and Theorem 6)

$$\begin{aligned} B_o &= aI + bS^1_{\mathbb{R},\theta} : L^N_p(\mathbb{R}, |x|^\alpha) \to L^N_p(\mathbb{R}, |x|^\alpha)\,, \\ B_+ &= cI + dS^1_{\mathbb{R}^+,\theta} : L^N_p(\mathbb{R}^+, x^\alpha) \to L^N_p(\mathbb{R}^+, x^\alpha)\,,\ 0 < \theta < \pi\,, \\ &-1 < \alpha < p-1\,, \qquad 1 < p < \infty\,, \qquad a(x) = a_-\chi_-(x) + a_+\chi_+(x)\,, \\ b(x) &= b_-\chi_-(x) + b_+\chi_+(x), \quad a_\pm, b_\pm, c, d \in \mathbb{C}, \quad \chi_\pm(x) = \frac{1}{2}(1 \pm \operatorname{sgn} x)\,. \end{aligned} \tag{23}$$

For this we notice that $S^1_{\mathbb{R},\theta}$, $S^1_{\mathbb{R}^+,\theta}$ represent Fourier convolution operators with discontinuous symbols

$$\begin{aligned} S^1_{\mathbb{R},\theta}\varphi &= W^0_{S_\theta}\varphi := \mathcal{F}^{-1}S_\theta\mathcal{F}\varphi\,, \\ S^1_{\mathbb{R}^+,\theta}\varphi &= r_+W^0_{S_\theta}\varphi := W_{S_\theta}\varphi\,, \end{aligned} \tag{24}$$

where r_+ is the restriction $r_+\varphi = \varphi|_{\mathbb{R}^+}$ and

$$\begin{aligned} S_\theta(\xi) &= \mathcal{F}g(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, \frac{(-x)dx}{\pi i(x^2 - \exp 2i\theta)} = \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{x-\exp i\theta} + \frac{1}{x+\exp i\theta}\right)e^{i\xi x}dx = \\ &= -\exp(i\xi\exp i\theta)\chi_+(\xi) + \exp(-i\xi\exp i\theta)\chi_-(\xi) = -\mathrm{sgn}\xi\,\exp(i|\xi|\exp i\theta)\,. \end{aligned} \tag{25}$$

Notice that the image of the function $S_\theta(\xi)$ on the complex plane $\mathbb{C}$ represents two spiral–like curves which start at -1 and $+1$ and twist around the origin (see Fig. 1 and 2 for different values of θ). For $\theta = \pi/2$ the curve degenerates into the interval $[-1, 1]$.

LEMMA 8. *Let $N = 1$. The operator B_o in* (23) *is invertible if and only if the following conditions hold* (*see* (4))*:*

(i) $a_\pm + S_\theta(\xi)b_\pm \neq 0 \qquad \xi \in \mathbb{R}$,
(ii) $g_{\beta_t}(a, b; t, \xi) \neq 0, \qquad \xi \in \mathbb{R}$,
(iii) $[\arg h_{1/2}(a, b; t, \xi)]_{\xi\in\mathbb{R}} + [\arg g_{\beta_t}(a, b; t, \xi)]_{\xi\in\mathbb{R}} = 0$,

where

$$\begin{aligned} h_{1/2}(a, b; t, \mu) &:= [a(t+0) + S_\theta(\xi)b(t+0)][a(t-0) + S_\theta(\xi)b(t-0)]^{-1}\,, \quad \xi \in \mathbb{R}\,, \\ g_{\beta_t}(a, b; t, \xi) &:= \frac{1}{2}[1 + \coth\pi(i\beta_t + \xi)]c(t+0)d^{-1}(t+0) + \\ &+ \frac{1}{2}[1 - \coth\pi(i\beta_t + \xi)]c(t-0)d^{-1}(t-0)\,, \quad \xi \in \mathbb{R} \end{aligned} \tag{26}$$

with (*see* (5))

$$c(t) = a(t) + b(t)\,, \quad d(t) = a(t) - b(t)\,, \quad \beta_t = \frac{1+\alpha_t}{p}\,, \quad \xi \in \mathbb{R}\,. \tag{27}$$

FIGURE 1. $\theta = 10^o$ and $\theta = 20^o$

Proof. The operator B_o can be represented as follows

$$B_o = \chi_- W^0_{a_- + \mathcal{S}_\theta b_-} + \chi_+ W^0_{a_+ + \mathcal{S}_\theta b_+}\,. \tag{28}$$

FIGURE 2. $\theta = 45^o$ and $\theta = 85^o$

From the results on paired convolution equations with scalar discontinuous presymbols, proved in [7], we get easily that the invertibility conditions for the operator (28) coincide with (i)–(iii). ∎

LEMMA 9. *If $a(x) \equiv a_o$, $b(x) \equiv b_o$ are constant $N \times N$ matrices then the operator B_o in* (23) *is invertible if and only if*

$$\det(a_o + S_\theta(\xi) b_o) \neq 0\,, \quad \xi \in \mathbb{R}\,. \tag{29}$$

Proof follows immediately since $B_o = W^0_{a_o + \mathcal{S}_\theta b_o}$ and (29) is well–known invertibility condition for this operator (see, e.g., [7, 14]). ∎

LEMMA 10. *Let $c, d \in \mathbb{C}$ (i.e. $N = 1$). The operator B_+ in* (23) *is invertible if and only if*

(i) $c + S_\theta(\xi) d \neq 0$, $\xi \in \mathbb{R}$;
(ii) $c - \coth \pi\,(i\beta + \xi)\, d \neq 0$, $\quad \beta = (1+\alpha)/p$, $\xi \in \mathbb{R}$;
(iii) $[\arg\{c + S_\theta(\xi) d\}]_{\xi\in\mathbb{R}} + [\arg\{c - \coth \pi\,(i\beta + \xi)\, d\}]_{\xi\in\mathbb{R}} = 0$.

Proof follows from the results of [7], since (see (24))

$$B_+ = r_+ W^0_{c + dS_\theta} = W_{c + d\mathcal{S}_\theta}$$

and the symbol $c + d\mathcal{S}_\theta(\xi)$ is piecewise–continuous with discontinuity at $\xi = 0$. ∎

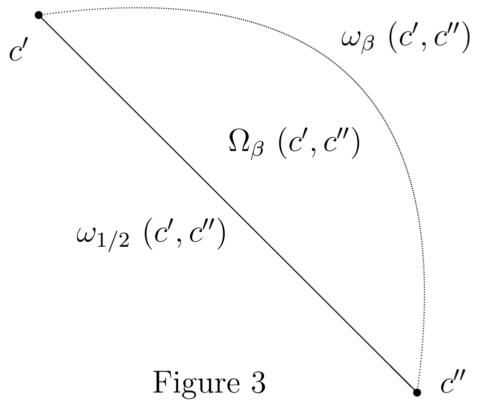

Figure 3

Let $c', c'' \in \mathbf{C}$, $0 < \beta < 1$, and let $\Omega_\beta(c', c'')$ denote the segment of the circle bounded by the straight line $\omega_{1/2}(c', c'')$ and by the part of the circular arc (see Fig. 3)

$$\omega_\beta(c', c'') = \Big\{\zeta \in \mathbf{C} : \zeta = \frac{c' + c''}{2} - \frac{c' - c''}{2} \coth \pi(i\beta + \xi),\ \xi \in \mathbb{R}\Big\}.$$

COROLLARY 11. *Let $\theta = \pi/2$ and $c, d \in \mathbb{C}$ (i.e. $N = 1$). Then the operator B_+ in (23) is invertible if and only if $0 \notin \Omega_\beta(c+d, c-d)$, $\beta = (1+\alpha)/p$.*

THEOREM 12. *Let Γ be a smooth closed curve and $a, b \in C^{N\times N}(\Gamma)$. Suppose the operator A in (2) is invertible in the space $L_p^N(\Gamma, \varrho)$; (see (1)). The sequence $\{A_\varepsilon\}_\varepsilon$ of Fredholm operators in (3) is stable if and only if the symbol $\sigma_A(x,\xi) = a(t) + b(t) sgn\,\xi$ $(t \in \Gamma,\ \xi \in \mathbb{R})$ satisfies the following condition:*

$$\inf\{|\det[a(t) + S_\theta(\xi)b(t)]| : t \in \Gamma\,,\ \xi \in \mathbb{R}\} > \nvdash\,. \tag{30}$$

Proof follows from Theorem 1 and Lemma 9. ∎

THEOREM 13. *Let Γ be as in Section 1, $\theta(t) \equiv \pi/2$, $a, b \in C^{N\times N}(\Gamma)$, and $p = 2$, $\varrho(t) \equiv 1$ if $N > 1$ or*

$$1 < p < \infty\,, \quad \frac{1+\alpha_j}{p} = \frac{1}{2}\,, \qquad j = 1, 2, ..., 2m \tag{31}$$

if $N = 1$. Suppose the operator A in (2) is invertible in the space $L_p^N(\Gamma, \varrho)$. The sequence $\{A_\varepsilon\}_\varepsilon$ of operators in (3) is stable if and only if the following condition holds

$$\inf\{|\det[a(t) + \mu b(t)]| : t \in \Gamma\,,\ -1 \le \mu \le 1\}\ . \tag{32}$$

Proof. Due to Theorem 1 we have to check the invertibility conditions for the operators $A^1_{t,\pi/2} = a(t) + b(t)S^1_{\mathbb{R},\pi/\nvdash} = W^o_{g_t}$, $g_t(\xi) := a(t) + S^1_{\pi/2}(\xi)b(t)$ (see (24)) in $L_p^N(\mathbb{R}, |\curvearrowleft|^{\alpha\approx})$ for $t \ne c_1, ..., c_{2m}$ and for the operators

$$A^1_{c_j,\pi/2} = a(t) + b(t)S^1_{\mathbb{R}^+,\pi/\nvdash} = W_{g_{c_j}}\ ,\ g_{c_j}(\xi) := a(c_j) + (-1)^{j+1}S_{\pi/2}(\xi)b(c_j)$$

in $L_p^N(\mathbb{R}^+, \curvearrowleft^{\alpha\daleth})$ for $j = 1, 2, ..., 2m$.

Since $S_{\pi/2}(\xi) = -\mathrm{sgn}\xi e^{-|\xi|} \equiv \mu \in [-1, 1]$, condition (32) reads

$$\inf\{|\det g_t(\xi)| : \xi \in \mathbb{R}\} > \nvdash \quad \text{for all} \quad \approx \in \Gamma\,. \tag{33}$$

For the operators $W^o_{g_t}$ the invertibility is ensured by (33) (see [7, 14]).

For the Wiener–Hopf operator $W_{g_{c_j}}$ in $L_p^N(\mathbb{R}^+, \curvearrowleft^{\alpha\daleth})$ condition (33) is only necessary, but not sufficient. For Fredholmness we have to impose the following restriction (see [8, 9])

$$\inf\{|\det h_j(\lambda)| : \lambda \in \mathbb{R}\} > \nvdash\,, \tag{34}$$

where

$$\begin{aligned} h_j(\lambda) &= \frac{1}{2}[1 - \coth\pi(i\beta_j + \lambda)]g_{c_j}(0-0) + \frac{1}{2}[1 + \coth\pi(i\beta_j + \lambda)]g_{c_j}(0+0) \\ &= a(c_j) + \coth\pi(i\beta_j + \lambda)b(c_j) = a(c_j) + \mu b(c_j)\,, \end{aligned}$$

since $\beta_j = 1/2$ (see (31)) and $\coth\pi(i/2 + \lambda) \equiv \mu \in [-1, 1]$. Therefore (34) coincides with (32).

For the index $\mathrm{Ind}W_{g_{c_j}}$ we have the formula (see [7, 8, 9])

$$\mathrm{Ind}W_{g_{c_j}} = -\frac{1}{2\pi}[\arg\det g_{c_j}(\xi)]_{\xi\in\mathbb{R}} + \frac{1}{2\pi}[\arg\det h_j(\lambda)]_{\xi\in\mathbb{R}} = 0\,.$$

This already yields the invertibility of the operator $W_{g_{c_j}}$ in the scalar case $N = 1$ (see [7]).

For the operator $W_{g_{c_j}}$ in the space $L_2^N(\mathbb{R}^+)$ we apply the strong ellipticity property: if (32) holds, then

$$Re\, e^{i\theta_o}(g_{c_j}(\xi)\eta,\eta) \geq c_o|\eta|^2 \tag{35}$$

for any $\xi \in \mathbb{R}$, $\eta \in \mathbb{C}^{\mathbb{N}}$ and some constants $0 \leq \theta_o < 2\pi$, $c_o > 0$. If we insert $\eta = \psi \in L_2^N(\mathbb{R})$, from (35) it follows after integrating that

$$Re\, e^{i\theta_o}(g_{c_j}\psi,\psi) \geq c_o||\psi|L_2^N(\mathbb{R})||^{\nvDash} \tag{36}$$

where $(\cdot,\cdot)$ stands now for the scalar product in $L_2^N(\mathbb{R})$.

Let ℓ_o be the extension operator by zero from $\mathbb{R}^+$ to $\mathbb{R}$. Then $\ell_o\varphi \in L_2^N(\mathbb{R})$ for any $\varphi \in L_2^N(\mathbb{R}^+)$ and we proceed with the help of (24) and (36) as follows

$$\begin{aligned} Re\, e^{i\theta_o}(W_{g_{c_j}}\varphi,\varphi) &= Re\, e^{i\theta_o}(r_+\mathcal{F}^{-1}g_{c_j}\mathcal{F}\ell_o\varphi,\varphi) \\ = Re\, e^{i\theta_o}(\mathcal{F}^{-1}g_{c_j}\mathcal{F}\ell_o\varphi,\ell_o\psi) = Re\, e^{i\theta_o}&(g_{c_j}\mathcal{F}\ell_o\varphi,\mathcal{F}\ell_o\psi) \geq 2\pi c_o||\varphi|L_2^N(\mathbb{R}^+)||^{\nvDash} \end{aligned} \tag{37}$$

since due to Parseval's equality we have

$$||\mathcal{F}\ell_o\psi|L_2^N(\mathbb{R})|| = \sqrt{\nvDash\pi}||\ell_{\rtimes}\varphi|\mathbb{L}_{\nvDash}^{\mathbb{N}}(\mathbb{R})|| = \sqrt{\nvDash\pi}||\varphi|\mathbb{L}_{\nvDash}^{\mathbb{N}}(\mathbb{R}^+)||\,.$$

The obtained inequality already implies that Ker $W_{g_{c_j}} = \{0\}$ and $W_{g_{c_j}}$ is normally solvable (i.e. the image $W_{g_{c_j}}L_2^N(\mathbb{R}^+)$ is closed). In fact, if one of these two properties fails there exists a sequence $\{\varphi_n\}_1^\infty \subset L_2^N(\mathbb{R}^+)$, $||\varphi_n|L_2^N(\mathbb{R}^+)|| = \Bbbk$ $(n = 1, 2, ...)$ such that $\lim_n W_{g_{c_j}}\varphi_n = 0$ (we can take $\varphi = \varphi_1 = \varphi_2 = ...$, $\varphi \in \mathrm{Ker}\, W_{g_{c_j}}$ if the latter is non–trivial). This leads to a contradiction, since (37) implies

$$||e^{i\theta_o}W_{g_{c_j}}\varphi|L_2^N(\mathbb{R}^+)|| \geq \sqrt{\rtimes}||\varphi|\mathbb{L}_{\nvDash}^{\mathbb{N}}(\mathbb{R}^+)||\,.$$

The adjoint operator $W^*_{g_{c_j}} = W_{\overline{g}_{c_j}}$ has a similar estimation. Therefore Coker $W_{g_{c_j}} \simeq$ Ker $W^*_{g_{c_j}} = \{0\}$ and $W_{g_{c_j}}$ is invertible in $L_2^N(\mathbb{R}^+)$. ∎

COROLLARY 14. *Let the conditions of Theorem 12 hold. The sequence* $\{A_\varepsilon\}_\varepsilon$ *of Fredholm operators in* (3) *is stable for any* $0 < \theta(t) < \pi$ *if and only if*

$$\det[a(t) + \zeta b(t)] \neq 0 \tag{38}$$

for all $t \in \Gamma$ *and* $\zeta \in \{\pm 1\} \cup \{\zeta \in \mathbb{C} : |\zeta| < \Bbbk\}$.

For $n = 1$ *condition* (38) *can be rewritten as follows:*

$$g_\pm(t) := a(t) \pm b(t) \neq 0 \quad \textit{and} \quad |\arg g_+(t) - \arg g_-(t)| \leq \frac{\pi}{2}$$

for all $t \in \Gamma$.

REMARK 15. *See also* [6, 17] *for the factorization of strongly elliptic matrix–functions and* [10, Sect. 3.6] *for more general assertions on pseudodifferential operators with local–sectorial symbols.*

4. SOME REMARKS ON ERROR ESTIMATES

Since, for fixed $\varepsilon > 0$, (3) is a Fredholm integral equation with smooth kernel, a wide variety of approximation methods applies to the numerical solution of equation (3), e.g., projection methods (such as Galerkin or collocation methods) and quadrature (Nyström) methods (see e.g. [1, 3, 4, 13, 16, 23]).

Assume that such an approximation method is given by the sequence of equations

$$A_\varepsilon^{(n)} \psi_{\varepsilon,n} = f_n \quad (n \in \mathbb{N}) \tag{39}$$

where $f_n \in X_n$ is known and $\psi_{\varepsilon,n} \in X_n$ is the approximate solution of equation (3) with X_n being a closed subspace of $L_p^N(\Gamma, \varrho)$. Then $\psi_{\varepsilon,n}$ can be viewed as an approximate solution of equation (2), too.

THEOREM 16. *Suppose the sequence $\{A_\varepsilon\}_\varepsilon$ and, for any fixed ε, the sequence $\{A_\varepsilon^{(n)}\}_n$ are stable. Assume $P_n : L_p^N(\Gamma, \varrho) \to X_n$ is a projection. Then for the solutions of the equations* (2)*,* (3) *and* (39) *the error estimate*

$$||\varphi - \psi_{\varepsilon,n}|| \leq c||A\varphi - A_\varepsilon\varphi|| + ||\psi - P_n\psi|| + C_\varepsilon(||f - f_n|| + ||A_\varepsilon\psi - A_\varepsilon^{(n)} P_n \psi||) \tag{40}$$

holds with

$$c = \sup_\varepsilon ||A_\varepsilon^{-1}||\,, \quad C_\varepsilon = \sup_n ||[A_\varepsilon^{(n)}]^{-1}||\,.$$

Proof follows immediately from the identities

$$\begin{aligned} \varphi - \psi &= A_\varepsilon^{-1}(A_\varepsilon\varphi - f)\,, \\ P_n\psi - \psi_{\varepsilon,n} &= [A_\varepsilon^{(n)}]^{-1}(A_\varepsilon^{(n)} P_n\psi - f_n) \end{aligned}$$

and the triangle inequality

$$||\varphi - \psi_{\varepsilon,n}|| \leq ||\varphi - \psi|| + ||\psi - \psi_{\varepsilon,n}||\,.$$

∎

Since for the aforementioned approximation methods estimates of the last three terms are known (see e.g. [1, 3, 4, 13, 16, 23]) the problem of estimating $||\varphi - \psi_{\varepsilon,n}||$ is reduced to estimating the term $||A\varphi - A_\varepsilon\varphi||$.

The following lemma gives a corresponding estimate in the particular case of a closed curve Γ. Notice that in this case the solution φ of (2) has the same regularity as f provided a, b and Γ are sufficiently smooth.

LEMMA 17. *Assume that Γ is a closed curve and $\varphi \in C^1(\Gamma)$. Then there is a positive constant C such that*

$$\max_{t\in\Gamma} |A\varphi(t) - A_\varepsilon\varphi(t)| \leq \varepsilon C \max_{t\in\Gamma} |\varphi'(t)|\,.$$

Proof. Since Γ is closed we have the relation

$$S_{\Gamma,\varepsilon}\psi(t) - S_\Gamma\psi(t) = \frac{\varepsilon^2 n^2(t)}{\pi i} \int_\Gamma \frac{1}{(\tau - t)^2 - n^2(t)\varepsilon^2} \frac{\psi(\tau) - \psi(t)}{\tau - t}\, d\tau\,. \tag{41}$$

Thus, it remains to estimate the integral

$$\int_\Gamma \frac{|d\tau|}{|(\tau - t)^2 - n^2(t)\varepsilon^2|} \leq C_1 \int_\Gamma \frac{|d\tau|}{|\tau - t|^2 + \varepsilon^2}\,.$$

Without loss of generality we may assume that, e.g., $l/4 < s = |t| < 3l/4$ where l is the length of the curve Γ. Hence we get

$$\int_\Gamma \frac{|d\tau|}{|\tau - t|^2 + \varepsilon^2} \le C_2 \int_0^l \frac{dy}{(y-s)^2 + \varepsilon^2} = \frac{C_2}{\varepsilon} \left[\text{arc tan } \frac{l-s}{\varepsilon} + \text{arc tan } \frac{s}{\varepsilon} \right] \le C_2 \pi / \varepsilon \,.$$

∎

Applying Hölder's inequality to (41), one obtains in a similar manner that

$$\max_{t \in \Gamma} |A\varphi(t) - A_\varepsilon \varphi(t)| \le \varepsilon^{1-\frac{1}{p}} C ||\varphi'||_{L_p^N(\Gamma)}$$

provided $\varphi' \in L_p^N(\Gamma)$ exists.

Concluding remarks. It was not the aim of this paper to give optimal estimates for the term $||A\varphi - A_\varepsilon \varphi||$. In a forthcoming paper we will compare by numerical experiments the efficiency of the method studied in the present paper with the efficiency of other well known methods for approximately solving equation (2) (see e.g. [23]).

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