Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 97

On estimates of the Boltzmann collision operator with cutoff

Roland Duduchava, Ralf Kirsch and Sergej Rjasanow

Saarbrücken 2003

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 97
submitted: October 29, 2003

On estimates of the Boltzmann collision operator with cutoff

Roland Duduchava
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
M. Alexidze Str. 1
380093 Tbilisi
Georgia
duduch@rmi.acnet.ge

Ralf Kirsch
Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
kirsch@num.uni-sb.de

Sergej Rjasanow
Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
rjasanow@num.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

On estimates of the Boltzmann collision operator with cutoff

R. Duduchava[*], R. Kirsch & S. Rjasanow

Abstract

We present new estimates of the Boltzmann collision operator in weighted Lebesgue and Bessel potential spaces. The main focus is put on hard potentials under the assumption that the angular part of the collision kernel fulfills some weighted integrability condition. In addition, the proofs for some previously known $\mathbb{L}_p$-estimates were considerably shortened and carried out by elementary methods. For a class of metric spaces, the collision integral is seen to be a continous operator into the same space.
Furthermore, we give a new pointwise lower bound as well as asymptotic estimates for the loss term without requiring that the entropy is finite.

1 Introduction

The classical Boltzmann equation

$$\partial_t f + (v, \nabla_x f) = Q(f, f) \tag{1.1}$$

describes the time evolution of the distribution density $f(t, x, v)$

$$f \ : \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3 \to \mathbb{R}_+.$$

for a monoatomic dilute gas of particles (see [CIP]). Here $\mathbb{R}_+$ denotes the set of non-negative real numbers and $\Omega \subset \mathbb{R}^3$ is a domain in physical space. The right-hand side of equation (1.1), known as the collision integral or the collision term is a quadratic operator:

$$Q(f, f)(v) = \int\limits_{\mathbb{R}^3} \int\limits_{S^2} B(v, w, e)\Big(f(v')f(w') - f(v)f(w)\Big)\, de\, dw\,. \tag{1.2}$$

Note that $Q(f, f)$ depends on t and x only as parameters, so we have omitted this dependence in (1.2) for conciseness. The symmetric form of the bilinear Q is given by

$$\mathcal{Q}(f, g)(v) =$$
$$\frac{1}{2}\int\limits_{\mathbb{R}^3} \int\limits_{S^2} B(v, w, e)\Big(f(v')g(w') + f(w')g(v') - f(v)g(w) - f(w)g(v)\Big)\, de\, dw\,. \tag{1.3}$$

[*]supported by the German Research Foundation under the project Sa 77/71-1

The following notations have been used in (1.2) and (1.3): $v, w \in \mathbb{R}^3$ are the pre-collisional velocities, $e \in S^2 \subset \mathbb{R}^3$ is a unit vector, $v', w' \in \mathbb{R}^3$ are the post-collisional velocities and $B(v, w, e)$ is the collision kernel. The operator $Q(f, f)$ represents the change of the distribution function $f(t, x, v)$ due to the binary collisions between particles. A single collision results in a change of the velocities of the colliding partners $v, w \rightarrow v', w'$ with

$$v' = \frac{1}{2}\Big(v + w + |u|\, e\Big) \quad , \quad w' = \frac{1}{2}\Big(v + w - |u|\, e\Big) , \tag{1.4}$$

where $u = v - w$ denotes the relative speed. Note that this describes elastic collisions, i.e. momentum, energy and the Euclidean norm of the relative speed are preserved:

$$v' + w' = v + w \quad , \quad |v'|^2 + |w'|^2 = |v|^2 + |w|^2 \quad , \quad |v' - w'| = |v - w| . \tag{1.5}$$

The Boltzmann equation (1.1) is subjected to an initial condition

$$f(0, x, v) = f_0(x, v), \; x \in \Omega, \; v \in \mathbb{R}^3 \tag{1.6}$$

and to the boundary conditions on $\Gamma = \partial\Omega$. If the distribution function does not depend on x, (1.1) and (1.6) reduce to the initial value problem for the spatially homogeneous Boltzmann equation

$$\begin{aligned} \partial_t f &= Q(f, f) \\ f(0, v) &= f_0(v) . \end{aligned} \tag{1.7}$$

The kernel $B(v, w, e)$ reads

$$B(v, w, e) = B(|u|, \mu) = |u|\, \sigma(|u|, \mu), \; \mu = \cos\,\theta = \frac{(u, e)}{|u|} . \tag{1.8}$$

The function $\sigma : \mathbb{R}_+ \times [-1, 1] \rightarrow \mathbb{R}_+$ is the differential cross-section and θ is the scattering angle. We briefly present some types of collision kernels frequently considered in the Boltzmann theory:

1. **Inverse power potentials**. In this model, the particle interaction is described by potentials of order m and the kernel acquires the form

   $$B(|u|, \mu) = b_\lambda(\mu)\, |u|^\lambda , \qquad \lambda = 1 - \frac{4}{m} , \qquad m > 1 . \tag{1.9}$$

   $b_\lambda(\mu)$ is a continuous function on the semi-closed interval $[-1, 1)$ and has a non-integrable singularity at $\mu = 1$ of order

   $$b_\lambda(\mu) = \mathcal{O}\left((1 - \mu)^{(\lambda - 1)/2}\right) , \quad -3 < \lambda \leq 1 . \tag{1.10}$$

   The condition (1.10) implies that the function $b_\lambda(\mu)(1 - \mu)$ is integrable on the interval $[-1, 1]$.
   For $0 < \lambda \leq 1$ the potential is called hard, for $-3 < \lambda < 0$ the potential is called soft.

   The special case $\lambda = 0$ in (1.9) (or $m = 4$) corresponds to the **Maxwell pseudo-molecules** with

   $$B(|u|, \mu) = b_0(\mu) . \tag{1.11}$$

   The collision kernel $B(|u|, \mu)$ here does not depend on the relative speed $|u|$.

2. The **Grad's cutoff assumption** (see [Gr]) supposes the function b_λ to be integrable, i.e.

$$C_\lambda = 2\pi \int_{-1}^{1} b_\lambda(\mu)\, d\mu < \infty\,. \tag{1.12}$$

3. In many practical applications, the interaction is considered to be independent of the scattering angle, as described by the **Variable Hard Spheres model** (VHS)

$$B(|u|,\mu) = \frac{C_\lambda}{4\pi}|u|^\lambda\,,\ -3 < \lambda \le 1. \tag{1.13}$$

The model includes as particular cases the **hard spheres model** for $\lambda = 1$

$$B(|u|,\mu) = \frac{d^2}{4}|u|\,, \tag{1.14}$$

where d denotes the diameter of the particles and the **isotropic Maxwell pseudo-molecules** for $\lambda = 0$.

REMARK 1.1 *For the study of the bilinear operator, it is common (see e.g. [Gu1], [Gu2] and [MV]) to impose angular cutoff conditions on $b_\lambda(\mu)$. By this, we mean that both frontal and grazing collisions are neglected:*

$$\text{supp}\, b_\lambda \subset [-1+\delta, 1-\delta] \quad , \quad \delta > 0\,.$$

For $Q(f,g)$ we will impose the following growth condition instead, which is clearly fulfilled for angular cutoff kernels:

$$C^{(1)}_{\lambda,p} = 2^{(\lambda+5-3/p)/2}\pi \int_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{3(1-1/p)/2}}\, d\mu < \infty\,,\ 1 \le p \le \infty\,. \tag{1.15}$$

It is worth noting that the quadratic operator $Q(f,f)$ and the symmetrized operator (1.3) *can be rewritten with b_λ replaced by*

$$\tilde{b}_\lambda(\mu) = \big(b_\lambda(\mu) + b_\lambda(-\mu)\big)\mathbf{1}_{[0,1]}(\mu)\,. \tag{1.16}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A (see e.g. [ADVW]). Assuming (1.15) *for $\tilde{b}_\lambda$ is obviously not more restrictive than Grad's cutoff assumption for b_λ, so the condition*

$$\tilde{C}^{(1)}_{\lambda,p} = 2^{(\lambda+5-3/p)/2}\pi \int_{-1}^{1} \frac{\tilde{b}_\lambda(\mu)}{(1+\mu)^{3(1-1/p)/2}}\, d\mu < \infty\,,\ 1 \le p \le \infty\,. \tag{1.17}$$

is then fulfilled automatically. Furthermore, we have $C^{(1)}_{\lambda,1} = 2^{\lambda/2}C_\lambda$.
Another estimate for $\mathcal{Q}(f,g)$ can be obtained by the more restrictive assumption

$$C^{(2)}_{\lambda,p} = 2^{(\lambda+3-3/p)/2} \max\left(2, 2^{3(1-1/p)/2}\right)\pi \int_{-1}^{1} \frac{b_\lambda(\mu)}{(1-\mu^2)^{3(1-1/p)/2}}\, d\mu < \infty\,. \tag{1.18}$$

Clearly, the condition (1.18) *implies* (1.15) *which implies* (1.12).
For practically relevant kernels such as the VHS model, conditions (1.15) *and* (1.18) *are fulfilled only for* $1 \leq p < 3$, *but the case* $p = 2$, *which is very interesting for numerical considerations, is included. If one uses the modification* (1.16) *for the collision kernel, then thanks to* (1.17)*, this constraint on* p *does not occur neither for the quadratic nor the symmetric form of the collision operator.*

In the cutoff case, the collision integral (1.2), decomposes into the natural gain and loss part

$$\mathcal{Q}(f,g)(v) = \mathcal{Q}_+(f,g)(v) - \mathcal{Q}_-(f,g)(v)\,, \tag{1.19}$$

where

$$\mathcal{Q}_+(f,g)(v) = \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu)\Big(f(v')g(w') + f(w')g(v')\Big)\,de\,dw \tag{1.20}$$

and

$$\mathcal{Q}_-(f,g)(v) = \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu)\Big(f(v)g(w) + f(w)g(v)\Big)\,de\,dw\,, \tag{1.21}$$

and analogously for the non-symmetric operator $Q(f,g)$.

Before we start the study of the mapping properties of the operators $Q_+(f)$ and $Q_-(f)$, we discuss results known from the literature.
Thanks to the pre-post-collisional change of integration variables,

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} F(v,w,(u,e))\,de\,dw\,dv = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} F(v',w',(u,e))\,de\,dw\,dv\,, \tag{1.22}$$

the weak form of the bilinear collision integral is given by

$$\int\limits_{\mathbb{R}^3} Q(f,g)(v)\,\varphi(v)\,dv = \tag{1.23}$$
$$\frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu)f(v)g(w)\big(\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w)\big)\,de\,dv\,dw\ .$$

A very important property of the collision operator is the invariance under rotations and translations in velocity space.

The mapping properties of the collision operator were studied by several authors.

This investigation started in [Ar], [Gu1] and [Gu2] in the setting of weighted Lebesgue spaces for kernels with angular or Grad's cutoff condition.

Because of the structure of the collision transformation (1.4), especially the gain part has ever drawn the attention of researchers to itself. In [Li], the application of results known from the theory of pseudodifferential operators made it possible to prove that the gain term has a smoothing effect, if the collision kernel B is infinitely smooth and compactly supported with respect to μ and the Euclidean norm of the relative speed u.
In the sequel, the assumptions on the kernel were relaxed. In [We], by use of the weighted

Radon transform it was shown that similar results hold also for kernels without finite support in u, if $1/2 < \lambda \leq 1$ and with additional integrability conditions for the distribution function. In [BD], it was assumed that for some constant K

$$|B(|u|,\mu)| + |\partial_\mu B(|u|,\mu)| \leq K(1+|u|)\,,$$

with continous $\partial_\mu B$. Similar assumptions were made in [Lu], where, in contrast to the previously mentioned papers, it was not required that the distribution function is in $\mathbb{L}_1$.
In [DR], smoothness of b_λ was only assumed in the sense of the Bessel potential space $\mathbb{H}_1^a([-1,1])$ for some $a > 1$ (see Theorem 3.6 for an extension of the main result there).
Finally, the recent work [MV] requires only Grad's cutoff assumption together with the condition, that there exists some $\delta > 0$ such that

$$\left| \int_{-1}^{1} b_\lambda(\mu)\,d\mu - \int_{-1+\varepsilon}^{1-\varepsilon} b_\lambda(\mu)\,d\mu \right| \leq C_{b_\lambda}\,\varepsilon^\delta \quad , \quad \varepsilon > 0\,,$$

to develop an elaborate regularity theory for the spatially homogeneous equation. Note that in contrast to (1.15), this condition is independent of p.

The motivation for finding estimates on the collision operator is to study integrablity and regularity properties of the solutions to the initial value problem (1.7). For example, the fact that the loss term has no smoothing effect implies, that no appearance of smoothness can be expected in finite time. However, if the initial datum is smooth, then the smoothness propagates ([MV]).The question arises if the situation is different for non-cutoff kernels. First steps towards that direction have been done in [DW], still for kernels with quite technical additional constraints.

Also, for the theoretical foundation of deterministic numerical schemes, knowledge about the mapping properties of Q can help to find suitable approximating functions and to establish corresponding consistency results.

The plan of the paper is as follows:

In section 2 we give the definition of the function spaces considered in this paper and collect some auxiliary results frequently used in the following sections.

In section 3 we investigate the mapping properties for the gain part of the collision operator for hard potentials with $0 \leq \lambda \leq 1$. A class of metric spaces is introduced, for which the gain term is a continous mapping into the same space. We recall one of the main results given in [DR] which also includes some soft potentials and present an extension. Using interpolation results for Bessel potential spaces we derive a heterogeneous mapping property for the weighted Lebesgue spaces.

Analogous estimates are then proved for the loss term in section 4 and combined with the results for the gain term to retrieve mapping properties for the full collision operator, which is the main goal of this paper. We also tried to make all constants in the estimates explicit, except for those appearing by interpolation.
We conclude with the presentation of a pointwise lower bound for the linear part of Q_- - the so-called collision frequency - which holds without the assumption, that the entropy is finite.

2 Function spaces and auxiliary results

We start this section with the definition of the function spaces under consideration using the following standard notation for the weight function

$$\langle v\rangle^{\nu} = (1+|v|^2)^{\nu/2}\,, \qquad v\in\mathbb{R}^n\,, \qquad \nu\in\mathbb{R}\,. \tag{2.24}$$

$\mathbb{L}_p^{\langle\nu\rangle}(\mathbb{R}^n)$, $1\le p\le\infty$, $\nu\in\mathbb{R}$, stands for the weighted Lebesgue space endowed with the norm

$$\big\|f\,|\,\mathbb{L}_p^{\langle\nu\rangle}\big\| = \left(\int\limits_{\mathbb{R}^n} \big|\langle v\rangle^{\nu} f(v)\big|^p\,dv\right)^{1/p} \tag{2.25}$$

for $1\le p<\infty$ and

$$\big\|f\,|\,\mathbb{L}_\infty^{\langle\nu\rangle}\big\| = \text{ess.}\sup_{v\in\mathbb{R}^n} |\langle v\rangle^{\nu} f(v)|\,.$$

As usual, we will write $\mathbb{L}_p(\mathbb{R}^n)$ instead of $\mathbb{L}_p^{\langle 0\rangle}(\mathbb{R}^n)$ and by default we will assume $\mathbb{R}^3$ as the underlying Euclidean space.
$\mathbb{S}$ denotes the Schwartz space of all infinitely smooth and rapidly decreasing functions, the elements of dual space $\mathbb{S}'$ are called tempered distributions.
The weighted Bessel potential space $\mathbb{H}_p^s = \mathbb{H}_p^{s,\langle\nu\rangle}$, $s,\nu\in\mathbb{R}$, $1\le p\le\infty$, is defined by

$$\mathbb{H}_p^{s,\langle\nu\rangle} = \mathbb{H}_p^{s,\langle\nu\rangle}(\mathbb{R}^3) = \Big\{\varphi\in\mathbb{S}' \;:\; \big\|\varphi\,|\,\mathbb{H}_p^{s,\langle\nu\rangle}\big\| = \big\|\langle\cdot\rangle^{\nu}\varphi\,|\,\mathbb{H}_p^{s}\big\| < \infty\Big\}\,, \tag{2.26}$$

where $\big\|\varphi\,|\,\mathbb{H}_p^s\big\|$ is the norm in the Bessel potential space $\mathbb{H}_p^s = \mathbb{H}_p^{s,\langle 0\rangle}$ without weight

$$\big\|\varphi\,|\,\mathbb{H}_p^s\big\| = \left(\int\limits_{\mathbb{R}^3} \Big|\mathcal{F}^{-1}_{\xi\to y}\Big[\langle\xi\rangle^s\hat\varphi(\xi)\Big](y)\Big|^p\,dy\right)^{1/p}$$

and $\hat\varphi(\xi) = \mathcal{F}_{v\to\xi}[\varphi(v)]$, $\mathcal{F}^{-1}_{\xi\to v}[\psi(\xi)]$ are the Fourier transforms

$$\mathcal{F}_{v\to\xi}[\varphi(v)](\xi) = \int\limits_{\mathbb{R}^3} \varphi(v)e^{\imath(v,\,\xi)}\,dv\,,$$

$$\mathcal{F}^{-1}_{\xi\to v}[\psi(\xi)](v) = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} \psi(\xi)e^{-\imath(\xi,\,v)}\psi(\xi)\,d\xi\,.$$

In a particular case $p=2$, $\nu=0$, due to the norm in the space $\mathbb{H}^s = \mathbb{H}_2^s$ acquires the simpler form

$$\|\varphi\,|\,\mathbb{H}^s\| = (2\pi)^{-3/2}\left(\int\limits_{\mathbb{R}^3} \langle\xi\rangle^{2s}|\hat\varphi(\xi)|^2\,d\xi\right)^{1/2}\,. \tag{2.27}$$

For non-negative integers $m=0,1,2,\ldots$ the weighted Bessel potential space $\mathbb{H}_p^{s,\langle\nu\rangle}$ is isomorphic to the classical weighted Sobolev space $\mathbb{W}_p^m = \mathbb{W}_p^{m,\langle\nu\rangle}$:

$$\mathbb{W}_p^{m,\langle\nu\rangle} = \mathbb{W}_p^{m,\langle\nu\rangle} = \Big\{g:\ \mathbb{R}^3\to\mathbb{C}\,,\ \partial^\alpha g\in\mathbb{L}_p^{\langle\nu\rangle}\,,\ \forall\alpha:\ |\alpha|\le m\Big\}\,, \tag{2.28}$$

endowed with the norm

$$\left\| g \,|\, \mathbb{W}_p^{m,\langle\nu\rangle} \right\| = \left(\sum_{|\alpha|\le m} \left\| \partial^\alpha g \,|\, \mathbb{L}_p^{\langle\nu\rangle} \right\|^p \right)^{1/p} . \tag{2.29}$$

Finally, we denote $\mathbb{W}^m_{\infty,\mathrm{loc}} = \bigcup_{r>0} \mathbb{W}^m_\infty(B_r)$, where

$$B_r = \{ v \in \mathbb{R}^3 \;:\; |v| \le r \}.$$

We collect some properties of the Boltzmann operator as well as some inequalities which we shall frequently use.

LEMMA 2.1 (see [DR, Lemma 3]). The invariance under Galilean transformations implies

$$\partial^\alpha Q_\pm(f,g)(v) = \sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Q_\pm(\partial^\beta f, \partial^{\alpha-\beta} g)(v)\,, \tag{2.30}$$

$$\partial^\alpha Q(f,g)(v) = \sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Q(\partial^\beta f, \partial^{\alpha-\beta} g)(v)\,. \tag{2.31}$$

and yield the following boundedness properties:

$$Q_+\,,\; Q_-\,,\; Q \;:\; \mathbb{S}\times\mathbb{S} \to \mathbb{S}\,. \tag{2.32}$$

The following lemma is a collection of some useful elementary inequalities:

LEMMA 2.2 *For all $v, w \in \mathbb{R}^3$ the following inequalities hold:*

a) Peetre's inequality:

$$\forall \nu \in \mathbb{R} \;:\; \langle v \pm w \rangle^\nu \le 2^{|\nu|/2} \langle v \rangle^\nu \langle w \rangle^{|\nu|}\,, \tag{2.33}$$

or equivalently

$$\forall \nu \ge 0 \;:\; \langle v \rangle^{-\nu} \langle w \rangle^{-\nu} \le 2^{\nu/2} \langle v \pm w \rangle^{-\nu}\,. \tag{2.34}$$

b) Let v', w' be the post-collisional velocities given by (1.4)*, then*

$$\forall \nu \ge 0 \quad:\quad \langle v' \rangle^\nu \le \langle v \rangle^\nu \langle w \rangle^\nu \,,\; \langle v \rangle^\nu \le \langle v' \rangle^\nu \langle w' \rangle^\nu\,. \tag{2.35}$$

c)

$$\forall \nu \ge 0 \;: \qquad |v|^\nu \le \langle v \rangle^\nu \le M_\nu (1 + |v|^\nu) \tag{2.36}$$

$$1 + |v|^\nu \le m_\nu \langle v \rangle^\nu, \tag{2.37}$$

where $m_\nu = \max(1, 2^{1-\nu/2})$ and $M_\nu = \max(1, 2^{\nu/2-1})$.

Results similar to the following one are standard in the context of the Boltzmann equation (see e.g. the "Cancellation lemma" in [ADVW]). For the sake of self-consistency, we present the proof:

LEMMA 2.3 *Let the collision kernel be of the form* (1.9) *and w' be defined as in* (1.4) *with $v \in \mathbb{R}^3$ fixed. Then for all $\gamma \in \mathbb{R}$ and any function F the following identity holds*

$$\int_{S^2}\int_{\mathbb{R}^3} b_\lambda(\mu)|v-w|^\gamma F(w')\,dw\,de = 2\pi\, 2^{(\gamma+3)/2}\int_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{(\gamma+3)/2}}\,d\mu \int_{\mathbb{R}^3} |v-z|^\gamma F(z)\,dz\,,$$

provided that the integrals on the right-hand side exist.

Proof. Let $v \in \mathbb{R}^3$ be fixed and define

$$\mu(w,e) = \frac{(v-w,e)}{|v-w|} \quad \text{and} \quad z = w' = \frac{1}{2}\Big(v + w - |v-w|e\Big)\,.$$

One computes

$$\frac{dz}{dw} = \frac{1}{2}\left(I + \frac{v-w}{|v-w|}\,e^T\right) \Rightarrow \det\left(\frac{dz}{dw}\right) = \frac{1}{8}(1+\mu(w,e))\,.$$

We have

$$2|v-z|^2 = 2|v-w-(z-w)|^2 = \frac{|v-w|^2}{2}\left|\frac{v-w}{|v-w|}+e\right|^2 = |v-w|^2(1+\mu(w,e))\,.$$

Multiplying the identity

$$2(v-z) = (v-w) + |v-w|e$$

with e and taking the square one finds

$$4|v-z|^2\mu(z,e)^2 = |v-w|^2(1+\mu(w,e))^2\,,$$

giving

$$\mu(w,e) = 2\mu(z,e)^2 - 1\,,\ |v-w| = \frac{|v-z|}{\mu(z,e)} \quad \text{and} \quad \det\left(\frac{dw}{dz}\right) = 4\mu(z,e)^{-2}\,.$$

With this we can change the integration variable from w to z noting that as w varies over $\mathbb{R}^3$, z varies (for given e) over the half space given by the condition $\mu(z,e) \geq 0$.

$$\begin{aligned}
&\int_{S^2}\int_{\mathbb{R}^3} b_\lambda(\mu)|v-w|^\gamma F(w')\,dw\,de \\
&= 4\int_{S^2}\int_{\{z:\mu(z,e)\geq 0\}} \frac{b_\lambda(2\mu(z,e)^2-1)}{\mu(z,e)^2}\frac{|v-z|^\gamma}{\mu(z,e)^\gamma}F(z)\,dz\,de \\
&= 4\int_{\mathbb{R}^3} |v-z|^\gamma F(z) \int_{\{e:\mu(z,e)\geq 0\}} \frac{b_\lambda(2\mu(z,e)^2-1)}{\mu(z,e)^{\gamma+2}}\,de\,dz\,.
\end{aligned}$$

To treat the inner integral we parametrize the half sphere $\{e : \mu(z,e) \geq 0\}$ as follows: Let U be an orthogonal 3×3-matrix with $(v-z)/|v-z|$ being the third column and

$$e = \frac{1}{\sqrt{2}}U\begin{pmatrix} \cos\phi\sqrt{1-\mu} \\ \sin\phi\sqrt{1-\mu} \\ \sqrt{1+\mu}\end{pmatrix} \quad \text{for} \quad 0 \leq \phi < 2\pi \quad , \quad -1 \leq \mu \leq 1\,.$$

It follows

$$\mu(z,e) = \sqrt{\frac{1+\mu}{2}} \quad , \quad de = \frac{2^{-3/2}}{\sqrt{1+\mu}}\, d\phi\, d\mu\,,$$

and we retrieve the desired result:

$$4 \int\limits_{\{e:\mu(z,e)\geq 0\}} \frac{b_\lambda(2\mu(z,e)^2-1)}{\mu(z,e)^{2+\gamma}}\, de = 2\pi\, 2^{(\gamma+3)/2} \int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{(\gamma+3)/2}}\, d\mu \qquad \blacksquare$$

3 The gain part of the collision operator

In the present section we will obtain estimates for the gain part of the collision kernel, assuming that the collision kernel satisfies the cutoff conditions presented in Remark 1.1 or Grad's cutoff.

For the sake of conciseness we will frequently use the notation

$$f_\nu(v) = \langle v\rangle^\nu f(v)\,. \tag{3.1}$$

Estimates as given in the following theorem have also been proven in [Gu1], [Gu2] and [MV, Section 2] for kernels with angular cutoff. To our knowledge, the proof presented here is the most elementary one.

THEOREM 3.1 *Let the condition* (1.15) *hold,* $1 \leq p \leq \infty$*, and* $\nu, s \geq 0$*. Then*

$$\big\|Q_+(f,g)\,\big|\,\mathbb{H}_p^{s,\langle\nu\rangle}\big\| \leq C_s\, C_{\lambda,p}^{(1)} \big\|g\,\big|\,\mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\big\|\big\|f\,\big|\,\mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| \tag{3.2}$$

with a constant C_s *depending only on the smoothness parameter* s *and such that* $C_0 = 1$*.*

Proof. We start with the simplest case $s = 0$ and $p = 1$. Using (1.15), the weak form of the gain part and inequalities (2.35), (2.36) and (2.33) we find

$$\begin{aligned}
\big\|Q_+(f,g)\,\big|\,\mathbb{L}_1^{\langle\nu\rangle}\big\| &\leq \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu)|f(v)||g(w)|\langle v'\rangle^\nu\, de\, dw\, dv \\
&\leq \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu)|v-w|^\lambda \langle v\rangle^\nu\langle w\rangle^\nu |f(v)|\,|g(w)|\, de\, dw\, dv \\
&\leq 2^{\lambda/2} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu)|f_{\nu+\lambda}(v)|\,|g_{\nu+\lambda}(w)|\, de\, dw\, dv \\
&\leq C_{\lambda,1}^{(1)} \big\|f\,\big|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\|\big\|g\,\big|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\|.
\end{aligned} \tag{3.3}$$

For $p > 1$ we will use the inequality

$$\frac{\langle v\rangle^\nu |v-w|^\lambda}{\langle v'\rangle^{\nu+\lambda}\langle w'\rangle^{\nu+\lambda}} \leq 2^{\lambda/2}\,, \tag{3.4}$$

which is immediately deduced from (1.5), (2.33) and (2.35). For the case $p = \infty$ we find

$$\big\|Q_+(f,g)\,\big|\,\mathbb{L}_\infty^{\langle\nu\rangle}\big\| = \sup_{v\in\mathbb{R}^3} \left| \langle v\rangle^\nu \int\limits_{\mathbb{R}^3}\int\limits_{S^2} \frac{|v-w|^\lambda}{\langle v'\rangle^{\nu+\lambda}\langle w'\rangle^{\nu+\lambda}} b_\lambda(\mu) f_{\nu+\lambda}(v') g_{\nu+\lambda}(w')\, de\, dw \right|$$

$$\le 2^{\lambda/2} \sup_{v'\in\mathbb{R}^3} |f_{\nu+\lambda}(v')| \sup_{v\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3}\int\limits_{S^2} |b_\lambda(\mu)||g_{\nu+\lambda}(w')|\, de\, dw$$

$$= 2^{\lambda/2}\big\|f\,|\mathbb{L}_\infty^{\langle\nu+\lambda\rangle}\big\| \sup_{v\in\mathbb{R}^3} \int\limits_{S^2}\int\limits_{\mathbb{R}^3} |b_\lambda(\mu)||g_{\nu+\lambda}(w')|\, dw\, de\,. \tag{3.5}$$

Since we assume (1.15), we can apply Lemma 2.3 with $\gamma = 0$ to the last expression and obtain

$$\big\|Q_+(f,f)\,|\,\mathbb{L}_\infty^{\langle\nu\rangle}\big\| \le 2^{(\lambda+5)/2}\pi \big\|f\,|\,\mathbb{L}_\infty^{\langle\nu+\lambda\rangle}\big\|\big\|g\,|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\| \int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{3/2}}\, d\mu$$

$$= C^{(1)}_{\lambda,\infty}\big\|f\,|\,\mathbb{L}_\infty^{\langle\nu+\lambda\rangle}\big\|\big\|g\,|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\|\,. \tag{3.6}$$

The case $p = \infty$ is also completed.
Now let $1 < p < \infty$ and set $p' = p/(p-1)$ so that $1/p + 1/p' = 1$. Using (3.4) and the Hölder inequality we proceed as follows

$$\big\|Q_+(f,g)\,|\,\mathbb{L}_p^{\langle\nu\rangle}\big\|^p = \int\limits_{\mathbb{R}^3} \left| \int\limits_{\mathbb{R}^3}\int\limits_{S^2} \frac{\langle v\rangle^\nu |v-w|^\lambda}{\langle v'\rangle^{\nu+\lambda}\langle w'\rangle^{\nu+\lambda}} b_\lambda(\mu) f_{\nu+\lambda}(v') g_{\nu+\lambda}(w')\, de\, dw \right|^p dv$$

$$\le 2^{\lambda p/2} \int\limits_{\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} \big|(1+\mu)^{3/(2p)} b_\lambda(\mu) g_{\nu+\lambda}(w')\big|^{1/p'}$$

$$\times \Big|(1+\mu)^{-3/(2p')} b_\lambda(\mu) g_{\nu+\lambda}(w') |f_{\nu+\lambda}(v')|^p\Big|^{1/p}\, de\, dw \Bigg)^p dv$$

$$\le 2^{\lambda p/2} \int\limits_{\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} (1+\mu)^{3/(2p)} b_\lambda(\mu) |g_{\nu+\lambda}(w')| de\, dw \Bigg)^{p/p'}$$

$$\times \int\limits_{\mathbb{R}^3}\int\limits_{S^2} (1+\mu)^{-3/(2p')} b_\lambda(\mu) |g_{\nu+\lambda}(w')||f_{\nu+\lambda}(v')|^p\, de\, dw\, dv$$

$$\le 2^{\lambda p/2} \sup_{v\in\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} (1+\mu)^{3/(2p)} |b_\lambda(\mu)||g_{\nu+\lambda}(w')|\, de\, dw \Bigg)^{p/p'}$$

$$\times \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} (1+\mu)^{-3/(2p')} |b_\lambda(\mu)||g_{\nu+\lambda}(w')||f_{\nu+\lambda}(v')|^p\, de\, dw\, dv\,. \tag{3.7}$$

Applying Lemma 2.3 to the first factor in the last expression and using (3.3) for the second one, we obtain

$$\big\|Q_+(f,g)\,|\,\mathbb{L}_p^{\langle\nu\rangle}\big\|^p \le 2^{\lambda p/2} \sup_{v\in\mathbb{R}^3} \Bigg(2^{5/2}\pi \big\|g\,|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\| \int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{3(1-1/p)/2}}\, d\mu \Bigg)^{p/p'}$$

$$\times 2\pi \big\|g\,|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\|\big\|f\,|\,\mathbb{L}_p^{\langle\nu+\lambda\rangle}\big\|^p \int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{3(1-1/p)/2}}\, d\mu$$

$$\le \Big(C^{(1)}_{\lambda,p} \big\|g\,|\,\mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\|\big\|f\,|\,\mathbb{L}_p^{\langle\nu+\lambda\rangle}\big\| \Big)^p \tag{3.8}$$

and (3.2) follows for the case $1 < p < \infty$ as well. The case $s = m = 1, 2, \dots$ is reduced to the case $s = 0$ if we apply an equivalent norm in the Sobolev space

$$\big\|\varphi \,|\, \mathbb{H}_p^{m,\langle\nu\rangle}\big\| \le C'_m \sum_{|\alpha|\le m} \big\|\langle\cdot\rangle^\nu \partial^\alpha \varphi \,|\, \mathbb{L}_p\big\|$$

and (2.30):

$$\begin{aligned}
\big\|Q_+(f,g) \,|\, \mathbb{H}_p^{m,\langle\nu\rangle}\big\| &\le C'_m \sum_{|\alpha|\le m} \big\|\langle\cdot\rangle^\nu \partial^\alpha Q_+(f,g) \,|\, \mathbb{L}_p\big\| \\
&\le C'_m \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \big\|Q_+(\partial^\beta f, \partial^{\alpha-\beta} g) \,|\, \mathbb{L}_p^{\langle\nu\rangle}\big\| \\
&\le C'_m C^{(1)}_{\lambda,p} \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \big\|\partial^{\alpha-\beta} g \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\| \big\|\partial^\beta f \,|\, \mathbb{L}_p^{\langle\nu+\lambda\rangle}\big\| \\
&\le C'_m C^{(1)}_{\lambda,p} \sum_{|\alpha|\le m} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \big\|g \,|\, \mathbb{H}_1^{m,\langle\nu+\lambda\rangle}\big\| \big\|f \,|\, \mathbb{H}_p^{m,\langle\nu+\lambda\rangle}\big\| \\
&\le 2^{3m-3} C'_m C^{(1)}_{\lambda,p} \big\|\partial^{\alpha-\beta} g \,|\, \mathbb{H}_1^{m,\langle\nu+\lambda\rangle}\big\| \big\|\partial^\beta f \,|\, \mathbb{H}_p^{m,\langle\nu+\lambda\rangle}\big\|\,, \qquad (3.9)
\end{aligned}$$

because

$$\sum_{\substack{\alpha\in\mathbb{N}_0^3 \\ |\alpha|\le m}} \sum_{\beta\le\alpha} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \sum_{\substack{\alpha\in\mathbb{N}_0^3 \\ |\alpha|\le m}} (1+1)^\alpha = 2^{3(m-1)}\,. \qquad (3.10)$$

We got (3.2) with $C_s = s^3 2^{2s} C'_s$ for any integer $s = m \in \mathbb{N}_0$. For $s > 0$, $s \ne 1, 2, \dots$ we apply the interpolation (cf. [DR, Theorem 10] for a similar considerations). ■

By Remark 1.1, we immediately obtain the following continuity statement for the symmetrized gain term:

COROLLARY 3.2 *Let $\mathcal{Q}_+$ be given by* (1.20) *and the condition* (1.17) *hold. Then for $1 \le p \le \infty$ and $\nu, s \ge 0$ the mapping*

$$\mathcal{Q}_+ : \Big(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\Big) \times \Big(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\Big) \longrightarrow \mathbb{H}_p^{s,\langle\nu\rangle} \qquad (3.11)$$

is continous and the estimate

$$\begin{aligned}
\big\|\mathcal{Q}_+(f,g) \,|\, \mathbb{H}_p^{s,\langle\nu\rangle}\big\| \le \frac{1}{2} C_s\, \tilde{C}^{(1)}_{\lambda,p} &\Bigg(\Big\|f \,|\, \mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\Big\| + \Big\|g \,|\, \mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\Big\| \Bigg) \\
\times &\Bigg(\big\|f \,|\, \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| + \big\|g \,|\, \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| \Bigg) \qquad (3.12)
\end{aligned}$$

holds.

The second continuity result for $\mathcal{Q}_+$ is obtained directly under stronger assumptions on the angular part of the collision kernel but we are rewarded with some refinement with respect to the norms of f and g. Note that because of the strong condition (1.18), this result is suitable only if $f \ne g$.

THEOREM 3.3 *Let the condition* (1.18) *hold,* $1 \le p \le \infty$, $\nu, s \ge 0$. *Then the symmetrized operator*

$$\mathcal{Q}_+ \quad : \quad \left(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\right) \times \left(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\right) \longrightarrow \mathbb{H}_p^{s,\langle\nu\rangle} \tag{3.13}$$

given by (1.20) *is bounded*

$$\big\|\mathcal{Q}_+(f,g) \,|\, \mathbb{H}_p^{s,\langle\nu\rangle}\big\| \le C_s\, C_{\lambda,p}^{(2)} \,\|f \,|\, \mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\| \; \|g \,|\, \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| \tag{3.14}$$

$$\big\|\mathcal{Q}_+(f,g) \,|\, \mathbb{H}_p^{s,\langle\nu\rangle}\big\| \le C_s\, C_{\lambda,p}^{(2)} \,\|g \,|\, \mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\| \; \|f \,|\, \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| \tag{3.15}$$

with a constant C_s *depending only on the smoothness parameter* s *and such that* $C_0 = 1$.

Proof. Due to symmetry it is enough to prove only one of inequalities (3.14) or (3.15). In the second summand of

$$\mathcal{Q}_+(f,g)(v) = \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu)|v-w|^\lambda \Big(f(v')g(w') + f(w')g(v')\Big)\, de\, dw$$

we replace the variable e by $-e$ which implies that μ is replaced by $-\mu$ and v' and w' are interchanged (this is indeed the idea behind the symmetrization leading to (1.16)). For $1 < p \le \infty$, this together with (3.4) gives as in the proof of Theorem 3.1

$$\begin{aligned}
&\big\|\mathcal{Q}_+(f,g) \,|\mathbb{L}_p^{\langle\nu\rangle}\big\|^p \\
&\le 2^{(\lambda/2-1)p}\int\limits_{\mathbb{R}^3}\left(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} |b_\lambda(\mu)|\Big(|f_{\nu+\lambda}(v')||g_{\nu+\lambda}(w')| + |g_{\nu+\lambda}(v')||f_{\nu+\lambda}(w')|\Big)\, de\, dw\right)^p dv \\
&\le 2^{(\lambda/2-1)p}\int\limits_{\mathbb{R}^3}\left(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} \Big(|b_\lambda(\mu)| + |b_\lambda(-\mu)|\Big)|f_{\nu+\lambda}(v')||g_{\nu+\lambda}(w')|\, de\, dw\right)^p dv \\
&\le \Bigg(2^{(\lambda+3-3/p)/2}\pi\, \|f \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle}\| \; \|g \,|\, \mathbb{L}_p^{\langle\nu+\lambda\rangle}\| \\
&\qquad \times\left(\int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1+\mu)^{3(1-1/p)/2}} d\mu + \int\limits_{-1}^{1} \frac{b_\lambda(\mu)}{(1-\mu)^{3(1-1/p)/2}} d\mu\right)\Bigg)^p \\
&\le \left(C_{\lambda,p}^2 \|f \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle}\|\|g \,|\, \mathbb{L}_p^{\langle\nu+\lambda\rangle}\|\right)^p,
\end{aligned} \tag{3.16}$$

because

$$\frac{1}{(1+\mu)^{3(1-1/p)/2}} + \frac{1}{(1-\mu)^{3(1-1/p)/2}} \le \frac{\max\left(2, 2^{3(1-1/p)/2}\right)}{(1-\mu^2)^{3(1-1/p)/2}}\,.$$

For $s \neq 0$ the proof is completed as in Theorem 3.1. ∎

The following result gives a pure $\mathbb{H}_p^{s,\langle\nu\rangle}$-estimate for the gain term. Note that we need stronger conditions on the weight but the reward is that we require only Grad's cutoff assumption for the collision kernel and therefore the result holds in particular for the VHS model for all p. The case $p = 1$ is already treated in Theorem 3.1 so it is omitted.

THEOREM 3.4 *Let the condition* (1.12) *hold,* $1 < p \leq \infty$ *and* $\nu, s \geq 0$*. Then for any* $\gamma > 3/p' + \lambda$ *with* $p' = p/(p-1)$ *the mapping*

$$Q_+ \,:\, \mathbb{H}_p^{s,\langle \nu+\gamma\rangle} \times \mathbb{H}_p^{s,\langle \nu+\gamma\rangle} \longrightarrow \mathbb{H}_p^{s,\langle \nu\rangle} \tag{3.17}$$

is continous and the following estimate holds

$$\|Q_+(f,g) \mid \mathbb{H}_p^{s,\langle\nu\rangle}\| \leq C_s\, 2^{\lambda/2}\, C_\lambda\, C_{\lambda,\gamma,p}\, \|f \mid \mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\|\, \|g \mid \mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\| \tag{3.18}$$

where the constant $C_{\lambda,\gamma,p}$ *depends only on the indicated quantities and* C_s *as in Theorem* 3.1.

Proof. Since both f and g are assumed to belong to the same space, it is enough for the proof to consider the non-symmetric form of Q_+. Similarly to the previous proofs we have for $1 < p < \infty$ using (2.33)

$$\begin{aligned}
\|Q_+(f,g) \mid \mathbb{L}_p^{\langle\nu\rangle}\|^p &\leq \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) |v-w|^\lambda\, |f(v')|\, |g(w')|\, de\, dw \Bigg)^p dv \\
&\leq 2^{p\lambda/2} \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) \langle v'\rangle^\lambda \langle w'\rangle^\lambda |f(v')|\, |g(w')|\, de\, dw \Bigg)^p dv \\
&= 2^{p\lambda/2} \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) \langle v'\rangle^{\lambda-\gamma} \langle w'\rangle^{\lambda-\gamma} |f_\gamma(v')|\, |g_\gamma(w')|\, de\, dw \Bigg)^p dv \\
&\leq 2^{p\lambda/2} \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) \langle w\rangle^{\lambda-\gamma} |f_\gamma(v')|\, |g_\gamma(w')|\, de\, dw \Bigg)^p dv\,,
\end{aligned}$$

where we used that $\gamma - \lambda > 0$ and so by (2.35) we have the inequality

$$\langle w\rangle^{\gamma-\lambda} \leq \langle v'\rangle^{\gamma-\lambda} \langle w'\rangle^{\gamma-\lambda}\,.$$

Writing $b_\lambda(\mu) = b_\lambda(\mu)^{1/p}\, b_\lambda(\mu)^{1/p'}$ and applying Hölder's inequality to the inner integral we obtain

$$\begin{aligned}
\|Q_+(f,g) \mid \mathbb{L}_p^{\langle\nu\rangle}\|^p \leq 2^{p\lambda/2} \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} &\Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) |f_\gamma(v')|^p\, |g_\gamma(w')|^p\, de\, dw \Bigg) \\
&\times \Bigg(\int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) \langle w\rangle^{p'(\lambda-\gamma)}\, de\, dw \Bigg)^{p-1} dv
\end{aligned} \tag{3.19}$$

The second integral in (3.19) decomposes into the integral over S^2, controlled by Grad's cutoff assumption, and the integral with respect to w, which is finite whenever $p'(\lambda-\gamma) < -3$ as it was assumed,

$$C_{\lambda,\gamma,p} = \Big(\int\limits_{\mathbb{R}^3} \langle w\rangle^{p'(\lambda-\gamma)}\, dw \Big)^{1/p'} < \infty\,. \tag{3.20}$$

Thus we have

$$\begin{aligned}
&\|Q_+(f,g) \mid \mathbb{L}_p^{\langle\nu\rangle}\|^p \\
&\le 2^{p\lambda/2}\, C_\lambda^{p-1}\, C_{\lambda,\gamma,p}^p \int\limits_{\mathbb{R}^3} \langle v\rangle^{p\nu} \int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu) |f_\gamma(v')|^p\, |g_\gamma(w')|^p\, de\, dw\, dv
\end{aligned}$$

and the estimate follows using (3.3).
The case $p=\infty$ is simpler:

$$\begin{aligned}
&|\langle v\rangle^\nu Q_+(f,g)(v)| \le 2^{\lambda/2} \langle v\rangle^\nu \int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu)\langle v'\rangle^{\lambda-\gamma}\langle w'\rangle^{\lambda-\gamma} |f_\gamma(v')||g_\gamma(w')|\, de\, dw \\
&\le 2^{\lambda/2} \|f_{\nu+\gamma} \mid \mathbb{L}_\infty\|\, \|g_{\nu+\gamma} \mid \mathbb{L}_\infty\| \int\limits_{\mathbb{R}^3}\int\limits_{S^2} b_\lambda(\mu)\langle w\rangle^{\lambda-\gamma}\, de\, dw\,, \qquad (3.21)
\end{aligned}$$

and the integral with respect to w is bounded since for $p=\infty$ we assume $\lambda-\gamma<-3$.
The result for the Bessel potential spaces follows as in the proof of Theorem 3.1. ■
Let us consider the spaces

$$\mathbb{L}_p^{\langle\infty\rangle} = \bigcap_{m=0}^{\infty} \mathbb{L}_p^{\langle m\rangle} = \bigcap_{\nu\ge 0} \mathbb{L}_p^{\langle\nu\rangle}\,, \qquad 1\le p\le\infty\,, \tag{3.22}$$

$$\mathbb{H}_p^{s,\langle\infty\rangle} = \bigcap_{m=0}^{\infty} \mathbb{H}_p^{s,\langle m\rangle} = \bigcap_{\nu\ge 0} \mathbb{H}_p^{s,\langle\nu\rangle}\,, \qquad s\in\mathbb{R}\,, \tag{3.23}$$

which can be endowed with the standard metric

$$\rho_p(f,g) = \sum_{m=0}^{\infty} \frac{\|f-g \,|\, \mathbb{L}_p^{\langle m\rangle}\|}{1+\|f-g \,|\, \mathbb{L}_p^{\langle m\rangle}\|} 2^{-m}\,,$$

$$\rho_{s,p}(f,g) = \sum_{m=0}^{\infty} \frac{\|f-g \,|\, \mathbb{H}_p^{s,\langle m\rangle}\|}{1+\|f-g \,|\, \mathbb{H}_p^{s,\langle m\rangle}\|} 2^{-m}\,.$$

A continuity result holds also for these metric spaces:

THEOREM 3.5 *Let the condition* (1.18) *hold,* $1\le p\le\infty$ *and* $s\ge 0$. *Then the operator*

$$Q_+ \;:\; \mathbb{H}_1^{s,\langle\infty\rangle}\times\mathbb{H}_p^{s,\langle\infty\rangle} \to \mathbb{H}_p^{s,\langle\infty\rangle} \tag{3.24}$$

is continuous: For any sequences $\{f_k\}$, $\{g_k\}$ *with*

$$\lim_{k\to\infty} \rho_{s,p}(f_k,f) = 0\,, \quad \lim_{k\to\infty} \rho_{s,1}(g_k,g) = 0$$

it follows

$$\lim_{k\to\infty} \rho_{s,p}(Q_+(g_k,f_k), Q_+(g,f)) = 0\,. \tag{3.25}$$

In particular, $Q_+ \;:\; \mathbb{L}_1^{\langle\infty\rangle}\times\mathbb{L}_p^{\langle\infty\rangle}\to\mathbb{L}_p^{\langle\infty\rangle}$ *is continuous.*

Proof. The convergence (3.25) follows from the equality

$$Q_+(g_k,f_k)-Q_+(g,f)=Q_+(g_k,f_k)-Q_+(g,f_k)+Q_+(g,f_k)-Q_+(g,f)$$
$$=Q_+(g_k-g,f_k)+Q_+(g,f_k-f)\,,$$

the estimates (3.14), (3.15) and the fact that the sequence $g_k - g$ converges to 0 and therefore is uniformly bounded in each $\mathbb{H}_p^{s,\langle\nu\rangle}$. ■

Requiring some moderate smoothness for the angular part of the collision kernel, one deduces the following smoothing properties of the gain term:

THEOREM 3.6 *Let Grad's cutoff condition* (1.15) *hold and, in addition,*

$$b_\lambda \in \mathbb{H}_1^a([-1,1])\,, \qquad a>1\,. \tag{3.26}$$

1. *If*

$$-\frac{1}{2}<\lambda\le 1\,,\quad \gamma>\frac{3}{2}+\lambda\,,\quad \nu\ge 0\,,\qquad \sigma\ge 0\,,\quad -\sigma\le s\le\sigma\,, \tag{3.27}$$

then the operator

$$Q_+ \quad : \quad \mathbb{H}^{\sigma,\langle\nu+\gamma\rangle}\times\mathbb{H}^{s,\langle\nu+\gamma\rangle}\to\mathbb{H}^{s+1,\langle\nu\rangle} \tag{3.28}$$

is continuous;

2. *If*

$$-2<\lambda\le 1\,,\quad \gamma>3+\lambda\,,\quad \nu\ge 0\,,\qquad \sigma\ge 0\,,\quad -\sigma\le s\le\sigma\,, \tag{3.29}$$

then the operator

$$Q_+ \quad : \quad \mathbb{H}_\infty^{\sigma,\langle\nu+\gamma\rangle}\times\mathbb{H}^{s,\langle\nu+\gamma\rangle}\to\mathbb{H}^{s+1,\langle\nu\rangle}\,, \tag{3.30}$$

is continuous.

Proof. By definition (2.26), we have

$$f\in\mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\Leftrightarrow f_\nu\in\mathbb{H}_p^{s,\langle\gamma\rangle}\quad,\quad p=2,\infty$$

and from (2.35) we deduce for $s=0$

$$\|Q_+(f,g)\mid\mathbb{L}_2^{\langle\nu+\gamma\rangle}\|=\|\langle\cdot\rangle^\nu Q_+(f,g)\mid\mathbb{L}_2^{\langle\gamma\rangle}\|\le\|Q_+(|f_\nu|,|g_\nu|)\mid\mathbb{L}_2^{\langle\gamma\rangle}\|\,. \tag{3.31}$$

So the problem is reduced to the case $\nu=0$ which was proven in detail in [DR, Theorems 7 and 10] by use of estimates for the adjoint operator Q_+^*. ■

REMARK 3.7 *Note that the continuity property* (3.28) *implies the following continuity result*

$$Q_+ \quad : \quad \left(\mathbb{H}_1^{\sigma,\langle\nu+\gamma\rangle}\cap\mathbb{H}_\infty^{\sigma,\langle\nu+\gamma\rangle}\right)\times\mathbb{H}^{s,\langle\nu+\gamma\rangle}\to\mathbb{H}^{s+1,\langle\nu\rangle}\,. \tag{3.32}$$

In fact, (3.32) *for* $\nu=s=0$ *follows from* (3.28), $\nu=s=0$ *since*

$$\|f\mid\mathbb{L}_2^{\langle\gamma\rangle}\|\le\|f\mid\mathbb{L}_\infty^{\langle\gamma\rangle}\|^{1/2}\|f\mid\mathbb{L}_1^{\langle\gamma\rangle}\|^{1/2}\,. \tag{3.33}$$

The case $\nu\neq 0$ *is tackled by splitting the weight as in Theorem* 3.6 *above and the case* $s\neq 0$ *by the interpolation as in [DR, Theorem 10].*

From the smoothing property (3.32) we derive a heterogeneous continuity statement for the gain term. In a different context a similar result was proved in [MV, Theorem 4.1].

COROLLARY 3.8 *Let* $1 < p < \infty$ *and the conditions* (1.15)*,* (3.26)*, and* (3.27) *hold. Then the operator*

$$Q_+ \quad : \quad \left(\mathbb{L}_1^{\langle\nu+\gamma\rangle} \cap \mathbb{L}_\infty^{\langle\nu+\gamma\rangle}\right) \times \mathbb{L}_q^{\langle\nu+\gamma\rangle} \to \mathbb{L}_r^{\langle\nu\rangle} \tag{3.34}$$

is continuous: There exists a constant $C_{\lambda,p,q,\nu,r}$ *such that*

$$\left\|Q_+(f,g)\,\middle|\mathbb{L}_r^{\langle\nu\rangle}\right\| \leq C_{\lambda,p,q,\nu,r}\left\|f\,\middle|\mathbb{L}_1^{\langle\nu+\gamma\rangle}\right\|^{1/2}\left\|f\,\middle|\mathbb{L}_\infty^{\langle\nu+\gamma\rangle}\right\|^{1/2}\left\|g\,\middle|\mathbb{L}_q^{\langle\nu+\gamma\rangle}\right\| \tag{3.35}$$

holds, provided

$$1 < q < p\,, \qquad q < r \leq \frac{q}{1-\Theta\, q/3}\,, \qquad \Theta = \frac{2(p-q)}{q(p-2)}\,, \quad 0 < \Theta < 1\,. \tag{3.36}$$

Proof. From (3.32) with $\nu = s = 0$ we see that

$$\langle\cdot\rangle^{-\gamma}Q_+ \;:\; (\mathbb{L}_1^{\langle\gamma\rangle} \cap \mathbb{L}_\infty^{\langle\gamma\rangle}) \times \mathbb{L}_2^{\langle\gamma\rangle} \to \mathbb{H}^{1,\langle\gamma\rangle} \tag{3.37}$$

and the continuity (3.2) with $\nu = s = 0$ yields

$$\langle\cdot\rangle^{-\gamma}Q_+ \;:\; \mathbb{L}_1^{\langle\gamma\rangle} \times \mathbb{L}_p^{\langle\gamma\rangle} \to \mathbb{L}_p^{\langle\gamma\rangle}\,. \tag{3.38}$$

Note that the latter continuity holds already for $\gamma \geq \lambda$. By applying the complex interpolation

$$\left(\mathbb{H}_{q_0}^{s_0}(\mathbb{R}^n), \mathbb{H}_{q_1}^{s_1}(\mathbb{R}^n)\right)_\Theta = \mathbb{H}_q^s(\mathbb{R}^n)\,, \qquad 0 \leq \Theta \leq 1 \tag{3.39}$$

$$0 < q_0, q_1 \leq \infty\,, \quad s = (1-\Theta)s_0 + \Theta\, s_1\,, \quad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$$

(see [Tr, 2.4.7(2) and 2.3.5(2)] we derive the continuity

$$\langle\cdot\rangle^{-\gamma}Q_+ \;:\; (\mathbb{L}_1^{\langle\gamma\rangle} \cap \mathbb{L}_\infty^{\langle\gamma\rangle}) \times \mathbb{L}_q^{\langle\gamma\rangle} \to \mathbb{H}_q^{\Theta,\langle\gamma\rangle}\,, \qquad \Theta = \frac{2(p-q)}{q(p-2)}\,. \tag{3.40}$$

The embedding

$$s_0 - \frac{n}{q_0} \geq s_1 - \frac{n}{q_1} \quad \Longrightarrow \quad \mathbb{H}_{q_0}^{s_0,\langle\gamma\rangle}(\mathbb{R}^n) \subset \mathbb{H}_{q_1}^{s_1,\langle\gamma\rangle}(\mathbb{R}^n) \tag{3.41}$$

(see [Tr, 2.7.1(2)]) adapted to the present situation gives

$$\Theta - \frac{3}{q} \geq -\frac{3}{r} \quad \text{or} \quad r \leq \frac{q}{1-\Theta\, q/3} \quad \Longrightarrow \quad \mathbb{H}_q^{\Theta,\langle\gamma\rangle} \subset \mathbb{L}_r^{\langle\gamma\rangle}\,, \tag{3.42}$$

which holds under the condition (3.36) (see [Tr, 2.7.1(2)]. Combining this with the continuity (3.40) yields

$$\langle\cdot\rangle^{-\gamma}Q_+ \;:\; (\mathbb{L}_1^{\langle\gamma\rangle} \cap \mathbb{L}_\infty^{\langle\gamma\rangle}) \times \mathbb{L}_q^{\langle\gamma\rangle} \to \mathbb{L}_r^{\langle\gamma\rangle} \tag{3.43}$$

which is the same as the continuity (3.34) for $p \neq 2$ and $\nu = 0$. The constant $C_{\lambda,p,q,\nu,r}$ contains the constants of the continuity estimates (3.37), (3.38) and the embedding (3.42). For $p = 2$ we get the continuity (3.34), $\nu = 0$ directly from (3.37) and the embedding (3.41).
For $\nu > 0$ the continuity (3.34) follows if we apply the weight splitting as in (3.31). ■

4 The loss part of the collision operator

In the present section we will obtain upper and lower estimates on the loss part $Q_-(f,g)$ of the collision operator. For collision kernels with cutoff the loss part can be written as

$$Q_-(f,g)(v) = C_\lambda \int\limits_{\mathbb{R}^3} |v-w|^\lambda f(v) g(w)\, dw = f(v) Lg(v)\,, \tag{4.1}$$

where L, also called **collision frequency**, is a linear convolution operator

$$Lg(v) = C_\lambda \int\limits_{\mathbb{R}^3} |v-w|^\lambda g(w)\, dw\,, \qquad -3 < \lambda \le 1\,. \tag{4.2}$$

THEOREM 4.1 *Let the condition* (1.12) *hold,* $1 \le p \le \infty$, $\nu \ge 0$ *and* $s \ge 0$. *Then the operator*

$$Q_- \quad : \quad \mathbb{H}_p^{s,\langle \nu+\lambda\rangle} \times \mathbb{H}_1^{s,\langle \nu+\lambda\rangle} \to \mathbb{H}_p^{s,\langle \nu\rangle} \tag{4.3}$$

is continuous

$$\big\| Q_-(f,g) \,|\, \mathbb{H}_p^{s,\langle\nu\rangle} \big\| \le 2^{\lambda/2}\, C_s\, C_\lambda \big\| g \,|\, \mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\big\| \big\| f \,|\, \mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\big\| \tag{4.4}$$

with a constant C_s depending only on the smoothness parameter s and such that $C_0 = 1$. Similarly to (3.24) *and* (3.25) *the operator*

$$Q_- \quad : \quad \mathbb{H}_p^{s,\langle\infty\rangle} \times \mathbb{H}_1^{s,\langle\infty\rangle} \to \mathbb{H}_p^{s,\langle\infty\rangle} \tag{4.5}$$

is continuous.

Proof. The proof is similar to the case of the operator Q_+ but yet simpler (cf. Theorem 3.1), with obvious sup-norm modification for $p=\infty$:

$$\begin{aligned}
\big\| Q_-(f,g) \,|\, \mathbb{L}_p^{\langle\nu\rangle}\big\| &= \Bigg(\int\limits_{\mathbb{R}^3} \Bigg| C_\lambda \int\limits_{\mathbb{R}^3} \frac{\langle v\rangle^\nu |v-w|^\lambda}{\langle v\rangle^{\nu+\lambda}\langle w\rangle^{\nu+\lambda}} f_{\nu+\lambda}(v) g_{\nu+\lambda}(w)\, dw \Bigg|^p dv \Bigg)^{1/p} \\
&\le 2^{\lambda/2} C_\lambda \Bigg(\int\limits_{\mathbb{R}^3} |g_{\nu+\lambda}(w)|\, dw \Bigg) \Bigg(\int\limits_{\mathbb{R}^3} |f_{\nu+\lambda}(v)|^p\, dv \Bigg)^{1/p} \\
&\le 2^{\lambda/2} C_\lambda \big\| g \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle}\big\| \big\| f \,|\, \mathbb{L}_p^{\langle\nu+\lambda\rangle}\big\|\,.
\end{aligned} \tag{4.6}$$

The case $s > 0$ is derived from this in the same way as in the proof of Theorem 3.1. ■

We now consider the symmetrized form of the collision operator. Combining Theorems 3.3, 3.5, 4.1 we immediately obtain for the full collision operator:

COROLLARY 4.2 *Let the condition* (1.12) *hold,* $1 \le p \le \infty$ *and* $s, \nu \ge 0$. *Then the collision operator*

$$\mathcal{Q} \colon \Big(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle} \Big) \times \Big(\mathbb{H}_1^{s,\langle\nu+\lambda\rangle} \cap \mathbb{H}_p^{s,\langle\nu+\lambda\rangle} \Big) \longrightarrow \mathbb{H}_p^{s,\langle\nu\rangle} \tag{4.7}$$

is continuous

$$\left\|\mathcal{Q}(f,g)\,|\,\mathbb{H}_p^{s,\langle\nu\rangle}\right\| \le \frac{1}{2}C_s\left(2^{\lambda/2}\,C_\lambda + \tilde{C}^{(1)}_{\lambda,p}\right)\Big(\left\|f\,|\,\mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\right\|\left\|g\,|\,\mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\right\| + \left\|g\,|\,\mathbb{H}_1^{s,\langle\nu+\lambda\rangle}\right\|\left\|f\,|\,\mathbb{H}_p^{s,\langle\nu+\lambda\rangle}\right\|\Big) \tag{4.8}$$

with a constant C_s depending only on the smoothness parameter s and such that $C_0 = 1$. Similarly to (3.24)-(3.25)*, the operator*

$$\mathcal{Q} \;:\; \mathbb{H}_p^{s,\langle\infty\rangle}\times\mathbb{H}_1^{s,\langle\infty\rangle}\to\mathbb{H}_p^{s,\langle\infty\rangle} \tag{4.9}$$

is continuous.

LEMMA 4.3 *Let $1 \le q \le \infty$, $q' = \dfrac{q}{q-1}$ and*

$$\nu > 3 - \frac{3}{q} + |\lambda - m| = \frac{3}{q'} + |\lambda - m| \ge \frac{3}{q'} + \lambda - m > 0 \tag{4.10}$$

(cf. (4.2)*) for some $m = 0,1,2,3$. Then*

$$L \,:\, \mathbb{H}_q^{s,\langle\nu\rangle} \to \mathbb{H}_\infty^{s+m,\langle-\lambda\rangle} \tag{4.11}$$

is continuous for all $s \ge 0$ and the inequality

$$\left\|Lg\,|\,\mathbb{H}_\infty^{s+m,\langle-\lambda\rangle}\right\| \le C_{1,\lambda,\nu,q,s,m}\left\|g\,|\,\mathbb{H}_q^{s,\langle\nu\rangle}\right\| \tag{4.12}$$

holds for all $g \in \mathbb{H}_q^{s,\langle\nu\rangle}$.

Proof. For $m = 0$ the proposition was proven in [DR, Corollary 13]). For $m = 1,2,3$ we apply m derivatives to the operator and get a similar operator yet with the kernel having the upper bound

$$\prod_{i=0}^{m-1}|\lambda - 2i|\,|v-w|^{\lambda-m}\,.$$

Replacing λ by $\lambda - m$ we can apply the part already proven. ■

LEMMA 4.4 (see [DR, Remark 15]) *The operator*

$$L \;:\; \mathbb{H}^s_{q,com} \to \mathbb{H}^{s+3+\lambda}_{q,loc} \tag{4.13}$$

is continuous for arbitrary $s \in \mathbb{R}$.

The next result is the counterpart of Theorem 3.4 for the loss term. Again it turns out that the proof is much simpler.

THEOREM 4.5 *Let the condition* (1.12) *hold, $1 < p \le \infty$ and $s \ge 0$. Then for any $\gamma > 3/p' + \lambda$ with $p' = p/(p-1)$ the following holds:*

a) The collision frequency L is continous for the setting

$$L \;:\; \mathbb{L}_p^{\langle\gamma\rangle} \longrightarrow \mathbb{L}_\infty^{\langle-\lambda\rangle}\,,$$

with the estimate

$$\|Lg \mid \mathbb{L}_\infty^{\langle-\lambda\rangle}\| \le 2^{\lambda/2}C_\lambda C_{\lambda,\gamma,p}\,\|g \mid \mathbb{L}_p^{\langle\gamma\rangle}\|\,, \tag{4.14}$$

where the constant $C_{\lambda,\gamma,p}$ is given by (3.20).

b) The loss term Q_-

$$Q_- \ : \ \mathbb{H}_p^{s,\langle\gamma\rangle} \times \mathbb{H}_p^{s,\langle\gamma\rangle} \longrightarrow \mathbb{H}_p^{s,\langle\gamma-\lambda\rangle} \,,$$

is continous and the following estimate holds:

$$\|Q_-(f,g) \mid \mathbb{H}_p^{s,\langle\gamma-\lambda\rangle}\| \leq 2^{\lambda/2} C_s \, C_\lambda \, C_{\lambda,\gamma,p} \, \|f \mid \mathbb{H}_p^{s,\langle\gamma\rangle}\| \, \|g \mid \mathbb{H}_p^{s,\langle\gamma\rangle}\| \,. \quad (4.15)$$

Proof.

a) This is a simple consequence of (2.36), Peetre's and Hölder's inequality, we have

$$\begin{aligned} |\langle v\rangle^{-\lambda} Lg(v)| &\leq C_\lambda \int_{\mathbb{R}^3} \langle v\rangle^{-\lambda} |v-w|^\lambda |g(w)| \, dw \\ &\leq 2^{\lambda/2} C_\lambda \int_{\mathbb{R}^3} \langle w\rangle^{\lambda-\gamma} |g_\gamma(w)| \, dw \\ &\leq 2^{\lambda/2} C_\lambda \Bigg(\int_{\mathbb{R}^3} \langle w\rangle^{p'(\lambda-\gamma)} \, dw \Bigg)^{1/p'} \Bigg(\int_{\mathbb{R}^3} |g_\gamma(w)|^p \, dw \Bigg)^{1/p} , \end{aligned} \quad (4.16)$$

and the first integral is finite by assumption on γ.

b) As for the gain term it is sufficient to consider the non-symmetric form of Q_-:

$$\begin{aligned} \|Q_-(f,g) \mid \mathbb{L}_p^{\langle\gamma-\lambda\rangle}\|^p &= \int_{\mathbb{R}^3} \langle v\rangle^{p(\gamma-\lambda)} |f(v)|^p \, |Lg(v)|^p \, dv \\ &\leq \|Lg \mid \mathbb{L}_\infty^{\langle-\lambda\rangle}\|^p \int_{\mathbb{R}^3} \langle v\rangle^{p\gamma} \, |f(v)|^p \, dv \end{aligned} \quad (4.17)$$

and the result follows by inserting (4.14). The statement for the Bessel potential spaces follows once again by first treating the case for positive integers s and subsequent interpolation. ■

Recalling that we have $\gamma > 3/p' + \lambda \geq \lambda$ we see that under the conditions of Theorems 3.4 and 4.5 the loss term maps into a subspace of the space containing the image of the gain term. Taking into account the proof of Theorem 3.5, we conclude:

COROLLARY 4.6 *Let the condition* (1.12) *hold,* $1 < p \leq \infty$ *and* $\nu, s \geq 0$. *Then for any* $\gamma > 3/p' + \lambda$ *with* $p' = p/(p-1)$ *the collision operator*

$$Q \ : \ \mathbb{H}_p^{s,\langle\nu+\gamma\rangle} \times \mathbb{H}_p^{s,\langle\nu+\gamma\rangle} \longrightarrow \mathbb{H}_p^{s,\langle\nu\rangle} \,,$$

is continous with the estimate

$$\|Q(f,g) \mid \mathbb{H}_p^{s,\langle\nu\rangle}\| \leq 2^{\lambda/2} \, C_s \, C_\lambda \, C_{\lambda,\gamma,p} \, \|f \mid \mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\| \, \|g \mid \mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\| \,. \quad (4.18)$$

Furthermore, the collision operator is a continous mapping in the following sense:

$$Q \ : \ \mathbb{H}_p^{s,\langle\infty\rangle} \times \mathbb{H}_p^{s,\langle\infty\rangle} \longrightarrow \mathbb{H}_p^{s,\langle\infty\rangle} \,. \quad (4.19)$$

We give a new pointwise lower bound for the collision frequency. It is given in a form which may be useful also in the context of the spatially inhomogeneous Boltzmann equation. For preparation, we denote by

$$\begin{pmatrix} \rho \\ \mathbf{m} \\ \mathbf{r} \end{pmatrix} = \int\limits_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ v|v|^2/2 \end{pmatrix} dv$$

the density, the momentum and the energy flux, respectively.

Lemma 4.7 *Let the collision kernel* $B(|u|, \mu)$ *satisfy* (1.12)*,* $0 < \lambda \le 2$ *and the function* $f \in \mathbb{L}_1$ *be a.e. non-negative with finite density, momentum and energy flux. Then the inequality*

$$Lf(v) \ge C_\lambda \Big(\rho(\langle v \rangle^\lambda - 1) - 2^{1-\lambda/2} \lambda \langle v \rangle^{\lambda-2} (v, \mathbf{m} + 2\mathbf{r}) \Big), \quad v \in \mathbb{R}^3 \tag{4.20}$$

holds.

Proof. By inequality (2.36) we see that

$$\begin{aligned} Lf(v) &= C_\lambda \int\limits_{\mathbb{R}^3} \big(1 + |v-w|^\lambda\big) f(w)\, dw - C_\lambda \rho \\ &\ge C_\lambda \int\limits_{\mathbb{R}^3} \langle v - w \rangle^\lambda f(w)\, dw - C_\lambda \rho\,. \end{aligned} \tag{4.21}$$

Using Lagrange's theorem for the weight function $\langle \cdot \rangle^\lambda$ we can estimate from below as follows: There exists $0 < \vartheta < 1$ such that

$$\begin{aligned} \langle v - w \rangle^\lambda &= \langle v \rangle^\lambda - \lambda \langle v - \vartheta w \rangle^{\lambda-2} (v - \vartheta w, w) \\ &= \langle v \rangle^\lambda + \vartheta \lambda \langle v - \vartheta w \rangle^{\lambda-2} |w|^2 - \lambda \langle v - \vartheta w \rangle^{\lambda-2} (v, w) \end{aligned} \tag{4.22}$$

The first summand is non-negative and for the second one, we invoke Peetre's inequality and the fact, that the weight function is increasing for non-negative exponents:

$$\langle v - \vartheta w \rangle^{\lambda-2} \le 2^{1-\lambda/2} \langle v \rangle^{\lambda-2} \langle \vartheta w \rangle^{2-\lambda} \le 2^{1-\lambda/2} \langle v \rangle^{\lambda-2} \langle w \rangle^2\,.$$

Thus, by definition of the weight function we have

$$\langle v - w \rangle^\lambda \ge \langle v \rangle^\lambda - 1 - 2^{1-\lambda/2} \lambda \langle v \rangle^{\lambda-2} (1 + |w|^2)(v, w)\,.$$

Inserting this into (4.21) we obtain

$$\begin{aligned} Lf(v) &\ge C_\lambda \rho \Big(\langle v \rangle^\lambda - 1 \Big) - 2^{1-\lambda/2} \lambda\, C_\lambda \langle v \rangle^{\lambda-2} (v, \int\limits_{\mathbb{R}^3} w(1 + |w|^2) f(w)\, dw) \\ &= C_\lambda \Big(\rho(\langle v \rangle^\lambda - 1) - 2^{1-\lambda/2} \lambda \langle v \rangle^{\lambda-2} (v, \mathbf{m} + 2\mathbf{r}) \Big) \end{aligned} \tag{4.23}$$

as claimed. ■

REMARK 4.8 *1. Lower bounds similar to* (4.20)*, are known but with the additional condition of finite entropy: If*

$$f(v) \geq 0\,, \qquad \mathcal{H}_f = \int\limits_{\mathbb{R}^3} [1 + |\log\, f(v)| + |v|^2] f(v)\, dv < \infty\,,$$

then (see [Ar, Lemma 4] and [Mas, Lemma 3.1])

$$Lf(v) \geq M(1 + |v|^\lambda)\,,$$

where the constant $M = M(\lambda, \mathcal{H}_f)$ depends on λ, $\mathcal{H}_f$ and $\|f \mid \mathbb{L}_1^{\langle 2\rangle}\|$.

2. *For the spatially homogeneous initial value problem, it suffices to consider such distribution functions for which the density is 1 and the average velocity is 0. Instead of using Peetre's inequality for the second summand in* (4.22)*, one can estimate*

$$\langle v\rangle^\lambda + \vartheta\lambda\langle v - \vartheta w\rangle^{\lambda-2}|w|^2 - \lambda\langle v - \vartheta w\rangle^{\lambda-2}\,(v, w) \geq \langle v\rangle^\lambda - \lambda\,(v, w)\,,$$

since $\langle\cdot\rangle^{\lambda-2}$ is bounded from above by 1. This gives

$$Lf(v) \geq \langle v\rangle^\lambda - 1\,.$$

With this, one can prove the production of $\mathbb{L}_1$-moments by the method in [We2] without assuming finite entropy.

3. *In general, the lower bound in Lemma* 4.7 *is of course positive for large $|v|$. Without assuming finite entropy, it seems rather difficult to find a strictly positive lower bound. However, an alternative estimate without the requirement of finite entropy was given in [Bo] for the special case of hard spheres* (1.14)*: Let $f(t,\cdot)$ be a solution of the initial value problem* (1.7)*, then*

$$Lf(t, v) \geq \frac{1}{\sqrt{7}} Lf_0(v)\,.$$

The next result shows that the estimate obtained in (4.20) is asymptotically precise.

LEMMA 4.9 *If $0 \leq \lambda < \infty$ and $f \in \mathbb{L}_1^{\langle\lambda\rangle}(\mathbb{R}^3)$, then*

$$\lim_{|v|\to\infty} \langle v\rangle^{-\lambda} Lf(v) = C_\lambda \int\limits_{\mathbb{R}^3} f(w)\, dw \tag{4.24}$$

or, equivalently,

$$Lf(v) = C_\lambda \rho \langle v\rangle^{-\lambda} + o(\langle v\rangle^{-\lambda}) \qquad \text{as} \quad |v| \to \infty\,. \tag{4.25}$$

Proof. Let $f_n(v) = f(v)$ for $|v| \leq n$ and $f_n(v) = 0$ for $|v| > n$. Then

$$\begin{aligned} \lim_{|v|\to\infty} \langle v\rangle^{-\lambda} Lf_n(v) &= C_\lambda \lim_{|v|\to\infty} \int\limits_{|v|\leq n} \frac{|v - w|^\lambda}{\langle v\rangle^\lambda} f(w)\, dw \\ &= C_\lambda \int\limits_{|v|\leq n} f(w)\, dw = C_\lambda \int\limits_{\mathbb{R}^3} f_n(w)\, dw\,. \end{aligned} \tag{4.26}$$

Obviously,

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} f_n(w)\, dw = C_\lambda \int\limits_{\mathbb{R}^3} f(w)\, dw\,. \tag{4.27}$$

On the other hand, applying Peetre's inequality (2.33) we find

$$\begin{aligned} \lim_{|v|\to\infty} \langle v\rangle^{-\lambda} L(f-f_n)(v) &= C_\lambda \lim_{|v|\to\infty} \int\limits_{|v|\geq n} \frac{|v-w|^\lambda}{\langle v\rangle^\lambda \langle w\rangle^\lambda} f_\lambda(w)\, dw \\ &\leq 2^{\lambda/2} C_\lambda \lim_{|v|\to\infty} \int\limits_{|v|\geq n} f_\lambda(w)\, dw = 0 \end{aligned} \tag{4.28}$$

where $f_\lambda \in \mathbb{L}_1$ defined as in (3.1). The equalities (4.26)-(4.28) imply (4.25). ■

References

[Al] R. Alexandre. Around 3D Boltzmann non linear operator without cutoff, a new formulation. *M2AN Math. Model. Numer. Anal.* **34** (2000), No. 3, 575-590.

[ADVW] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg. Entropy Dissipation and Long-Range Interactions. *Arch. Rational Mech. Anal.* **152** (2002), 327-355

[Ar] L. Arkeryd. L^∞ estimates for the spatially homogeneous Boltzmann equation. *J. Stat. Phys.* **31** (1983), no. 2, 347-361.

[Bo] A.V. Bobylev. Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *J. Statist. Phys.* **88** (1997), No. 5-6, 1183-1214.

[BD] F. Bouchut and L. Desvillettes. A proof of the smoothing properties of the positive part of Boltzmann's kernel, *Rev. Mat. Iberoamericana*, **14(1)** (1998), 47-61.

[CIP] C. Cercignani, R. Illner, and M. Pulvirenti. *The Mathematical Theory of Dilute Gases*. Springer, New York, 1994.

[DW] L. Desvillettes, B. Wennberg. Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. Preprint (2003).

[DR] R. Duduchava, S. Rjasanow. Mapping properties of the Boltzmann collision operator, *Universität des Saarlandes, Fachrichtung 6.1 – Mathematik*, Preprint **32**, 1-30, Saarbrücken, 2001. To appear in *Integral Equations and Operator Theory*.

[Gr] H. Grad. *Handbuch der Physik*, volume 12, pages 205-294. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.

[Gu1] T. Gustafsson. L^p-estimates for the nonlinear spatially homogeneous Boltzmann equation. *Arch. Rational Mech. Anal.* , **92** (1986), 23-57.

[Gu2] T. Gustafsson. Global L^p–properties for the spatially homogeneous Boltzmann equation. *Arch. Rational Mech. Anal.* , **103** (1988), 1-38.

[Li] P.L. Lions. Compactness in Boltzmann's equation via Fourier integral operators and applications I. *J. Math. Kyoto Univ.*, **34(2)** (1994), 391-427.

[Lu] X. Lu. A direct method for the regularity of the gain term in the Boltzmann equation. *J. Math. Anal. Appl.*, **228(2)** (1998), 409-435.

[Mas] N. Maslova. *Nonlinear Evolution Equations (Kinetic Approach)*. World Scientific, Singapure, 1993.

[MV] C. Mouhot, C. Villani. Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. Preprint (2003).

[Tr] H. Triebel. *Theory of Function Spaces*. Birkhäuser, Basel, 1978.

[We] B. Wennberg. Regularity in the Boltzmann equation and the Radon transform. *Commun. in partial differential equations*, **19 (11/12)** (1994), 2057-2074.

[We2] B. Wennberg. Entropy dissipation and moment production for the Boltzmann equation. *J. Stat. Phys*, **86 (5/6)** (1997), 1053-1066.