

**BASIC BOUNDARY VALUE PROBLEMS OF  
THERMOELASTICITY FOR ANISOTROPIC BODIES  
WITH CUTS II**

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AND

**ABSTRACT.** In the first part [1] of the paper the basic boundary value problems of the mathematical theory of elasticity for three-dimensional anisotropic bodies with cuts were formulated. It is assumed that the two-dimensional surface of a cut is a smooth manifold of an arbitrary configuration with a smooth boundary. The existence and uniqueness theorems for boundary value problems were formulated in the Besov ( $B_{p,q}^s$ ) and Bessel-potential ( $H_i^\sim$ ) spaces. In the present part we give the proofs of the main results (Theorems 7 and 8) using the classical potential theory and the nonclassical theory of pseudodifferential equations on manifolds with a boundary.

This paper continues [1]. After recalling some auxiliary results, we prove Theorems 7 and 8 formulated in §3.

**§ 4. AUXILIARY RESULTS**

**4.1. Convolution Operators.**  $\mathbb{S}(\mathbb{R}^\times)$  denotes the space of  $C^\infty$ -smooth fast decaying functions, while  $\mathbb{S}'(\mathbb{R}^\times)$  stands for the dual space of tempered distributions. The Fourier transform and its inverse

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^\times} e^{i\xi x} \varphi(x) dx, \quad \mathcal{F}^{-1}\varphi(\xi) = (\infty)^{-1} \int_{\mathbb{R}^\times} e^{-i\xi x} \varphi(x) dx$$

are continuous operators in both spaces  $\mathbb{S}(\mathbb{R}^\times)$  and  $\mathbb{S}'(\mathbb{R}^\times)$ . Hence the convolution operator

$$\mathbf{a}(\mathbf{D})\varphi = \mathcal{F}^{-1}\mathcal{F}\varphi, \quad \varphi \in \mathbb{S}'(\mathbb{R}^\times), \quad \mathbf{a}(\mathbf{D})\varphi \in \mathbb{S}(\mathbb{R}^\times) \quad (4.1)$$

is a continuous transformation

$$\mathbf{a}(\mathbf{D}) : \mathbb{S}(\mathbb{R}^\times) \rightarrow \mathbb{S}'(\mathbb{R}^\times)$$

(cf. [2], [3]).

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If operator (4.1) has a bounded extension

$$\mathbf{a}(\mathbf{D}) : \mathbb{L}_i(\mathbb{R}^\kappa) \rightarrow \mathbb{L}_i(\mathbb{R}^\kappa), \quad \mathbb{M} \leq i \leq \infty,$$

we write  $a \in M_p(\mathbb{R}^\kappa)$  and  $a(\xi)$  is called the (Fourier)  $L_p$ -multiplier. Let

$$M_p^{(r)}(\mathbb{R}^\kappa) = \{(\mathbb{M} + |\xi|^\kappa)^{\wedge/\kappa} \mathcal{D}(\xi) : \mathcal{D} \in \mathbb{M}_i(\mathbb{R}^\kappa)\}.$$

Recall that the Bessel potential space  $\mathbb{H}_i^\sim(\mathbb{R}^\kappa)$  is defined as a subset of  $\mathbb{S}'(\mathbb{R}^\kappa)$  endowed with the norm

$$\begin{aligned} \|u|\mathbb{H}_i^\sim(\mathbb{R}^\kappa)\| &= \|\mathcal{I}^s(\mathbf{D})\mathbf{u}|\mathbb{L}_i(\mathbb{R}^\kappa)\|, \\ \mathcal{I}^s(\xi) &:= (\mathbf{1} + |\xi|^2)^{s/2}. \end{aligned} \quad (4.2)$$

Therefore due to the obvious property

$$\mathbf{a}_1(\mathbf{D})\mathbf{a}_2(\mathbf{D}) = (\mathbf{a}_1\mathbf{a}_2)(\mathbf{D}), \quad \mathbf{a}_j \in \mathbb{M}_p^{(r_j)}(\mathbb{R}^\kappa) \quad (4.3)$$

we easily find that the operator

$$\mathbf{a}(\mathbf{D}) : \mathbb{H}_i^\sim(\mathbb{R}^\kappa) \rightarrow \mathbb{H}_i^{\sim-\wedge}(\mathbb{R}^\kappa), \quad \sim, \wedge \in \mathbb{R}, \quad \mathbb{M} \leq i \leq \infty, \quad (4.4)$$

is bounded if and only if  $a \in M_p^{(r)}(\mathbb{R}^\kappa)$ .

The interpolation property

$$\begin{aligned} \mathbb{B}_{i,\mathbb{M}}^\sim(\mathbb{R}^\kappa) &= [\mathbb{H}_i^{\sim\mathbb{M}}(\mathbb{R}^\kappa), \mathbb{H}_i^{\sim\mathbb{M}}(\mathbb{R}^\kappa)]_{\theta,\mathbb{M}}, \\ 1 < p < \infty, \quad 1 \leq p \leq \infty, \quad s_1, s_2 \in \mathbb{R}, \\ s &= (1 - \theta)s_1 + \theta s_2, \quad 0 \leq \theta \leq 1 \end{aligned} \quad (4.5)$$

(see [4], [5]),  $a \in M_p^{(r)}(\mathbb{R}^\kappa)$  ensures the boundedness of the operator

$$\mathbf{a}(\mathbf{D}) : \mathbb{B}_{i,\mathbb{M}}^\sim(\mathbb{R}^\kappa) \rightarrow \mathbb{B}_{i,\mathbb{M}}^{\sim-\wedge}(\mathbb{R}^\kappa), \quad \mathbb{M} \leq \mathbb{M} \leq \infty. \quad (4.6)$$

Equality (4.2) and boundedness (4.4) imply that the operator

$$\mathcal{I}^r : \mathbb{H}_i^\sim(\mathbb{R}^\kappa) \rightarrow \mathbb{H}_i^{\sim-\wedge}(\mathbb{R}^\kappa) \quad (4.7)$$

arranges an isometrical isomorphism.

Further, it is well known that the operators

$$\begin{aligned} \mathcal{I}_+^r : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^\kappa) &\rightarrow \widetilde{\mathbb{H}}_i^{\sim-\wedge}(\mathbb{R}_+^\kappa), \quad \mathbb{R}_+^\kappa = \mathbb{R}^{\kappa-\mathbb{M}} \times \mathbb{R}^+, \quad \mathbb{R}^+ = [\mathbb{M}, +\infty), \\ \mathcal{I}_-^r : \mathbb{H}_i^\sim(\mathbb{R}_+^\kappa) &\rightarrow \mathbb{H}_i^{\sim-\wedge}(\mathbb{R}_+^\kappa), \quad \mathcal{I}_\pm^r(\xi) = (\xi_n \pm i|\xi'| \pm i)^r, \\ \xi &= (\xi', \xi^n) \in \mathbb{R}^\kappa, \quad \xi' \in \mathbb{R}^{\kappa-\mathbb{M}}, \end{aligned} \quad (4.8)$$

also arrange isomorphisms (though not isometrical ones; see, for example, [3], [6]). Isomorphisms similar to (4.8) exist for any smooth manifold with a Lipschitz boundary (for details see [3], [7]).

The equality  $M_2(\mathbb{R}^\times) = \mathbb{L}_\infty(\mathbb{R}^\times)$  is well-known and trivial. A reasonable description of the class  $M_p^r(\mathbb{R}^\times)$  for  $p \neq 2$  is less trivial and the problem still remains unsolved.

**Theorem 12** (see [8], Theorem 7.9.5; [9]). *Let  $1 < p < \infty$  and*

$$\sum_{\substack{|\beta| < [n/2] + 1 \\ 0 \leq \beta \leq 1}} \sup \{ |\xi^\beta D^\beta a(\xi)|, \xi \in \mathbb{R}^\times \} \leq M < \infty,$$

where for the multi-index  $\beta = (\beta_1, \dots, \beta_n)$  the inequality  $0 \leq \beta \leq 1$  reads as  $0 \leq \beta_j \leq 1$ ,  $j = 1, \dots, n$ . Then  $a \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^\times)$ .

If  $a \in M_p^{(r)}(\mathbb{R}^\times)$ , the operators

$$\begin{aligned} \mathbf{r}_+ \mathbf{a}(\mathbf{D}) : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^\times) &\rightarrow \mathbb{H}_i^{\sim - \searrow}(\mathbb{R}_+^\times) \\ &: \widetilde{\mathbb{B}}_{p,q}^s(\mathbb{R}_+^\times) \rightarrow \mathbb{B}_{i,ii}^{\sim - \searrow}(\mathbb{R}_+^\times) \end{aligned} \quad (4.9)$$

are bounded ( $1 < p < \infty$ ,  $s, r \in \mathbb{R}$ ,  $1 \leq q \leq \infty$ ); here  $\mathbf{r}_+ \varphi = \varphi|_{\mathbb{R}_+^\times}$  denotes the restriction operator.

An equality similar to (4.3)

$$\mathbf{r}_+ \mathbf{a}_1(\mathbf{D}) \ell_0 \mathbf{r}_+ \mathbf{a}_2(\mathbf{D}) = \mathbf{r}_+ (\mathbf{a}_1 \mathbf{a}_2)(\mathbf{D}), \quad \mathbf{a}_j \in \mathbf{M}_p^{(\mathbf{r}_j)}(\mathbb{R}^\times), \quad (4.10)$$

where  $\ell_0$  is extension by 0 from  $\mathbb{R}_+^\times$  to  $\mathbb{R}^\times$ , fails to be fulfilled in general. However, (4.10) holds if there is an analytic extension either  $a_1(\xi', \xi_n - i\lambda)$  or  $a_2(\xi', \xi_n + i\lambda)$ , which can be estimated from above by  $C(1 + |\xi| + \lambda)^N$  with  $N > 0$ ,  $\lambda > 0$ ,  $C = \text{const}$ .

**4.2. Pseudodifferential operators.** If the symbol  $a(x, \xi)$  depends on the variable  $x$ , the corresponding convolution (cf. (4.1))

$$\mathbf{a}(\mathbf{x}, \mathbf{D}) \varphi(\mathbf{x}) := \mathcal{F}_{\xi \rightarrow \mathbf{x}}^{-\infty} \dashv(\mathbf{x}, \cdot) \mathcal{F}_{\mathbf{x} \rightarrow \xi} \varphi(\xi) \quad (4.11)$$

is called the pseudodifferential operator ( $\varphi \in \mathbb{S}(\mathbb{R}^\times)$ ,  $|a(x, \xi)| < C(1 + |\xi|)^N$ ,  $N > 0$ ,  $C = \text{const}$ ).

Let  $M_p^{(s, s-r)}(\mathbb{R}^\times \times \mathbb{R}^\times)$  denote a class of symbols  $a(x, \xi)$  for which operator (4.11) can be extended to the bounded mapping

$$\mathbf{a}(\mathbf{x}, \mathbf{D}) : \mathbb{H}_i^\sim(\mathbb{R}^\times) \rightarrow \mathbb{H}_i^{\sim - \searrow}(\mathbb{R}^\times). \quad (4.12)$$

By  $S^r(\Omega \times \mathbb{R}^\times)$  ( $\Omega \subset \mathbb{R}^\times$ ,  $r \in \mathbb{R}$ ) is denoted the Hörmander class of symbols  $a(x, \xi)$  if

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq M_{\alpha, \beta} (1 + |\xi|)^{r - |\beta|}, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^\times, \quad (4.13)$$

where  $M_{\alpha, \beta}$  is independent of  $x$  and  $\xi$ .

By  $S_r^{l, m}(\Omega \times \mathbb{R}^\times)$  ( $\Omega \subset \mathbb{R}^\times$ ,  $l, m \in \mathbb{Z}^+$ ,  $r \in \mathbb{R}$ ) we denote the class of symbols  $a(x, \xi)$  having the estimations

$$\int_{\Omega} |D_x^\alpha (\xi D_\xi)^\beta a(x, \xi)| dx \leq M'_{\alpha, \beta} (1 + |\xi|)^r$$

$$\forall \xi \in \mathbb{R}^\times, \quad |\alpha| \leq \ll, \quad |\beta| \leq \gg,$$

where

$$(\xi D_\xi)^\beta := (\xi_1 D_{\xi_1})^{\beta_1} \dots (\xi_n D_{\xi_n})^{\beta_n}.$$

If  $\Omega \subset \mathbb{R}^\times$  is compact, then  $S^r(\Omega \times \mathbb{R}^\times) \subset \mathbb{S}_<^{<, >}(\Omega \times \mathbb{R}^\times)$ . Such an inclusion does not hold for non-compact  $\Omega$ .

**Theorem 13.** *Let  $s, r \in \mathbb{R}$ ,  $l, m \in \mathbb{Z}^+$ ,  $m > [n/2] + 1$ ; then*

$$S^r(\mathbb{R}^\times \times \mathbb{R}^\times) \subset \mathbb{M}_l^{(\sim, \sim - \sim)}(\mathbb{R}^\times \times \mathbb{R}^\times).$$

*If, additionally,  $-l + 1 + 1/p < s - r < l + 1/p$ , then*

$$S_r^{l+n, m}(\mathbb{R}^\times \times \mathbb{R}^\times) \subset \mathbb{M}_l^{(\sim, \sim - \sim)}(\mathbb{R}^\times \times \mathbb{R}^\times).$$

*Proof.* When a symbol  $a \in S^0(\mathbb{R}^\times \times \mathbb{R}^\times)$  has a compact support with respect to  $x$ , then the continuity of  $\mathbf{a}(\mathbf{x}, \mathbf{D})$  in  $\mathbb{L}_l(\mathbb{R}^\times)$  follows from Theorem 12, as shown in [10].

For an arbitrary  $a \in S^0(\mathbb{R}^\times \times \mathbb{R}^\times)$  the above statement is proved for  $\mathbb{L}_l(\mathbb{R}^\times)$  using the arguments involved in the proof of Theorem 3.5 from [12]. In the general case the continuity of the mapping  $\mathbb{H}_l^\sim(\mathbb{R}^\times) \rightarrow \mathbb{H}_l^{\sim - \sim}(\mathbb{R}^\times)$  is established with the aid of the order reduction operator (4.7) (see [4], [10]), while the continuity of the mapping  $\mathbf{a}(\mathbf{x}, \mathbf{D}) : \mathbb{B}_{l, ll}^\sim(\mathbb{R}^\times) \rightarrow \mathbb{B}_{l, ll}^{\sim - \sim}(\mathbb{R}^\times)$  is proved by interpolation (see [4]).

For a different proof of the first claim see [11].

To prove the second claim we shall introduce some notation. For a multi-index  $\mu = (\mu_1, \dots, \mu_n)$ ,  $0 \leq \mu \leq 1$  we define

$$dx^\mu := \prod_{\substack{\mu_j=1 \\ j=1,2,\dots,n}} dx_j, \quad (x, h)_\mu := (z_1, \dots, z_n),$$

$$z_j = \begin{cases} x_j, & \text{if } \mu_j = 1, \\ h_j, & \text{if } \mu_j = 0, \end{cases} \quad x, h \in \mathbb{R}^\times.$$

Let

$$a_{(\alpha)}(x, \xi) := D_x^\alpha a(x, \xi).$$

By virtue of Theorem 12 the inclusion  $a \in S_r^{l,m}(\mathbb{R}^\times \times \mathbb{R}^\times)$  implies

$$\int_{\mathbb{R}^\times} \|D_x^\alpha a(x, \cdot) | M_p^{(r)}(\mathbb{R}^\times)\| \cap < \infty, \quad |\alpha| \leq < + \times.$$

From this finiteness and Fubini's theorem we get

$$\text{mes}_{\mathbb{R}^\times} \Delta_{\mu, \gamma} = 0 \quad \text{for any } 0 \leq \mu \leq 1, \quad |\gamma| \leq l,$$

where

$$\Delta_{\mu, \gamma} := \left\{ h \in \mathbb{R}^\times : \int_{\mathbb{R}^{|\mu|}} \|\mathcal{D}_{(\mu+\gamma)}((\cap, \bar{\gamma})_\mu, \cdot) | \mathbb{M}_i^{(\cap)}(\mathbb{R}^\times)\| \cap^\mu = \infty \right\}.$$

If now

$$\Delta = \bigcup_{\substack{0 \leq \mu \leq 1 \\ |\gamma| \leq l}} \Delta_{\mu, \gamma}$$

then, obviously,  $\text{mes}_{\mathbb{R}^\times} \Delta = 0$ . There exists a vector  $h_0 \in \mathbb{R}^\times \setminus \Delta$ . Then we have

$$\int_{\mathbb{R}^\times} \|a_{(\mu+\gamma)}((y, h_0)_\mu, \cdot) | M_p^{(r)}(\mathbb{R}^\times)\| \cap^\mu < \infty.$$

With these conditions we can use Theorem 5.1 and Remark 5.5 from [20] where the claimed inclusion  $a \in M_p^{(s, s-r)}(\mathbb{R}^\times \times \mathbb{R}^\times)$  is proved.  $\square$

Let

$$\mathbf{A}, \mathbf{B} : \mathbb{H}_i^\sim(\mathbb{R}^\times) \rightarrow \mathbb{H}_i^{\sim-\sim}(\mathbb{R}^\times)$$

be the bounded operators; they are called to be locally equivalent at  $x_0 \in \mathbb{R}^\times$  (see [3], [13]) if

$$\inf \{ \|\chi(\mathbf{A} - \mathbf{B})\| : \chi \in \mathbf{C}_{\mathbf{x}_0}(\mathbb{R}^\times) \} = \inf \{ \|(\mathbf{A} - \mathbf{B})\chi \mathbf{I}\| : \chi \in \mathbf{C}_{\mathbf{x}_0}(\mathbb{R}^\times) \} = \nu,$$

where  $\mathbf{I}$  is the identity operator and  $C_{x_0}(\mathbb{R}^\times) = \{\chi \in \mathbb{C}_\nu^\infty(\mathbb{R}^\times) : \chi(\curvearrowleft) = \mathbb{1}\}$  in some neighbourhood of  $x_0$ . In such a case we write  $\mathbf{A} \xrightarrow{x_0} \mathbf{B}$ . In a similar manner we define the equivalence  $\mathbf{A}_0 \xrightarrow{x_0} \mathbf{B}_0$  for operators

$$\mathbf{A}_0, \mathbf{B}_0 : \tilde{\mathbb{H}}_p^s(\mathbb{R}_+^\times) \rightarrow \mathbb{H}_i^{\sim - \sim}(\mathbb{R}_+^\times).$$

Assume now that  $\bar{S} = S \cup \partial S$  is a compact  $n$ -dimensional  $C^\infty$ -smooth manifold with a  $C^\infty$ -smooth boundary  $\partial S$  and

$$S = \bigcup_{j=1}^N V_j, \quad \varkappa_j : X_j \rightarrow V_j, \quad X_j \subset \mathbb{R}_+^\times \quad (4.14)$$

are coordinate diffeomorphisms. Let  $\{\chi_j\}_1^N \subset C_0^\infty(S)$  be a partition of the unity subordinated to the covering of  $S$  in (4.14); also let

$$\varkappa_{j*} \varphi(t) = \chi_j^0 \varphi(\chi_j(t)), \quad \varkappa_j^{-1} \psi(x) = \chi_j \psi(\varkappa_j^{-1}(x)),$$

where  $\chi_j^0(t) := \chi_j(\varkappa_j(t))$ ,  $t \in \mathbb{R}_+^\times$ ,  $x \in S$ . The following mapping properties

$$\begin{aligned} \varkappa_{j*} : \mathbb{H}_i^\sim(\mathbb{S}) &\rightarrow \mathbb{H}_i^\sim(\mathbb{R}_+^\times), \quad \text{supp } \varkappa_j^{-\mathbb{1}} \cap \partial \mathbb{S} \neq \emptyset, \\ \varkappa_{j*} : \tilde{\mathbb{H}}_p^r(S) &\rightarrow \tilde{\mathbb{H}}_p^r(\mathbb{R}_+^\times), \quad \text{supp } \varkappa_j^{-\mathbb{1}} \cap \partial \mathbb{S} \neq \emptyset, \\ \varkappa_{j*} : \mathbb{H}_i^\sim(\mathbb{S}) &\rightarrow \mathbb{H}_i^\sim(\mathbb{R}^\times), \quad \text{supp } \varkappa_j^{-\mathbb{1}} \cap \partial \mathbb{S} = \emptyset. \end{aligned} \quad (4.15)$$

are almost evident.

A bounded operator

$$\mathbf{A} : \tilde{\mathbb{H}}_p^\nu(\mathbf{S}) \rightarrow \mathbb{H}_i^{\nu - \sim}(\mathbb{S}) \quad (4.16)$$

is called pseudodifferential (of order  $r$ ) if:

- (i)  $\chi_1 \mathbf{A} \chi_2 \mathbf{I}$  is a compact operator in  $\tilde{\mathbb{H}}_p^r(S) \rightarrow \mathbb{H}_i^{\nu - \sim}(\mathbb{S})$  for any  $\chi_1, \chi_2 \in C_0^\infty(S)$  with disjoint supports  $\text{supp } \chi_1 \cap \text{supp } \chi_2 = \emptyset$ ;
- (ii)

$$\begin{aligned} \varkappa_{j*} \mathbf{A} \varkappa_{j*}^{-1} &\xrightarrow{x_0} \mathbf{a}(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \mathbf{S}, \\ \varkappa_{j*} \mathbf{A} \varkappa_{j*}^{-1} &\xrightarrow{x_0} \mathbf{r}_+ \mathbf{a}(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \partial \mathbf{S}, \end{aligned} \quad (4.17)$$

where  $a(x_0, \cdot) \in M_p^{(r)}(\mathbb{R}^\times)$  for any  $x_0 \in \bar{S}$ .

**Example 14** (see [3], Example 3.19]). Let  $\bar{\Omega} \subset \mathbb{R}^\times$  be a compact domain with a smooth boundary  $\partial \Omega \neq \emptyset$ .

The operator  $\mathbf{r}_\Omega \mathbf{a}(\mathbf{x}, \mathbf{D})$ , where  $a(x, \xi) \in S^r(\Omega \times \mathbb{R}^\times)$  and  $\mathbf{r}_\Omega \varphi = \varphi|_\Omega$  denotes the restriction, is a pseudodifferential one of order  $r$  and

$$\begin{aligned} \mathbf{r}_\Omega \mathbf{a}(\mathbf{x}, \mathbf{D}) &\xrightarrow{x_0} \mathbf{a}(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \notin \partial \Omega, \\ \mathbf{r}_\Omega \mathbf{a}(\mathbf{x}, \mathbf{D}) &\xrightarrow{x_0} \mathbf{r}_+ \mathbf{a}(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \partial \Omega. \end{aligned} \quad (4.18)$$

If  $a(x_0, \xi)$  has the radial limits

$$a^\infty(x_0, \xi) = \lim_{\lambda \rightarrow \infty} \lambda^{-r} a(x_0, \lambda \xi) \quad (4.19)$$

which are non-trivial bounded functions of  $\xi$ , then  $a^\infty(x_0, \xi)$  is a homogeneous function of order  $r$  with respect to  $\xi$ :

$$a^\infty(x_0, \lambda \xi) = \lambda^r a^\infty(x_0, \xi), \quad \lambda > 0.$$

Let

$$a^0(x_0, \xi) = a^\infty(x_0, (1 + |\xi'|)|\xi'|^{-1}\xi', \xi_n) \quad (4.20)$$

represent the modified symbol (see [6], Section 3). Assume that  $a^0 \in M_p^{(r)}(\mathbb{R}^\times)$ ; then using (4.17) and the relation

$$\lim_{R \rightarrow \infty} \sup_{|\xi| \geq R} |\xi|^{-r} |a(x_0, \xi) - a^0(x_0, \xi)| = 0$$

we obtain

$$\begin{aligned} \varkappa_{j_*} \mathbf{A} \varkappa_{j_*}^{-1} &\xrightarrow{x_0} \mathbf{a}^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \notin \Omega, \\ \varkappa_{j_*} \mathbf{A} \varkappa_{j_*}^{-1} &\xrightarrow{x_0} \mathbf{r}_+ \mathbf{a}^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \Omega. \end{aligned} \quad (4.21)$$

Thus the operators  $\chi[\mathbf{a}(\mathbf{x}_0, \mathbf{D}) - \mathbf{a}^0(\mathbf{x}_0, \mathbf{D})]$ ,  $[\mathbf{a}(\mathbf{x}_0, \mathbf{D}) - \mathbf{a}^0(\mathbf{x}_0, \mathbf{D})]\chi \mathbf{I}$  with  $\chi \in C_0^\infty(\mathbb{R}^\times)$  are compact in  $\mathbb{H}_i^\nu(\mathbb{R}^\times) \rightarrow \mathbb{H}_i^{\nu-\infty}(\mathbb{R}^\times)$  (see [3]). As for the compact operator  $\mathbf{T} : \mathbb{H}_i^\nu(\mathbb{R}^\times) \rightarrow \mathbb{H}_i^{\nu-\infty}(\mathbb{R}^\times)$ , the equivalence  $\mathbf{T} \xrightarrow{x_0} \mathbf{0}$  holds automatically.

The functions  $a^\infty(x_0, \xi)$  (see (4.19)) and  $a^0(x_0, \xi)$  (see (4.20)) are respectively called the homogeneous principal symbol and the modified principal symbol of the operator  $\mathbf{A}$ .

**Theorem 15** (see [3]). *Let (4.16) be a pseudodifferential operator ( $r, \nu \in \mathbb{R}, 1 < \nu < \infty$ ).  $\mathbf{A}$  is a Fredholm operator if and only if the following conditions are fulfilled:*

- (i)  $\inf\{|\det a^\infty(x_0, \xi)| : x_0 \in \bar{S}, \xi \in \mathbb{R}^\times\} > 0$ ;
- (ii)  $\mathbf{r}_+ \mathbf{a}_{\nu, \mathbf{r}}(\mathbf{x}_0, \mathbf{D})$  is a Fredholm operator in the space  $\mathbb{L}_i(\mathbb{R}_+^\times)$  for any  $x_0 \in \partial S$ , where

$$\begin{aligned} a_{\nu, r}(x_0, \xi) &= (\xi_n - i|\xi'| - i)^{\nu-r} a^0(x_0, \xi) (\xi_n + i|\xi'| + i)^{-\nu}, \\ \xi &= (\xi', \xi_n), \quad \xi' \in \mathbb{R}^{\times-\#}. \end{aligned}$$

**Theorem 16** (see [3]). *Let  $\mathbf{a}(\mathbf{x}, \mathbf{D})$  be a pseudodifferential operator of the order  $r \in \mathbb{R}$  with the  $N \times N$  matrix symbol  $a(x, \cdot) \in S^r(\mathbb{R}^\times)$  for any*

$x \in \overline{S}$ . If  $a(x, \xi)$  is positive definite, i.e.

$$(a(x, \xi)\eta, \eta) \geq \delta_0 |\xi|^r |\eta|^2 \quad \text{for some } \delta_0 > 0 \quad (4.22)$$

and any  $\xi \in \mathbb{R}^\times$ ,  $\eta \in \overline{\mathbb{S}}$ ,  $\eta \in \mathbb{C}^N$ .

Then

$$\mathbf{a}(\mathbf{x}, \mathbf{D}) : \widetilde{\mathbb{H}}_2^{\frac{r}{2} + \nu}(\mathbf{S}) \rightarrow \mathbb{H}_\neq^{-\frac{r}{2} + \nu}(\mathbb{S}) \quad (4.23)$$

is a Fredholm operator for any  $|\nu| < \frac{1}{2}$  and

$$\text{Ind } \mathbf{a}(\mathbf{x}, \mathbf{D}) = \mathbf{0}. \quad (4.24)$$

**4.3. Further Auxiliary Results.** Let  $\mathcal{H}^\nabla(\mathbb{R}^\times)$  denote the class of functions with the properties

- (i)  $a(\lambda\xi) = \lambda^r a(\xi)$ ,  $\lambda > 0$ ,  $\xi \in \mathbb{R}^\times$ ;
- (ii)  $a(\omega) \in C^\infty(S^{n-1})$ ,  $S^{n-1} = \{\omega \in \mathbb{R}^\times : |\omega| = 1\}$ ;
- (iii) if  $a(\xi) = a_0(\omega', t, \xi_n)$ , where  $\omega' = |\xi'|^{-1}\xi'$ ,  $t = |\xi'|$ ,  $\xi = (\xi', \xi_n) \in \mathbb{R}^\times$ , then

$$\lim_{t \rightarrow 0} D_t^k a_0(\omega', t, -1) = (-1)^k \lim_{t \rightarrow 0} D_t^k a_0(\omega', t, 1), \quad (4.25)$$

$\omega' \in S^{n-2}$ ,  $k = 0, 1, 2, \dots$

For  $r = 0$  condition (4.25) coincides with the well-known transmission property (see [6,14]).

**Lemma 17.** *Let  $a \in \mathcal{H}^\nabla(\mathbb{R}^\times)$  be a positive definite  $N \times N$  matrix-function (cf. (4.22))*

$$(a(\xi)\eta, \eta) \geq \delta_0 |\xi|^r |\eta|^2 \quad \text{for some } \delta_0 > 0 \quad (4.26)$$

and any  $\xi \in \mathbb{R}^\times$ ,  $\eta \in \mathbb{C}^N$ .

Then  $a(\xi)$  admits the factorization

$$a(\xi) = a_-(\xi)a_+(\xi), \quad a_\pm(\xi) = (\xi_n \pm i|\xi'|)^{-\frac{r}{2}} b_\pm(\xi), \quad (4.27)$$

where  $b_\pm^{\pm 1}(\xi', \xi_n + i\lambda)$ ,  $b_\pm^\pm(\xi', \xi_n - i\lambda)$  have uniformly bounded analytic extensions for  $\lambda > 0$ ,  $\xi' \in \mathbb{R}^{\times - \neq}$ ,  $\xi_n \in \mathbb{R}$  and

$$\sum_{|\alpha| \leq m} \sup \{ |\xi^\alpha D^\alpha b_\pm^{\pm 1}(\xi)| : \xi \in \mathbb{R}^\times \} \leq \mathbb{M}_> < \infty, \quad (4.28)$$

$$m = 0, 1, 2, \dots$$

*Proof.* For the proof of this lemma see [2,9,15].  $\square$

*Remark 18.* A lemma similar to the above one but for a general elliptic symbol was proved in [2,9] (see [6] for the scalar case  $N = 1$ ). In [15, §2] a similar but more general assertion is proved when  $a(x, \xi)$  depends smoothly on a parameter  $x \in S$ .

A pair of Banach spaces  $\{\mathbb{X}_\nu, \mathbb{X}_\mu\}$  embedded in some topological space  $\mathbb{E}$  is called an interpolation pair. For such a pair we can introduce the following two spaces  $\mathbb{X}_{\min} = \mathbb{X}_\nu \cap \mathbb{X}_\mu$  and  $\mathbb{X}_{\max} = \mathbb{X}_\nu + \mathbb{X}_\mu := \{\cdot \in \mathbb{E} : \cdot = \cdot_\nu + \cdot_\mu, \cdot_\nu, \cdot_\mu \in \mathbb{X}_\nu, \nu = \nu, \mu = \mu\}$ ;  $\mathbb{X}_{\min}$  and  $\mathbb{X}_{\max}$  become Banach spaces if they are endowed with the norms

$$\begin{aligned} \|x|\mathbb{X}_{\min}\| &= \max \{ \|\cdot_\nu|\mathbb{X}_\nu\|, \|\cdot_\mu|\mathbb{X}_\mu\| \}, \\ \|x|\mathbb{X}_{\max}\| &= \inf \{ \|\cdot_\nu|\mathbb{X}_\nu\| + \|\cdot_\mu|\mathbb{X}_\mu\| : \cdot = \cdot_\nu + \cdot_\mu, \cdot_\nu \in \mathbb{X}_\nu, \nu = \nu, \cdot_\mu \in \mathbb{X}_\mu, \mu = \mu \}, \end{aligned}$$

respectively.

Besides, we have the continuous embeddings

$$\mathbb{X}_{\min} \subset \mathbb{X}_\nu, \mathbb{X}_\mu \subset \mathbb{X}_{\max}. \quad (4.29)$$

For any interpolation pairs  $\{\mathbb{X}_\nu, \mathbb{X}_\mu\}$  and  $\{\mathbb{Y}_\nu, \mathbb{Y}_\mu\}$  the space  $\mathcal{L}(\{\mathbb{X}_\nu, \mathbb{X}_\mu\}, \{\mathbb{Y}_\nu, \mathbb{Y}_\mu\})$  consists of all linear operators from  $\mathbb{X}_{\max}$  into  $\mathbb{Y}_{\max}$  whose restrictions to  $\mathbb{X}_j$  belong to  $\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)$  ( $j = 0, 1$ ). The notation  $\mathcal{L}(\mathbb{X}, \mathbb{Y})$  is used for the space of all linear bounded operators  $\mathbf{A} : \mathbb{X} \rightarrow \mathbb{Y}$ .

**Lemma 19.** *Assume  $\{\mathbb{X}_\nu, \mathbb{X}_\mu\}$  and  $\{\mathbb{Y}_\nu, \mathbb{Y}_\mu\}$  to be interpolation pairs and the embeddings  $\mathbb{X}_{\min} \subset \mathbb{X}_{\max}$ ,  $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$  to be dense. Let an operator  $\mathbf{A} \in \mathcal{L}(\mathbb{X}_\nu, \mathbb{Y}_\nu) \cap \mathcal{L}(\mathbb{X}_\mu, \mathbb{Y}_\mu)$  have a common regularizer:  $\mathbf{R} \in \mathcal{L}(\mathbb{Y}_\nu, \mathbb{X}_\nu) \cap \mathcal{L}(\mathbb{Y}_\mu, \mathbb{X}_\mu)$  and  $\mathbf{R}\mathbf{A} - \mathbf{I} \in \mathcal{L}(\mathbb{X}_\nu, \mathbb{X}_\nu) \cap \mathcal{L}(\mathbb{X}_\mu, \mathbb{X}_\mu)$  be compact. Then*

$$\mathbf{A} : \mathbb{X}_{\min} \rightarrow \mathbb{Y}_{\min}, \quad \mathbf{A} : \mathbb{X}_{>0} \rightarrow \mathbb{Y}_{\max}$$

are Fredholm operators and

$$\text{Ind}_{\mathbb{X}_{\min} \rightarrow \mathbb{Y}_{\min}} \mathbf{A} = \text{Ind}_{\mathbb{X}_{\max} \rightarrow \mathbb{Y}_{\max}} \mathbf{A} = \text{Ind}_{\mathbb{X}_j \rightarrow \mathbb{X}_j} \mathbf{A}, \quad j = 0, 1. \quad (4.30)$$

If  $y \in \mathbb{Y}_j$ , then any solution  $x \in \mathbb{X}_{\max}$  of the equation  $\mathbf{A}x = y$  belongs to  $\mathbb{X}_j$ . In particular,

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} = \ker_{\mathbb{X}_j} \mathbf{A} = \ker_{\mathbb{X}_{\max}} \mathbf{A}, \quad j = 0, 1. \quad (4.31)$$

*Proof.* We begin by noting that the definition of a norm in  $\mathbb{X}_{\min}, \dots, \mathbb{Y}_{\max}$  implies

$$\begin{aligned} \|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})\| &\leq \max \{ \|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = \nu, \mu \}, \\ \|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max})\| &\leq \max \{ \|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = \nu, \mu \}. \end{aligned}$$

Whence we find

$$\mathcal{L}(\mathbb{X}_\mathcal{K}, \mathbb{Y}_\mathcal{K}) \cap \mathcal{L}(\mathbb{X}_\mathcal{K}, \mathbb{Y}_\mathcal{K}) \subset \mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \cap \mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}).$$

Next we shall prove that  $\mathbf{A}$  is a Fredholm operator in the spaces  $\mathbb{X}_{\min} \rightarrow \mathbb{Y}_{\min}$  and  $\mathbb{X}_{\max} \rightarrow \mathbb{Y}_{\max}$ . For this it suffices to show that  $\mathbf{A}\mathbf{R} - \mathbf{I}$ ,  $\mathbf{R}\mathbf{A} - \mathbf{I}$  are compact in the spaces  $\mathbb{X}_{\min}$  and  $\mathbb{X}_{\max}$ , since by the conditions of the lemma they are compact in  $\mathbb{X}_\mathcal{K}$  and  $\mathbb{X}_\mathcal{K}$ . Let us prove a more general inclusion

$$\text{Com}(\mathbb{X}_\mathcal{K}, \mathbb{Y}_\mathcal{K}) \cap \text{Com}(\mathbb{X}_\mathcal{K}, \mathbb{Y}_\mathcal{K}) \subset \text{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \text{Com}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}),$$

that implies the claimed assertion.

Assume  $\mathbf{T} : \mathbb{X}_j \rightarrow \mathbb{Y}_j$  ( $j = 0, 1$ ) to be compact and  $\{x_k\}_{k \in \mathbb{N}}$  to be an arbitrary bounded sequence in  $\mathbb{X}_{\min}$ . Then  $\{x_k\}_{k \in \mathbb{N}}$  is bounded in both spaces  $\mathbb{X}_\mathcal{K}$  and  $\mathbb{X}_\mathcal{K}$ . It can be assumed without loss of generality that the sequences  $\{\mathbf{T}x_k\}_{k \in \mathbb{N}}$  are convergent in both  $\mathbb{Y}_\mathcal{K}$  and  $\mathbb{Y}_\mathcal{K}$  (otherwise we can select subsequences). Then  $\{\mathbf{T}x_k\}_{k \in \mathbb{N}}$  is convergent in  $\mathbb{Y}_{\min}$  and therefore  $\mathbf{T} \in \text{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})$ .

If  $S_0$ ,  $S_1$  and  $S_{\max}$  denote the unit balls in  $\mathbb{X}_\mathcal{K}$ ,  $\mathbb{X}_\mathcal{K}$  and  $\mathbb{X}_{\max}$ , respectively, then  $S_{\max} \subset S_0 + S_1$ . Due to the compactness of  $\mathbf{T} : \mathbb{X}_j \rightarrow \mathbb{Y}_j$  ( $j = 0, 1$ ), there exist  $\varepsilon/2$ -grids  $\{y_k^{(j)}\}_{k=1}^{m_j} \subset \mathbf{T}(S_j)$  ( $j = 0, 1$ ),  $\varepsilon = 0$ . Then  $\{y_k^{(0)} + y_n^{(1)}\}_{k,n} \subset \mathbf{T}(S_0) + \mathbf{T}(S_1)$  defines an  $\varepsilon$ -grid in  $\mathbf{T}(S_{\max})$  ( $\subset \mathbf{T}(S_0) + \mathbf{T}(S_1)$ ). Since  $\varepsilon > 0$  is arbitrary,  $\mathbf{T} : \mathbb{X}_{\max} \rightarrow \mathbb{Y}_{\max}$  is compact.

Now we shall show that the density of the embedding  $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$  implies the density of  $\mathbb{Y}_{\min} \subset \mathbb{Y}_j$  ( $j = 0, 1$ ). For the sake of definiteness assume that  $j = 0$ . By the condition of the lemma for any  $\varepsilon > 0$  and  $a \in \mathbb{Y}_\mathcal{K}$  there exists  $h \in \mathbb{Y}_{\min}$  with the property

$$\|(a - b)|\mathbb{Y}_{\max}\| < \varepsilon;$$

i.e. there exist  $a_0 \in \mathbb{Y}_\mathcal{K}$ ,  $a_1 \in \mathbb{Y}_\mathcal{K}$  such that  $a - b = a_0 + a_1$ ,

$$\|a_0|\mathbb{Y}_\mathcal{K}\| + \|\partial_\mathcal{K}|\mathbb{Y}_\mathcal{K}\| < \varepsilon.$$

Since  $a \in \mathbb{Y}_\mathcal{K}$  and  $b \in \mathbb{Y}_{\min} \subset \mathbb{Y}_\mathcal{K}$ , we obtain  $a - b \in \mathbb{Y}_\mathcal{K}$  and  $a_1 = (a - b) - a_0 \in \mathbb{Y}_\mathcal{K}$ . So that  $a_1 \in \mathbb{Y}_\mathcal{K} \cap \mathbb{Y}_\mathcal{K} = \mathbb{Y}_{\min}$  and  $a_1 + b \in \mathbb{Y}_{\min}$ . Therefore

$$\|[a - (a_1 + b)]|\mathbb{Y}_\mathcal{K}\| = \|\partial_\mathcal{K}|\mathbb{Y}_\mathcal{K}\| < \varepsilon,$$

which proves that the embedding  $\mathbb{Y}_{\min} \subset \mathbb{Y}_\mathcal{K}$  is dense.

The density of the embeddings  $\mathbb{Y}_{\min} \subset \mathbb{Y}_j \subset \mathbb{Y}_{\max}$ ,  $j = 0, 1$ , yields

$$\mathbb{Y}_{\max}^* \subset \mathbb{Y}_j^* \subset \mathbb{Y}_{\min}^*, \quad j = \mathcal{K}, \mathcal{K}.$$

Since  $\mathbb{X}_{\min} \subset \mathbb{X}_j \subset \mathbb{X}_{\max}$  and  $\mathbf{A}^* : \mathbb{Y}_j^* \rightarrow \mathbb{X}_j^*$  ( $j = 0, 1$ ),  $\mathbf{A}^* : \mathbb{Y}_{\min}^* \rightarrow \mathbb{X}_{\min}^*$ ,  $\mathbf{A}^* : \mathbb{Y}_{\max}^* \rightarrow \mathbb{X}_{\max}^*$  are Fredholm, we have

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} \subset \ker_{\mathbb{X}_j} \mathbf{A} \subset \ker_{\mathbb{X}_{\max}} \mathbf{A}, \quad (4.32)$$

$$\ker_{\mathbb{Y}_{\max}^*} \mathbf{A}^* \subset \ker_{\mathbb{Y}_j^*} \mathbf{A}^* \subset \ker_{\mathbb{Y}_{\min}^*} \mathbf{A}^*. \quad (4.33)$$

The dimensions of the kernels ( $\dim \ker \mathbf{A}$ ) in appropriate spaces will be denoted by  $\alpha_{\min}$ ,  $\alpha_j$ ,  $\alpha_{\max}$ , while the notation  $\beta_{\min}$ ,  $\beta_j$ ,  $\beta_{\max}$  will be used for the dimensions of cokernels ( $\dim \text{Coker } \mathbf{A}$ ). Note that for a Fredholm operator we have

$$\dim \text{Coker } \mathbf{A} = \dim \ker \mathbf{A}^*.$$

Embeddings (4.32) and (4.33) imply

$$\alpha_{\min} \leq \alpha_j \leq \alpha_{\max}, \quad j = 0, 1, \quad (4.34)$$

$$\beta_{\max} \leq \beta_j \leq \beta_{\min}, \quad j = 0, 1. \quad (4.35)$$

By the definition of  $\text{Ind } \mathbf{A}$  we obtain

$$\text{Ind}_{\mathbb{X}_{\min} \rightarrow \mathbb{Y}_{\min}} \mathbf{A} \leq \text{Ind}_{\mathbb{X}_j \rightarrow \mathbb{Y}_j} \mathbf{A} \leq \text{Ind}_{\mathbb{X}_{\max} \rightarrow \mathbb{Y}_{\max}} \mathbf{A}. \quad (4.36)$$

A similar inequality for indices of the regularizer  $\mathbf{R}$  is proved just in the same manner. Since  $\text{Ind } \mathbf{R} = -\text{Ind } \mathbf{A}$ , the inequalities inverse to (4.36) are valid and therefore (4.30) holds. Now from (4.34) and (4.35) we obtain  $\alpha_{\min} = \alpha_j = \alpha_{\max}$ . The latter equality and (4.32) give (4.31).  $\square$

*Remark 20.* Similar statements under different conditions on spaces and operators are well-known (see, for example, [16], [17], [18]).

## § 5. PROOFS OF THEOREMS

**5.1. Proof of Theorem 7.** In the first place we shall prove that  $\mathbf{P}_{\mathbf{S}}^1$  (see (3.2), (3.6), (3.7)) is a pseudodifferential operator according to the definition given in Subsection 4.2.

Let  $U_1, \dots, U_N$  be a covering of  $S \subset \mathbb{R}^{\sharp}$  (see (4.14) where  $n = 2$ ),  $\varkappa_1, \dots, \varkappa_N$  be coordinate diffeomorphisms and

$$\begin{aligned} \tilde{\varkappa}_j : \tilde{X}_j &\rightarrow \tilde{U}_j, \quad \tilde{X}_j, \tilde{U}_j \subset \mathbb{R}^{\sharp}, \quad \tilde{U}_j \cap \mathbb{S} = \mathbb{V}_j, \\ \tilde{X}_j &= (-\varepsilon, \varepsilon) \times X_j, \quad \tilde{\varkappa}_j|_{X_j} = \varkappa_j, \quad j = 1, \dots, N, \end{aligned} \quad (5.1)$$

be extensions of diffeomorphisms (4.14). By  $d\varkappa_j(t) = \varkappa'_j(t)$  and  $d\tilde{\varkappa}_j(\tilde{t}) = \tilde{\varkappa}'_j(\tilde{t})$  ( $t = (t_1, t_2) \in \mathbb{R}_+^{\sharp} := \mathbb{R} \times \mathbb{R}^+$ ,  $\tilde{t} = (\tilde{t}_1, \tilde{t}_2) \in \mathbb{R}_+^{\sharp} := \mathbb{R}^{\sharp} \times \mathbb{R}^+$ ) we denote the corresponding Jacobian matrices of orders  $3 \times 2$  and  $3 \times 3$ .  $\varkappa'_j(t)$  will coincide with  $\tilde{\varkappa}'_j(0, t)$  ( $t \in X_j \subset \mathbb{R}_+^{\sharp}$ ) if the first column in these matrices is deleted.

Let further

$$\Gamma_{\varkappa_j}(t) = (\det \|(\partial_k \varkappa_j, \partial_l \varkappa_j)\|_{2 \times 2})^{1/2}, \quad \partial_k \varkappa_j = (\partial_k \varkappa_{j1}, \partial_k \varkappa_{j2}, \partial_k \varkappa_{j3})$$

denote the square root of the Gramm determinant of the vector-function  $\varkappa_j = (\varkappa_{j1}, \varkappa_{j2}, \varkappa_{j3})$ .

If the operator  $\mathbf{P}_S^1$  is lifted locally from the manifold  $S$  onto the half-space  $\mathbb{R}_+^\kappa$  by means of operators (4.15), then we obtain the operator (cf. (4.17))

$$\begin{aligned} \mathbf{P}_{s, \varkappa_j}^1 \mathbf{v}(\mathbf{t}) &= \varkappa_{j*} \mathbf{P}_s^1 \varkappa_{j*}^{-1} \mathbf{v}(\mathbf{t}) = \chi_j^0(\mathbf{t}) \int_{\mathbb{R}_+^\kappa} \Phi((\varkappa_j(\mathbf{t}) - \\ &- \varkappa_j(\theta), \tau) \chi_j^0(\theta) \Gamma_{\varkappa_j}(\theta) v(\theta) d\theta, \quad t \in \mathbb{R}_+^\kappa, \quad \chi_j^\kappa \in \mathbb{C}^\infty(\overline{\mathbb{R}}_+^\kappa). \end{aligned}$$

From the last equality it follows that operator (3.7) is bounded. Besides

$$\begin{aligned} \mathbf{K}_j \mathbf{v}(\mathbf{t}) &:= \chi_j^0(\mathbf{t}) \int_{\mathbb{R}_+^\kappa} [\Phi(\varkappa_j(\mathbf{t}) - \varkappa_j(\theta), \tau) \Gamma_{\varkappa_j}(\theta) - \\ &- \Phi(\varkappa'_j(t)(t - \theta), \tau) \Gamma_{\varkappa_j}(t)] \chi_j^0(\theta) v(\theta) d\theta \end{aligned}$$

has the order  $-2$ , i.e. the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}_p^\nu(\mathbb{R}_+^\kappa) \rightarrow \mathbb{H}_i^{\nu+\kappa}(\mathbb{R}_+^\kappa) \quad (5.2)$$

is bounded for any  $\nu \in \mathbb{R}$  (see [19, Section 33.2 and Theorem 13]). Due to (5.2) the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}_p^\nu(\mathbb{R}_+^\kappa) \rightarrow \mathbb{H}_i^{\nu+\kappa}(\mathbb{R}_+^\kappa) \quad (5.3)$$

is compact, since  $\chi_j^0 \in C_0^\infty(\overline{\mathbb{R}}_+^2)$  (see [4.19]). From (5.3), Example 14 and (2.1) it follows that the symbol of the pseudodifferential operator  $\mathbf{P}_S^1$  reads as ( $x \in \overline{S}, \xi \in \mathbb{R}^\kappa$ )

$$\begin{aligned} \mathcal{P}_S^\infty(\xi, \xi) &= \Gamma_{\varkappa_j}(\square) \int_{\mathbb{R}^\kappa} \square^{\langle \xi, \eta \rangle} \Phi(\varkappa'_j(\square) \eta, \tau) \lceil \eta = \\ &= \Gamma_{\varkappa_j}(t) \int_{\mathbb{R}^\kappa} e^{i\xi \eta} \Phi(\widetilde{\varkappa}'_j(0, t)(0, \eta), \tau) d\eta = \\ &= \frac{\Gamma_{\varkappa_j}(t)}{(2\pi)^3} \int_{\mathbb{R}^\kappa} e^{i\xi \eta} \int_{\mathbb{R}^\kappa} e^{-i(\widetilde{\varkappa}'_j(0, t)(0, \eta), \widetilde{y})} \mathcal{A}^{-\infty}(\widetilde{\mathbf{t}}, \tau) \lceil \widetilde{\mathbf{t}} \lceil \eta = \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma_{\kappa_j}(t)}{(2\pi)^3 \det \tilde{\kappa}'_j(0, t)} \int_{\mathbb{R}^k} e^{i\xi\eta} \int_{\mathbb{R}^k} e^{-i\eta y} \int_{-\infty}^{\infty} \mathcal{A}^{-\infty} \left( [(\tilde{\kappa}'_j(\iota, \square))^\mathcal{T}]^{-\infty} \tilde{\zeta}, \tau \right) \lceil \dagger, \lceil \dagger \lceil \eta = \\
&= \frac{\Gamma_{\kappa_j}(t)}{2\pi \det \tilde{\kappa}'_j(0, t)} \int_{-\infty}^{\infty} \mathcal{A}^{-\infty} \left( [(\tilde{\kappa}'_j(\iota, \square))^\mathcal{T}]^{-\infty} \zeta, \tau \right) \lceil \dagger,.
\end{aligned} \tag{5.4}$$

for  $t = \kappa_j^{-1}(x)$ ,  $x \in S$ ,  $t \in \mathbb{R}_+^k$ ,  $\xi \in \mathbb{R}^k$ ,  $\tilde{y} = (y_0, y) \in \mathbb{R}^{k+1}$ ,  $\zeta = (y_0, \xi)$ . By (2.3) the principal homogeneous symbol of  $\mathbf{P}_S^1$  (see (2.18)) is written in the form

$$(\mathcal{P}_S^\infty)^\infty(\xi, \xi) = \frac{\Gamma_{\kappa_j}(\square)}{\varepsilon \pi \det \tilde{\kappa}'_j(\iota, \square)} \int_{-\infty}^{\infty} \mathcal{A}_r^{-\infty} \left( [(\tilde{\kappa}'_j(\iota, \square))^\mathcal{T}]^{-\infty} \zeta \right) \lceil \dagger, \tag{5.5}$$

$$x \in \overline{S}, \quad \xi \in \mathbb{R}^k, \quad \approx = \kappa_j^{-k}(\curvearrowright) \in \mathbb{R}_+^k, \quad \zeta = (\curvearrowright, \xi),$$

$$\mathcal{A}_r^{-\infty}(\tilde{\xi}) = \begin{vmatrix} \mathcal{C}^{-\infty}(\tilde{\xi}) & 0 \\ 0 & \Lambda^{-1}(-i\tilde{\xi}) \end{vmatrix}, \quad \tilde{\xi} \in \mathbb{R}^k, \tag{5.6}$$

where  $\mathcal{C}(\tilde{\xi})$  and  $\Lambda(\tilde{\xi})$  are defined by (2.4). Since  $-\mathcal{C}(\tilde{\xi})$  and  $-\Lambda(-i\tilde{\xi})$  are positive-definite (see (1.12) and (1.14)), the same is true for  $-\mathcal{A}_r^{-\infty}(\tilde{\xi})$ :

$$(-\mathcal{A}_r^{-\infty}(\tilde{\xi})\eta, \eta) \geq \delta_\varepsilon |\eta|^\varepsilon |\tilde{\xi}|^{-\varepsilon}, \quad \delta_\varepsilon > 1, \quad \eta \in \mathbb{C}^k, \quad \tilde{\xi} \in \mathbb{R}^k.$$

Applying this fact, we proceed as follows:

$$\begin{aligned}
&((-\mathcal{P}_S^\infty)^\infty(\xi, \xi)\eta, \eta) = \\
&= \frac{\Gamma_{\kappa_j}(t)}{2\pi \det \tilde{\kappa}'_j(t)} \int_{-\infty}^{+\infty} \left( -\mathcal{A}_r^{-\infty} \left( [(\tilde{\kappa}'_j(\iota, \square))^\mathcal{T}]^{-\infty} \zeta \right) \eta, \eta \right) \lceil \dagger, \geq \\
&\geq \delta_2 |\eta|^2 \int_{-\infty}^{+\infty} |\tilde{\kappa}'_j(0, t)\zeta|^{-2} dy_0 \geq \\
&\geq \delta_3 |\eta|^2 \int_{-\infty}^{+\infty} \frac{dy_0}{y_0^2 + |\xi|^2} = \delta_4 |\eta|^2 |\xi|^{-1},
\end{aligned} \tag{5.7}$$

$$\eta \in \mathbb{C}^k, \quad \xi \in \mathbb{R}^k, \quad \zeta = (\curvearrowright, \xi), \quad \delta_\varepsilon = \varepsilon \kappa_j^{-k}(\curvearrowright), \quad \varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}.$$

Formulas (1.6), (5.5) and (5.6) also imply

$$\begin{aligned}
D_x^\alpha D_{\xi_1}^m (\mathcal{P}_S^\infty)^\infty(\xi, \lambda\xi) &= |\lambda|^{-\infty} \lambda^{-\frac{m}{2}} \mathcal{D}_\xi^\alpha \mathcal{D}_{\xi_1}^{\frac{m}{2}} (\mathcal{P}_S^\infty)^\infty(\xi, \xi), \\
|\alpha| &< \infty, \quad m = 0, 1, \dots, \quad \xi \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}.
\end{aligned} \tag{5.8}$$

Hence we have the equivalences (see (4.18), (4.21), (5.1), (5.2))

$$\begin{aligned}\varkappa_{j*} \mathbf{P}_{\mathbf{S}}^1 \varkappa_{j*}^{-1} &\stackrel{\mathbf{x}_0}{\approx} (\mathbf{P}_{\mathbf{S}}^1)^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \mathbf{U}_j \subset \mathbf{S}, \quad \mathbf{x}_0 \notin \partial \mathbf{S}, \\ \varkappa_{j*} \mathbf{P}_{\mathbf{S}}^1 \varkappa_{j*}^{-1} &\stackrel{\mathbf{x}_0}{\approx} \mathbf{r}_+(\mathbf{P}_{\mathbf{S}}^1)^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \mathbf{U}_j \cap \partial \mathbf{S},\end{aligned}$$

where (see (4.20))

$$(\mathcal{P}_{\mathcal{S}}^\infty)'(\xi, \xi) := (\mathcal{P}_{\mathcal{S}}^\infty)^\infty(\xi, (\infty + |\xi_\infty|)|\xi_\infty|^{-\infty} \xi_\infty, \xi_\infty).$$

Due to (5.7) the symbol  $(\mathcal{P}_{\mathcal{S}}^\infty)'(\xi, \xi)$  is an elliptic one

$$\inf\{|\det(\mathcal{P}_{\mathcal{S}}^\infty)^\infty(\xi, \xi)| : \xi \in \overline{\mathcal{S}}, \quad |\xi| = \infty\} > 1.$$

Since condition (5.8) implies the continuity property (4.25) for the symbol  $(\mathcal{P}_{\mathcal{S}}^\infty)^\infty(\xi, \xi)$ , by virtue of Lemma 17 it admits the factorization

$$\begin{aligned}(\mathcal{P}_{\mathcal{S}}^\infty)'(\xi, \xi) &= [(\xi_\infty - )|\xi_\infty| - )^{-\infty/\epsilon} \mathcal{P}_-(\xi, \xi)] [(\xi_\infty + )|\xi'| + )^{-\infty/\epsilon} \mathcal{P}_+(\xi, \xi)], \\ \mathcal{P}_-^{\pm\infty}(\xi, \cdot), \quad \mathcal{P}_+^{\pm\infty}(\xi, \cdot) &\in \mathcal{M}_{\sqrt{\epsilon}}(\mathbb{R}^k), \quad \sqrt{\epsilon} \in \partial \mathcal{S},\end{aligned}$$

where  $\mathcal{P}_-^{\pm\infty}(\xi, \xi_\infty - )\lambda$ ,  $\mathcal{P}_+^{\pm\infty}(\xi, \xi_\infty + )\lambda$  have bounded analytic extensions for  $\lambda > 0$ . According to Theorem 15 operator (3.7) is a Fredholm one if and only if the operators  $\mathbf{r}_+(\mathbf{P}_{\mathbf{S}}^1)_{\nu, -1}(\mathbf{x}_0, \mathbf{D})$  are Fredholm ones in  $\mathbb{L}_1(\mathbb{R}_+^k)$  for all  $x_0 \in \partial S$ , where

$$\begin{aligned}(\mathcal{P}_{\mathcal{S}}^\infty)_{\nu, -\infty}(\xi, \xi) &= \frac{(\xi_\infty - )|\xi_\infty| - )^{\nu+\infty}}{(\xi_\infty + )|\xi_\infty| + )^\nu} (\mathcal{P}_{\mathcal{S}}^\infty)'(\xi, \xi) = \\ &= \left( \frac{\xi_2 - i|\xi_1| - i}{\xi_2 + i|\xi_1| + i} \right)^{\nu+1/2} \mathcal{P}_-(\xi, \xi) \mathcal{P}_+(\xi, \xi), \quad \xi \in \partial \mathcal{S}.\end{aligned} \quad (5.9)$$

Therefore (see (4.10), (5.9))

$$\mathbf{r}_+(\mathbf{P}_{\mathbf{S}}^1)_{\nu, -1}(\mathbf{x}_0, \mathbf{D}) = \mathbf{r}_+ \mathbf{P}_-(\mathbf{x}_0, \mathbf{D}) \ell_0 \mathbf{r}_+ \mathbf{G}_\nu(\mathbf{D}) \mathbf{P}_+(\mathbf{x}_0, \mathbf{D}), \quad (5.10)$$

with

$$\mathcal{G}_\nu(\xi) = \left( \frac{\xi_\infty - )|\xi_\infty| - )}{\xi_\infty + )|\xi_\infty| + )} \right)^{\nu+\infty/\epsilon} \quad (5.11)$$

and since  $\mathbf{r}_+ \mathbf{P}_\pm(\mathbf{x}_0, \mathbf{D})$  are invertible (according to (4.10) the inverses read as  $\mathbf{r}_+ \mathbf{P}_\pm^{-1}(\mathbf{x}, \mathbf{D})$ ). The proof will be completed if we find invertibility conditions for  $\mathbf{r}_+ \mathbf{G}_\nu(\mathbf{D})$  in  $\mathbb{L}_1(\mathbb{R}_+^k)$ ; the latter is invertible if and only if

$$1/p - 1 < \nu + 1/2 < 1/p \quad (5.12)$$

and the inverse reads as  $(\mathbf{r}_+ \mathbf{G}_\nu(\mathbf{D}))^{-1} = \mathcal{I}_+^{\nu+\nu/\epsilon}(\mathcal{D}) \ell_0 \mathbf{r}_+ \mathcal{I}_\nu^{-\nu-\infty/\epsilon}(\mathcal{D})$  (see [2], §2). Conditions (5.12) coincide with (3.8).

The local inverses to  $\mathbf{P}_S^1 : \tilde{\mathbb{H}}_p^\nu(\mathbf{S}) \rightarrow \mathbb{H}_i^{\nu+\mu}(\mathbb{S})$  are, therefore, independent of the parameters  $p$  and  $\nu$  if conditions (3.8) are fulfilled.

In fact, the operator

$$(\mathbf{r}_+ \mathbf{P}_S^1)_{\nu, -1}^{-1}(\mathbf{x}_0, \mathbf{D}) := \mathbf{P}_+^{-1}(\mathbf{x}_0, \mathbf{D}) \mathcal{I}_+^{\nu+\infty/\epsilon}(\mathcal{D}) \ell, \mathbf{r}_+ \mathcal{I}_-^{-\nu-\infty/\epsilon}(\mathcal{D}) \mathbf{P}_+^{-1}(\mathbf{x}_0, \mathbf{D})$$

is inverse to  $(\mathbf{r}_+ \mathbf{P}_S^1)_{\nu, -1}(\mathbf{x}_0, \mathbf{D})$  in  $L_p(\mathbb{R}_+^\mu)$ ; if we "lift" these operators from the space  $L_p(\mathbb{R}_+^\mu)$  to the Bessel potential spaces by means of the Bessel potentials  $\mathcal{I}_\pm^\mu(\mathcal{D})$  defined by (4.8), we shall come to the following conclusion: if (3.8) holds, the operator

$$\begin{aligned} & \mathcal{I}_+^{-\nu}(\mathcal{D}) \ell, (\mathbf{r}_+ \mathbf{P}_S^1)_{\nu, -1}^{-1}(\mathbf{x}_0, \mathbf{D}) \mathcal{I}_-^{\nu+\infty}(\mathcal{D}) = \\ & = \mathbf{P}_+^{-1}(\mathbf{x}_0, \mathbf{D}) \mathcal{I}_+^{\infty/\epsilon}(\mathcal{D}) \ell, \mathbf{r}_+ \mathcal{I}_-^{\infty/\epsilon} \mathbf{P}_+^{-1}(\mathbf{x}_0, \mathbf{D}) \end{aligned}$$

inverts the operator

$$\begin{aligned} & \mathcal{I}_+^{-\nu}(\mathcal{D}) \ell, (\mathbf{r}_+ \mathbf{P}_S^1)_{\nu, -1}(\mathbf{x}_0, \mathbf{D}) \mathcal{I}_-^{\nu+\infty}(\mathcal{D}) = \\ & = \mathbf{P}_S^1(\mathbf{x}_0, \mathbf{D}) : \tilde{\mathbb{H}}_p^\nu(\mathbb{R}_+^\mu) \rightarrow \mathbb{H}_i^{\nu+\mu}(\mathbb{R}_+^\mu), \quad \nu \in \partial \mathbb{S} \end{aligned}$$

which is a local representation of  $\mathbf{P}_S^1 = \mathbf{P}_S^1(\mathbf{x}, \mathbf{D})$  ( $x \in S$ ,  $x_0 \in \partial S$ ).

Thus the regularizer constructed by means of the local inverses (see, for example [2], [3], [13]) can be chosen independent of  $p$  and  $\nu$  if (3.8) holds. Now we can take  $p = 2$  and by Theorem 16 and Lemma 19 we get  $\text{Ind } \mathbf{P}_S^1 = \mathbf{0}$ .

To complete the proof for the space  $H_p^\nu(S)$  it remains to check that  $\ker \mathbf{P}_S^1 = \mathbf{0}$ . We need to do this only for  $\nu = -1/2$  and  $p = 2$ , since  $\ker \mathbf{P}_S^1$  is also independent of the parameters  $p$  and  $\nu$  (see Lemma 19).

The equality  $\ker \mathbf{P}_S^1 = \mathbf{0}$ , in turn, follows from the triviality of a solution of the homogeneous Problem  $D$ . Actually, formula (1.17) implies that for any solution  $U = (u_1, \dots, u_4)$  of the homogeneous Problem  $D$  we have

$$\int_{\mathbb{R}_S^\mu} \left\{ c_{ijkl} D_l u_k D_j \bar{u}_i + \rho \tau^2 u_k \bar{u}_k + \frac{1}{\tau T_0} \lambda_{ij} D_j u_4 D_i \bar{u}_4 + \frac{c_0}{T_0} u_4 \bar{u}_4 \right\} dx = 0;$$

recalling that  $\tau = \sigma + i\omega$  and separating the real and the imaginary part, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_S^3} \left\{ c_{ijkl} D_l u_k D_j \bar{u}_i + \rho(\sigma^2 - \omega^2) u_k \bar{u}_k + \right. \\ & \left. + \frac{\sigma}{|\tau|^2 T_0} \lambda_{ij} D_j u_4 D_i \bar{u}_4 + \frac{c_0}{T_0} u_4 \bar{u}_4 \right\} dx = 0, \end{aligned} \quad (5.13)$$

$$\frac{\omega}{T_0} \int_{R_S^3} \left\{ 2\sigma T_0 u_k \bar{u}_k + \lambda_{ij} D_j u_4 D_i \bar{u}_4 \right\} dx = 0.$$

Whence by (1.12) and (1.14) we find  $U = 0$  for an arbitrary  $\tau$  with  $\operatorname{Re} \tau > 0$ . For  $\tau = 0$  we obtain

$$D_j u_k(x) + D_k u_j(x) = 0, \quad u_4 = 0, \quad k, j = 1, 3, \quad x \in \mathbb{R}_S^{\mathbb{M}}. \quad (5.14)$$

The general solution of this system is (see [1])

$$U = [a \times x] + b,$$

where  $a$  and  $b$  are the constant three-dimensional vectors with complex entries and  $[\cdot \times \cdot]$  denotes the vector product of two vectors. From conditions (1.10) and (5.14) it follows that  $U = 0$ .

Thus the homogeneous Problem D has only a trivial solution and  $\ker \mathbf{P}_S^1 = \{\mathbf{0}\}$ .

To prove the theorem for the Besov space  $\mathbb{B}_{i,1}^{\nu}(\mathbb{S})$  recall the following interpolation property from (4.5):

If  $\mathbf{A} : \tilde{\mathbb{H}}_{\mathbf{p}}^{\nu}(\mathbb{S}) \rightarrow \mathbb{H}_i^{\nu+\infty}(\mathbb{S})$  is bounded for any  $\nu_0 < \nu < \nu_1$  and some  $1 < p < \infty$ , then the operator  $\mathbf{A} : \tilde{\mathbb{B}}_{\mathbf{p},\mathbf{q}}^{\nu}(\mathbb{S}) \rightarrow \mathbb{B}_{i,1}^{\nu+\infty}(\mathbb{S})$  is also bounded for any  $\nu < \nu < \nu_1$ ,  $1 \leq q \leq \infty$ .

Let conditions (3.8) be fulfilled. Then the operator  $\mathbf{P}_S^1 : \tilde{\mathbb{H}}_{\mathbf{p}}^{\nu}(\mathbb{S}) \rightarrow \mathbb{H}_i^{\nu+\infty}(\mathbb{S})$  has the bounded inverse  $(\mathbf{P}_S^1)^{-1} : \mathbb{H}_i^{\nu+\infty}(\mathbb{S}) \rightarrow \tilde{\mathbb{H}}_{\mathbf{p}}^{\nu}(\mathbb{S})$ ; due to the above-mentioned interpolation property the operator  $(\mathbf{P}_S^1)^{-1} : \mathbb{B}_{i,1}^{\nu+\infty}(\mathbb{S}) \rightarrow \tilde{\mathbb{B}}_{i,1}^{\nu}(\mathbb{S})$  will also be bounded and therefore the operator  $\mathbf{P}_S^1$  in (3.6) has the bounded inverse.

**5.2. Proof of Theorem 8.** After the localization and local transformation of variables (see (5.1)–(5.9)) we obtain the equivalences

$$\begin{aligned} \varkappa_{j*} \mathbf{P}_S^4 \varkappa_{j*}^{-1} \xrightarrow{x_0} (\mathbf{P}_S^4)^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \mathbf{U}_j \subset \mathbf{S}, \quad \mathbf{x}_0 \notin \partial \mathbf{S}, \\ \varkappa_{j*} \mathbf{P}_S^4 \varkappa_{j*}^{-1} \xrightarrow{x_0} \mathbf{r}_+(\mathbf{P}_S^4)^0(\mathbf{x}_0, \mathbf{D}), \quad \mathbf{x}_0 \in \mathbf{U}_j \cap \partial \mathbf{S}, \end{aligned}$$

where

$$(\mathcal{P}_S^{\Delta})'(\xi, \xi) = \mathcal{B}'(\xi, \xi)(\mathcal{P}_S^{\infty})'(\xi, \xi)(\mathcal{B}')^T(\xi, \xi) \quad (5.15)$$

and  $\mathcal{B}'(\xi, \xi)$  represents the modified principal symbol of the operators  $\mathbf{B}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}(\mathbf{x}))$  and  $\mathbf{Q}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}(\mathbf{x}))$  (whose principal symbols coincide). The order of  $\mathcal{B}'(\xi, \xi)$  is 1 and therefore (5.15), (5.7) yield

$$((\mathcal{P}_S^{\Delta})^{\infty}(\xi, \xi)\eta, \eta) \geq \delta_{\nabla} |\xi| |\eta|^{\epsilon}, \quad \xi \in \mathbb{R}^{\mathbb{M}}, \quad \eta \in \mathbb{C}^{\mathbb{M}}, \quad \delta_{\nabla} > \mathbb{M}.$$

The homogeneity property

$$D_{\xi_1}^m D_x^\alpha (\mathcal{P}_S^\Delta)^\infty(\xi, \lambda \xi) = |\lambda| \lambda^{-\frac{m}{2}} \mathcal{D}_{\xi_\infty}^{\frac{m}{2}} \mathcal{D}_\xi^\alpha (\mathcal{P}_S^\Delta)^\infty(\xi, \xi),$$

$$|\alpha| < \infty, \quad m = 0, 1, \dots, \quad \xi \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}$$

holds as well (see (5.8)).

Thus the symbol  $(\mathcal{P}_S^\Delta)^\infty(\xi, \xi)$  is elliptic

$$\inf\{|\det(\mathcal{P}_S^\Delta)^\infty(\xi, \xi)| : \xi \in \bar{\mathcal{S}}, |\xi| = \infty\} > 0$$

and operator (3.10) is a Fredholm one if and only if the operators  $\mathbf{r}_+(\mathbf{P}_S^4)_{\nu+1,1}^0(\mathbf{x}_0, \mathbf{D})$  are Fredholm in  $\mathbb{L}_+(\mathbb{R}_+^k)$  for all  $x_0 \in \partial S$ ; here

$$(\mathcal{P}_S^\Delta)'_{\nu+\infty, \infty}(\xi, \xi) = \frac{(\xi_\infty - \lambda|\xi_\infty| - \lambda)^{\nu}}{(\xi_\infty + \lambda|\xi_\infty| + \lambda)^{\nu+\infty}} (\mathcal{P}_S^\Delta)'(\xi, \xi) =$$

$$= \left( \frac{\xi_2 - i|\xi_1| - i}{\xi_2 + i|\xi_1| + i} \right)^{\nu+1/2} \mathcal{P}_-^\Delta(\xi, \xi) \mathcal{P}_+^\Delta(\xi, \xi),$$

$$(\mathcal{P}_+^\Delta)^{\pm\infty}(\xi, \cdot), (\mathcal{P}_-^\Delta)^{\pm\infty}(\xi, \cdot) \in \mathcal{M}_\nu(\mathbb{R}^k), \quad \nu \in \partial \mathbb{S},$$

and  $(\mathcal{P}_+^\Delta)^{\pm\infty}(\xi, \xi_\infty, \xi_\infty + \lambda), (\mathcal{P}_-^\Delta)^{\pm\infty}(\xi, \xi_\infty, \xi_\infty - \lambda)$  have bounded analytic restrictions for  $\lambda > 0$ . The proof is completed similarly to that of Theorem 7.  $\square$

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