Asymptotics of Potentials in the Edge Calculus

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Abstract. Boundary value problems on manifolds with conical singularities or edges contain potential operators as well as trace and Green operators which play a similar role as the corresponding operators in (pseudo-differential) boundary value problems on a smooth manifold. There is then a specific asymptotic behaviour of these operators close to the singularities. We characterise potential operators in terms of actions of cone or edge pseudo-differential operators (in the neighbouring space) on densities supported by submanifolds which also have conical or edge singularities. As a byproduct we show the continuity of such potentials as continuous operators between cone or edge Sobolev spaces and subspaces with asymptotics.

Key words: Surface potentials with asymptotics, edge Sobolev spaces, operators on manifolds with conical and edge singularities

2000 Mathematics Subject Classification: 35B40, 35S15, 35J30

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Introduction

Potential operators are known to belong to the structure of parametrices of elliptic boundary value problems, e.g., potentials with respect to Poisson kernels in problems with smooth boundary, cf. Agmon, Douglis and Nirenberg [1], Boutet de Monvel [2]. If the boundary is not smooth and has, for instance, geometric singularities (say, edges, as is the case in crack configurations in mechanics) asymptotic phenomena play a specific role, and it is interesting to analyse the interplay between geometric singularities and (local and global) asymptotic contributions of the involved operators. The present paper is aimed at representing potential operators on configurations with conical (or edge) singularities with boundary in terms of operators from the respective cone (or edge) calculus in the neighbouring space by their actions on corresponding 'surface densities'. As a consequence we obtain asymptotics of potentials wchich we express in the framework of weighted Sobolev spaces. Other aspects of surface potentials with asymptotics have been studied before by Chkadua and Duduchava in [3].

By asymptotics (in simplest form) we understand a behaviour of (say, C^∞) functions $u(r)$ on $\mathbb{R}_+$ of the form

$$u(r) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} r^{-p_j} \log^k r \tag{1}$$

as $r \to 0$, with exponents $p_j \in \mathbb{C}$, $\operatorname{Re} p_j \to \infty$ as $j \to \infty$, which means that for every $M > 0$ there is an $N(M) \in \mathbb{N}$ such that $u(r) - \sum_{j=0}^{N} \sum_{k=0}^{\infty} c_{jk} r^{-p_j} \log^k r$ is flat of order M at $r = 0$ for every $N \geq N(M)$. As is well known by Kondratyev's work [8], solutions to elliptic equations of Fuchs type on a manifold with conical singularities (locally modelled on a (stretched) cone $X^\wedge := \mathbb{R}_+ \times X$ with base X) have asymptotics of the form (1) (in this case the coefficients c_{jk} belong to $C^\infty(X)$).

The nature of asymptotics of solutions to elliptic equations on a configuration with edges (locally modelled on a (stretched) wedge $X^\wedge \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open) requires more explanation. Denoting the points of $X^\wedge \times \Omega$ by (r, x, y), the asymptotic expansion of solutions $u(r, x, y)$ contains y-depending coefficients $c_{jk}(x, y)$ where the Sobolev smoothness in y depends on $\operatorname{Re} p_j$. For instance, in the case of a 'trivial' wedge $\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^q$ with edge $\mathbb{R}^q$, elliptic regularity may refer to standard Sobolev spaces $H^s(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^q)$. Writing elements of that space in polar coordinates $(r, x) \in \mathbb{R}_+ \times S^n$ with respect to the variables $\tilde{x} \in \mathbb{R}^{n+1} \backslash \{0\}$, for $s > \frac{n+1}{2}$, the Taylor coefficients at r^j belong to $C^\infty(S^n, H^{s-j-\frac{n+1}{2}}(\mathbb{R}^q))$ for $0 \leq j < s - \frac{n+1}{2}$ (in this case we have $p_j = -j$ and $m_j = 0$ for all j).

For non-trivial wedges, say, $X^\wedge \times \mathbb{R}^q$ with an arbitrary base X, the operators in question are assumed to be edge-degenerate, cf. the notation below. Among these operators are Laplace-Beltrami operators to wedge metrics of the form $\mathrm{d}r^2 + r^2 g_X(r, y) + \mathrm{d}y^2$, when g_X is a family of Riemannian metrics, smoothly depending on (r, y) (up to $r = 0$). Solvability of elliptic equations can be described in weighted edge Sobolev spaces $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ and subspaces with asymptotics, cf. [12] or [13]. A similar calculus is known for boundary value problems on manifolds with edges and boundary, cf. [9] and [6]. In particular, this theory yields parametrices of elliptic crack problems. In this connection it is natural to ask the asymptotics of potentials of 'densities', supported by a hypersurface with boundary and with 'edge-asymptotics' at the boundary. The potential refers to a parametrix of a given elliptic operator, and the task is to characterise the asymptotics of the potential in the neighbouring space close to the boundary.

In this paper we give the answer in terms of operators in the edge algebra from [13] under the assumption on y- independence of the asymptotic data. A similar structure in simpler form (for smooth boundary and Taylor asymptotics) is known from boundary value problems with the transmission property at the boundary, cf. [2], and in fact, we also employ some information from that case on the smooth part of our surface. The case of non-constant exponents of the asymptotics could be embedded into the framework of continuous asymptotics, cf. [13], but this is voluminous and will not be treated in the present paper explicitly.

1 Asymptotics in weighted edge Sobolev spaces

1.1 Cone and edge Sobolev spaces

We first establish basic facts on differential operators in so called cone and edge Sobolev spaces. Given a differential operator

$$A = \sum_{|\alpha|\le\mu} a_\alpha(\tilde{x}) D_{\tilde{x}}^\alpha$$

in $\mathbb{R}^m \ni \tilde{x}$ with coefficients $a_\alpha \in C^\infty(\mathbb{R}^m)$ we can interpret a hypersurface $\mathbb{R}^q$ in $\mathbb{R}^m$ as a fictitious edge and reformulate A as an edge-degenerate operator. That means, writing $\mathbb{R}^m = \mathbb{R}^{n+1} \times \mathbb{R}^q$ for $m = n+1+q$, and inserting polar coordinates (r, φ) in $\mathbb{R}^{n+1} \setminus \{0\}$, the operator takes the form

$$A = r^{-\mu} \sum_{j+|\beta|\le\mu} a_{j\beta}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\beta \tag{2}$$

with coefficients $a_{j\beta}(r,y) \in C^\infty(\overline{\mathbb{R}}\times\mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\alpha|)}(S^n))$; here S^n is the unit sphere in $\mathbb{R}^{n+1}$, and $\mathrm{Diff}^\nu(M)$ for a C^∞ manifold M denotes the space of all differential operators of order ν on M with smooth coefficients (the space $\mathrm{Diff}^\nu(M)$ is Fréchet in a natural way). A differential operator of the form (2) will be called edge-degenerate (clearly such operators are much more general than the ones induced by smooth operators via polar coordinates). Note that when the operator A is elliptic in $\mathbb{R}^m$ in the sense that the homogeneous principal symbol in $(\tilde{x}, \tilde{\xi})$ does not vanish for $\tilde{\xi} \neq 0$ and all $\tilde{x}$, the homogeneous principal symbol $\sigma_\psi(A)(r, x, y, \varrho, \xi, \eta)$ of (2) in the variables (r, x, y) and covariables (ϱ, ξ, η) (in local coordinates x on S^n) is elliptic in the 'edge-degenerate' sense, i.e.,

$$r^\mu \sigma_\psi(A)(r, x, y, r^{-1}\varrho, \xi, r^{-1}\eta) \neq 0$$

for all $(\varrho, \xi, \eta) \neq 0$ and (r, x, y), up to $r = 0$.

Instead of S^n it also makes sense to insert any other compact C^∞ manifold X, and y may vary in any open set $\Omega \subseteq \mathbb{R}^q$; then edge degeneracy refers to the splitting of variables (r, x, y) in the (open stretched) wedge $\mathbb{R}_+ \times X \times \Omega$ with edge Ω and (stretched) model cone $X^\wedge := \mathbb{R}_+ \times X$. The operator A can be written as a pseudo-differential operator

$$A = \mathrm{Op}_y(a)$$

with an operator-valued amplitude function

$$a(y,\eta) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha, \tag{3}$$

$\mathrm{Op}_y(a)u(y) = \iint e^{i(y-y')\eta}a(y,\eta)u(y')\mathrm{d}y'đ\eta$, $đ\eta = (2\pi)^{-q}\mathrm{d}\eta$. To study the nature of the operator function (3) we need some notation on the Mellin transform and weighted Sobolev spaces.

The Mellin transform will be used in its classical form, namely,

$$Mu(z) = \int_0^\infty r^{z-1}u(r)\mathrm{d}r,$$

first on functions $u \in C_0^\infty(\mathbb{R}_+)$, $z \in \mathbb{C}$ and then extended to various larger function and distribution spaces, also vector-valued ones. Then z usually varies on a 'weight line'

$$\Gamma_\beta = \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$$

for some appropriate $\beta \in \mathbb{R}$. Function spaces on Γ_β, e.g., the Schwartz or Sobolev spaces in the real variable $\mathrm{Im}\, z \in \Gamma_\beta$, will be denoted by $\mathcal{S}(\Gamma_\beta)$, $H^s(\Gamma_\beta)$, etc. Recall that the map $M_\gamma : u \to Mu|_{\Gamma_{\frac{1}{2}-\gamma}}$, $C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$ extends to an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$$

with inverse $M_\gamma^{-1}g(r) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z}g(z)\mathrm{d}z$. Let us set

$$\mathrm{op}_M^\gamma(f)u(r) = \iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\varrho)} f(r,r',z)u(r')\frac{\mathrm{d}r'}{r'}đ\varrho, \tag{4}$$

$z = \frac{1}{2} - \gamma + i\varrho \in \Gamma_{\frac{1}{2}-\gamma}$, interpreted as a pseudo-differential operator with respect to the weighted Mellin transform M_γ. Here, in the scalar case, the amplitude function belongs to $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}))$ with $S^\mu(\Gamma_{\frac{1}{2}-\gamma})$ being Hörmander's space of symbols (with constant coefficients) of order μ in the covariable $\varrho = \mathrm{Im} z$, $z \in \Gamma_{\frac{1}{2}-\gamma}$. We will mainly need (4) for operator-valued amplitude functions, namely, parameter-dependent pseudo-differential operators on a C^∞ manifold X with the parameter as covariable.

In general, by $L^\mu_{(\mathrm{cl})}(X;\mathbb{R}^l)$ we denote the space of families $A(\lambda)$ of pseudo-differential operators of order μ on X, dependent on a parameter $\lambda \in \mathbb{R}^l$ (subscript '(cl)' means that corresponding considerations are valid both for the classical or non-classical elements, and we write 'cl', if we talk about the classical case). By definition, we have $L^{-\infty}(X;\mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ which is the Schwartz space of functions in $\mathbb{R}^l$ with values in $L^{-\infty}(X)$, the space of smoothing operators on X. The elements $A(\lambda) \in L^\mu_{(\mathrm{cl})}(X;\mathbb{R}^l)$ are defined by local (classical or non-classical) amplitude functions in covariables $(\xi,\lambda) \in \mathbb{R}^{n+l}$, $n = \mathrm{dim} X$, modulo $L^{-\infty}(X;\mathbb{R}^l)$.

Let us now assume that X is a closed compact C^∞ manifold, and let $H^s(X)$ denote the standard Sobolev space of smoothness s on X.

It is well known that for every $\mu \in \mathbb{R}$ there exists an element $R^\mu(\lambda) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l)$ that induces isomorphisms $R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X)$ for all $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$.

Let us fix such an $R^\mu(z) \in L^\mu_{\mathrm{cl}}(X;\Gamma_{\frac{n+1}{2}-\gamma})$ for $\mu = s$; then $\mathcal{H}^{s,\gamma}(X^\wedge)$ denotes the completion of the space $C_0^\infty(X^\wedge)$ with respect to the norm

$$\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(z)Mu(z)||^2_{L^2(X)}\mathrm{d}z\}^{\frac{1}{2}}.$$

The space $L^2(X)$ refers to a measure associated with a fixed Riemannian metric on X. Clearly the specific choice of $R^s(z)$ only affects the norm of $\mathcal{H}^{s,\gamma}(X^\wedge)$ up to equivalence.

Let us also consider the space $H^s_{\text{cone}}(X^\wedge)$ which is for the case $X = S^n$ the subspace of all $u \in H^s_{\text{loc}}(\mathbb{R} \times S^n)|_{\mathbb{R}_+ \times S^n}$ such that $\chi u \in H^s(\mathbb{R}^{n+1})$ for any excision function χ in $\mathbb{R}^{n+1}$ (i.e., C^∞, vanishing for $|\tilde{x}| < R_0$, and equal to 1 for $|\tilde{x}| > R_1$ for some $0 < R_0 < R_1$). In the latter relations $\mathbb{R}_+ \times S^n$ is identified with $\mathbb{R}^{n+1}\setminus\{0\}$ via polar coordinates.

For X in general we can define $H^s_{\text{cone}}(X^\wedge)$ by a simple localisation procedure on subsets $\mathbb{R}_+ \times U$ for coordinate neighbourhoods U on X such that $\mathbb{R}_+ \times U$ is diffeomorphic to a conical subset of $\mathbb{R}^{n+1}\setminus\{0\}$; more details may be found in [12].

In this paper, a cut-off function on the half-axis is any real-valued $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of zero.

We now define the space

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge),\ v \in H^s_{\text{cone}}(X^\wedge)\}, \tag{5}$$

where ω is any cut-off function. Clearly this space is independent of the specific choice of ω. The space (5) is endowed with the structure of a non-direct sum of the completions of $\omega\mathcal{H}^{s,\gamma}(X^\wedge)$ and $(1-\omega)H^s_{\text{cone}}(X^\wedge)$ in the respective spaces (cf. [13] for the general definition of non-direct sums).

The spaces (5) play a crucial role in future. Concerning more details cf. [12] or [13]. Let us only mention here the relations

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X),$$

with L^2 referring to $\mathrm{d}r\mathrm{d}x$. Moreover, we have

$$\omega r^\delta \mathcal{K}^{s,\gamma}(X^\wedge) = \omega\mathcal{K}^{s,\gamma+\delta}(X^\wedge)$$

for arbitrary $s, \gamma, \delta \in \mathbb{R}$ and any cut-off function ω.

Remark 1.1 *The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are Hilbert spaces with suitable scalar products. Setting*

$$(\kappa_\lambda u)(r,x) := \lambda^{\frac{n+1}{2}} u(\lambda r, x), \qquad \lambda \in \mathbb{R}_+,$$

$n = \dim X$, on the space $\mathcal{K}^{s,\gamma}(X^\wedge)$ we obtain a strongly continuous group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms. If necessary we also write $\kappa_\lambda^{(n)}$ instead of κ_λ.

Remark 1.2 *Assume that the coefficients $a_{j\beta}$ in (2) are independent of r for $r > R$ for some $R > 0$. Set*

$$a(y,\eta) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\beta}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\beta.$$

Then

$$a(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{6}$$

is a family of continuous operators for every $s \in \mathbb{R}$, smoothly dependent on (y,η) in the operator norm.

It will be essential in the following to interpret (6) as an operator-valued symbol in (y,η), according to the following definition.

Definition 1.3 (i) *Given a Hilbert space E with a group $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, of isomorphisms, $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$, and strongly continuous in λ, i.e., $\kappa_\lambda e \in C(\mathbb{R}_+, E)$ for every $e \in E$, we say that E is endowed with a group action.*

(ii) *If E and $\tilde{E}$ are Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively, the space $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ of operator-valued symbols for open $U \subseteq \mathbb{R}^q$, $\mu \in \mathbb{R}$, is defined as the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ such that*

$$\sup_{(y,\eta) \in K \times \mathbb{R}^q} \langle \eta \rangle^{-\mu + |\beta|} ||\tilde{\kappa}^{-1}_{\langle \eta \rangle} \{ D^\alpha_y D^\beta_\eta a(y,\eta) \} \kappa_{\langle \eta \rangle} ||_{\mathcal{L}(E,\tilde{E})}$$

is finite for every $K \Subset U$, and multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$.

(iii) *The space $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ of classical operator-valued symbols is defined as the subspace of all $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ such that there are $a_{(\mu-j)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \tilde{E}))$, $j \in \mathbb{N}$, which are homogeneous of order $\mu - j$ in $\eta \neq 0$ in the sense*

$$a_{(\mu-j)}(y, \lambda\eta) = \lambda^{\mu-j} \tilde{\kappa}_\lambda a_{(\mu-j)}(y,\eta) \kappa^{-1}_\lambda$$

for all $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$, such that

$$a(y,\eta) - \chi(\eta) \sum_{j=0}^{N} a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \tilde{E})$$

for all $N \in \mathbb{N}$ and any excision function χ.

Example 1.4 *The operator function (6) represents an element $a(y,\eta) \in S^\mu(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ for every $s, \gamma \in \mathbb{R}$. The symbol $a(y,\eta)$ is classical, if the coefficients $a_{j\beta}$ are independent of r.*

Definition 1.5 *Let E be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$. Then the space $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm*

$$\left\{ \int \langle \eta \rangle^{2s} ||\kappa^{-1}_{\langle \eta \rangle} \hat{u}(\eta)||^2_E \mathrm{d}\eta \right\}^{\frac{1}{2}}$$

(with $\hat{u}(\eta) = Fu(\eta)$ being the Fourier transform of u). We call $\mathcal{W}^s(\mathbb{R}^q, E)$ an ('abstract') edge Sobolev space of smoothness s, where $\mathbb{R}^q$ is the edge.

Remark 1.6 *The definitions, both of edge Sobolev spaces and symbol spaces have an immediate generalisation to the case of Fréchet spaces E or $\tilde{E}$ with group actions, cf.* [13], [6]. *By a Fréchet space with group action we mean that E is written as a projective limit of Hilbert spaces $\varprojlim_{k \in \mathbb{N}} E^k$ with continuous embeddings $E^{k+1} \hookrightarrow E^k \hookrightarrow \ldots \hookrightarrow E^0$ for all k, such that there is a group action on E^0 which restricts to group actions on E^k for all k.*

Example 1.7 (i) *For $E = H^s(\mathbb{R}^{1+n})$ with $(\kappa_\lambda u)(\tilde{x}) = \lambda^{\frac{n+1}{2}} u(\lambda \tilde{x})$, $\lambda \in \mathbb{R}$, we have*

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^{1+n})) = H^s(\mathbb{R}^{1+n+q})$$

for every $s \in \mathbb{R}$.

(ii) *The case* $\{\kappa_\lambda\} = \mathrm{id}_E$ *for all* $\lambda \in \mathbb{R}_+$ *is an admitted (trivial) choice of a group action in* E*; in this case we write* $H^s(\mathbb{R}^q, E)$ *instead of* $\mathcal{W}^s(\mathbb{R}^q, E)$*. Also note that the space* $\mathcal{W}^\infty(\mathbb{R}^q, E)$ *is independent of the choice of* $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ *and* $\mathcal{W}^\infty(\mathbb{R}^q, E) = H^\infty(\mathbb{R}^q, E)$*.*

(iii) *For* $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ *endowed with the group action from Remark* 1.1 *we set* $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$*, called a weighted edge space of smoothness* $s \in \mathbb{R}$ *and weight* $\gamma \in \mathbb{R}$*, with respect to the edge* $\mathbb{R}^q$ *and with (stretched) model cone* $X^\wedge$*.*

Similarly as the 'comp' and 'loc' version of standard Sobolev spaces we have spaces of the kind $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ and $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ for any open set $\Omega \subseteq \mathbb{R}^q$. The following continuity is similar to a corresponding result in the scalar case (cf. [12] or [13]):

Proposition 1.8 *Let* $\Omega \subseteq \mathbb{R}^q$ *be an open set, and let* $a(y, y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$*. Then* $\mathrm{Op}_y(a)$ *induces continuous operators*

$$\mathrm{Op}_y(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$$

for every $s \in \mathbb{R}$*. If* a *is independent of* y, y' *then we obtain continuous operators*

$$\mathrm{Op}_y(a) : \mathcal{W}^s(\mathbb{R}, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}, \tilde{E})$$

for all $s \in \mathbb{R}$*.*

1.2 Edge asymptotics

In this section we single out subspaces $\mathcal{W}^{s,\gamma}_P(X^\wedge \times \mathbb{R}^q)$ of $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \ni u(r, x, y)$ with so called discrete asymptotics for $r \to 0$ of type P. This will be formulated in terms of corresponding subspaces $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ of $\mathcal{K}^{s,\gamma}(X^\wedge)$ with asymptotics. By an asymptotic type P we understand a sequence

$$P = \{(p_j, m_j, L_j)\}_{j=0,\dots,N} \tag{7}$$

for an $N = N(P) \in \mathbb{N} \cup \{\infty\}$, such that following properties hold: $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and L_j is a subspace of $C^\infty(X)$ of finite dimension. Moreover, $\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\dots,N}$ is required to be contained in $\{z : \mathrm{Re}\, z < \frac{n+1}{2} - \gamma\}$ for some weight γ, and $\pi_{\mathbb{C}} P \cap \{z \in \mathbb{C} : c \le \mathrm{Re}\, z < \frac{n+1}{2} - \gamma\}$ is a finite set for every $c < \frac{n+1}{2} - \gamma$. If $\pi_{\mathbb{C}} P$ is finite and contained in a strip $\{z \in \mathbb{C} : \frac{n+1}{2} - \gamma - \theta < \mathrm{Re}\, z < \frac{n+1}{2} - \gamma\}$ for some $-\infty \le \theta < 0$ we say that P is associated with the weight data $\boldsymbol{g} = (\gamma, \Theta)$ for the weight strip $\Theta = (\theta, 0]$. Let $\mathrm{As}(X, \boldsymbol{g})$ denote the set of all P associated with $\boldsymbol{g}$. For the case $\mathrm{dim} X = 0$ we simply write $\mathrm{As}(\boldsymbol{g})$.

Let us set

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\theta-\varepsilon}(X^\wedge)$$

considered in the Fréchet topology of projective limit. Moreover, for $P \in \mathrm{As}(X, \boldsymbol{g})$ and finite Θ, let $\mathcal{E}_P(X^\wedge)$ denote the linear span of all functions $c(x) r^{-p} \log^k r\, \omega(r)$ for arbitrary

$$(p, m, L(p)) \in P, \quad k \le m, \quad c \in L(p)$$

and some fixed choice of a cut-off function ω. The space $\mathcal{E}_P(X^\wedge)$ is then of finite dimension and has the properties

$$\mathcal{E}_P(X^\wedge) \subset \mathcal{K}^{\infty,\gamma+\delta}(X^\wedge)$$

for some $0 < \delta < \mathrm{dist}(\pi_{\mathbb{C}} P, \Gamma_{\frac{n+1}{2}-\gamma})$, furthermore $\mathcal{E}_P(X^\wedge) \cap \mathcal{K}^{s,\gamma}(X^\wedge) = \{0\}$. We now define

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_P(X^\wedge) \tag{8}$$

with the Fréchet topology of the direct sum. For the case $\Theta = (-\infty, 0]$ we choose any sequence of numbers $\theta_k < 0$, $\theta_k \to -\infty$ as $k \to \infty$ and form the set $P_k := \{(p, m, L) \in P : \frac{n+1}{2} - \gamma + \theta_k < \mathrm{Re}\, p < \frac{n+1}{2} - \gamma\}$. We then have continuous embeddings $\mathcal{K}_{P_{k+1}}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}_{P_k}^{s,\gamma}(X^\wedge)$ for all $k \in \mathbb{N}$. Then

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) := \varprojlim_{k \in \mathbb{N}} \mathcal{K}_{P_k}^{s,\gamma}(X^\wedge)$$

is a Fréchet space in the topology of the projective limit.

For the purposes below we set

$$\mathcal{S}_P^\gamma(X^\wedge) := \varprojlim_{k \in \mathbb{N}} \langle r \rangle^{-k} \mathcal{K}_P^{k,\gamma}(X^\wedge) \tag{9}$$

in the corresponding Fréchet topology.

Remark 1.9 *The space $\mathcal{K}_P^{s,\gamma}(X^\wedge)$, $P \in \mathrm{As}(X, \boldsymbol{g})$, can be written as a projective limit of Hilbert spaces $E^k \subset \mathcal{K}^{s,\gamma}(X^\wedge)$, $k \in \mathbb{N}$ with group action, induced by that of Remark 1.1, and continuous embeddings $E^{k+1} \hookrightarrow E^k \hookrightarrow \cdots \hookrightarrow E^0 = \mathcal{K}^{s,\gamma}(X^\wedge)$ for all k. A similar remark is true of the spaces* (9).

We now introduce subspaces of $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \ni u(r, x, y)$ with asymptotics for $r \to 0$, which are discrete and constant with respect to the edge variable y.

Using Remark 1.9 we can write $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ as a projective limit of $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$-invariant Hilbert spaces E^k, $k \in \mathbb{N}$, which gives us the edge spaces $\mathcal{W}^s(\mathbb{R}^q, E^k)$ with continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{k+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^k)$ for all k, and then we define

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)) \tag{10}$$

as the projective limit $\varprojlim_{k \in \mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^k)$ with the corresponding Fréchet structure. It can easily be proved that (10) is independent of the specific choice of the sequence $\{E^k\}_{k \in \mathbb{N}}$ with the properties of Remark 1.9. Similarly as the 'comp' and 'loc' version of abstract edge Sobolev spaces on an open set $\Omega \subseteq \mathbb{R}^q$ we have the spaces

$$\mathcal{W}_{\mathrm{comp}(y)}^{s,\gamma}(X^\wedge \times \Omega) := \mathcal{W}_{\mathrm{comp}}^s(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge))$$

as well as those with $\mathrm{loc}(y)$ and subspaces with asymptotics $\mathcal{W}_{\mathrm{comp}(y),P}^{s,\gamma}(X^\wedge \times \Omega)$, etc..

To characterise the singular functions of the edge asymptotics we first observe that when E is a Hilbert (or Fréchet space) with group action, we have canonical isomorphisms

$$T(\eta) := F^{-1} \kappa_{\langle \eta \rangle}^{-1} F : \mathcal{W}^s(\mathbb{R}^q, E) \to H^s(\mathbb{R}^q, E)$$

for all $s \in \mathbb{R}$, cf. Example 1.7 (ii) and [12]. Let $E = E_0 \oplus E_1$ be a direct decomposition of E into closed subspaces, not necessarily invariant under the group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ on E. We then obtain $H^s(\mathbb{R}^q, E) = H^s(\mathbb{R}^q, E_0) \oplus H^s(\mathbb{R}^q, E_1)$ which generates a direct decomposition

$$\mathcal{W}^s(\mathbb{R}^q, E) = T^{-1} H^s(\mathbb{R}^q, E_0) \oplus T^{-1} H^s(\mathbb{R}^q, E_1) \tag{11}$$

into closed subspaces.

Let us apply this construction to the space (8) for an element $P \in \mathrm{As}(X, \boldsymbol{g})$, $g = (\gamma, \Theta)$, where the weight interval Θ is finite. The space $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ is closed with respect to $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$; this gives us

$$T^{-1} H^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)),$$

also denoted by $\mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. However, $\mathcal{E}_P(X^\wedge)$ is not preserved under the group action, but we can form

$$\mathcal{V}_P^s(X^\wedge \times \mathbb{R}^q) := T^{-1} \mathcal{E}_P(X^\wedge)$$

which is as a closed subspace of $\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. In other words, we have a direct decomposition

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q) + \mathcal{V}_P^s(X^\wedge \times \mathbb{R}^q)$$

into a component of distributions of edge-flatness Θ and a space of singular functions with discrete (and constant in y) edge asymptotics of type P.

Remark 1.10 *Every $f(r, x, y) \in \mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for a (discrete) asymptotic type $P = \{(p_j, m_j, L_j)\}_{j=0,\dots,N(P)} \in \mathrm{As}(X, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \Theta)$, $\Theta = (\theta, 0]$ finite (i.e., $N(P) < \infty$) can be written in the form*

$$f(r, x, y) = f_{\mathrm{sing}}(r, x, y) + f_\Theta(r, x, y)$$

for singular functions

$$f_{\mathrm{sing}}(r, x, y) = \sum_{j=0}^{N(P)} \sum_{k=0}^{m_j} F_{\eta \to y}^{-1} [\eta]^{\frac{n+1}{2}} \omega(r[\eta]) c_{jk}(x) (r[\eta])^{-p_j} \log^k (r[\eta]) \hat{v}_{jk}(\eta)$$

with suitable $v_{jk} \in H^s(\mathbb{R}^q)$, coefficients $c_{jk} \in L_j$, $0 \le k \le m_j$, for all j, and a flat remainder $f_\Theta(r, x, y) \in \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. Note that in the case $s = \infty$ we may write

$$f_{\mathrm{sing}}(r, x, y) = \sum_{j=0}^{N(P)} \sum_{k=0}^{m_j} \omega(r) c_{jk}(x) w_{jk}(y) r^{-p_j} \log^k r$$

mod $\mathcal{W}_\Theta^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = H^\infty(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge))$ *with elements $w_{jk} \in H^\infty(\mathbb{R}^q)$, cf. also Example* 1.7 (ii).

One may ask to what extent our notation of singular functions of the edge asymptotics depends on the choice of the function $\eta \to [\eta]$. One can prove, cf. [6], that when $p(\eta)$ is any other element of $C^\infty(\mathbb{R}^q)$ such that $c_1[\eta] \le p(\eta) \le c_2[\eta]$ for all $\eta \in \mathbb{R}^q$, with constants $c_1 < c_2$, then $f_{\mathrm{sing}}(r, x, y)$ can be reformulated into an equivalent expression with $p(\eta)$ in place of $[\eta]$ and other coefficients c_{jk}, v_{jk}, mod $\mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. Also the choice of ω is unessential modulo such flat remainders.

2 Mellin representation of parametrices

2.1 Mellin operators in spaces with asymptotics

First, let X be a closed compact C^∞ manifold, and introduce operator-valued Mellin symbols with asymptotics. Let us start from the case of discrete asymptotics. A sequence

$$R = \{(p_j, m_j, L_j)\}_{j \in \mathbb{Z}} \tag{12}$$

with $\pi_{\mathbb{C}} R = \{p_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$, $m_j \in \mathbb{N}$, is called a discrete asymptotic type of Mellin symbols if

$$\pi_{\mathbb{C}} R \cap \{z : c \le \operatorname{Re} z \le c'\}$$

is finite for every $c \le c'$, and L_j is a finite-dimensional subspace of operators in $L^{-\infty}(X)$ of finite rank. Let $\mathbf{As}(X)$ denote the set of all such sequences R.

If $U \subseteq \mathbb{C}$ is an open set and E a Fréchet space by $\mathcal{A}(U, E)$ we denote the space of all holomorphic functions in U with values in E.

Definition 2.1 (i) *The space $M_{\mathcal{O}}^{\mu}(X)$ for $\mu \in \mathbb{R}$ is defined as the set of all $h(z) \in \mathcal{A}(\mathbb{C}, L_{\mathrm{cl}}^{\mu}(X))$ such that*

$$h(\beta + i\varrho) \in L_{\mathrm{cl}}^{\mu}(X; \mathbb{R}_{\varrho})$$

holds for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$, for arbitrary $c \le c'$.

(ii) *$M_R^{-\infty}(X)$ is the space of all $f(z) \in \mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}} R, L^{-\infty}(X))$ that are meromorphic with poles at p_j of multiplicities $m_j + 1$ and Laurent coefficients at $(z - p_j)^{-(k+1)}$ in L_j, $0 \le k \le m_j$, for all $j \in \mathbb{Z}$, and satisfy*

$$\chi_R(\beta + i\varrho) f(\beta + i\varrho) \in L^{-\infty}(X; \mathbb{R}_{\varrho})$$

for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$, for arbitrary $c \le c'$; here $\chi_R(z)$ is any $\pi_{\mathbb{C}} R$-excision function.

The spaces in Definition 2.1 (i), (ii) are Fréchet in a natural way.

Let us set

$$M_R^{\mu}(X) := M_{\mathcal{O}}^{\mu}(X) + M_R^{-\infty}(X) \tag{13}$$

in the Fréchet topology of the non-direct sum. The elements of $M_R^{\mu}(X)$ are interpreted as Mellin symbols, i.e., amplitude functions in corresponding Mellin pseudo-differential operators.

Theorem 2.2 [12] *Let $f \in M_R^{\mu}(X)$, and let $\omega(r), \tilde{\omega}(r)$ be cut-off functions. Then $\omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}$ induces continuous operators*

$$\omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

and

$$\omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma}(X^\wedge)$$

for all $s \in \mathbb{R}$, and every discrete asymptotic type $P \in \mathrm{As}(X, (\gamma, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma, \Theta))$ (not depending on s).

In order to discuss potential operators we now assume that X is a compact C^∞ manifold with boundary ∂X. Moreover, let X be embedded into its double $2X$ (i.e., two copies $X_{\pm}$ of X, glued together along their common boundary ∂X, with X being identified with the positive side X_+), where ∂X is an interface in $2X$ of codimension 1.

We want to apply Mellin pseudo-differential operators on $(2X)^\wedge$ in the sense of Theorem 2.2 to surface densities u on $(\partial X)^\wedge$ and then restrict the result to $(\mathrm{int}X)^\wedge$. The problem to be discussed here is to what extent asymptotics of u near $r = 0$ is inherited by such potentials.

The problem will be treated for the Mellin symbols with the transmission property at ∂X. Let us denote this class by

$$M_R^\mu(2X)_{\mathrm{tr}},$$

$\mu \in \mathbb{Z}$. The calculus of boundary value problems in a cone $X^\wedge$ with boundary $(\partial X)^\wedge$ contains potential operators that are also connected with Mellin symbols with asymptotic types.

Let $B^\nu(X, \partial X)$ denote the space of all potential operators

$$K : H^s(\partial X) \to H^{s-\nu}(X)$$

of order $\nu \in \mathbb{R}$ in Boutet de Monvel's calculus in X. There is then a natural parameter-dependent version, namely, $B^\nu(X, \partial X; \mathbb{R}^l)$ of families $K(\lambda)$ of potential operators with parameter $\lambda \in \mathbb{R}^l$ cf. [2], see also [13, Chapter 4]. For the definition of potential Mellin symbols we need corresponding asymptotic types. A sequence (12) is called a discrete asymptotic type of Mellin potential symbols if (p_j, m_j) are as before, while L_j is a finite-dimensional subspace of operators of finite rank, with kernels in $C^\infty(X \times \partial X)$. Similarly as before, $\mathbf{As}(X, \partial X)$ will denote the set of all such asymptotic types of Mellin potential symbols.

Definition 2.3 (i) *$M_{\mathcal{O}}^\nu(X, \partial X)$ for $\nu \in \mathbb{R}$ is defined as the set of all $h(z) \in \mathcal{A}(\mathbb{C}, B^\nu(X, \partial X))$ such that*

$$h(\beta + i\varrho) \in B^\nu(X, \partial X; \mathbb{R}_\varrho)$$

holds for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$, for arbitrary $c \leq c'$.

(ii) *$M_R^{-\infty}(X, \partial X)$ is the space of all $f(z) \in \mathcal{A}(\mathbb{C} \backslash \pi_{\mathbb{C}} R, B^{-\infty}(X, \partial X))$ that are meromorphic with poles at p_j of multiplicities $m_j + 1$ and Laurent coefficients at $(z - p_j)^{-(k+1)}$ in L_j, $0 \leq k \leq m_j$, for all $j \in \mathbb{Z}$, and satisfy*

$$\chi_R(\beta + i\varrho) f(\beta + i\varrho) \in \mathcal{S}(\mathbb{R}_\varrho, C^\infty(X \times \partial X))$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$, for arbitrary $c \leq c'$ and for any $\pi_{\mathbb{C}} R$-excision function χ_R.

Similarly as before, the spaces $M_{\mathcal{O}}^\nu(X, \partial X)$ and $M_R^{-\infty}(X, \partial X)$ are Fréchet in a natural way, and we set

$$M_R^\nu(X, \partial X) := M_{\mathcal{O}}^\nu(X, \partial X) + M_R^{-\infty}(X, \partial X) \tag{14}$$

in the Fréchet topology of the non-direct sum.

Theorem 2.4 *Let $f \in M_R^\nu(X, \partial X)$, let $\omega(r), \tilde{\omega}(r)$ be cut-off functions, and assume $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{n}{2} - \gamma} = \emptyset$. Then $\omega \mathrm{op}_M^{\gamma - \frac{n}{2}}(f) \tilde{\omega}$ induces continuous operators*

$$\omega \mathrm{op}_M^{\gamma - \frac{n}{2}}(f) \tilde{\omega} : \mathcal{K}^{s, \gamma - \frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}^{s - \nu, \gamma}(X^\wedge)$$

and

$$\omega \mathrm{op}_M^{\gamma - \frac{n}{2}}(f) \tilde{\omega} : \mathcal{K}_P^{s, \gamma - \frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}_Q^{s - \nu, \gamma}(X^\wedge)$$

for all $s \in \mathbb{R}$, and every discrete asymptotic type $P \in \mathrm{As}(\partial X, (\gamma - \frac{1}{2}, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma, \Theta))$ (not depending on s).

A proof of Theorem 2.4 may be found in [11], see also [6].

Let us now consider an element

$$f(z) \in M_R^\mu(2X)_{\mathrm{tr}}, \quad \mu \in \mathbb{Z}$$

for a Mellin asymptotic type $R \in \mathbf{As}(2X)$. By definition, we have

$$f(z) \in \mathcal{A}(\mathbb{C} \backslash \pi_{\mathbb{C}} R, L_{\mathrm{cl}}^\mu(2X)_{\mathrm{tr}})$$

where subscript 'tr' indicates the subspace of all elements of $L_{\mathrm{cl}}^\mu(2X)$ with the transmission property at ∂X as well as the other properties from Definition 2.1.

According to (13) we write

$$f(z) = h(z) + l(z) \tag{15}$$

for $h \in M_{\mathcal{O}}^\mu(2X)_{\mathrm{tr}}$, $l \in M_R^{-\infty}(2X)$.

A basic result from boundary value problems is the following observation.

Let $A \in L_{\mathrm{cl}}^\mu(2X)_{\mathrm{tr}}$, $\mu \in \mathbb{Z}$, and apply the operator to an element $u \in \mathcal{E}'(2X)$ defined by a surface density $v \otimes \delta_{\partial X}$ for some $v \in C^\infty(\partial X)$, i.e.,

$$u \, : \, \varphi \to \langle u, \varphi \rangle = \int_{\partial X} v(x')(\varphi|_{\partial X})(x')\mathrm{d}x'$$

$\varphi \in C^\infty(2X)$. Then

$$K : v \to A(v \otimes \delta_{\partial X})|_{\mathrm{int} X}$$

defines a continuous operator

$$K : C^\infty(\partial X) \to C^\infty(X)$$

which belongs to $B^{\mu+\frac{1}{2}}(X, \partial X)$.

In other words, $A \to K$ gives us a map

$$L_{\mathrm{cl}}^\mu(2X)_{\mathrm{tr}} \to B^{\mu+\frac{1}{2}}(X, \partial X). \tag{16}$$

We want to apply (16) to $f \in M_{\mathcal{O}}^\mu(2X)_{\mathrm{tr}}$ which is a holomorphic z-dependent family of elements in $L_{\mathrm{cl}}^\mu(2X)_{\mathrm{tr}}$.

Theorem 2.5 *The correspondence* (16)*, z-wise applied to* $f \in M_R^\mu(2X)_{\mathrm{tr}}$ *for an* $R \in \mathbf{As}(2X)$*, defines a continuous operator*

$$M_R^\mu(2X)_{\mathrm{tr}} \to M_{R'}^{\mu+\frac{1}{2}}(X, \partial X) \tag{17}$$

for a resulting asymptotic type $R' \in \mathbf{As}(X, \partial X)$*. In particular, the map* (17) *induces a continuous operator*

$$M_{\mathcal{O}}^\mu(2X)_{\mathrm{tr}} \to M_{\mathcal{O}}^{\mu+\frac{1}{2}}(X, \partial X).$$

Proof. Let us write f in the form (15). Let $\{U_1, \dots, U_N\}$ be a cover of $2X$ by coordinate neighbourhoods, $\{\varphi_1, \dots, \varphi_N\}$ a subordinate partition of unity, and $\psi_1, \dots, \psi_N$ a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi \equiv 1$ on $\mathrm{supp}\, \varphi_j$ for all j. We then have

$$f(z) = \sum_{j=1}^N \varphi_j f(z) \psi_j + m(z) \tag{18}$$

for a certain element $m \in M_{\mathcal{O}}^{-\infty}(2X)$. The discussion of $m(z)$ will be postponed to the consideration in connection with $l(z)$ in the relation (15). So we concentrate on the terms $\varphi_j f(z)\psi_j$ which can be expressed in local coordinates in the form

$$\int e^{i(x-\tilde{x})\xi} a(x,\tilde{x},z,\xi)u(\tilde{x})\mathrm{d}\tilde{x}đ\xi \tag{19}$$

for a symbol $a(x,\tilde{x},z,\xi) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}_{x,\tilde{x}}^{2n} \times \mathbb{C} \times \mathbb{R}_{\xi}^{n})$. The meaning of the latter notation is as follows. By $S_{\mathrm{cl}}^{\mu}(U \times \mathbb{C} \times \mathbb{R}_{\xi}^{n})$ for any open set $U \subseteq \mathbb{R}^q$ we denote the set of all $a(x,z,\xi) \in \mathcal{A}(\mathbb{C}, C^{\infty}(U \times \mathbb{R}_{\xi}^{n}))$ such that $a(x,\beta+i\varrho,\xi) \in S_{\mathrm{cl}}^{\mu}(U \times \mathbb{R}_{\varrho,\xi}^{1+n})$ for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c \leq c'$. Since we want to restrict the operator to surface densities on ∂X it suffices to consider such charts $\chi : U \to \mathbb{R}^n \ni x = (x_1,\ldots,x_n)$ on X for which $U \cap \partial X \neq \emptyset$. Without loss of generality we then assume that X induces by restriction to $U' = U \cap \partial X$ a diffeomorphism $\chi' : U' \to \mathbb{R}^{n-1}$ for $x' = (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}$, where x_n corresponds to the normal direction to ∂X. Let $\xi = (\xi',\xi_n)$ be the associated covariables. Because of the special form of the summands on the right hand side of (18) we may draw the factors ψ_j in local coordinates to the argument function u. Then it suffices to consider the case of symbols $a(x,z,\xi) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}_x^n \times \mathbb{C} \times \mathbb{R}_{\xi}^{n})_{\mathrm{tr}}$. We now obtain for (19)

$$\iint e^{i(x'-\tilde{x}')\xi'} e^{i(x_n-\tilde{x}_n)\xi_n} a(x',x_n,z,\xi',\xi_n)u(\tilde{x}',\tilde{x}_n)\mathrm{d}\tilde{x}'\mathrm{d}\tilde{x}_n đ\xi' đ\xi_n$$

which we apply to distributions $u(\tilde{x}',\tilde{x}_n) = v(\tilde{x}') \otimes \delta_{\mathbb{R}^{n-1}}$. To express the potential operator we have to consider

$$\mathrm{Op}_{x'}(k)(z)v$$

for the operator-valued symbol

$$k(x',z,\xi')c := \int e^{ix_n\xi_n} a(x',x_n,z,\xi',\xi_n)đ\xi_n|_{x_n>0} \cdot c, \tag{20}$$

acting on scalars $c \in \mathbb{C}$. By construction, $a(x',x_n,z,\xi',\xi_n)$ has compact support in $x = (x',x_n)$. Let us first ignore the dependence on x_n, i.e., look at $a(x',z,\xi',\xi_n)$. The general case with non-trivial dependence on x_n will be discussed afterwards. To analyse the structure of (20) it is helpful to compose the expression from the left with $\kappa_{\langle\varrho,\xi'\rangle}^{-1}$ for $\varrho = \operatorname{Im} z$ with respect to the variable x_n in the exponent, i.e., to pass to

$$\kappa_{\langle\varrho,\xi'\rangle}^{-1} k(x',z,\xi') = \langle\varrho,\xi'\rangle^{-\frac{1}{2}} \int e^{i\langle\varrho,\xi'\rangle^{-1}x_n\xi_n} a(x',z,\xi',\xi_n)đ\xi_n|_{x_n>0} =$$

$$\langle\varrho,\xi'\rangle^{\frac{1}{2}} \int e^{ix_n\xi_n} a(x',z,\xi',\langle\varrho,\xi'\rangle\xi_n)đ\xi_n|_{x_n>0}. \tag{21}$$

Let us fix for a moment $\beta = \operatorname{Re} z$. Then using a standard property of symbols with the transmission property, cf., for instance, [6], we obtain that $\langle\varrho,\xi'\rangle^{\frac{1}{2}} a(x',\beta+i\varrho,\xi',\langle\varrho,\xi'\rangle\xi_n)$ belongs to the space

$$S_{\mathrm{cl}}^{\mu+\frac{1}{2}}(\mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{\varrho,\xi'}^{n})\hat{\otimes}_{\pi} S_{\mathrm{cl}}^{\mu}(\mathbb{R}_{\xi_n})_{\mathrm{tr}}$$

with $S_{\mathrm{cl}}^{\mu}(\mathbb{R}_{\xi_n})_{\mathrm{tr}}$ being the corresponding space of classical symbols with the transmission property in ξ_n (at $x_n = 0$) with constant coefficients. This gives us for (21) an element $g(x',\varrho,\xi';x_n)$ of the space

$$S_{\mathrm{cl}}^{\mu+\frac{1}{2}}(\mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{\varrho,\xi'}^{n})\hat{\otimes}_{\pi} \mathcal{S}(\overline{\mathbb{R}}_{+,\xi_n}).$$

In other words, it follows that

$$k(x', \beta + i\varrho, \xi')c = \langle \varrho, \xi' \rangle^{\frac{1}{2}} g(x', \varrho, \xi'; \langle \varrho, \xi' \rangle x_n)c.$$

This proves that $k(x', \beta + i\varrho, \xi')$ is a potential symbol in the calculus with the transmission property, and we obtain continuity in the sense

$$\mathrm{Op}_{x'}(k)(z) : H^s_{\mathrm{comp}}(\mathbb{R}^{n-1}) \to H^{s-(\mu+\frac{1}{2})}_{\mathrm{loc}(x')}(\mathbb{R}^n_+)$$

for all $s \in \mathbb{R}$.

The operator-valued symbols $k(x', \beta + i\varrho; \xi')$ run over a bounded set in the space

$$S^{\mu+\frac{1}{2}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^n_{\varrho,\xi'}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+))$$

when β varies in a compact interval. Together with the relation (20) which shows the holomorphic dependence on z it follows that

$$k(x', z, \xi') \in S^{\mu+\frac{1}{2}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{C} \times \mathbb{R}^n_{\xi'}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)),$$

cf. Definition 1.3 and Remark 1.6, where $E = \mathbb{C}$ is endowed with the trivial group action and $\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}} \langle x \rangle^{-k} H^k(\mathbb{R}_+)$, with κ_λ acting as $u(x_n) \to \lambda^{\frac{1}{2}} u(\lambda x_n)$, $\lambda > 0$. In other words, we proved our assertion when the original symbol a is independent on x_n. For the general case we obtain a function in

$$C_0^\infty(\overline{\mathbb{R}}_{+,x_n}, S^{\mu+\frac{1}{2}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^n_{\varrho,\xi'}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)))$$

and it can easily be verified, cf. [6, Section 1.2.4], that this can be replaced by a function without x_n-dependence, modulo a smoothing potential operator of infinite flatness at $x_n = 0$.

To complete the proof, it remains to consider an arbitrary $l \in M^{-\infty}_R(2X)$, $R \in \mathbf{As}(2X)$, and to interpret the application of the z-depending smoothing operator family on $2X$ to a function on ∂X, combined with the restriction to X, as a map

$$M^{-\infty}_R(2X) \to M^{-\infty}_{R'}(X, \partial X) \tag{22}$$

for a resulting asymptotic type $R' \in \mathbf{As}(X, \partial X)$. □

Remark 2.6 *Composing Mellin potential symbols of the class $M^\nu_{R'}(X, \partial X)$ from the left by the operator of restriction to the boundary we obtain elements of $M^{\nu+\frac{1}{2}}_{R''}(\partial X)$, Mellin symbols with asymptotics R'' (in the sense of the cone algebra with base manifold ∂X, cf.* [12]*).*

Corollary 2.7 *Let $f \in M^\mu_R(2X)_{\mathrm{tr}}$ be a Mellin symbol with asymptotics of type R, and let $\pi_\mathbb{C} R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. Then $\omega \mathrm{op}^{\gamma-\frac{n}{2}}_M(f)\tilde{\omega}$, first interpreted as a map $\mathcal{E}'((2X)^\wedge) \to \mathcal{D}'((2X)^\wedge)$ restricts to a continuous operators*

$$\omega \mathrm{op}^{\gamma-\frac{n}{2}}_M(f)\tilde{\omega} : \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

and

$$\omega \mathrm{op}^{\gamma-\frac{n}{2}}_M(f)\tilde{\omega} : \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}_P((\partial X)^\wedge) \to \mathcal{K}^{s-\mu,\gamma}_Q(X^\wedge)$$

for all $s \in \mathbb{R}$ and every $P \in \mathrm{As}(\partial X, (\gamma + \frac{1}{2}, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma, \Theta))$.

The relation (16) can be generalised to pseudo-differential operators on a manifold M with conical exit to infinity. Let $L^{\mu;\delta}_{(\mathrm{cl})}(M)$ denote the set of (classical) pseudo-differential operators of order $\mu \in \mathbb{R}$ on M, with exit condition and weight δ at infinity. If M is written as a double $2X$ for a C^∞ manifold X with boundary and conical exit to infinity we have also Boutet de Monvel's calculus $\mathcal{B}^{\mu,d;\delta}(X)$ on X (of classical operators of order $\mu \in \mathbb{Z}$, type $d \in \mathbb{N}$ and weight $\delta \in \mathbb{R}$ at infinity).

Let $B^{\mu,d;\delta}(X)$ denote the space of upper left corners A of the 2×2 block matrix space $\mathcal{B}^{\mu,d;\delta}(X)$ also containing trace and potential entries. The operators $A \in B^{\mu,d;\delta}(X)$ are continuous

$$A : H^{s;\beta}(X) \to H^{s-\mu;\beta-\delta}(X) \tag{23}$$

for every $s, \beta \in \mathbb{R}$, $s > d - \frac{1}{2}$. (The Sobolev spaces in (23) are defined as $H^{s;\beta}(X) = H^{s;\beta}(M)|_{\mathrm{int}X}$, with the usual Sobolev smoothness $s \in \mathbb{R}$ and a power weight $\beta \in \mathbb{R}$ at infinity, cf. [6].) The space of potential operators K of order $\mu + \frac{1}{2}$ in this calculus will be denoted by $B^{\mu+\frac{1}{2};\delta}(X, \partial X)$; those operators are continuous in the sense

$$H^{s+\frac{1}{2};\beta}(\partial X) \to H^{s-\mu;\beta-\delta}(X) \tag{24}$$

for all $s \in \mathbb{R}$.

Theorem 2.8 *Let $A \in L^{\mu;\delta}_{\mathrm{cl}}(2X)_{\mathrm{tr}}$ be an operator of order $\mu \in \mathbb{Z}$ and weight $\delta \in \mathbb{R}$ at infinity* (*subscript* 'tr' *means operators with the transmission property at ∂X, and* 'cl' *indicates classical operators both in variables and covariables*). *Then $K : v \to A(v \otimes \delta_{\partial X})|_{\mathrm{int}X}$ defines a map*

$$L^{\mu;\delta}_{\mathrm{cl}}(2X)_{\mathrm{tr}} \to B^{\mu+\frac{1}{2};\delta}(X, \partial X).$$

Proof. It is obviously sufficient to assume $\delta = 0$. The space $L^{-\infty;0}(2X)_{\mathrm{tr}}$ coincides with the space of all operators with C^∞-kernels which are Schwartz functions in direction to the conical exit of $2X$ to infinity. It is then clear that our potential operator in this case belongs to $B^{-\infty;0}(X, \partial X)$. For the non-smoothing part we may consider the local situation where $2X$ is replaced by $\mathbb{R}^n$ (for $n = \mathrm{dim}X$) and X by $\overline{\mathbb{R}}^n_+$; then $\partial X = \mathbb{R}^{n-1}_{x'}$. If $a(x', x_n, \xi', \xi_n)$ is a classical symbol of order μ in $\mathbb{R}^n$ with exit order 0 (and also classical of order 0 in (x', x_n)-variables) we first have

$$Au(x', x_n) = \iint e^{i(x'-\tilde{x}')\xi'} e^{i(x_n-\tilde{x}_n)\xi_n} a(x', x_n, \xi', \xi_n) u(\tilde{x}', \tilde{x}_n) d\tilde{x}' d\tilde{x}_n đ\xi' đ\xi_n.$$

Then, similarly as in the proof of Theorem 2.5 we obtain

$$\kappa^{-1}_{\langle\xi'\rangle} k(x', \xi') = \langle\xi'\rangle^{-\frac{1}{2}} \int e^{i\langle\xi'\rangle^{-1} x_n \xi_n} a(x', \xi', \xi_n) đ\xi_n|_{x_n>0} =$$

$$\langle\xi'\rangle^{\frac{1}{2}} \int e^{i x_n \xi_n} a(x', \xi', \langle\xi'\rangle\xi_n) đ\xi_n|_{x_n>0}$$

Let us first assume that a is independent of x_n. In the present case we have

$$\langle\xi'\rangle^{\frac{1}{2}} a(x', \xi', \langle\xi'\rangle\xi_n) \in S^{\mu+\frac{1}{2}}_{\mathrm{cl}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^n_{\xi'}) \hat{\otimes}_\pi S^\mu_{\mathrm{cl}}(\mathbb{R}_{\xi_n})_{\mathrm{tr}},$$

where the subscript 'cl' means classical in ξ' as well as in x' (of order 0). Now the remaining part of the proof is similar as before in the proof of Theorem 2.5. It remains to note that the consideration of the x_n-dependent case does not affect the character of the final potential operator in the sense of its exit symbolic structure. □

Remark 2.9 *Observe that when we compose a potential operator in* $B^{\mu+\frac{1}{2};\delta}(X, \partial X)$ *from the left with the restriction operator to* ∂X *we obtain an element in* $L_{\mathrm{cl}}^{\mu+1;\delta}(\partial X)$ (*which is also classical in variables and covariables*).

2.2 Example

Remark 2.10 *Theorem* 2.5 *can easily be generalised to the case of transmission configurations, i.e., when we interpret the boundary* ∂X *as an interface in* $2X = X_- \cup X_+$ *where we distinguish between the minus- and the plus- side of the boundary. In that sense* $2X \backslash \partial X$ *can be completed to a manifold* $2\mathbb{X}$ *with interior boundary* $\partial X_- \cup \partial X_+$. *The analogue of the map* (16)

$$L_{\mathrm{cl}}^{\mu}(2X)_{\mathrm{tr}} \to B^{\mu+\frac{1}{2}}(2\mathbb{X}, \partial X)$$

means that the pseudo-differential operators on $2X$ *with the transmission property at* ∂X, *restricted to densities on* ∂X, *generate an element in* $B^{\mu+\frac{1}{2}}(2\mathbb{X}, \partial X)$. *Instead of* (17) *we then have a corresponding map*

$$M_R^{\mu}(2X)_{\mathrm{tr}} \to M_{R'}^{\mu+\frac{1}{2}}(2\mathbb{X}, \partial X). \tag{25}$$

If we replace $2X$ by a circle S^1 and ∂X by any $p \in S^1$ we obtain a similar situation. In this case we first consider $S^1 \backslash \{p\}$ and then add two different end points such that the new configuration can be identified with an interval $\mathbb{S}^1 = \{\phi : 0 \leq \phi \leq 2\pi\}$ where the point p is replaced by 0 and 2π. Analogously to (25) we then have a corresponding map

$$M_R^{\mu}(S^1)_{\mathrm{tr}} \to M_{R'}^{\mu+\frac{1}{2}}(\mathbb{S}^1, \{p\}). \tag{26}$$

$\mathbb{S}^1$ may be regarded as the base of a cone K obtained from the slit plane $\mathbb{R}^2 \backslash \mathbb{R}_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ by distinguishing two copies of $\mathbb{R}_+$, which correspond to limits for $x_2 \to 0$ from $x_2 > 0$ or $x_2 < 0$. Polar coordinates $(r, \phi) \in \mathbb{R}_+ \times [0, 2\pi]$ then give us an identification

$$K \cong (\overline{\mathbb{R}}_+ \times \mathbb{S}^1)/(\{0\} \times \mathbb{S}^1). \tag{27}$$

The corresponding set of discrete asymptotic types associated with weight data $\boldsymbol{g} = (\gamma, \Theta)$ is then denoted by $\mathrm{As}(\mathbb{S}^1, \boldsymbol{g})$.

Note that K can be regarded as a branch of a Riemannian surface, obtained by gluing together two copies of (27), with an identification of the $\pm$-half-axis of the first copy with the $\mp$-half-axis of the second copy.

Let us consider a fundamental solution of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{28}$$

represented in polar coordinates. First, (28) has the form

$$\Delta = r^{-2}\{(-r\partial r)^2 + \partial_\phi^2\} = r^{-2}\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)$$

for $h(z) := z^2 + \partial_\phi^2$ which is a holomorphic family of Fredholm operators

$$h(z) : H^s(S^1) \to H^{s-2}(S^1). \tag{29}$$

(29) is bijective for all $z \in \mathbb{C} \backslash D$ for $D = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\}$.

The Laplace operator induces a continuous map

$$H_\gamma := r^{-2}\mathrm{op}_M^{\gamma-\frac{1}{2}}(h) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times S^1) \to \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+ \times S^1)$$

for all $s, \gamma \in \mathbb{R}$. The operator $H_\gamma^{-1} := \mathrm{op}_M^{\gamma-\frac{1}{2}}(h^{-1})r^2$ for $0 < \gamma < \frac{1}{4}$ is a parametrix of Δ in $\mathbb{R}^2$ in the standard sense, i.e., we have for every $\varphi \in (\pi^{-1})^* C_0^\infty(\mathbb{R}^2)$

$$H_\gamma H_\gamma^{-1}\varphi = \varphi \quad \text{and} \quad H_\gamma^{-1} H_\gamma \varphi = \varphi.$$

Here $\pi : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}_+ \times S^1$ is the map connected with the polar coordinates $(x_1, x_2) \to (r, \phi)$.

Let us now choose cut-off functions $\omega(r)$, $\tilde{\omega}(r)$, $\tilde{\tilde{\omega}}(r)$ such that $\tilde{\omega} \equiv 1$ on supp ω and $\omega \equiv 1$ on supp $\tilde{\tilde{\omega}}$. Then

$$B := \omega(r)\mathrm{op}_M^{\gamma-\frac{1}{2}}(h^{-1})r^2\tilde{\omega}(r) + (1-\omega(r))\mathrm{op}_{x_1,x_2}(\chi(\xi_1,\xi_2)|\xi_1,\xi_2|^{-2})(1-\tilde{\tilde{\omega}}(r)) \quad (30)$$

is a parametrix of Δ in $\mathbb{R}^2 \backslash \{0\}$ for every $\gamma \in \mathbb{R} \backslash D$. The operator B belongs to the cone algebra on $\mathbb{R}_+ \times S^1$ (in the terminology of [13]). Thus it defines a continuous operator

$$B : \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+ \times S^1) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times S^1)$$

for all $s \in \mathbb{R}$, and B restricts to continuous operators between subspaces with asymptotics

$$B : \mathcal{K}_{P_0}^{s-2,\gamma-2}(\mathbb{R}_+ \times S^1) \to \mathcal{K}_{Q_0}^{s,\gamma}(\mathbb{R}_+ \times S^1)$$

for every $P_0 \in \mathrm{As}(S^1, (\gamma-2, \Theta))$ with some resulting $Q_0 \in \mathrm{As}(S^1, (\gamma, \Theta))$ for every $\Theta = (\theta, 0]$, $-\infty \leq \theta < 0$. Note that the second summand in (30) maps to flat functions. For the first summand we apply a version of Theorem 2.4 in connection with (26). This gives us a continuous map

$$B : \mathcal{K}_P^{s-2,\gamma-2}(\mathbb{R}_+) \to \mathcal{K}_Q^{s+\frac{1}{2},\gamma+\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{S}^1)$$

for every $P \in \mathrm{As}(\gamma-2, \Theta)$ with some resulting $Q \in \mathrm{As}(\mathbb{S}^1, (\gamma+\frac{1}{2}, \Theta))$.

2.3 Potentials of operators in the cone algebra

We now study potentials of surface distributions with respect to operators in the cone algebra on the (infinite stretched) cone $(2X)^\wedge$.

Let us fix weight data $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ for $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$. Recall that the operators in the cone algebra on $(2X)^\wedge$ with discrete asymptotics, associated with $\boldsymbol{g}$, are defined as the set of all operators

$$A = r^{-\mu}\omega(r)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega}(r) + (1-\omega(r))A_\infty(1-\tilde{\tilde{\omega}}) + \\ r^{-\mu}\omega(r)\sum_{j=0}^{k} r^j \mathrm{op}_M^{\gamma_j-\frac{n}{2}}(f_j)\tilde{\omega}(r) + G. \quad (31)$$

Here we assume $h \in M_{\mathcal{O}}^\mu(2X)$, $f_j \in M_{R_j}^{-\infty}(2X)$, $\gamma_j \in \mathbb{R}$ are weights such that $\gamma_j \leq \gamma \leq j + \gamma_j$ for all j, moreover, G is a Green operator with discrete asymptotics, and A_∞ is an element in $L_{\mathrm{cl}}^{\mu;0}((2X)^\wedge)$; the latter notation means the space of all operators $A|_{(2X)^\wedge}$ for arbitrary $A \in L_{\mathrm{cl}}^{\mu;0}((\mathbb{R}_+ \times (2X))_\asymp)$ where $(\mathbb{R} \times (2X))_\asymp \ni (r, x)$ is the manifold with conical exits for $r \to \pm\infty$ modelled on the infinite cylinder $\mathbb{R} \times (2X)$.

Finally, a Green operator, associated with weight data $\boldsymbol{g}$ is an operator such that both

$$G : \mathcal{K}^{s,\gamma}((2X)^\wedge) \to \mathcal{S}_P^{\gamma-\mu}((2X)^\wedge)$$

and

$$G^* : \mathcal{K}^{s,-\gamma+\mu}((2X)^\wedge) \to \mathcal{S}_Q^{-\gamma}((2X)^\wedge)$$

are continuous for all $s \in \mathbb{R}$, with (G-dependent) asymptotic types P and Q, cf. the formula (9). The formal adjoint G^* refers to the $\mathcal{K}^{0,0}$-scalar product. Details about the cone algebra with closed base of the cone may be found in [12] and [13]. The cone algebra of boundary value problems with the transmission property is developed in [10], [11], see also [6] in a new variant with classical symbols.

Theorem 2.11 *Let A be an operator in the cone algebra on $(2X)^\wedge$ of order $\mu \in \mathbb{Z}$ associated with the weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, and assume that A has the transmission property at $(\partial X)^\wedge$. Then A, first interpreted as a map $\mathcal{E}'((2X)^\wedge) \to \mathcal{D}'((2X)^\wedge)$, restricts to continuous operators*

$$A' : \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

and

$$A' : \mathcal{K}_P^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma-\mu}(X^\wedge)$$

for all $s \in \mathbb{R}$ and every $P \in \mathrm{As}(\partial X, (\gamma + \frac{1}{2}, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma, \Theta))$ for all $s \in \mathbb{R}$.

Proof. If A consists of a Green operator G the assertion is obvious. Concerning the second summand on the right of (31) we can apply Theorem 2.8 to A_∞. Moreover, for the Mellin operators contained in (31) we can apply Theorem 2.5. Thus the operator $A|_{\mathcal{E}'((2X)^\wedge)}$ extends to a potential operator in the cone algebra with discrete asymptotics on the infinite cone $X^\wedge$ with boundary $(\partial X)^\wedge$, cf. [11]. This gives us the asserted continuity. □

Remark 2.12 *If A satisfies the assumptions of Theorem* 2.11*, the map*

$$K : v \to A(v \otimes \delta_{(\partial X)^\wedge})|_{\mathrm{int} X^\wedge}$$

defines a operator

$$L_{\mathrm{cl}}^{\mu;\delta}((2X)^\wedge)_{\mathrm{tr}} \to B^{\mu+\frac{1}{2};\delta}(X^\wedge, (\partial X)^\wedge).$$

Remark 2.13 *It can be proved that each potential operator of the cone algebra of boundary value problems* (*say, in the classical variant* [6]) *can be obtained as in Theorem* 2.11 *for a suitable cone operator A.*

Remark 2.14 *Theorem* 2.11 *can be generalised to families of operators $A(y)$ in the cone algebra smoothly depending on a parameter $y \in U$ for some open $U \subset \mathbb{R}^p$. The smoothness can be defined by assuming C^∞ dependence on y of the involved symbols and C^∞ dependence of the Green operators* (*every Green operator with fixed asymptotic types belongs to a corresponding Fréchet space*)*. It follows then that also $A'(y)$ is smooth in y, where the smoothness of A' in a parameter can be defined in a similar manner as before since A' is an element* (*a potential operator*) *of the cone algebra of boundary value problems.*

2.4 Elements of the edge calculus

In the application below we need a parameter-dependent analogue of Definition 2.1.

Definition 2.15 (i) *Let X be a closed compact C^∞ manifold. Then $M^\mu_{\mathcal{O}}(X;\mathbb{R}^q)$ for $\mu \in \mathbb{R}$ is defined as the set of all $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q))$ such that*

$$h(\beta + i\varrho, \eta) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}_\varrho \times \mathbb{R}^q_\eta)$$

holds for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$, for arbitrary $c \le c'$.

(ii) *If X is a compact C^∞ manifold with boundary ∂X the space $M^\nu_{\mathcal{O}}(X, \partial X; \mathbb{R}^q)$ for $\nu \in \mathbb{R}$ is defined as the set of all $h(z) \in \mathcal{A}(\mathbb{C}, B^\nu(X, \partial X; \mathbb{R}^q))$ such that*

$$h(\beta + i\varrho, \eta) \in B^\nu(X, \partial X; \mathbb{R}_\varrho \times \mathbb{R}^q)$$

holds for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$, for arbitrary $c \le c'$.

In addition applying Definition 2.15 (i) to $2X$ we have the subspace

$$M^\mu_{\mathcal{O}}(2X;\mathbb{R}^q)_{\mathrm{tr}}$$

of $M^\mu_{\mathcal{O}}(2X;\mathbb{R}^q)$ of operator families with the transmission property at ∂X. As a corollary of the arguments in the proof of Theorem 2.5 we obtain the following result:

Theorem 2.16 *Let X be a compact C^∞ manifold with boundary ∂X. Then the correspondence* (16)*, pointwise applied for every $(z,\eta) \in \mathbb{C} \times \mathbb{R}^q$, induces a continuous operator*

$$M^\mu_{\mathcal{O}}(2X;\mathbb{R}^q)_{\mathrm{tr}} \to M^{\mu+\frac{1}{2}}_{\mathcal{O}}(X, \partial X; \mathbb{R}^q)$$

for every $\mu \in \mathbb{Z}$.

Proof. It suffices to modify the constructions in the proof by formally replacing ξ' by (ξ', η). □

Let $\omega(r), \tilde{\omega}(r)$ be two cut-off functions, and let $\eta \to [\eta]$ be any strictly positive function in $C^\infty(\mathbb{R}^q)$ that is equal to η for $|\eta| \ge c$ for some $c > 0$. Let

$$\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(2X;\mathbb{R}^q_{\tilde{\eta}})_{\mathrm{tr}}) \tag{32}$$

for any open set $\Omega \subseteq \mathbb{R}^q$, $h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta)$ and form the operator families

$$a_{\mathcal{O}}(y,\eta) = r^{-\mu}\omega(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}(r[\eta]).$$

Moreover, consider an element

$$\tilde{p}_\infty(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(2X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}})_{\mathrm{tr}})$$

for any open $\Omega \subseteq \mathbb{R}^q$, and set $p_\infty(r,y,\varrho,\eta) = \tilde{p}_\infty(r,y,r\varrho,r\eta)$. We use the fact that there exists an operator family of the form (32) such that

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta) = \mathrm{op}_r(p_\infty)(y,\eta) \mod C^\infty(\Omega, L^{-\infty}((2X)^\wedge;\mathbb{R}^q_\eta))$$

for every $\gamma \in \mathbb{R}$, cf. [13], see also [4]. Let us assume that $\tilde{\omega}$ is equal to 1 on supp ω and choose another cut-off function $\tilde{\tilde{\omega}}$ such that ω is equal to 1 on supp $\tilde{\tilde{\omega}}$. Setting

$$a_1(y,\eta) = r^{-\mu}(1 - \omega(r[\eta]))\mathrm{op}_r(p_\infty)(y,\eta)(1 - \tilde{\tilde{\omega}}(r[\eta]))$$

we now form

$$a(y,\eta) = \sigma(r)\{a_{\mathcal{O}}(y,\eta) + a_1(y,\eta)\}\tilde{\sigma}(r) \tag{33}$$

with arbitrary fixed cut-off functions σ and $\tilde{\sigma}$. Applying the mapping (16) to

$$h(r,y,z,\eta) \qquad \text{and} \qquad p_\infty(r,y,\varrho,\eta)$$

for fixed (r,y,z,η) and (r,y,ϱ,η), respectively, we obtain corresponding families

$$h'(r,y,z,\eta) \quad \text{and} \quad p'_\infty(r,y,\varrho,\eta)$$

belonging to $B^{\mu+\frac{1}{2}}(X,\partial X)$. Note that there is an

$$\tilde{h}'(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{\mu+\frac{1}{2}}(X,\partial X;\mathbb{R}^q_{\tilde{\eta}}))$$

such that

$$h'(r,y,z,\eta) = \tilde{h}'(r,y,z,r\eta).$$

Applying $\mathrm{op}_M^{\gamma-\frac{n}{2}}$ to h' and op_r to p'_∞, from $a(y,\eta)$ we obtain a family

$$a'(y,\eta) = \sigma(r)\{a'_{\mathcal{O}}(y,\eta) + a'_1(y,\eta)\}\tilde{\sigma}(r) \tag{34}$$

for

$$a'_{\mathcal{O}}(y,\eta) = r^{-\mu}\omega(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h')(y,\eta)\tilde{\omega}(r[\eta]),$$

and

$$a'_1(y,\eta) = r^{-\mu}(1-\omega(r[\eta]))\mathrm{op}_r(p'_\infty)(y,\eta)(1-\tilde{\tilde{\omega}}(r[\eta])).$$

Summing up, the mapping (16) generates a correspondence

$$a(y,\eta) \to a'(y,\eta) \tag{35}$$

for every operator function of the form (33) with a resulting expression (34).

Theorem 2.17 (i) *The operator family $a(y,\eta)$ represents symbols*

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}((2X)^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}((2X)^\wedge))$$

as well as

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}_{\tilde{P}}((2X)^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}_{\tilde{Q}}((2X)^\wedge))$$

for all $s \in \mathbb{R}$, and for every $\tilde{P} \in \mathrm{As}(2X,(\gamma,\Theta))$ with some resulting $\tilde{Q} \in \mathrm{As}(2X,(\gamma-\mu,\Theta))$.

(ii) *The operator family $a'(y,\eta)$ represents symbols*

$$a'(y,\eta) \in S^{\mu+\frac{1}{2}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

as well as

$$a'(y,\eta) \in S^{\mu+\frac{1}{2}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}_P((\partial X)^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}_Q(X^\wedge))$$

for all $s \in \mathbb{R}$, and for every $P \in \mathrm{As}(\partial X,(\gamma+\frac{1}{2},\Theta))$ with some resulting $Q \in \mathrm{As}(X,(\gamma-\mu,\Theta))$.

Proof. Part (i) of Theorem 2.17 is proved in [6, Theorem 4.4.20]. For Part (ii) we have to consider the ingredients of $a(y,\eta)$ separately. Let us first assume that the function $\tilde{h}$ which is involved in $a_{\mathcal{O}}(y,\eta)$ is independent of r. Then it is suffices to observe the homogeneity

$$\sigma a'_{\mathcal{O}}(y,\lambda\eta)\tilde{\sigma} = \lambda^{\mu+\frac{1}{2}}\kappa_\lambda^{(n)}\sigma a'_{\mathcal{O}}(y,\eta)\tilde{\sigma}(\kappa_\lambda^{(n-1)})^{-1} \tag{36}$$

for all $\lambda \geq 1$, $|\eta| \geq c$ for some $c > 0$ which entails

$$\sigma a'_{\mathcal{O}}(y,\eta)\tilde{\sigma} \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}_{(P)}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge), \mathcal{K}_{(Q)}^{s-\mu,\gamma-\mu}(X^\wedge))); \tag{37}$$

here subscripts '(P)' and '(Q)' at the spaces mean that the considerations are valid both for spaces without asymptotics as well as with asymptotics of the corresponding types. The homogeneity of (36) is a consequence of a corresponding relation for $\sigma a_{\mathcal{O}}\tilde{\sigma}$ itself, with $\{\kappa_\lambda^{(n)}\}_{\lambda\in\mathbb{R}_+}$ on both sides. The smoothness (37) follows from Remark 2.14. If $\tilde{h}$ depends on the variable r we can apply a simple tensor product argument to obtain the assertion in general.

For the operator function $\sigma a_1(y,\eta)\tilde{\sigma}$ we have to recall some arguments which yield the properties of Theorem 2.17 (i). For simplicity, assume that the symbol $\tilde{p}_\infty$ contained in a_1 is independent of r; the general case then follows again by a tensor product argument.

The operators

$$\sigma a_1(y,\eta)\tilde{\sigma} : \mathcal{K}^{s,\gamma}((2X)^\wedge) \to \mathcal{K}_\Theta^{s-\mu,\gamma-\mu}((2X)^\wedge)$$

smoothly depend on $(y,\eta) \in \Omega \times \mathbb{R}^q$; then also

$$\sigma a'_1(y,\eta)\tilde{\sigma} : \mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}_\Theta^{s-\mu,\gamma-\mu}(X^\wedge)$$

are smooth in $(y,\eta) \in \Omega \times \mathbb{R}^q$, cf. Remark 2.14. Another step of the proof for the symbol property on $(2X)^\wedge$ is that for any excision function $\chi(\eta)$ the operator family

$$b(y,\eta) := \chi(\eta)r^{-\mu}(1-\omega(r[\eta]))\mathrm{op}_r(p_\infty)(y,\eta)(1-\tilde{\tilde{\omega}}(r[\eta]))$$

is a classical symbol of order μ because we have again $\kappa_\lambda^{(n)}$- homogeneity in η for large $|\eta|$. The operator function $b(y,\eta)$ is non-trivial only for $\eta \neq 0$; therefore $r^{-\mu}\mathrm{op}_r(p_\infty)(y,\eta)$ for every fixed y and $\eta \neq 0$ (combined with the excision $(1-\omega(r[\eta]))$ and $(1-\tilde{\tilde{\omega}}(r[\eta]))$) is an operator on $X^\wedge$ in the class $L_{\mathrm{cl}}^{\mu;0}(X^\wedge)$, cf. [6, Section 3.1.2]. Thus, we can apply Theorem 2.8 and obtain that

$$b'(y,\eta) := \chi(\eta)r^{-\mu}(1-\omega(r[\eta]))\mathrm{op}_r(p'_\infty)(y,\eta)(1-\tilde{\tilde{\omega}}(r[\eta]))$$

is a C^∞ family of continuous operators

$$\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}_\Theta^{s-\mu,\gamma-\mu}(X^\wedge).$$

Similarly as (36) we have homogeneity of $b'(y,\eta)$ of order $\mu + \frac{1}{2}$ for large $|\eta|$, and hence $b'(y,\eta)$ is an operator-valued symbol between the spaces in question. Since the operators of multiplication by σ and $\tilde{\sigma}$ behave as (non-classical) operator-valued symbols of order zero we obtain the symbol property also for $\sigma b'(y,\eta)\tilde{\sigma}$. Because of

$$\sigma a'_1(y,\eta)\tilde{\sigma} = \sigma b'(y,\eta)\tilde{\sigma} + \sigma b''(y,\eta)\tilde{\sigma} \tag{38}$$

for $b''(y,\eta) = (1-\chi(\eta))r^{-\mu}(1-\omega(r[\eta]))\mathrm{op}_r(p'_\infty)(y,\eta)(1-\tilde{\tilde{\omega}}(r[\eta]))$ where also $\sigma b''(y,\eta)\tilde{\tilde{\sigma}}$ is a C^∞ function of continuous operators between our spaces and with compact support in η, for (38) we obtain that $\sigma a'_1(y,\eta)\tilde{\sigma}$ has the desired properties. □

Remark 2.18 *The operator-valued symbols in Theorem* 2.17 *are classical, if the function* (32) *is independent of* r.

Let us fix a weight interval $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, and consider functions

$$l_{j\alpha}(y,z) \in C^\infty(\Omega, M^{-\infty}_{R_{j\alpha}}(2X)), \qquad l'_{j\alpha}(y,z) \in C^\infty(\Omega, M^{-\infty}_{R'_{j\alpha}}(X,\partial X))$$

for $0 \le j \le k$, with (for simplicity) y-independent discrete asymptotic types $R_{j\alpha} \in \mathbf{As}(2X)$ and $R'_{j\alpha} \in \mathbf{As}(X,\partial X)$, respectively. Moreover, assume that

$$\pi_{\mathbb{C}} R_{j\alpha} \cap \Gamma_{\frac{n+1}{2}-\gamma_j} = \pi_{\mathbb{C}} R'_{j\alpha} \cap \Gamma_{\frac{n+1}{2}-\gamma'_j} = \emptyset$$

with certain weights $\gamma_j, \gamma'_j \in \mathbb{R}$, satisfying

$$\gamma_j \le \gamma \le j + \gamma_j, \quad \gamma'_j \le \gamma \le j + \gamma'_j$$

for some fixed weight $\gamma \in \mathbb{R}$, $0 \le j \le k$.

Let us form

$$m(y,\eta) = r^{-\mu}\omega(r[\eta]) \sum_{j=0}^{k} \sum_{|\alpha| \le j} r^j \mathrm{op}_M^{\gamma_j - \frac{n}{2}}(l_{j\alpha})(y)\eta^\alpha \tilde{\omega}(r[\eta]) \tag{39}$$

and

$$m'(y,\eta) = r^{-\mu}\omega(r[\eta]) \sum_{j=0}^{k} \sum_{|\alpha| \le j} r^j \mathrm{op}_M^{\gamma'_j - \frac{n}{2}}(l'_{j\alpha})(y)\eta^\alpha \tilde{\omega}(r[\eta]). \tag{40}$$

Applying the map (22) to the Mellin symbols $l_{j\alpha}$ in (39) we obtain corresponding Mellin symbols $l'_{j\alpha}$. This gives us a map

$$m(y,\eta) \to m'(y,\eta) \tag{41}$$

from operator families (39) to associated operator families (40).

Proposition 2.19 *Let* $(E, \tilde{E})$ *denote one of the following pairs of spaces*

$$(\mathcal{K}^{s,\gamma}((2X)^\wedge), \mathcal{K}^{\infty,\gamma-\mu}((2X)^\wedge)), \quad \text{or} \quad (\mathcal{K}^{s,\gamma}_{\tilde{P}}((2X)^\wedge), \mathcal{S}^{\gamma-\mu}_{\tilde{Q}}((2X)^\wedge)), \tag{42}$$

or

$$(\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)), \quad \text{or} \quad (\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}_P((\partial X)^\wedge), \mathcal{S}^{\gamma-\mu}_Q(X^\wedge)). \tag{43}$$

Then $m(y,\eta)$ *belongs to* $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ *for the pairs* $(E,\tilde{E})$ *from* (42), $s \in \mathbb{R}$ (*in the second case for every* $\tilde{P}$ *with some resulting* $\tilde{Q}$, *depending on* m), *and* $m'(y,\eta)$ *belongs to* $S^{\mu+\frac{1}{2}}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ *for the pairs* $(E,\tilde{E})$ *from* (43), $s \in \mathbb{R}$ (*in the second case for every* P *with some resulting* Q, *depending on* m').

Proof. The assertion concerning (42) is known, see, for instance, [6]. Let us write the operator function (39) in the form $\sum_{j=0}^k m_j(y,\eta)$ where $m_j(y,\eta)$ is the j^{th} summand in (39). Then we have

$$m_j(y,\lambda\eta) = \lambda^{\mu-j}\kappa^{(n)}_\lambda m_j(y,\eta)(\kappa^{(n)}_\lambda)^{-1}$$

for all $\lambda \geq 1$, $|\eta| \geq c$ for some $c > 0$. This gives us for the j^{th} summand $m'_j(y, \eta)$ on the right of (40) the homogeneity

$$m'_j(y, \lambda\eta) = \lambda^{\mu+\frac{1}{2}} \kappa_\lambda^{(n)} m'_j(y, \eta)(\kappa_\lambda^{(n-1)})^{-1}$$

for all $l \geq 1$, $|\eta| \geq c$. On the other hand, similarly as for Mellin symbols of the kind $m_j(y, \eta)$ the operator families $m'_j(y, \eta)$ between the spaces in (43) are C^∞ in $(y, \eta) \in \Omega \times \mathbb{R}^q$. Homogeneity together with the latter property yields the assertion on $m'_j(y, \eta)$ and then also for $m'(y, \eta)$. $\square$

2.5 Edge symbols of Green type

As in the beginning of Section 1.2 for convenience we assume that X is a closed compact C^∞ manifold. We now define a specific class of operator-valued symbols of the edge calculus, called Green symbols, here with discrete asymptotic types $P \in \mathrm{As}(X, (\delta, \Theta))$ and $Q \in \mathrm{As}(X, (-\gamma, \Theta))$ for some choice of weights $\gamma, \delta \in \mathbb{R}$ and an arbitrary weight interval $\Theta = (\theta, 0]$. At the same time we define what is called trace and potential symbols with respect to the edge. These objects are entries of a 2×2 block matrix operator function

$$g(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{\infty,\delta}(X^\wedge) \oplus \mathbb{C}^{j_+}))$$

for some dimensions $j_\pm \in \mathbb{N}$. These are classical symbols in the sense that $g_0(y, \eta) = \mathrm{diag}(1, \langle\eta\rangle^{\frac{n+1}{2}})g(y, \eta)\mathrm{diag}(1, \langle\eta\rangle^{-\frac{n+1}{2}})$ are symbols

$$g_0(y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{S}^\delta_P(X^\wedge) \oplus \mathbb{C}^{j_+}) \tag{44}$$

such that

$$g_0^*(y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\delta}(X^\wedge) \oplus \mathbb{C}^{j_+}, \mathcal{S}^{-\gamma}_Q(X^\wedge) \oplus \mathbb{C}^{j_-}). \tag{45}$$

Here $s \in \mathbb{R}$ is arbitrary and fixed, and the formal adjoint g^* is taken (y, η)-wise with respect to the scalar products of $\mathcal{K}^{0,0}(X^\wedge) \oplus \mathbb{C}^{j_\pm}$.

Recall that Definition 1.3 has a version for Fréchet spaces, here with $\mathcal{S}^\delta_P(X^\wedge)$ and $\mathcal{S}^{-\gamma}_Q(X^\wedge)$, respectively, and that these spaces can be represented as projective limits of Hilbert spaces with group action, cf. Remark 1.9.

In our application the weight interval will be $\Theta = (-\infty, 0]$. In this case there is a useful explicit representation of Green symbols.

If F is a Fréchet space with its countable system of semi-norms $(\pi_j)_{j \in \mathbb{N}}$, by $S^\mu(\Omega \times \mathbb{R}^q, F)$ we denote the space of all $f(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, F)$ such that

$$\pi_j(D^\alpha_y D^\beta_\eta f(y, \eta)) \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^q$, all $(y, \eta) \in K \times \mathbb{R}^q$, $K \Subset \Omega$, and all $j \in \mathbb{N}$, with constants $c(\alpha, \beta, K; j) > 0$. We also have the subspace $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q, F)$ of classical symbols with values in F, defined by means of sequences of homogeneous components $f_{(\mu-k)}(y, \eta)$, $k \in \mathbb{N}$, analogously to the standard context.

We will apply this to the projective tensor product

$$F = (\mathcal{S}^\delta_P(X^\wedge) \oplus \mathbb{C}^{j_+}) \hat{\otimes}_\pi (\mathcal{S}^{-\gamma}_{\overline{Q}}(X^\wedge) \oplus \mathbb{C}^{j_-}) \tag{46}$$

for discrete asymptotic types P, Q (with $\overline{Q}$ being the complex conjugate of Q).

Elements of $S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, F)$ for the space (46) are 2×2 block matrices $(f_{ij}(y,\eta))_{i,j=1,2}$, with a corresponding dependence on (r,x), $(r',x')\in X^\wedge$ in the entries. For $f(y,\eta)$ we also write

$$f(y,\eta;r,x,r',x')=\begin{pmatrix} f_{11}(y,\eta;r,x,r',x') & f_{12}(y,\eta;r,x) \\ f_{21}(y,\eta;r',x') & f_{22}(y,\eta)\end{pmatrix}.$$

Theorem 2.20 *For every Green symbol $g(y,\eta)$ with asymptotic types P,Q there is an element*

$$f(y,\eta)=(f_{ij}(y,\eta))_{i,j=1,2} \quad \text{with} \quad f_{ij}(y,\eta)\in S^{\mu_{ij}}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, F_{ij})$$

for $\mu_{11}=\mu+1$, $\mu_{12}=\mu+\frac{n+1}{2}$, $\mu_{21}=\mu-\frac{n-1}{2}$, $\mu_{22}=\mu$ and $F_{11}=\mathcal{S}^{\delta}_P(X^\wedge)\hat{\otimes}_\pi\mathcal{S}^{-\gamma}_{\overline{Q}}(X^\wedge)$, $F_{12}=\mathcal{S}^{\delta}_P(X^\wedge)\otimes\mathbb{C}^{j_-}$, $F_{21}=\mathbb{C}^{j_+}\otimes\mathcal{S}^{-\gamma}_{\overline{Q}}$, $F_{22}=\mathbb{C}^{j_+}\otimes\mathbb{C}^{j_-}$, such that

$$g(y,\eta)\begin{pmatrix} u(r,x) \\ c\end{pmatrix}=\iint_0^\infty f(y,\eta;[\eta]r,x,[\eta]r',x')\begin{pmatrix} u(r',x') \\ c\end{pmatrix}\mathrm{d}r'\mathrm{d}x' \tag{47}$$

for every $u\oplus c\in\mathcal{K}^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^{j_-}$.

The notation in (47) means that f is interpreted as a 2×2 block matrix, applied to the vector $u(r',x')\oplus c$, where

$$f(y,\eta;[\eta]r,x,[\eta]r',x')=\begin{pmatrix} f_{11}(y,\eta;[\eta]r,x,[\eta]r',x') & f_{12}(y,\eta;[\eta]r,x) \\ f_{21}(y,\eta;[\eta]r',x') & f_{22}(y,\eta),\end{pmatrix}$$

and the integration with respect to (r',x') only concerns the first column while the second column is a simple algebraic composition.

A proof of Theorem 2.20 may be found in [14], based on tensor product representations of Green operators of the cone algebra from [16], see also [15].

Let us now return to the original context and apply the notation and constructions of this section to $2X$ for a compact C^∞ manifold X with boundary. Then the representation of Green symbols $g(y,\eta)$ in terms of kernels (46) allows us to restrict

$$g(y,\eta)\in S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}((2X)^\wedge)\oplus\mathbb{C}^{j_-},\mathcal{S}^{\delta}_P((2X)^\wedge)\oplus\mathbb{C}^{j_+})$$

(y,η)-wise to argument of $\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge)\oplus\mathbb{C}^{j_-}$ combined with the restriction in the first component of the image to X. Let $g'(y,\eta)$ denote the resulting family of operators

$$g'(y,\eta):\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge)\oplus\mathbb{C}^{j_-}\to\mathcal{S}^{\delta}_{P'}(X^\wedge)\oplus\mathbb{C}^{j_+}$$

(here P' is an asymptotic type referring to X, uniquely determined by P in an obvious way). Then for the correspondence between upper left corners of $g(y,\eta)$

$$g_{11}(y,\eta)\to g'_{11}(y,\eta) \tag{48}$$

we obtain

$$g'_{11}(y,\eta)\in S^{\mu+\frac{1}{2}}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s+\frac{1}{2},\gamma+\frac{1}{2}}((\partial X)^\wedge),\mathcal{S}^{\delta}_{P'}(X^\wedge)) \tag{49}$$

for all $s\in\mathbb{R}$. For the other entries we have a similar correspondence, namely $g_{ij}(y,\eta)\to g'_{ij}(y,\eta)$ where

$$g'_{12}(y,\eta)\in S^{\mu-\frac{n+1}{2}}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathbb{C}^{j_-},\mathcal{S}^{\delta}_{P'}(X^\wedge)),$$

$$g'_{21}(y,\eta)\in S^{\mu+\frac{n}{2}}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}((\partial X)^\wedge),\mathbb{C}^{j_+})$$

while g_{22} remains untouched, i.e., is equal to g'_{22}.

3 Potentials with asymptotics

3.1 Edge potential operators

In this section we consider edge pseudo-differential operators, based on (operator-valued) edge amplitude functions. We first recall a few elements of the general 'edge algebra'; the requirements and constructions for the definition can be interpreted as results about the 'Mellin-edge' behaviour of parametrices of elliptic operators.

Let $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; (j_-, j_+))$ for $\Omega \subseteq \mathbb{R}^q$ open, and weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, be the space of all edge amplitude functions $a(y, \eta)$.

Here $\gamma \in \mathbb{R}$ has the meaning of a weight, $\mu \in \mathbb{R}$ of an order, and $\Theta = (-(k+1), 0]$ is a weight interval of length $k + 1 \in \mathbb{N} \cup \{+\infty\}$.

Let us first consider the case $j_- = j_+ = 0$. Then, by definition, we have

$$a(y, \eta) = \sigma(r)\{\omega(r) r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h)(y, \eta) \tilde{\omega}(r) \\ + (1 - \omega(r)) r^{-\mu} \mathrm{op}_r(p)(y, \eta)(1 - \tilde{\tilde{\omega}}(r))\} \tilde{\sigma}(r) + m(y, \eta) + g(y, \eta) \tag{50}$$

where p is a finite linear combination of expressions of the form

$$(\chi^{-1})_* \mathrm{op}_x(a)(r, y, \varrho, \eta)$$

for a chart $\chi : U \to \Sigma$ on M, $\Sigma \subseteq \mathbb{R}^n$ open, and a classical symbol $a(r, x, y, \varrho, \xi, \eta)$ on $\mathbb{R}_+ \times \Sigma \times \Omega$ of the form

$$a(r, x, y, \varrho, \xi, \eta) = \tilde{a}(r, x, y, r\varrho, \xi, r\eta),$$

for some $\tilde{a}(r, x, y, \tilde{\varrho}, \xi, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$. Moreover, h has the form

$$h(r, y, z, \eta) = \tilde{h}(r, y, z, r\eta)$$

for an element $\tilde{h}(r, y, z, \tilde{\eta}) \in C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(M; \mathbb{R}^q_{\tilde{\eta}}))$, cf. Section 2.4, which has the property

$$\mathrm{op}_M^{\gamma - \frac{n}{2}}(h_0)(y, \eta) = \mathrm{op}_r(p_0)(y, \eta)$$

mod $C^\infty(\Omega, L^{-\infty}(M^\wedge; \mathbb{R}^q))$, when we set

$$h_0(r, y, z, \eta) := \tilde{h}(0, y, z, r\eta),$$

and define $p_0(r, y, \varrho, \eta)$ similarly as $p(r, y, \varrho, \eta)$ but in terms of the symbols

$$a_0(r, x, y, \varrho, \xi, \eta) := \tilde{a}(0, x, y, r\varrho, \xi, r\eta).$$

The cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$, $\omega(r)$, $\tilde{\omega}(r)$, $\tilde{\tilde{\omega}}(r)$ in (50) are arbitrary, except for the condition that $\tilde{\omega} \equiv 1$ on $\mathrm{supp}\, \omega$ and $\omega \equiv 1$ on $\mathrm{supp}\, \tilde{\tilde{\omega}}$. The smoothing Mellin symbols $m(y, \eta)$ and the Green symbols $g(y, \eta)$ in the expression (50) (for arbitrary j_-, j_+) are described in Sections 2.4 and 2.5.

Now the most specific contribution of our program to analyse potentials consists of pseudo-differential operators $\mathrm{Op}(a)$ with amplitude functions of the kind (50), where $\mathrm{Op} = \mathrm{Op}_y$ denotes the standard pseudo-differential operator convention, based on the Fourier transform in $\mathbb{R}^q \ni y$, i.e., $\mathrm{Op}_y(a)u(y) = \iint e^{i(y-y')\eta} a(y, \eta) u(y') dy' đ\eta$.

In this case the dimensions $j_\pm$ are zero; we then denote by $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ the space of all operator functions of the kind (50).

Every $a(y,\eta) \in R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ represents a family of operators

$$a(y,\eta) \in C^\infty(\Omega, L^\mu_{\mathrm{cl}}(M^\wedge; \mathbb{R}^q)).$$

From now on we assume $M = 2X$ and denote by $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})_{\mathrm{tr}}$ the subspace of all elements of $R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ which have the transmission property at the interface ∂X.

Then the mappings (35), (41) and (48) induce a correspondence $a(y,\eta) \to a'(y,\eta)$,

$$R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})_{\mathrm{tr}} \to C^\infty(\Omega, B^{\mu+\frac{1}{2}}(X^\wedge, (\partial X)^\wedge; \mathbb{R}^q)) \tag{51}$$

where $B^{\mu+\frac{1}{2}}(D, \partial D; \mathbb{R}^q)$ for a C^∞ manifold D (not necessarily compact) with boundary denotes the space of potential operators in Boutet de Monvel's calculus of order $\mu + \frac{1}{2}$ with parameters $\eta \in \mathbb{R}^q$. Recall that the parameter-dependence is defined by local potential amplitude functions which contain η as an additional covariable, and the smoothing operators are given by

$$B^{-\infty}(D, \partial D; \mathbb{R}^q) = \mathcal{S}(\mathbb{R}^q, B^{-\infty}(D, \partial D)).$$

For every $a(y,\eta) \in R^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, we have $A := \mathrm{Op}_y(a) \in L^\mu_{\mathrm{cl}}((2X)^\wedge \times \Omega)$, and

$$\mathrm{Op}_y(a') \in B^{\mu+\frac{1}{2}}(X^\wedge, \partial X \times \Omega). \tag{52}$$

As a pseudo-differential operator A induces a map

$$\mathcal{E}'((2X)^\wedge \times \Omega) \to \mathcal{D}'((2X)^\wedge \times \Omega). \tag{53}$$

Setting $A' = \mathrm{Op}_y(a')$ with a' being related to a we obtain a correspondence $A \to A'$. From (52) we have the continuity property

$$A' : H^{s+\frac{1}{2}}_{\mathrm{comp}}((\partial X)^\wedge \times \Omega) \to H^{s-\mu}_{\mathrm{loc}}(X^\wedge \times \Omega). \tag{54}$$

What we obtain in the present situation are continuity results between spaces with edge asymptotics on the wedges $(\partial X)^\wedge \times \Omega$ and $X^\wedge \times \Omega$, respectively.

Theorem 3.1 *Let $a = a(\eta) \in R^\mu(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ be independent of $y \in \mathbb{R}^q$. Then the associated potential operator $A' = \mathrm{Op}_y(a')$ extends to continuous operators*

$$A' : \mathcal{W}^{s+\frac{1}{2}, \gamma+\frac{1}{2}}_{(P)}((\partial X)^\wedge \times \mathbb{R}^q) \to \mathcal{W}^{s-\mu, \gamma-\mu}_{(Q)}(X^\wedge \times \mathbb{R}^q)$$

for all $s \in \mathbb{R}$, for every discrete asymptotic type $P \in \mathrm{As}(\partial X, (\gamma + \frac{1}{2}, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma - \mu, \Theta))$ (depending on P and a not on s).

Proof. We apply Proposition 1.8 for the case $\Omega = \mathbb{R}^q$ and the pairs of spaces

$$(E, \tilde{E}) = (\mathcal{K}^{s+\frac{1}{2}, \gamma+\frac{1}{2}}_{(P)}((\partial X)^\wedge), \mathcal{K}^{s-\mu, \gamma-\mu}_{(Q)}(X^\wedge))$$

for any fixed s, using Theorem 2.17 (ii), Proposition 1.8 and the relation (49). □

Remark 3.2 *Using the framework of the edge algebra of boundary value problems on a (say, compact and stretched) manifold $\mathbb{W}$ with boundary and edge Y in the sense of* [6] *it can be proved that every element A in the edge algebra on the double $2\mathbb{W}$ (which is a closed (stretched) manifold with edge Y) induces by restriction to 'surface densities' on the boundary of $\mathbb{W}$ a potential operator in the edge algebra. Conversely, every such operator can be obtained in this way. In this generality we need the concept of continuous asymptotics, but under suitable assumptions on the behaviour of coefficients of the involved amplitude functions along Y we have such a result in the framework of discrete asymptotics (or asymptotics in the sense of the following Section 3.2).*

3.2 A generalisation of edge asymptotics

Theorem 3.1 refers to symbols with constant coefficients in $y \in \Omega$. For potentials of surface densities in general this is, of course, too special, because the geometry may contribute non-constant coefficients. Therefore we now extend the concept of asymptotic types by admitting more general Laurent coefficients of corresponding meromorphic functions.

Let $\mathrm{as}(n, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \Theta)$, $n = \dim X$, denote the set of all sequences

$$\{(p_j, m_j)\}_{j=0,\dots,N}$$

with (p_j, m_j) as in (7) (in other words, $\mathrm{as}(n, \boldsymbol{g})$ is obtained from $\mathrm{As}(X, \boldsymbol{g})$ by omitting the spaces L_j). Let first Θ be finite, and let $\mathcal{E}_P(X^\wedge)$ be the linear span of all functions $c(x) r^{-p} \log^k r \omega(r)$ for some fixed cut-off function $\omega(r)$, for arbitrary $(p, k) \in P$ and $c \in C^\infty(X)$. Then $\mathcal{E}_P(X^\wedge)$ is isomorphic to a finite direct sum of copies of $C^\infty(X)$, and $\mathcal{E}_P(X^\wedge)$ is a closed subspace of $\mathcal{K}^{\infty,\gamma}(X^\wedge)$ which is direct to the space $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ of the flat functions. We can again form the space (8) in the Fréchet topology of the direct sum. This kind of asymptotics can be subsumed under the concept of continuous asymptotics, cf. [12] or [13], although we do not use this explicitly here. The present generalisation is much simpler and more specific; it admits, for instance, smoothly varying coefficient spaces L_j with respect to edge variable y.

Also the notation on edge Sobolev spaces with asymptotics $P \in \mathrm{as}(n, \boldsymbol{g})$ makes sense, i.e., we have the spaces $\mathcal{W}^{s,\gamma}_P(X^\wedge \times \mathbb{R}^q)$ by a similar construction as (11), including the analogue of the information on the nature of singular functions of edge asymptotics, cf. Remark 1.10. In the present case the singular functions have the form

$$f_{\mathrm{sing}}(r, x, y) = \sum_{j=0}^{N(P)} \sum_{k=0}^{m_j} F^{-1}_{\eta \to y} [\eta]^{\frac{n+1}{2}} \omega(r[\eta]) v_{jk}(x, \hat{\eta}) (r[\eta])^{-p_j} \log^k (r[\eta])$$

for arbitrary $v_{jk} \in C^\infty(X, H^s(\mathbb{R}^q))$, $v_{jk}(x, \hat{\eta}) = F_{y \to \eta} v_{jk}(x, y)$.

Moreover, we have a generalisation of meromorphic operator-valued Mellin symbols described by asymptotic types

$$R = \{(p_j, m_j)\}_{j \in \mathbb{Z}}$$

where p_j, m_j are as in (12), but we do not require any specific control of spaces L_j of Laurent coefficients. Let $\boldsymbol{as}$ denote this kind of Mellin asymptotic types.

Let $M^{-\infty}_R(2X)$ denote the space of all elements of $\mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, L^{-\infty}(2X))$ that are meromorphic with poles at p_j of multiplicities $m_j + 1$. In a similar manner we define $M^{-\infty}_R(X)$ where $L^{-\infty}(2X)$ is to be replaced by the space of operators with kernels in $C^\infty(X \times X)$ (recall that we fix Riemannian metrics on the manifolds under consideration). Analogously, we have $M^{-\infty}_R(X, \partial X)$, the space of all $f(z) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, C^\infty(X \times \partial X))$, meromorphic with poles at p_j of multiplicities $m_j + 1$, where $C^\infty(X \times \partial X)$ is identified with the corresponding space of operators with kernels in $C^\infty(X \times \partial X)$. All these spaces are Fréchet in a natural way, and we can form spaces of the kind (14) for arbitrary $R \in \boldsymbol{as}$.

Finally, the definition of Green symbols $g(y, \eta)$ has a straightforward generalisation to arbitrary asymptotic types $P \in \mathrm{as}(n, (\delta, \Theta))$ and $Q \in \mathrm{as}(n, (-\gamma, \Theta))$, cf. the relations (44), (45). Thus we have all ingredients to form edge amplitude functions $a(y, \eta)$ of the form (50) with $m(y, \eta)$ defined in terms of smoothing Mellin symbols

$l_{j\alpha}(y,z) \in C^\infty(\Omega, M^{-\infty}_{R_{j\alpha}}(2X))$, $R_{j\alpha} \in$ **as** and Green symbols $g(y,\eta)$ as mentioned before.

We now have again a correspondence $a(y,\eta) \to a'(y,\eta)$, cf. the relation (51), such that the associated operators $A' := \mathrm{Op}(a')$ have the meaning of the restriction of $A = \mathrm{Op}(a)$ to surface densities on $(\partial X)^\wedge \times \mathbb{R}^q$, combined with the restriction to $(\mathrm{int}\, X)^\wedge \times \mathbb{R}^q$.

Theorem 3.3 *Let $A = \mathrm{Op}(a)$ be an edge pseudo-differential operator on $(2X)^\wedge \times \mathbb{R}^q$ with amplitude function $a(y,\eta)$ (now with asymptotic data of general types). Then the potential operator (54) given by $A' = \mathrm{Op}_y(a')$ extends to continuous operators*

$$A' : \mathcal{W}^{s+\frac{1}{2},\gamma+\frac{1}{2}}_{\mathrm{comp}(y)}((\partial X)^\wedge \times \Omega) \to \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}(y)}(X^\wedge \times \Omega)$$

and

$$A' : \mathcal{W}^{s+\frac{1}{2},\gamma+\frac{1}{2}}_{\mathrm{comp}(y),P}((\partial X)^\wedge \times \Omega) \to \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}(y),Q}(X^\wedge \times \Omega)$$

for all $s \in \mathbb{R}$, for arbitrary $P \in \mathrm{as}(n-1,(\gamma+\frac{1}{2},\Theta))$ with some resulting $Q \in \mathrm{as}(n,(\gamma-\mu,\Theta))$ (depending on P and a not on s).

Proof. The arguments are analogous to those for Theorem 3.1. In order to apply Proposition 1.8 we need to verify that our amplitude functions $a'(y,\eta)$ are symbols in a similar sense as before with the only exception that for the consideration with asymptotic types we refer to the set-up of the present section. An inspection of the definitions and results shows that the necessary elements have immediate generalisations, in particular, we have analogues of Theorem 2.17, Proposition 1.8 and of the construction of Section 2.5. This gives us the desired continuity results. □

3.3 Examples

In this section we consider some examples which show a technique to calculate asymptotics of potentials with respect to a parametrix of an elliptic differential operator.

Let us first recall a general result on a relation between standard Sobolev spaces in $\mathbb{R}^{n+1}$ and weighted Sobolev spaces:

Proposition 3.4 *Let $t > \frac{n+1}{2}$ be a real number, and set*

$$H^t_0(\mathbb{R}^{n+1}) := \{u \in H^t(\mathbb{R}^{n+1}) : D^\alpha_{\tilde{x}} u(0) = 0 \text{ for all } \alpha \in \mathbb{N}^{n+1},\ |\alpha| < t - \frac{n+1}{2}\}.$$

Then for $t - \frac{n+1}{2} \notin \mathbb{N}$ we have a canonical isomorphism

$$H^t_0(\mathbb{R}^{n+1}) \cong \mathcal{K}^{t,t}(\mathbb{R}^{n+1}\setminus\{0\}),$$

cf. Dauge [5]*, Kondratyev* [8, §4]*, or the author's joint monograph* [6, Section 2.1.2]. *Moreover, for $-\frac{n+1}{2} < t < \frac{n+1}{2}$ we have*

$$H^t(\mathbb{R}^{n+1}) = \mathcal{K}^{t,t}(\mathbb{R}^{n+1}\setminus\{0\}).$$

Let us now consider the Laplace operator Δ in $\mathbb{R}^3$, given as a map

$$\Delta : H^s(\mathbb{R}^3) \to H^{s-2}(\mathbb{R}^3),$$

$s \in \mathbb{R}$. Let $\Phi \in L^{-2}(\mathbb{R}^3)$ be a (properly) supported parametrix of Δ (any other (pseudo-differential) parametrix of Δ is equal to Φ modulo an operator with smooth kernel). Therefore, for the evaluation of asymptotics of potentials, the specific choice of Φ is unessential.

For technical reasons we also consider the operator

$$\Delta - 1 : H^s(\mathbb{R}^3) \to H^{s-2}(\mathbb{R}^3)$$

which induces isomorphisms for all $s \in \mathbb{R}$. We then have $(\Delta - 1)^{-1} =: P \in L^{-2}_{\mathrm{cl}}(\mathbb{R}^3)$. Moreover, the relation $\Delta P = 1 + P$ allows us to reconstruct Φ as an asymptotic sum

$$\Phi \sim \sum_{j=0}^{\infty} (-1)^j P^{j+1}$$

(with $\sim$ denoting equivalence modulo smoothing operators). We want to show the behaviour of potentials with respect to P. The same method applies for P^{j+1} for arbitrary $j \in \mathbb{N}$; then, since $\mathrm{ord} P^{j+1} \to -\infty$ for every finite part of asymptotic expansions it suffices to look at a finite number of j. In other words, it is enough to discuss $\Delta - 1$ with its inverse P.

Let us write points in $\mathbb{R}^3$ as (x, y) for $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$. Set

$$a(\eta) := \Delta_2 - |\eta|^2 - 1$$

for $\Delta_2 := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Then $a(\eta)$ is an operator-valued symbol

$$a(\eta) : H^s(\mathbb{R}^2) \to H^{s-2}(\mathbb{R}^2)$$

which is invertible for every $\eta \in \mathbb{R}$, and we have

$$\Delta - 1 = \mathrm{Op}_y(a) \quad \text{and} \quad P = \mathrm{Op}_y(p)$$

for $p(\eta) = (\Delta_2 - |\eta| - 1)^{-1}$. Let us now fix an $0 < \varepsilon < \frac{1}{2}$ and set $s = 2 - \varepsilon$. The space $H^{2-\varepsilon}(\mathbb{R}^2)$ can be written as a direct sum

$$H^{2-\varepsilon}(\mathbb{R}^2) = H_0^{2-\varepsilon}(\mathbb{R}^2) + V(\eta) \tag{55}$$

for every $\eta \in \mathbb{R}$, where $H_0^{2-\varepsilon}(\mathbb{R}^2) = \{u \in H^{2-\varepsilon}(\mathbb{R}^2) : u(0) = 0\}$ and $V(\eta) = \{c[\eta]\omega([\eta]x) : c \in \mathbb{C}\}$ where ω is any fixed element of $C_0^\infty(\mathbb{R}^2)$ with $\omega(0) \neq 0$. The map

$$k(\eta) : c \to c[\eta]\omega([\eta]x)$$

represents a potential symbol of the class $S^0_{\mathrm{cl}}(\mathbb{R}; \mathbb{C}, \mathcal{S}(\mathbb{R}^2))$, cf. Definition 1.3 (iii), and we have $H^{2-\varepsilon}(\mathbb{R}^3) = \mathcal{W}^{2-\varepsilon}(\mathbb{R}, H_0^{2-\varepsilon}(\mathbb{R}^2)) + \mathrm{im} K$, where K is the potential operator

$$K = \mathrm{Op}_y(k) : H^{2-\varepsilon}(\mathbb{R}) \to \mathcal{W}^{2-\varepsilon}(\mathbb{R}, H^{2-\varepsilon}(\mathbb{R}^2)).$$

Let us form the families of isomorphisms

$$\begin{pmatrix} a(\eta) & a(\eta)k(\eta) \end{pmatrix} : \begin{array}{c} H_0^{2-\varepsilon}(\mathbb{R}^2) \\ \oplus \\ \mathbb{C} \end{array} \to H^{-\varepsilon}(\mathbb{R}^2),$$

$$(1 \quad k(\eta)) \ : \ \begin{array}{c} H_0^{2-\varepsilon}(\mathbb{R}^2) \\ \oplus \\ \mathbb{C} \end{array} \to H^{2-\varepsilon}(\mathbb{R}^2).$$

Then we have $a(\eta) == (a(\eta) \quad a(\eta)k(\eta)) \, (1 \quad k(\eta))^{-1}$. Let us set $\mathcal{K} := (1 \quad K)$, i.e., $\mathcal{K} = \mathrm{Op}_y(1 \quad k(\eta))$. Then we obtain isomorphisms

$$\mathcal{K}^{-1} = \mathrm{Op}_y\left((1 \quad k(\eta))^{-1}\right) : \mathcal{W}^{2-\varepsilon}(\mathbb{R}, H^{2-\varepsilon}(\mathbb{R}^2)) \xrightarrow{\cong} \begin{array}{c} \mathcal{W}^{2-\varepsilon}(\mathbb{R}, H_0^{2-\varepsilon}(\mathbb{R}^2)) \\ \oplus \\ H^{2-\varepsilon}(\mathbb{R}) \end{array},$$

$$(\Delta - 1)\,\mathcal{K} = \mathrm{Op}_y(a(\eta) \quad a(\eta)k(\eta)) : \begin{array}{c} \mathcal{W}^{2-\varepsilon}(\mathbb{R}, H_0^{2-\varepsilon}(\mathbb{R}^2)) \\ \oplus \\ H^{2-\varepsilon}(\mathbb{R}) \end{array} \xrightarrow{\cong} H^{-\varepsilon}(\mathbb{R}^3). \tag{56}$$

Writing $\Delta - 1 = (\Delta - 1)\mathcal{K}\mathcal{K}^{-1}$ it follows that $P = \mathcal{K}(\mathcal{K}^{-1}P)$.

We now concentrate on the operator (56) and express its inverse $\mathcal{K}^{-1}P$. First note that it is an elliptic element in the edge calculus with edge $\mathbb{R}$ and model cone $\mathbb{R}^2 \backslash \{0\}$.

Let us rewrite (56) in terms of edge Sobolev spaces, using the identifications

$$H_0^{2-\varepsilon}(\mathbb{R}^2) = \mathcal{K}^{2-\varepsilon, 2-\varepsilon}(\mathbb{R}^2 \backslash \{0\}), \quad H^{-\varepsilon}(\mathbb{R}^2) = \mathcal{K}^{-\varepsilon, -\varepsilon}(\mathbb{R}^2 \backslash \{0\}),$$

cf. Proposition 3.4. Then (56) takes the form

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{2-\varepsilon, 2-\varepsilon}((\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}) \\ \oplus \\ H^{2-\varepsilon}(\mathbb{R}) \end{array} \to \mathcal{W}^{-\varepsilon, -\varepsilon}((\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}).$$

The amplitude function of $\mathcal{A}$ is a row matrix as in (56) with

$$a(\eta) = r^{-2}\left(\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial}{\partial \phi^2} - r^2|\eta|^2 - r^2\right)$$

as the first component.

It belongs to the edge algebra and has constant coefficients with respect to the edge variables y. It is invertible as an operator function

$$(a(\eta) \quad a(\eta)k(\eta)) : \begin{array}{c} \mathcal{K}^{2-\varepsilon, 2-\varepsilon}(\mathbb{R}^2 \backslash \{0\}) \\ \oplus \\ \mathbb{C} \end{array} \to \mathcal{K}^{-\varepsilon, -\varepsilon}(\mathbb{R}^2 \backslash \{0\}),$$

for all $\eta \in \mathbb{R}$, and its homogeneous principal part (which is the principal edge symbol $\sigma_\wedge(\mathcal{A})(\eta)$) is invertible between those spaces for all $\eta \neq 0$. Under these circumstances, as is known from abstract pseudo-differential operators with operator-valued symbols, the inverse has the form

$$\mathcal{A}^{-1} = \mathrm{Op}\left((a(\eta) \quad a(\eta)k(\eta))^{-1}\right)$$

which is equal to $\mathcal{K}^{-1}P$ and has the form of a column matrix $\mathcal{A} = \begin{pmatrix} B \\ T \end{pmatrix}$. The operator $\mathcal{A}^{-1}$ belongs to the edge algebra (with constant discrete asymptotics) because the inverse of an elliptic and invertible edge symbol is again an edge symbol (of opposite order). The behaviour of asymptotics under the map

$$\mathcal{A}^{-1} : \mathcal{W}^{-\varepsilon,-\varepsilon}_{P_0}((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}) \to \begin{array}{c} \mathcal{W}^{2-\varepsilon,2-\varepsilon}_{Q_0}((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}) \\ \oplus \\ H^{2-\varepsilon}(\mathbb{R}) \end{array}$$

for every constant discrete asymptotic type $P_0 \in \mathrm{As}(S^1, (-\varepsilon, \Theta))$ with a corresponding resulting $Q_0 \in \mathrm{As}(S^1, (2-\varepsilon, \Theta))$ (for any $\Theta = (-(k+1), 0])$ follows completely from the non-bijectivity points of the principal conormal symbol

$$\sigma_M \sigma_\wedge(\mathcal{A})(z) = z^2 + \partial^2_\phi : H^s(S^1) \to H^{s-2}(S^1)$$

with respect to $z \in \mathbb{C}$. This was calculated in Section 2.2. Now we apply Theorem 3.1 to the situation as in Section 2.2, i.e., $X = \mathbb{S}^1$, $(\partial X)^\wedge = \mathbb{R}_+$, and $q = 1$, $s = \gamma = -\varepsilon$. Then, if we denote by B' the operator which is induced by the first component B of $\mathcal{A}^{-1}$ on the space

$$\mathcal{W}^{-\varepsilon+\frac{1}{2},-\varepsilon+\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}) \tag{57}$$

we obtain by restriction to the corresponding subspace with edge asymptotics a continuous operator

$$B' : \mathcal{W}^{-\varepsilon+\frac{1}{2},-\varepsilon+\frac{1}{2}}_R(\mathbb{R}_+ \times \mathbb{R}) \to \mathcal{W}^{2-\varepsilon,2-\varepsilon}_Q((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R})$$

for every $R \in \mathrm{As}(-\varepsilon + \frac{1}{2}, \Theta)$ with a corresponding $Q \in \mathrm{As}(\mathbb{S}^1, (2-\varepsilon, \Theta))$.

Let us now return to $P = \mathcal{K}\mathcal{A}^{-1} = B + KT$ for $\mathcal{K} = (1 \quad K)$ which defines a continuous operator

$$P : \mathcal{W}^{-\varepsilon,-\varepsilon}((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}) \to \mathcal{W}^{2-\varepsilon,2-\varepsilon}((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}) + K(H^{2-\varepsilon}(\mathbb{R})).$$

Similarly as before we form P', the operator on the space (57) induced by P. Then, summing up, we obtain the following result:

Theorem 3.5 *The potential with respect to P of edge distributions on the half-plane $\mathbb{R}_+ \times \mathbb{R} \ni (r, y)$ in $\mathbb{R}^3$ with asymptotics of type $R \in \mathrm{As}(-\varepsilon + \frac{1}{2}, \Theta)$ for $r \to 0$ defines a continuous operator*

$$P' : \mathcal{W}^{-\varepsilon+\frac{1}{2},-\varepsilon+\frac{1}{2}}_R(\mathbb{R}_+ \times \mathbb{R}) \to \mathcal{W}^{2-\varepsilon,2-\varepsilon}_Q((\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}) + K(H^{2-\varepsilon}(\mathbb{R}))$$

for a resulting asymptotic type $Q \in \mathrm{As}(\mathbb{S}^1, (2-\varepsilon, \Theta))$. Here $K(H^{2-\varepsilon}(\mathbb{R}))$ consists of all $u \in H^{2-\varepsilon}(\mathbb{R}^3)$ of the form

$$\{F^{-1}_{\eta\to y}\hat{v}(\eta)[\eta]\omega([\eta]x) : v(y) \in H^{2-\varepsilon}(\mathbb{R})\}$$

(*cf. the general shape of edge asymptotics of Remark* 1.10)

The method of this section to calculate asymptotics of potentials can be generalised to arbitrary elliptic equations (and systems) in $\mathbb{R}^m$, using a result of [7] on the edge algebra structure of elliptic operators with respect to a (smooth) hypersurface of any codimension in $\mathbb{R}^m$. In the case of non-constant coefficients we can either apply the concept of continuous asymptotics, or if the non-bijectivity points of conormal symbols remain fixed along the edge, the notion of asymptotics as discussed in Section 3.2.

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