The spline collocation method for Mellin convolution equations

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Abstract:

We consider Mellin convolution equations with additional Cauchy kernel in weighted Lebesgue and Lebesgue-Sobolev spaces. They contain, as an important subclass, singular integral equations with fixed singularities. Based on the Fredholm theory for these equations, on a smooth (sigmoidal) transformation (squeezing the underlying interval towards the endpoints) and on spectral methods in stability investigations by Banach algebra techniques, we obtain criteria for the stability of spline–collocation methods of the transformed equation in weighted Lebesgue–Sobolev spaces. We show that in weighted Hilbert spaces the approximation is stable provided that the principal symbol of the equation is strongly elliptic. Moreover, if the Cauchy kernel is absent, the stability holds in weighted Lebesgue–Sobolev spaces and the convergence of the approximate solutions is quasi–optimal.

Keywords:
Mellin convolutions with Cauchy kernels, weighted spline collocation, Stability, Weighted Sobolev spaces.

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Contents

Introduction

We consider the equation

$$
\begin{aligned}
\mathcal{A}u(x): \;&=\; c_0(x)u(x)\\
&+\; \frac{c_1(x)}{\pi i}\int_0^1 \frac{u(y)dy}{y-x} + \int_0^1 r\left(\frac{x}{y}\right)u(y)\frac{dy}{y} + \int_0^1 k(x,y)u(y)dy = f(x)\,, \quad (0.1)\\
x \;&\in \Im\,, \quad k(x,y) = \frac{k_0(x,y)}{|x-y|^{\mu}} \quad \text{with } k_0 \in C(\Im\times\Im), \quad 0<\mu<1\,.(0.2)
\end{aligned}
$$

Here c_0, c_1 are continuous functions on the unit interval $\Im = [0,1]$; the kernel r and its derivative have the following estimates: there exist $\gamma_0, \gamma_1 \in \mathbb{R} := (-\infty, \infty)$ such that

$$
\begin{aligned}
|r(x)| \le M_1 x^{\gamma_0}(1+x)^{\gamma_1}\,, \quad \left|\frac{d}{dx}r(x)\right| \le M_2 x^{\gamma_0-1}(1+x)^{\gamma_1}\,, \quad 0<x<\infty\,,\\
\gamma_0+\gamma_1 < -\nu_0 < \gamma_0\,, \quad \text{for some} \quad 0<\nu_0<1
\end{aligned}
\tag{0.3}
$$

and some constants $M_1, M_2 > 0$. Conditions(0.2) and (0.3) ensure the boundedness of the operator $\mathcal{A}$ in the weighted Lebesgue space $L_{p,\alpha,\beta}(\Im) = L_p(\Im, \varrho)$ with the weight

$$
\begin{aligned}
\varrho(x) := x^{\alpha}(1-x)^{\beta}\,,\\
\text{where} \quad \frac{1}{p}+\alpha = \nu_0\,, \quad 0<\nu_1 := \frac{1}{p}+\beta < 1\,, \quad 1<p<\infty\,;
\end{aligned}
\tag{0.4}
$$

the space is equipped with the norm

$$
\|\varphi\|_{p,\alpha,\beta} = \left(\int_0^1 |x^{\alpha}(1-x)^{\beta}\varphi(x)|^p dx\right)^{\frac{1}{p}}\,, \qquad \|\varphi\|_p := \|\varphi\|_{p,0,0}
$$

(see Lemma 1.5).

To justify restrictions on the kernel (0.3) let us mention that almost all kernels of equations like (0.1) arising in applications, satisfy conditions (0.3). Let us recall one particular but important subclass of equations (0.1)–singular integral equations with fixed singularities:

$$
\begin{aligned}
\mathcal{A}u(x) \;&:=\; c_0(x)u(x)\\
&+\; \frac{c_1(x)}{\pi i}\int_0^1 \frac{u(y)dy}{y-x} + \sum_{k=0}^{N}\frac{c_{k+2}x^{n_k}}{\pi i}\int_0^1 \frac{y^{k-n_k}u(y)dy}{(y+x)^{k+1}} + \int_0^1 k(x,y)u(y)dy\\
&=\; f(x) \quad \text{for} \quad x\in\Im \quad \text{with} \quad 0\le n_k\le k.
\end{aligned}
\tag{0.5}
$$

Such equations occur frequently in applications (see e.g. [9, 10, 12, 13, 32, 34, 40, 63]). These applications stimulated investigations of equation (0.1) in two directions: solvability (Fredholm theory) and approximation methods. The solvability theory in weighted Lebesgue and Lebesgue–Sobolev spaces, including index formulae as well as explicit solutions to characteristic equations, is rather complete (see [12, 13, 20, 37]). A profound investigation of the asymptotics of solutions of these equations and of pseudo–differential equations of Mellin type, was presented in [20]. Many of the above mentioned particular results have already been applied to elasticity problems and other problems of mathematical physics (see [9, 11, 12, 13, 20, 32, 34, 38, 63]).

Equations (0.1), (0.5) belong to the class of convolution equations (or Mellin convolution equations [5, 12, 27, 48], or Mellin pseudodifferential equations) of order 0 (see [20, 37]). In their recent book [27], R.Hagen, S.Roch and B.Silbermann presented a systematic study of the spectral theory of approximation methods for convolution equations with the help of Banach algebra techniques, covering almost all known results in the subject, essentially generalizing and extending them. We refer the reader to this book and to [49] for an exhaustive account of the history of the subject and its present state of art.

In [30, 31], a collocation quadrature method was developed for equation (0.5) in the special case $c_0(x) \equiv 0$, $c_1(x) \equiv 1$ with arbitrary c_2 and $c_k = 0$ for $k \geq 3$.

Approximation schemes for the formulation of computational methods have been developed intensively and mostly either for singular integral and integro–differential equations of Cauchy type or for a pure Mellin convolution equation with continuous symbol (i.e. without the Cauchy kernel, $c_1(x) \equiv 0$ in (0.1)) except in [27] and several papers which will be mentioned below. For corresponding collocation methods we refer the reader to [1, 2, 6, 7, 8, 9, 10, 11, 19, 21, 22, 42, 43, 47, 48, 49, 52, 53] which is only a part of the vast amount of work devoted to that topic.

In our investigations we shall take advantage of the following particular smooth (sigmoidal) transformation

$$\sigma_\theta(x) := \frac{x^\theta}{x^\theta + (1-x)^\theta}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty\,, \tag{0.6}$$

which is a C^∞–diffeomorphism of the unit interval $\sigma_\theta : \Im \to \Im$ (for other sigmoidal transformations see [17]) and which has the following almost obvious properties:

$$\begin{gathered}\sigma_\theta(\sigma_{\frac{1}{\theta}}(x)) \equiv x\,, \quad \mathcal{D}^k\sigma_\theta(x) = \emptyset\Big(x^{\theta-k}\Big) \quad \text{for} \quad x \to 0, \;\; 0 \leq k \leq \theta \\ \left.\begin{aligned}\sigma_\theta(x) &= 1 + \emptyset\Big((1-x)^\theta\Big) \\ \mathcal{D}^m\sigma_\theta(x) &= \emptyset\Big((1-x)^{\theta-m}\Big)\end{aligned}\right\} \quad \text{for} \quad x \to 1, \;\; 1 \leq m \leq \theta \\ \sigma_\theta(x) + \sigma_\theta(1-x) \equiv 1 \quad \text{for} \quad 0 \leq x \leq 1\,,\end{gathered} \tag{0.7}$$

where

$$\mathcal{D}^m\varphi(x) := \frac{d^m}{dx^m}\varphi(x). \tag{0.8}$$

The advantages of the transformation $\sigma_\theta(x)$ and of similar ones were exploited in [17, 18, 36] to solve a pure Mellin convolution equation (i.e. equation (0.1) with $c_1(x) \equiv 0$) approximately; the same method was applied in [42, 50] to the solution of equations (0.1) in $L_2(\Im)$ spaces by quadrature methods.

To illustrate the advantage of such a transformation, let us consider the following approximation of an integral by the Euler–Maclaurin formula (see [18, 36]):

$$I(\varphi) := \int_0^1 \varphi(y)dy\,, \quad \varphi(x) = x^{-\delta_0}(1-x)^{-\delta_1}\varphi_0(x)\,, \tag{0.9}$$
$$\text{with} \quad 0 \le \delta_0, \delta_1 < 1\,, \quad x^k(1-x)^k\mathcal{D}^k\varphi_0 \in C(\Im)\,. \quad k = 1, 2, ..., 2m-1$$

for some $m \in \mathbb{N}$. If we fix

$$\theta > \frac{2m}{\min\{1-\delta_0, 1-\delta_1\}}\,, \tag{0.10}$$

and write in the integral $y = \sigma_\theta(x)$, we find

$$I(\varphi) = \int_0^1 \varphi_\theta(y)dy\,, \quad \varphi_\theta(x) := \sigma'_\theta(x)\varphi\left(\sigma_\theta(x)\right)\,,$$

where

$$\mathcal{D}^k\varphi_\theta(0) = \mathcal{D}^k\varphi_\theta(1) = 0 \quad \text{for} \quad k = 1, 2, ..., 2m-1 \quad \text{and} \quad \int_0^1 \left|\mathcal{D}^{2m+1}\varphi_\theta(y)\right| dy < \infty\,.$$

Now we have a higher precision trapezoidal (Euler–Maclaurin) formula

$$I(\varphi) = \frac{1}{n}\sum_{j=1}^{n-1}\varphi_\theta\left(\frac{j}{n}\right) + R_n(\varphi) = \sum_{j=1}^{n-1}\sigma_{\theta,n,j}\varphi\left(x_{\theta,n,j}\right) + R_n(\varphi) \tag{0.11}$$
$$\text{where} \quad \sigma_{\theta,n,j} := \frac{1}{n}\sigma'_\theta\left(\frac{j}{n}\right)\,, \quad x_{\theta,n,j} := \sigma_\theta\left(\frac{j}{n}\right)\,,$$

with the error estimate

$$|R_n(\varphi)| \le \frac{M}{n^{2m+1}}\int_0^1 \left|\mathcal{D}^{2m+1}\varphi_\theta(y)\right| dy\,.$$

It is clear that (0.11) exploits nothing but a mesh refinement near the singular points of the integrand: the sigmoidal transformation squeezes the underlying interval towards the endpoints and suppresses the singularities of the integrand. The quadrature approximation in [18, 19, 36, 42, 50] is based on formula (0.11).

We shall exploit the singularity suppression property of the sigmoidal transformation (0.6): we change the variables $x = \sigma_\theta(t)$, $y = \sigma_\theta(\tau)$ in equation (0.1), replace the unknown function

$$u\left(\sigma_\theta(\tau)\right) \quad \text{by} \quad \tau^{-(\theta-1)\alpha}(1-\tau)^{-(\theta-1)\beta}\left[\sigma'_\theta(\tau)\right]^{-\frac{1}{p}}\varphi(\tau)\,,$$

introducing a new unknown φ, and multiply both sides of the equation by $t^{(\theta-1)\alpha}(1-t)^{(\theta-1)\beta}\left[\sigma'_\theta(t)\right]^{\frac{1}{p}}$ (see (1.13)). The transformed equation will be equivalent to the original one in the space $L_{p,\alpha,\beta}(\Im)$, but will have smoother solutions in the weighted Lebesgue–Sobolev spaces, provided that the right–hand side of the equation has appropriate smoothness (see Theorem **??**). This ensures a–priori smoothness of a solution and consequently better convergence for the spline collocation and quadrature methods.

In Theorem **??** we formulate a criterion for the convergence of the spline collocation method for the transformed equation in a weighted Lebesgue space. The criterion contains certain indefiniteness due to the presence of a compact operator which cannot be identified exactly. In the case when the Cauchy kernel is absent, the convergence holds in a weighted Lebesgue–Sobolev space. In Theorem **??** we formulate necessary conditions for such convergence, which is a direct consequence of Theorem **??** and of Theorem **??**.

Theorem **??** deals with the criterion for the stability of the spline collocation method in weighted Hilbert and Hilbert–Sobolev spaces. There we prove that for a Mellin convolution equation with a locally strongly elliptic symbol the spline collocation method is stable. The proof applies the techniques developed in [2, 10, 27, 48, 56].

Our assertions on spline collocation are formulated regardless of the spline orders (see Theorems **??**, **??** and **??**), whilst we know well that the stability conditions differ for the odd and even order splines (see [1, 53]). This independence is due to the choice of the collocation points: for the odd order splines we choose mid–point collocation, while for the even order splines–the break–point collocation, as proposed in [27] (see Section 2).

In what follows we outline the results. An extended version of the paper, with complete proofs, will appear elsewhere.

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1 Formulation of the main results

1.1 Solvability of Mellin convolution equations

Let us start by defining the symbol of equation (0.1) (and of (0.5)) which governs Fredholm properties and the index of the equation.

With the operator $\mathcal{A}$ in (0.1) and a pair (ν_0, ν_1) satisfying (0.3)–(0.4) we associate the symbol $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ for $\omega \in \mathcal{R} := \Im \times \mathbb{R}$, defined as follows:

$$\mathcal{A}_{\nu_0,\nu_1}(\omega) := \begin{cases} c_0(0) + c_1(0)\coth \pi(i\nu_0 - \lambda) \;+ & \mathcal{M}r(\nu_0 - i\lambda), \\ & \omega = (0,\lambda) \in \{0\} \times \mathbb{R}, \\ c_0(x) + c_1(x) \;\; \operatorname{sgn} x\,, & \omega = (x,\lambda) \in (0,1) \times \mathbb{R}, \\ c_0(1) - c_1(1)\coth \pi(i\nu_1 + \lambda)\,, & \omega = (1,\lambda) \in \{1\} \times \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathcal{M}r(z)$ denotes the Mellin transform of the kernel r,

$$\mathcal{M}r(z) := \int_0^\infty t^z r(t) \frac{dt}{t}\,, \qquad \text{and} \qquad z = \nu_0 - i\lambda\,, \quad \lambda \in \mathbb{R}\,. \tag{1.2}$$

Due to conditions (0.3) the symbol $\mathcal{M}r(\nu_0 - i\lambda)$ has the bounded total variation on $\mathbb{R}$ (see Lemma 2.3 and Theorem 2.4 below) and is analytic in some strip $\nu_0^- < \nu_0 < \nu_0^+$; moreover, the operator $\mathcal{A}$ is of the class $OP\Sigma^0_{\nu_0^-,\nu_0^+}(\Im)$ introduced in [20, Sect.4]; for such operators one finds there an explicit asymptotic behaviour of a solution in the vicinity of $x = 0, x = 1$ (see Theorem 1.3 below).

If the symbol $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ is elliptic, i.e. if

$$\mathbf{inf}\,\{|\mathcal{A}_{\nu_0,\nu_1}(\omega)| : \omega \in \mathcal{R}\} > 0 \tag{1.3}$$

we can define the index of the symbol

$$\text{ind}\ \mathcal{A}_{\nu_0,\nu_1} := \frac{1}{2\pi}\left[\arg \frac{\mathcal{A}_{\nu_0,\nu_1}(x,+1)}{\mathcal{A}_{\nu_0,\nu_1}(x,-1)}\right]_{x\in\Im} + \frac{1}{2\pi}\left[\arg \frac{\mathcal{A}_{\nu_0,\nu_1}(0,\lambda)}{\mathcal{A}_{\nu_0,\nu_1}(1,\lambda)}\right]_{\lambda\in\mathbb{R}}. \tag{1.4}$$

Since the limits

$$\lim_{\lambda\to\pm\infty} \coth \pi(i\nu_0 + \lambda) = \pm 1, \quad k = 0, 1, \quad \lim_{\lambda\to\pm\infty} \mathcal{M}r(\nu_0 - i\lambda) = 0$$

exist it is easy to ascertain that $\text{ind}\ \mathcal{A}_{\nu_0,\nu_1}$ gives the winding number of the continuous curve $\{\mathcal{A}_{\nu_0,\nu_1}(\omega) : \omega \in \mathcal{R}\}$ on the complex plane $\mathbb{C}$ around the origin $0 \in \mathbb{C}$ and is integer–valued.

For equation (0.5) the Mellin transform of the kernel $\mathcal{M}r(z)$ in can be written explicitly:

$$\mathcal{M}r(z) = \sum_{k=0}^{m} (-1)^k c_{k+2} \binom{z + n_k - 1}{k} \frac{1}{\sinh \pi(z + n_k)} \tag{1.5}$$

(see [12, Section 8]).

The next theorem was proved in [12] (see also [5, 27]).

Theorem 1.1 *Equation* (0.1) *is Fredholm in the space* $L_{p,\alpha,\beta}(\Im)$ *if and only if the symbol* $\mathcal{A}_{\frac{1}{p}+\alpha,\frac{1}{p}+\beta}(\omega)$ *is elliptic (see* (1.3)*) and, if this is the case,*

$$\text{Ind}\ \mathcal{A} = -\ \text{ind}\ \mathcal{A}_{\frac{1}{p}+\alpha,\frac{1}{p}+\beta}. \tag{1.6}$$

Remark 1.2 *If* $k(x,y) \equiv 0$ *and* $c_0(x) \equiv c_0$, $c_1(x) \equiv c_1$ *are constant, then either the homogeneous equation*

$$\mathcal{A}u = 0$$

or the conjugate homogeneous one,

$$\mathcal{A}^*\psi = 0, \qquad \psi \in L^*_{p,\alpha,\beta}(\Im) = L_{p',-\alpha,-\beta}(\Im)$$

have only the trivial solution: either $u = 0$ *or* $\psi = 0$.

Both solutions are trivial $u = \psi = 0$ *iff* $\text{ind}\ \mathcal{A}_{\frac{1}{p}+\alpha,\frac{1}{p}+\beta} = 0$.

Consider the operator

$$V_{\theta,\nu_0,\nu_1}\varphi(x) := x^{(\theta-1)\nu_0}(1-x)^{(\theta-1)\nu_1}\left[\sigma'_\theta(x)\right]^{\frac{1}{p}}\varphi\left(\sigma_\theta(x)\right), \quad x\in\Im, \quad 0<\theta<\infty, \quad (1.7)$$

where $\mu_0, \mu_1 \in \mathbb{R}$ and $\sigma_\theta(x)$ is the sigmoidal transformation from (0.6); then

$$\begin{aligned} V_{\theta,\mu_0,\mu_1}V_{\frac{1}{\theta},\mu_0,\mu_1}\varphi &= v_{\theta,\mu_0+\mu_1}\varphi\,, \\ v_{\theta,\mu}(x): &= \left[x^\theta+(1-x)^\theta\right]^{\left(1-\frac{1}{\theta}\right)\mu} \neq 0, \qquad v_{\theta,\mu}(0)=v_{\theta,\mu}(1)=1\,. \end{aligned}$$

Therefore

$$V^{-1}_{\theta,\mu_0,\mu_1}\varphi(x) = \frac{1}{v_{\theta,\mu_0+\mu_1}\left(\sigma_{\frac{1}{\theta}}(x)\right)}V_{\frac{1}{\theta},\mu_0,\mu_1}\varphi(x) = v_{\frac{1}{\theta},-\mu_0-\mu_1}(x)V_{\frac{1}{\theta},\mu_0,\mu_1}\varphi(x) \qquad (1.8)$$

and the mappings

$$V_{\theta,\alpha,\beta},\ \ V^{-1}_{\theta,\alpha,\beta}: L_{p,\alpha,\beta}(\Im)\rightarrow L_{p,\alpha,\beta}(\Im) \qquad (1.9)$$

are automorphisms of the Banach space $L_{p,\alpha,\beta}(\Im)$.

Let $m\in\mathbb{N}_0 := \{0,1,...\}$ and $\mathcal{J}\subset\mathbb{R}$ be an interval ($\mathcal{J}=\Im$, $\mathcal{J}=\mathbb{R}$ or $\mathcal{J}=\mathbb{R}^+ := [0,\infty)$); let $\varrho(x)$ be a locally integrable non–negative weight function on $\mathcal{J}$ (i.e. $\varrho(x)\geq 0$ is integrable on each compact subset of $\mathcal{J}$). Then, similarly to $L_p(\Im,\varrho)$ defined in the Introduction, we define the following Lebesgue space with the weight

$$L_p(\mathcal{J},\varrho) := \left\{\varphi(x) : \|\varphi\,|\,L_p(\mathcal{J},\varrho)\| = \left(\int\limits_{\mathcal{J}}|\varrho(x)\varphi(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\}.$$

Let further

$$\begin{aligned} C^m(\mathcal{J}) &:= \left\{\varphi\in C(\mathcal{J}) : \mathcal{D}^k\varphi\in C(\mathcal{J}), \quad k=1,...m\right\}, \\ W^m_p(\mathcal{J},\varrho) &:= \left\{\varphi\in L_p(\mathcal{J},\varrho) : \mathcal{D}^k\varphi\in L_p(\mathcal{J},\varrho), \quad k=1,...m\right\}, \end{aligned} \qquad (1.10)$$

where the derivatives $\mathcal{D}^k\varphi\in L_p(\mathcal{J},\varrho)$ are understood in the sense of distributions and $W^m_p(\mathcal{J},\varrho)$ is called the Lebesgue–Sobolev space with a weight.

If $\mathcal{J}=\Im$ and $\varrho(x)$ is defined as in (0.4), we shall use the notation $W^m_{p,\alpha,\beta}(\Im)$ and if $\mathcal{J}=\mathbb{R}^+$, $\varrho(x)=x^\alpha$, we shall write $W^m_{p,\alpha}(\mathbb{R}^+)$. If $\varrho(x)\equiv 1$ we shall write $W^m_p(\mathcal{J})$ rather than $W^m_p(\mathcal{J},1)$.

We need the following spaces:

$$\begin{aligned} \widetilde{W}^m_p(\Im,\varrho) &= \widetilde{W}^m_{p,\alpha,\beta}(\Im) \\ : &= \left\{u\in W^m_{p,\alpha,\beta}(\Im)\ :\ \mathcal{D}^k u(0)=\mathcal{D}^k u(1)=0\,,\ \ k=0,1,\dots,m-1\right\}, \\ \widetilde{W}^m_p(\mathbb{R}^+,x^\alpha) &= \widetilde{W}^m_{p,\alpha}(\mathbb{R}^+) \\ : &= \left\{u\in W^m_{p,\alpha}(\Im)\ :\ \mathcal{D}^k u(0)=0\,,\ \ k=0,1,\dots,m-1\right\}. \end{aligned} \qquad (1.11)$$

Due to the embeddings

$$\widetilde{W}^m_{p,\alpha,\beta}(\Im) \subset C^{m-1}(\Im)\,, \quad \widetilde{W}^m_{p,\alpha}(\mathbb{R}^+) \subset C^{m-1}(\mathbb{R}^+)\,, \quad \widetilde{W}^m_p(\mathbb{R}) \subset C^{m-1}(\mathbb{R})\,, \tag{1.12}$$

which are almost trivial cases of Sobolev's lemma (see [61, Section 28.1, Remark 2]), the values $\mathcal{D}^k u(j)$, $k = 1, \ldots, m-1$, $j = 0, 1$ in (1.11) are well–defined.

The norms in these spaces are defined as follows:

$$\begin{aligned}
\|\varphi\|^{(m)}_{p,\alpha,\beta} &= \|\varphi \,|\, \widetilde{W}^m_{p,\alpha,\beta}(\Im)\| := \sum_{k=0}^{m} \|\mathcal{D}^k\varphi \,|\, L_{p,\alpha,\beta}(\Im)\|\,,\\
\|\varphi\|^{(m)}_{p,\alpha,\beta} &= \|\varphi \,|\, W^m_{p,\alpha,\beta}(\Im)\| := \mathbf{inf}\left\{\sum_{k=0}^{m} \|\mathcal{D}^k\widetilde{\varphi} \,|\, L_{p,\alpha,\beta}(\mathbb{R})\| : \widetilde{\varphi} \in W^m_{p,\alpha,\beta}(\mathbb{R})\right\},
\end{aligned}$$

where the infimum is taken over all possible extensions $\widetilde{\varphi} \in W^m_{p,\alpha,\beta}(\mathbb{R})$ (see e.g. [61]).

The norms in $W^m_{p,\alpha}(\mathbb{R}^+)$ and in $\widetilde{W}^m_{p,\alpha}(\mathbb{R}^+)$ are defined similarly.

Due to (1.9) the next equation is equivalent to (0.1) in the space $L_{p,\alpha,\beta}(\Im)$:

$$\begin{aligned}
\mathcal{A}_0\varphi(x) &= V_{\theta,\alpha,\beta}\mathcal{A}V^{-1}_{\theta,\alpha,\beta}\varphi(x) = c_{0,\theta}(x)\varphi(x)\\
&+ \frac{c_{1,\theta}(x)}{\pi i}\int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)\alpha}\left(\frac{1-x}{1-y}\right)^{(\theta-1)\beta}\left[\frac{\sigma'_\theta(x)}{\sigma'_\theta(y)}\right]^{\frac{1}{p}}\frac{\sigma'_\theta(y)}{\sigma_\theta(y)-\sigma_\theta(x)}\varphi(y)dy\\
&+ \int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)\alpha}\left(\frac{1-x}{1-y}\right)^{(\theta-1)\beta}\left[\frac{\sigma'_\theta(x)}{\sigma'_\theta(y)}\right]^{\frac{1}{p}} r\left(\frac{\sigma_\theta(x)}{\sigma_\theta(y)}\right)\frac{\sigma'_\theta(y)}{\sigma_\theta(y)}\varphi(y)dy \qquad (1.13)\\
&+ \int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)\alpha}\left(\frac{1-x}{1-y}\right)^{(\theta-1)\beta}\left[\frac{\sigma'_\theta(x)}{\sigma'_\theta(y)}\right]^{\frac{1}{p}}\frac{\sigma'_\theta(y)k_{0,\theta}(x,y)}{|\sigma_\theta(y)-\sigma_\theta(x)|^\mu}\varphi(y)dy\\
&\qquad\qquad\qquad\qquad = f_0(x), \quad x \in \Im\,,
\end{aligned}$$

$$\begin{aligned}
c_{j,\theta}(x) :&= c_j(\sigma_\theta(x)), \quad k_{0,\theta}(x,y) := k_0(\sigma_\theta(x), \sigma_\theta(y))\,,\\
u &= V^{-1}_{\theta,\alpha,\beta}\varphi,\,, \qquad \varphi = V_{\theta,\alpha,\beta}u\,, \qquad f_0 = V^{-1}_{\theta,\alpha,\beta}f
\end{aligned} \tag{1.14}$$

(see (1.7), (1.8)).

Theorem 1.3 *Let* (0.3) *hold for all* $\nu_0 \in [\nu_0^-, \nu_0^+]$, *where* $0 < \nu_0^- < \nu_0^+ < 1$ *and:*

(i) *there exist* $0 < \nu_1^- < \nu_1^+ < 1$ *such that the symbol* $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ *of equation* (0.1) *is elliptic for all* $\nu_0^- < 1/p + \alpha = \nu_0 < \nu_0^+$ *and* $\nu_1^- < 1/p + \beta = \nu_1 < \nu_1^+$*;*

(ii) $x^{\alpha'+k}(1-x)^{\beta'+k}\mathcal{D}^k_x f \in L_p(\Im)$ *for some* $\alpha' < \alpha$, $\beta' < \beta$ *and all* $k = 0, 1, ..., m$, $m \in \mathbb{N}$*;*

(iii) $c_0, c_1 \in C^m(\Im)$ *and* $x^k(1-x)^k\mathcal{D}^k_x k_0(x,y) \in C(\Im \times \Im)$ *for all* $k = 0, 1, ..., m$*;*

(iv) $\theta(\nu_j - \nu_j^-) > m$ *for* $j = 0, 1$.

Then all solutions $\varphi \in L_{p,\alpha,\beta}(\Im)$ *of equation* (1.13) *belong to the space* $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$.

Proof: As already mentioned due to (1.9), equations (0.1) and (1.13) are equivalent. Therefore, due to the condition (iv), the inclusion $\varphi \in \widetilde{W}^m_{p,\alpha,\beta}(\Im)$ will follow if

$$x^{\alpha''+k}(1-x)^{\beta''+k}\mathcal{D}^k_x u \in L_p(\Im) \qquad \begin{array}{ll} \text{for some} & \alpha''<\alpha, \;\; \beta''<\beta \\ \text{and all} & k=0,1,...,m. \end{array} \tag{1.15}$$

The symbols $\mathcal{A}_{\nu_0,\nu_1}(0,\lambda)$ and $\mathcal{A}_{\nu_0,\nu_1}(1,\lambda)$ are analytic and elliptic (non–vanishing) in strips $\{\nu_0 - i\lambda : \nu_0^- < \nu_0 < \nu_0^+ \,,\; \lambda \in \mathbb{R}\} \subset \mathbb{C}$ and in $\{\nu_1 - i\lambda : \nu_1^- < \nu_1 < \nu_1^+ \,,\; \lambda \in \mathbb{R}\} \subset \mathbb{C}$ respectively.

The claimed inclusion (1.15) follows from [20, Theorem 4.10] and conditions (ii), (iii) of the theorem. □

Remark 1.4 *The foregoing theorem exploits very rough asymptotic property of solutions to equation* (0.1) *which, neverthless, suit our purposes: now we are able to apply the spline collocation method to the equation provided the right–hand side function* $f(x)$ *is chosen properly and the solution is continuous.*

For more refined asymptotic analyses we refer the reader to [20].

1.2 Spline collocation

Next we shall recall some well–known definitions and properties of smoothest polynomial splines from [3, 4, 27, 55]. More about splines will be exposed in Section 3.

Let

$$\Phi_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) := \begin{cases} 1 & \text{if } |x| \le \dfrac{1}{2}, \\ 0 & \text{elsewhere} \;, \end{cases} \tag{1.16}$$

where $\chi_{[c,d]}(x)$ denotes the characteristic function of the interval $[c,d] \subset \mathbb{R}$ and

$$\begin{aligned} \Phi_m(x) := \Phi_0 * \Phi_{m-1}(x) \;\; &= \Phi_{m-1} * \Phi_0(x) = \int\limits_{-\infty}^{\infty} \Phi_{m-1}(x-y)\Phi_0(y)dy \\ &= \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{m-1}(x-y)dy, \quad m=1,2,..., \quad x \in \mathbb{R}\,. \end{aligned} \tag{1.17}$$

From (1.17) it follows that $\Phi_m \in C^{m-1}_0(\mathbb{R})$, has piecewise–constant $m-$th derivative $\mathcal{D}^m_x\Phi_m(x)$ and

$$\text{supp}\;\Phi_m = \left[-\frac{m+1}{2}, \frac{m+1}{2}\right] \,. \tag{1.18}$$

For a fixed integer $n \in \mathbb{N}$ we define

$$\begin{aligned} S^{(n)}_m(\Im) := \;\; &\text{span}\; \left\{\Phi_{m,[\frac{m}{2}]+1}, \ldots, \Phi_{m,n-[\frac{m}{2}]-1}\right\} , \\ \text{where} \quad &\Phi^{(n)}_{mj}(x) := \Phi_m(nx-j), \quad j \in \mathbb{Z} := \{0, \pm 1, \ldots\} , \end{aligned} \tag{1.19}$$

and $[\nu] \in \mathbb{Z}$ denotes the largest integer less or equal to $\nu \in \mathbb{R}$.

The space $S_m^{(n)}(\Im)$ is $\left(n-2\left[\frac{m}{2}\right]-1\right)$ –dimensional and spanned by those splines which are supported inside the interval $\Im$ (see (1.18)).

The approximating power of smoothest polynomial splines is well–known:

$$\begin{aligned}\operatorname{dist}_{W_2^r(\Im)}(\varphi, S_m^{(n)}(\Im)) : &= \inf\left\{\|(\varphi-\psi)|W_2^r(\Im)\| : \psi \in S_m^{(n)}(\Im)\right\}\\ &\le Cn^{r-s}\|\varphi|W_2^s(\Im)\| \quad \text{provided} \quad 0 \le r < m+\frac{1}{2}, \ r \le s\end{aligned}$$

where the constant C is independ of $\varphi \in W_2^s(\Im)$ and n (see [49, Chapter 2, Theorem 2.26]). The approximation becomes better (i.e. converges faster) if special graded meshes are chosen (see e.g. $x_{\theta,j,n}$ in (0.11) and cf. [4, 21, 55], [49, Chapter 5, Lemma 5.23]).

Lemma 1.5 *If*

$$g_m(\eta) := \sum_{j=-\infty}^{\infty} \Phi_m(j)\exp(i\eta j) = \Phi_m(0) + \sum_{j=1}^{\left[\frac{m+1}{2}\right]} \Phi_m(j)\cos(\eta j)\,, \tag{1.20}$$

then

$$g_m(\eta) \neq 0 \qquad \text{for all} \quad 0 \le \eta \le 2\pi \qquad \text{and all} \quad m \in \mathbb{N}_0\,. \tag{1.21}$$

Proof. See [58, Theorem 2.2] and [39].

Thus, we can define the Fourier coefficients

$$\begin{aligned}\left(g_m^{-1}\right)_k : &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp(i\eta k)d\eta}{g_m(\eta)}\,, \quad k \in \mathbb{Z} := \{0, \pm 1, \dots\}\,,\\ \frac{1}{g_m(\eta)} &= \sum_{j=-\infty}^{\infty} (g_m^{-1})_k \exp(i\eta k)\,, \quad 0 \le \eta \le 2\pi.\end{aligned} \tag{1.22}$$

Consider the operators

$$\begin{aligned}\hat{P}_{\Im,m}^{(n)}\varphi(x) : &= \sum_{j,k=m_0}^{n-m_0} (g_m^{-1})_{j-k}\,\varphi\left(\frac{k}{n}\right)\Phi_{mj}^{(n)}(x)\,,\\ \hat{P}_{\Im,m}^{(n)} : & \qquad W_{p,\alpha}^k(\Im) \to S_m^{(n)}(\Im)\,,\end{aligned} \tag{1.23}$$

$$\text{where} \quad k = 1, \dots, m\,, \quad n = 1, 2, \dots\,, \quad m_0 := \left[\frac{m}{2}\right] + 1.$$

$\hat{P}_{\Im,m}^{(n)}$ are quasi–projections

$$\left(\hat{P}_{\Im,m}^{(n)}\right)^2\varphi(x) = \hat{P}_{\Im,m}^{(n)}\varphi(x) \qquad \text{provided} \qquad \frac{m_0}{n} < x < 1 - \frac{m_0}{n}\,, \tag{1.24}$$

but fail to be projections since (1.24) holds not for all $x \in (0,1)$, unless $m = 0, 1$ (see Section 3 for details).

Let us look for an approximate solution of equation (1.13) in the form

$$\mathcal{A}_{0,m}^{(n)}\varphi_m^{(n)} = f_{0,m}^{(n)}\,, \quad \varphi_m^{(n)} \in S_m^{(n)}(\Im)\,, \tag{1.25}$$

where (cf. (1.23))

$$f_{0,m}^{(n)} := \hat{P}_{\Im,m}^{(n)} f_0, \quad \mathcal{A}_{0,m}^{(n)} := \hat{P}_{\Im,m}^{(n)} \mathcal{A}_0 \Big|_{S_m^{(n)}(\Im)} .$$

Equation (1.25) can be rewritten as an equivalent $(n-2m_0+1)\times(n-2m_0+1)$ system of linear algebraic equations

$$c_{0,\theta}\left(\frac{j}{n}\right)\varphi\left(\frac{j}{n}\right) + \sum_{k=m_0}^{n-m_0} b_{m,j,k}^{(n)}\varphi\left(\frac{k}{n}\right) = f_0\left(\frac{j}{n}\right), \quad j = m_0, \dots, n-m_0 \,, \tag{1.26}$$

where

$$\begin{aligned} b_{m,j,k}^{(n)} : &= \sum_{l=m_0}^{n-m_0} \left(g_m^{-1}\right)_l d_{m,j,l}^{(n)} \,, \quad \sum_{j,k=m_0}^{n-m_0} \left(g_m^{-1}\right)_{j-k} \varphi\left(\frac{k}{n}\right) \Phi_{mj}^{(n)}(x) =: \varphi_m^{(n)}(x) \\ d_{m,j,l}^{(n)} : &= \int_0^1 \left(\frac{\frac{j}{n}}{y}\right)^{(\theta-1)\alpha} \left(\frac{1-\frac{j}{n}}{1-y}\right)^{(\theta-1)\beta} \left[\frac{\sigma_\theta'(\frac{j}{n})}{\sigma_\theta'(y)}\right]^{\frac{1}{p}} \left[\frac{c_{1,\theta}(\frac{j}{n})}{\pi i} \frac{\sigma_\theta'(y)}{\sigma_\theta(y)-\sigma_\theta(\frac{j}{n})}\right. \\ &+ \left. r\left(\frac{\sigma_\theta(\frac{j}{n})}{\sigma_\theta(y)}\right)\frac{\sigma_\theta'(y)}{\sigma_\theta(y)} + \frac{\sigma_\theta'(y)k_{0,\theta}(\frac{j}{n},y)}{\left|\sigma_\theta(y)-\sigma_\theta(\frac{j}{n})\right|^\mu}\right] \Phi_{mj}^{(n)}(y)dy \,. \end{aligned}$$

Here we should mention again that we follow [27], choosing the mid–point collocation for odd order splines, while for even order splines we choose the break–point collocation. This leads to unified formulations of Theorems **??**, **??** and **??** in both cases.

Theorem 1.6 *Let $m \in \mathbb{N}$ and solutions $\varphi \in L_{p,\alpha,\beta}(\Im)$ of equation* (1.13) *belong to the space $\widetilde{W}_{p,\alpha,\beta}^m(\Im)$ (see Theorem* **??***).*

For the stability of the approximation (1.25) *(i.e. of the approximation* (1.26)*) of Equation* (1.13) *in the space $L_{p,\alpha,\beta}(\Im)$ it is necessary that the following conditions hold:*

- (i) *equation* (1.13) *has a unique solution in $L_{p,\alpha,\beta}^m(\Im)$ (equivalently:* (0.1) *has a unique solution in $L_{p,\alpha,\beta}^m(\Im)$);*
- (ii) $\mathcal{A}_0(0,0,\lambda) := c_0(0) + c_1(0)\cot\pi(\frac{i}{p}+i\alpha+\lambda) + \mathcal{M}r(\frac{1}{p}+\alpha-i\lambda) \neq 0$ *for all* $\lambda \in \mathbb{R}$ *(see* (1.2) *for $\mathcal{M}r(\omega)$);*
- (iii) $\mathcal{A}_0(x,\mu) := c_0(x) + \mu c_1(x) \neq 0$ *for all* $0 \le x \le 1$ *and all* $-1 \le \mu \le 1$*;*
- (iv) $\mathcal{A}_0(1,0,\lambda) := c_0(1) - c_1(1)\cot\pi(\frac{i}{p}+i\beta+\lambda) \neq 0$ *for all* $\lambda \in \mathbb{R}$*;*
- (v) $[\arg \mathcal{A}_0(j,\mu)]_{\mu\in[-1,1]} + (-1)^j [\arg \mathcal{A}_0(j,0,\lambda)]_{\lambda\in\mathbb{R}} = 0$ *for* $j = 0, 1$.

If, in addition,

$$c_1(x) \equiv 0 \qquad \text{and} \qquad \theta\left(\gamma_0 + \frac{1}{p} + \alpha\right) - m \ge 0\,, \tag{1.27}$$

(i.e. equation (0.1) *is pure Mellin convolutional without the Cauchy kernel) then the conditions* (i)–(v) *are necessary for the stability of the spline collocation* (1.25) *in the Lebesque–Sobolev space $\widetilde{W}_{p,\alpha,\beta}^m(\Im)$.*

Proof follows directly from Theorems 1.9 and 1.10. □

In the general case $p \neq 2$ we have not a transparent criterium for stability of approximation (1.25) (see Theorem 1.9 below); but the next theorem gives sufficient conditions for the stability, which are applicable to a wide class of equations arising in applications.

The symbol $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ $(\omega \in \mathcal{R})$ of equation (0.1) will be reffered to as **locally strongly elliptic** if there exists a continuous function $\mu(x)$ $(x \in \Im)$ such that

$$\mathbf{inf}\{\ \mathrm{Re}\ [\mu(x)\mathcal{A}_{\nu_0,\nu_1}(x,\lambda)] : x \in \Im,\ \ \lambda \in \mathbb{R}\} > 0\,.$$

An equivalent reformulation of the "local strong ellipticity" property is the following **local sectorial** property: for every $x \in \Im$ there exists a constant $\gamma_x(\mathcal{A}_{\nu_0,\nu_1}) \in \mathbb{C}$ and a neighbourhood $U_x \subset \Im$ such that

$$\sup\{|\ \arg \mathcal{A}_{\nu_0,\nu_1}(y,\lambda) - \gamma_x(\mathcal{A}_{\nu_0,\nu_1})| : y \in U_x\,,\ \ \lambda \in \mathbb{R}\} < \frac{\pi}{2}\,. \tag{1.28}$$

Theorem 1.7 *Let* (0.3) *hold for all* $\nu_0 \in [\nu_0^-, \nu_0^+]$*, where* $0 < \nu_0^- < \nu_0^+ < 1$ *and:*

(i) *there exist* $0 < \nu_1^- < \nu_1^+ < 1$ *such that the symbol* $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ *of equation* (0.1) *is locally strongly elliptic for all* $\nu_0^- < 1/p + \alpha = \nu_0 < \nu_0^+$ *and* $\nu_1^- < 1/p + \beta = \nu_1 < \nu_1^+$*;*

(ii) $x^{\nu_0^- - \frac{1}{2} + k}(1-x)^{\nu_1^- - \frac{1}{2} + k}\mathcal{D}^k f \in L_2(\Im)$ *for all* $k = 0, 1, \ldots, m$ *and* $j = 0, 1$*;*

(iii) $c_0, c_1 \in C^m(\Im)$ *and* $x^k(1-x)^k\mathcal{D}_x^k k_0(x,y) \in C(\Im \times \Im)$ *for all* $k = 0, 1, ..., m$*;*

(iv) $\theta(\nu_j - \nu_j^-) > m$ *for* $j = 0, 1$.

(v) *equation* (0.1) *with f=0 (the homogeneous equation) has only the trivial solution* $u = 0$ *in the space* $L_{2,\alpha,\beta}(\Im)$*, where*

$$\alpha = \nu_0 - \frac{1}{2}\,, \qquad \beta = \nu_1 - \frac{1}{2}\,. \tag{1.29}$$

Then the approximation (1.24) *of Equation* (1.13) *is stable in the weighted Hilbert space* $L_{2,\alpha,\beta}(\Im)$*.*

If (1.27) *holds the approximation is stable in the Hilbert–Sobolev space* $\widetilde{W}^m_{2,\alpha,\beta}(\Im)$ *and the convergence of the approximate solutions* $\varphi_m^{(n)}$ *to the solution* φ *is quasi–optimal. That is,*

$$\|(\varphi - \varphi_m^{(n)})|\widetilde{W}^m_{2,\alpha,\beta}(\Im)\| \le C\mathbf{inf}\left\{\|(\varphi - v)|\widetilde{W}^m_{2,\alpha,\beta}(\Im)\| : v \in S_m^{(n)}(\Im)\right\} \tag{1.30}$$

with some constant $C > 0$*.*

Proof will be exposed in Section 4.

Remark 1.8 *Note that conditions* (ii)–(v) *in Theorem* **??** *coincide with local strong ellipticity of the symbol* $\mathcal{A}_{\nu_0,\nu_1}(\omega)$ *on the set* $(0,1]\times\mathbb{R}=\mathcal{R}\setminus\{\{0\}\times\mathbb{R}\}$ *even for* $p\neq 2$, *while on the remainder* $\{0\}\times\mathbb{R}$ *they differ slightly; namely the strong ellipticity at* $x=0$ *means that the convex hull of the continuous curve*

$$\mathcal{N}:=\{z\in\mathcal{A}_{\nu_0,\nu_1}(0,0,\lambda):\lambda\in\mathbb{R}\}\cup\{c_0(0)+c_1(0)\mu:-1<\mu<1\}$$

in the complex plane $\mathbb{C}$ *does not contain the origin* 0, *while the conditions of Theorem* **??** *state that the curve* $\mathcal{N}$ *does not cross the origin and must have winding number 0.*

If, for example, $r\equiv 0$ *(i.e. we have a pure singular integral equation with Cauchy kernel) then the necessary conditions of Theorem* **??** *and the sufficient conditions of Theorem* **??** *coincide. This case was first treated in [48].*

1.3 A criterium for stability

To formulate the conditions for stability of the approximation (1.24) we need some further definitions.

For a bounded function $a\in L_\infty([0,2\pi])$, the Töplitz operator T_a is composed of the Fourier coefficients of $a(\eta)$:

$$T_a:=\|a_{j-k}\|_{j,k\in\mathbb{N}}\,,\quad a_l:=\frac{1}{2\pi}\int\limits_0^{2\pi}\exp(i\eta l)a(\eta)d\eta\,,\quad l\in\mathbb{Z}\,. \tag{1.31}$$

If $l_{p,\alpha}(\mathbb{N})$ denotes the Banach space of sequences with the weighted norm

$$l_{p,\alpha}(\mathbb{N}):=\left\{\xi=\{\xi_j\}_{j=1}^\infty:\|\xi\,|l_{p,\alpha}(\mathbb{N})\|:=\left(\sum_{j=1}^\infty j^{\alpha p}\,|\xi_j|^p\right)^{\frac{1}{p}}\right\}, \tag{1.32}$$

then, according to Stechkin's theorem(see [5, 12]), the operator

$$T_a:l_{p,\alpha}(\mathbb{N})\to l_{p,\alpha}(\mathbb{N})$$

is bounded provided

$$a\in V_1([0,2\pi])\,,\quad 1<p<\infty\,,\quad 0<\frac{1}{p}+\alpha<1\,. \tag{1.33}$$

Here $V_1(\mathcal{J})$ denotes the space of functions with bounded total variation on $\mathcal{J}\subset\mathbb{R}$.

For a kernel function $r(x)$ (see (0.1), (0.3), (1.13)) let G_r denotes the discretized Mellin convolution operator

$$G_r:=\left\|r\left(\frac{j}{k}\right)\frac{1}{k}\right\|_{j,k\in\mathbb{N}} \tag{1.34}$$

(see [46, 47, 51] and [27, Section 2.4]). If conditions (0.3) and (0.4) hold, the operator

$$G_r:l_{p,\alpha}(\mathbb{N})\to l_{p,\alpha}(\mathbb{N})$$

is bounded (see Lemma 2.8 below).

Condition (1.20) ensures that the truncated Töplitz matrices

$$T^{(m_0)}_{g_m^{-1}} : \; = \; \left\| \left(g_m^{-1} \right)_{j-k} \right\|_{j,k=m_0}^{n-m_0} = Q_{m_0} T_{g_m^{-1}} Q_{m_0} \,,$$
$$\text{where} \quad Q_{m_0}\xi := \{0,\dots,\xi_{m_0},\dots,\xi_{n-m_0},0,\dots\} \,, \quad \xi = \{\xi_j\}_{j=1}^{\infty}$$

are all invertible. In fact,

$$\|T_a \,|l_2(\mathbb{N})\,\| = \sup\left\{|a(\eta)| \;:\; 0 \le \eta \le 2\pi\right\} \,, \quad \|Q_{m_0}\,|l_2(\mathbb{N})\,\| = 1$$

(see [5, 25]). Since $g_m(\eta) > 0$, there exist $\varepsilon > 0$ and $\mu_0 > 0$ such that

$$\mu_0 g_m^{-1} = 1 - \left(1 - \mu_0 g_m^{-1}\right) \,, \;\; \sup\left\{\left|1 - \mu_0 g_m^{-1}(\eta)\right| \;:\; 0 \le \eta \le 2\pi\right\} \le 1 - \varepsilon$$

Then

$$\left\|\left(I - \mu_0 T^{(m_0)}_{g_m^{-1}}\right) |Q_{m_0} l_2(\mathbb{N})\right\| = \left\|Q_{m_0}\left(I - \mu_0 T^{(m_0)}_{g_m^{-1}}\right) Q_{m_0}\,|l_2(\mathbb{N})\right\|$$
$$= \left\| \; Q_{m_0} T^{(m_0)}_{1-\mu_0 g_m^{-1}} Q_{m_0}\,|l_2(\mathbb{N})\right\| \le \left\|T^{(m_0)}_{1-\mu_0 g_m^{-1}}\,|l_2(\mathbb{N})\right\| \le 1 - \varepsilon < 1$$

and $\mu_0 T^{(m_0)}_{g_m^{-1}}$ is invertible due to Banach's theorem.

This invertibility

$$\det T^{(m_0)}_{g_m^{-1}} \ne 0 \tag{1.35}$$

is important since

$$\left\|b^{(n)}_{mjk}\right\|_{j,k=m_0}^{n-m_0} = T^{(m_0)}_{g_m^{-1}} \left\|d^{(n)}_{mjk}\right\|_{j,k=m_0}^{n-m_0} \tag{1.36}$$

(see (1.26)).

Obviously, $g_m \in C^\infty([0,2\pi])$ and $g_m(0) = g_m(2\pi)$ (i.e. g_m is periodic). In contrast to this the functions

$$a_m(\eta) := \sum_{k=-\infty}^{\infty} \frac{\exp(i\eta k)}{2\pi i} \int\limits_{-\frac{m+1}{2}}^{\frac{m+1}{2}} \frac{\Phi_m(y)dy}{y-k} \,, \quad a_m \in C^\infty([0,2\pi]) \tag{1.37}$$

are smooth $a_m \in C^\infty([0,2\pi])$ and non–periodic: $a_m(0) = -1$, $a_m(2\pi) = 1$ (see [27, Section 2.11]).

Theorem 1.9 *Let $m \in \mathbb{N}$ and solutions $\varphi \in L_{p,\alpha,\beta}(\Im)$ of equation* (1.13) *belong to the space $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ (see Theorem* **??***).*

For the stability of the spline collocation method (1.25) *(i.e. of* (1.26)*) in the space $L_{p,\alpha,\beta}(\Im)$ it is necessary and sufficient that the following conditions hold:*

(i) *equation* (1.13) *has a unique solution in $L_{p,\alpha,\beta}(\Im)$ (equivalently:* (0.1) *has a unique solution in $L_{p,\alpha,\beta}(\Im)$);*

(ii) $\mathcal{A}_0(0,0,\lambda) := c_0(0) + c_1(0)\cot\pi(\frac{i}{p} + i\alpha + \lambda) + \mathcal{M}r(\frac{1}{p} + \alpha - i\lambda) \neq 0$ *for all* $\lambda \in \mathbb{R}$ *(see* (1.2) *for* $\mathcal{M}r(\omega)$*);*

(iii) *the operator*

$$\mathcal{A}_0(0) := c_0(0) + c_1(0)T_{a_m} + G_r + K \tag{1.38}$$

is invertible in the space $l_{p,\alpha}(\mathbb{N})$ *with some (unidentified) compact operator* K *(see Remark 1.11 below);*

(iv) *the operator*

$$\mathcal{A}_0(1) := c_0(1)T_{g_m} - c_1(0)T_{a_m} + G_r + T_{g_m}^{-1}R_m^0 T_{g_m} \tag{1.39}$$

with the finite–dimentional projection $R_m^0\xi := \left\{\xi_1, \ldots, \xi_{\frac{m}{2}}, 0, \ldots\right\}$, $\xi = \{\xi_j\}_{j=1}^{\infty}$ *is invertible in* $l_{p,\alpha}(\mathbb{N})$.

If (1.27) *holds, the conditions of the Theorem are necessary and sufficient for the stability of the spline collocation method* (1.25) *in the Sobolev space* $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ *and the convergence of the approximate solutions is quasi–optimal. That is,*

$$\|(\varphi - \varphi_m^{(n)})|\widetilde{W}^m_{p,\alpha,\beta}(\Im)\| \leq C\mathbf{inf}\left\{\|(\varphi - v)|\widetilde{W}^m_{p,\alpha,\beta}(\Im)\| : v \in S_m^{(n)}(\Im)\right\}$$

with some constant $C > 0$.

Proof see in Section 4.

Theorem 1.10 *The operator* (1.38) *is Fredholm in the space* $l_{p,\alpha}(\mathbb{N})$ *if and only if the following properties hold:*

$$\mathcal{A}_0(0,\mu) \neq 0\,, \quad \mathcal{A}_0(0,0,\lambda) \neq 0\,, \qquad \text{for} \qquad -1 \leq \mu \leq 1\,, \quad \lambda \in \mathbb{R} \tag{1.40}$$

(see Theorem 1.6*). If* (1.40) *is the case, then*

$$\text{Ind}\,\mathcal{A}_0(0) = -\frac{1}{2\pi}\left[\ \arg\mathcal{A}_0(0,\mu)\right]_{\mu\in[-1,1]} - \frac{1}{2\pi}\left[\ \arg\mathcal{A}_0(0,0,\lambda)\right]_{\lambda\in\mathbb{R}}\,. \tag{1.41}$$

The operator (1.39) *is Fredholm in the space* $l_{p,\alpha}(\mathbb{N})$ *if and only if the following properties hold:*

$$\mathcal{A}_0(1,\mu) \neq 0\,, \quad \mathcal{A}_0(1,0,\lambda) \neq 0\,, \qquad \text{for} \qquad -1 \leq \mu \leq 1\,, \quad \lambda \in \mathbb{R} \tag{1.42}$$

(see Theorem 1.6). If (1.42) *is the case, then*

$$\text{Ind}\,\mathcal{A}_0(1) = -\frac{1}{2\pi}\left[\ \arg\mathcal{A}_0(1,\mu)\right]_{\mu\in[-1,1]} - \frac{1}{2\pi}\left[\ \arg\mathcal{A}_0(1,0,\lambda)\right]_{\lambda\in\mathbb{R}}\,. \tag{1.43}$$

Proof see in Section 4.

Remark 1.11 *Theorems 1.9 and 1.10 provide the stability conditions for the spline collocation method* (1.25) *generically, since a Fredholm operator with vanishing index being not invertible is exceptional; but this case still needs a careful treatment. Similar situations are considered in [27, Chapter 4] and in [45, 46, 50].*

In the case when (1.27) *holds, i.e. when* (0.1) *is a pure Mellin convolution equation without the Cauchy kernel, more precise results can be obtained by invoking the "singularity cut–off" technique, suggested in [7, 8, 21].*

2 Mellin convolution operators in Sobolev spaces

2.1 Sobolev spaces

The linear space $C_0^\infty(\mathbb{R})$ of infinitely differentiable functions with compact supports and the linear spaces

$$\begin{aligned} C_0^\infty(\mathbb{R}^+) &:= \{\varphi \in C_0^\infty(\mathbb{R}) : \ \ \mathrm{supp}\,\varphi \in (0,\infty)\}\,, \\ C_0^\infty(\Im) &:= \{\varphi \in C_0^\infty(\mathbb{R}) : \ \ \mathrm{supp}\,\varphi \in (0,1)\}\,, \end{aligned}$$

are dense in the spaces $W^m_{p,\alpha}(\mathbb{R})$, $\widetilde{W}^m_{p,\alpha}(\mathbb{R}^+)$ and in $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ respectively, provided (0.4) holds. The conditions in (0.4) ensure the inclusion $C_0^\infty(\mathbb{R}) \subset W^m_{p,\alpha}(\mathbb{R})$ etc. (see [61]).

The space $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ can also be defined in a different way by using the equivalent norm

$$\|u\|^{[m]}_{p,\alpha,\beta} := \|x^\alpha(1-x)^\beta u\|^{(m)}_p = \sum_{k=0}^m \|\mathcal{D}^k x^\alpha (1-x)^\beta u\|_p\,, \tag{2.1}$$

since the following Lemma holds.

Lemma 2.1 *The norms* (1.12) *and* (2.1) *are equivalent in the space* $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$*: there exists a positive constant* M *such that*

$$M^{-1}\|u\|^{[m]}_{p,\alpha,\beta} \le \|u\|^{(m)}_{p,\alpha,\beta} \le M\|u\|^{[m]}_{p,\alpha,\beta} \tag{2.2}$$

for all $u \in \widetilde{W}^m_{p,\alpha,\beta}(\Im)$.

Proof. Let us begin with an estimate between different weighted norms: let $0 < 1/p + \nu < 1$, $\varphi \in C_0^\infty(\Im)$ and $n \in \mathbb{N}$. Then

$$\|x^{\nu-n}\varphi\|_p \le \frac{\Gamma(\nu+1-n)}{\Gamma(\nu+1)}\|x^\nu \mathcal{D}^n\varphi\|_p\,. \tag{2.3}$$

Inequality (2.3) follows from Hardy's inequality

$$\int\limits_0^\infty |t^{\sigma-1}f(t)|^p dt \le \sigma^{-p}\int\limits_0^\infty |t^\sigma f'(t)|^p dt\,, \qquad \text{where} \quad 1<p<\infty, \quad 0<\sigma<\infty$$

(see [28, Theorem 330],[61, Section 3.2.6, Remark 1]). by n successive applications.

The proof of Lemma 2.1 is based on induction. For $m=0$, inequality (2.2) becomes the equality $\|u\|^{(0)}_{p,\alpha,\beta} = \|u\|^{[0]}_{p,\alpha,\beta} = \|u\|_{p,\alpha,\beta}$.

Suppose (2.2) to be valid for $m-1$. Then, due to (2.1), we have

$$\begin{aligned} \|u\|^{[m]}_{p,\alpha,\beta} &= \|u\|^{[m-1]}_{p,\alpha,\beta} + \|\mathcal{D}^m \varrho u\|_p \\ &\le M_1\|u\|^{(m)}_{p,\alpha,\beta} + M_2\sum_{k=0}^m\left(\left\|\frac{\varrho\mathcal{D}^k u}{x^{m-k}}\right\|_p + \left\|\frac{\varrho\mathcal{D}^k u}{(1-x)^{m-k}}\right\|_p\right) \end{aligned} \tag{2.4}$$

(see (0.1) for $\varrho(x)$). To the second terms we apply (2.3):

$$\left\|\varrho\frac{\mathcal{D}^k u}{x^{m-k}}\right\|_p \leq \left(\int\limits_0^{1/2}|x^{\alpha-m+k}(1-x)^{\beta}\mathcal{D}^k u(x)|^p dx\right)^{\frac{1}{p}} + \left(\int\limits_{1/2}^{1}|x^{\alpha-m+k}(1-x)^{\beta}\mathcal{D}^k u(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq M_3\left(\int\limits_0^{1/2}|x^{\alpha}(1-x)^{\beta}\mathcal{D}^m u(x)|^p dx\right)^{\frac{1}{p}} + 2^{m-k}\|\varrho\mathcal{D}^k\|_{p,\alpha,\beta} \leq M_4\|u\|^{(m)}_{p,\alpha,\beta}\,.$$

A similar estimate is valid at the other endpoint $x=1$ Hence, the left inequality in (2.1) is valid. To prove the right inequality, we apply (2.3) successively k times separately at $x=0$ and at $x=1$ under assumption that (2.2) is valid for $m-1$. After some calculations we get

$$\begin{aligned}\|u\|^{(m)}_{p,\alpha,\beta} &= \|u\|^{(m-1)}_{p,\alpha,\beta} + \|\varrho\mathcal{D}^m u\| \\ &\leq M\|u\|^{[m-1]}_{p,\alpha,\beta} + \|\mathcal{D}^m(\varrho u) - \sum_{k=1}^{m}\mathcal{D}^k\varrho\mathcal{D}^{m-k}u\|_p \\ &\leq (M+1)\|u\|^{[m]}_{p,\alpha,\beta} + M_5\sum_{k=1}^{m}\|\mathcal{D}^k(\varrho\mathcal{D}^{n-k}u)\|_p\,.\end{aligned} \tag{2.5}$$

Repeating successively the estimate in (2.5) $m-1$ times to functions $\varrho\mathcal{D}^{m-k}u$ for $k=1,2,\dots,m-1$, we derive the right inequality in (2.2). $\square$

Similarly to (2.2) it can readily be proved that the norm

$$\|v\|^{[m]}_{p,\alpha} = \sum_{k=0}^{m}\left(\int\limits_0^{\infty}|\mathcal{D}^k[x^{\alpha}v(x)]|^p dx\right)^{\frac{1}{p}} \tag{2.6}$$

is equivalent to $\|v\|^{(m)}_{p,\alpha}$ for any function $v\in\widetilde{W}^m_{p,\alpha}(\mathbb{R}^+)$.

2.2 Properties of the Mellin transform

A further equivalent norm can be introduced for the Hilbert space case, i.e in $\widetilde{W}^m_{2,\alpha,\beta}(\Im)$ with the help of the Mellin transform (see (1.3) for $\mathcal{M}$):

$$\begin{aligned}\|u\|^{\{m\}}_{2,\alpha,\beta}: &= \left\{\int\limits_{\mathbb{R}_{\alpha-m}}(1+|z|^2)^m|(\mathcal{M}v_0u)(z)|^2|dz|\right. \\ &\quad + \left.\int\limits_{\mathbb{R}_{\beta-m}}(1+|z|^2)^m|(\mathcal{M}\Im v_1u)(z)|^2|dz|\right\}^{\frac{1}{2}}.\end{aligned} \tag{2.7}$$

Here $v_0, v_1\in C^{\infty}(\Im)$, $v_1=1-v_0$ are cut–off functions with $v_0(x)=0$ ($v_0(x)=1$) in some neighbourhood of $x=1$ (of $x=0$),

$$\mathcal{I}v(x)=v(1-x) \qquad \text{whereas} \qquad \mathbb{R}_{\gamma}:=\frac{1}{2}+\gamma-i\mathbb{R}=\left\{\frac{1}{2}+\gamma-i\xi:\xi\in\mathbb{R}\right\}. \tag{2.8}$$

The equivalence of the norms in (2.1) (for $p = 2$) and in (2.7) is proved in [20, Section 3.4] and can also be derived from the forthcoming Lemma 2.2(b).

Most of the properties of the Mellin transform which are listed in the following lemma are well–known but dispersed in different publications. We collect them here for convenient use.

Lemma 2.2 *Suppose* (2.1) *and* $\alpha' \in \mathbb{R}\,; k \in \mathbb{N}_0 := \{0, 1, 2, ...\}$.

(a) *(Parseval's equality) If* $u, v \in L_2(\mathbb{R}^+)$ *then*

$$\langle u, v\rangle := \int_0^\infty u(x)\overline{v(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}_O} \mathcal{M}u(z)\overline{\mathcal{M}v(z)}dz$$

where $\mathbb{R}_0 = 1/2 - i\mathbb{R}$ *(cf.* (2.8)*) and* $\mathcal{M}$ *is the Mellin transform (see* (1.2)*).*

(b) *Let* $0 \le k \le m$ *and* $k - m \le \alpha' \le 0$. *Then*

$$\mathcal{M}[\mathcal{D}^k x^{\alpha'} v](z) = \prod_{j=1}^{k}(j - z)\mathcal{M}v(z + \alpha' - k)$$

and $\mathcal{M}[\mathcal{D}^k x^{\alpha'} v] \in L_2(\mathbb{R}_{\alpha-\alpha'+k-m}, (1 + |z|)^{m-k})$ *for any* $v \in \widetilde{W}^m_2(\mathbb{R}^+, x^\alpha)$.

(c) *Assume that*

$$\|r\|^{(\nu)} := \int_0^\infty x^\nu |r(x)|\frac{dx}{x} < \infty \quad \text{and} \quad 0 < \nu - m < 1, \quad m \in \mathbb{N}_0. \tag{2.9}$$

Then the operator

$$\mathcal{R}v(x) := \int_0^\infty r\left(\frac{x}{y}\right) v(y)\frac{dy}{y} \tag{2.10}$$

is bounded in the space : $\widetilde{W}^m_p(\mathbb{R}^+, x^\alpha)$ *provided* $\alpha = \nu - 1/p$. *Moreover,*

$$\mathcal{M}(\mathcal{R}v)(z) = \mathcal{M}r(z)\mathcal{M}v(z) \;\text{ for } v \in C_0^\infty(\mathbb{R}^+)$$
$$\textit{and} \mathcal{M}(\mathcal{R}v) \in L_2(\mathbb{R}_{\alpha-m}\,,\ (1 + |z|)^m)\, \textit{for any} v \in \widetilde{W}^m_2(\mathbb{R}^+, x^\alpha)\,.$$

(d) *The singular integral operator*

$$\begin{aligned} \mathcal{S}^{(m)}_{\mathbb{R}^+} &: \widetilde{W}^m_p(\mathbb{R}^+, x^\alpha) \to \widetilde{W}^m_p\left(\mathbb{R}^+, x^\alpha(1 + x)^{-m}\right) \quad \textit{with} \\ \mathcal{S}^{(m)}_{\mathbb{R}^+}v(x) &:= \frac{1}{\pi i}\int_{\mathbb{R}^+}\left(\frac{x}{y}\right)^m \frac{v(y)dy}{y - x}, \quad \mathcal{S}_{\mathbb{R}^+} := \mathcal{S}^{(0)}_{\mathbb{R}^+}, \quad 0 < \alpha + \frac{1}{p} < 1 \end{aligned} \tag{2.11}$$

is bounded,

$$\mathcal{M}(\mathcal{S}^{(m)}_{\mathbb{R}^+}v)(z) = \coth(\pi i z)\mathcal{M}v(z) \;\text{ for } v \in \widetilde{W}^m_2(\mathbb{R}^+, x^\alpha)$$

and $\mathcal{M}(\mathcal{S}^{(m)}_{\mathbb{R}^+}v) \in L_2(\mathbb{R}_{\alpha-m})$ *for any* $v \in \widetilde{W}^m_2(\mathbb{R}^+, x^\alpha)$.

Proof:

(a) Let $u, v \in C_0^\infty(\mathbb{R}^+)$ and recall Parseval's equality for the Fourier transform; then

$$\begin{aligned}
\langle u, v\rangle &= \int_{-\infty}^{\infty} e^{-\frac{t}{2}} u(e^{-t}) e^{-\frac{t}{2}} \overline{v(e^{-t})} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\lambda - \frac{\lambda}{2}} u(e^{-\lambda}) d\lambda \overline{\int_{-\infty}^{\infty} e^{it\lambda - \frac{\lambda}{2}} v(e^{-\lambda}) d\lambda} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} x^{\frac{1}{2-it}} u(x) \frac{dx}{x} \overline{\int_{0}^{\infty} x^{\frac{1}{2}-it} v(x) \frac{dx}{x}} dt \\
&= \frac{1}{2\pi} \int_{\Im_0} \mathcal{M}u(z) \overline{\mathcal{M}v(z)} dz\,.
\end{aligned}$$

Extension to any $u, v \in L_2(\mathbb{R}^+)$ follows by continuity.

(b) Let $u(x) := v(x) := x^\alpha u_0(x)$ and $u_0 \in L_2(\mathbb{R}^+, x^\alpha)$, then (see (a))

$$\|\mathcal{M}u_0 | L_2(\mathbb{R}_\alpha)\| = \|\mathcal{M}u | L_2(\mathbb{R}_0)\| = \|u\|_2 = \|u_0\|_{2,\alpha}\,. \tag{2.12}$$

Due to (2.1) $x^{n-m}\mathcal{D}^m v \in L_2(\mathbb{R}^+, x^\alpha)$ for $n = 0, 1, \ldots, m$; therefore,

$$w_0(x) = (\mathcal{D}^k x^{\alpha'} v)(x) = \sum_{n=0}^{k} c_n x^{\alpha' - (k-n)} \mathcal{D}^n v(x) = \sum_{n=0}^{k} c_n x^{\alpha' - (k-n)} \frac{\mathcal{D}^n v(x)}{x^{m-n}}$$

and since $\alpha' - (k - n) \le \alpha' \le 0$, we have $w_0 \in L_2(\mathbb{R}^+, x^\alpha)$. Then $\mathcal{M}w_0 \in L_2(\mathbb{R}_\alpha)$ due to (2.12). Integration by parts gives

$$\mathcal{M}(\mathcal{D}^k x^{\alpha'} v)(z) = \prod_{j=1}^{k} (j - z) \mathcal{M}(x^{\alpha'} v)(z - k) = \prod_{j=1}^{k} (j - z) \mathcal{M}v(z + \alpha' - k)\,.$$

If $k = m, \alpha' = 0$, this identity reads

$$\mathcal{M}(\mathcal{D}^m v)(z) = \prod_{j=1}^{m} (j - z) \mathcal{M}v(z - m)\,,$$

and, due to (2.12), $\mathcal{M}v \in L_2(\mathbb{R}_{\alpha-m}\,, (1 + |z|)^m)$, since $\mathcal{D}^m v \in L_2(\mathbb{R}^+, x^\alpha)$ by definition. This results in

$$\begin{aligned}
&(1 + |z - \alpha' + k|)^{m-k} \mathcal{M}(\mathcal{D}^k x^{\alpha'} v)(z - \alpha' + k) \\
&\quad = \frac{\prod_{j=1}^{k} (j - z + \alpha' - k)}{(1 + |z - \alpha' + k|)^k} (1 + |z - \alpha' + k|)^m \mathcal{M}v(z) \in L_2(\mathbb{R}_{\alpha-m})\,,
\end{aligned}$$

since the rational multiplier is bounded. Therefore,

$$(1 + |z|)^{m-k} \mathcal{M}(\mathcal{D}^k x^{\alpha'} v)(z) \in L_2(\mathbb{R}_{\alpha - \alpha' + k - m})\,.$$

(c) Let $v \in C_0^\infty(\mathbb{R}^+)$. Integration by parts yields for $0 < k \leq m$

$$
\begin{aligned}
\mathcal{D}^k \mathcal{R} v(x) &= \mathcal{D}^k \left[\int_0^\infty v(y) \frac{d}{dy} \left(\int_0^y r\left(\frac{x}{t}\right) \frac{dt}{t} \right) dy \right] \\
&= -\mathcal{D}^k \left[\int_0^\infty v'(y) dy \int_{x/y}^\infty r(\tau) \frac{d\tau}{\tau} \right] = \mathcal{D}^{k-1} \int_0^\infty \frac{y}{x} r\left(\frac{x}{y}\right) \mathcal{D} v(y) \frac{dy}{y} \\
&= \cdots = \int_0^\infty \left(\frac{y}{x}\right)^k r\left(\frac{x}{y}\right) \mathcal{D}^k v(y) \frac{dy}{y} := (\mathcal{R}_k \mathcal{D}^k v)(x) \,. \qquad (2.13)
\end{aligned}
$$

The kernel $r(x)$ can be approximated by smooth functions in the norm $\|\cdot\|_0^{(\alpha-m)}$. Due to (0.2) and (2.9) this implies $r(0) = \cdots = \mathcal{D}^{m-1} r(0) = 0$. Therefore,

$$
\mathcal{D}^k \mathcal{R} v(0) = (\mathcal{D}^k r)(0) \int_0^\infty \frac{v(y)}{y^k} \frac{dy}{y} = 0 \quad \text{for } k = 0, \ldots, m-1 \,.
$$

Since

$$
\|\mathcal{R} v\|_{p,\alpha} \leq \|r\|^{(\alpha)} \|v\|_{p,\alpha} \,,
$$

(see [12, Section 12]), we get

$$
\begin{aligned}
\|\mathcal{R} v\|_{p,\alpha}^{(m)} &= \sum_{k=0}^m \|\mathcal{D}^k \mathcal{R} v\|_{p,\alpha} = \sum_{k=0}^m \|\mathcal{R}_k \mathcal{D}^k v\|_{p,\alpha} \\
&\leq \sum_{k=0}^m \|r\|^{(\alpha-k)} \|\mathcal{D}^k v\|_{p,\alpha} \leq \|r\|^{(\alpha-m)} \|v\|_{p,\alpha}^{(m)} \,.
\end{aligned}
$$

The boundedness of $\mathcal{R}$ in (2.10) follows since $C_0^\infty(\mathbb{R}^+)$ is dense in $\widetilde{W}_p^m(\mathbb{R}^+, x^\alpha)$. Therefore, $\mathcal{R} v \in \widetilde{W}_2^m(\mathbb{R}^+, x^\alpha)$ for $v \in \widetilde{W}_2^m(\mathbb{R}^+, x^\alpha)$ and (b) for $\alpha' = k = 0$ yields $\mathcal{M}(\mathcal{R} v) \in L_2(\mathbb{R}_{\alpha-m}, (1+|z|)^m)$.

The identity $\mathcal{M}(\mathcal{R} v) = \mathcal{M} r \cdot \mathcal{M} v$ follows immediately from the definitions (0.2) and (2.10).

(d) From (2.3) we get

$$
\begin{aligned}
\left| \int_0^\infty \frac{v(y)}{y^{k+1}} dy \right| &\leq \int_0^1 \frac{|v(y)| dy}{y^{k+1}} + \int_1^\infty \frac{|v(y)| dy}{y^{k+1}} \\
&\leq \left(\int_0^1 \left| \frac{v(y)}{y^{k+1}} \right|^p dy \right)^{\frac{1}{p}} + \left(\int_1^\infty |y^\alpha v(y)|^p dy \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-p'(\alpha+k+1)} dy \right)^{\frac{1}{p'}} \\
&\leq C_1 \|v\|_{p,\alpha}^{(m)} \quad \text{for} \quad k = 0, \ldots, m-1 \quad \text{and } p' = 1 - 1/p \,.
\end{aligned}
$$

With the identity

$$
\mathcal{S}_{\mathbb{R}^+}^{(m)} u(x) = \frac{1}{\pi i} \int_0^\infty \frac{u(y) dy}{y - x} - \sum_{k=0}^{m-1} \frac{x^k}{\pi i} \int_0^\infty \frac{u(y)}{y^{k+1}} dy \,, \qquad (2.14)
$$

and together with the well–known boundedness of the Cauchy singular integral operator $\mathcal{S}_{\mathbb{R}^+} = \mathcal{S}^{(0)}_{\mathbb{R}^+}$ in $L_p(\mathbb{R}^+, x^\alpha)$ (see [24, Section 12]) this gives

$$\|(1+x)^{-m}\mathcal{S}^{(m)}_{\mathbb{R}^+}v\|_{p,\alpha} \le \|\mathcal{S}_{\mathbb{R}^+}v\|_{p,\alpha} + \frac{m}{\pi}C_1\|v\|^{(m)}_{p,\alpha} \le C_2\|v\|^{(m)}_{p,\alpha} \tag{2.15}$$

Equation (2.14) can be rewritten in the form

$$\mathcal{S}^{(m)}_{\mathbb{R}^+}v(x) = \mathcal{S}_{\mathbb{R}^+}v(x) - \sum_{k=0}^{m-1}\frac{1}{k!}x^k\mathcal{D}^k(\mathcal{S}_{\mathbb{R}^+}v)(0)\,, \tag{2.16}$$

and since

$$\mathcal{D}^k\mathcal{S}_{\mathbb{R}^+}v = \mathcal{S}_{\mathbb{R}^+}\mathcal{D}^k v \quad \text{ for } \ v \in C^\infty_0(\mathbb{R}^+)\,, \tag{2.17}$$

with (2.16) this yields the existence of $\mathcal{D}^k(\mathcal{S}^{(m)}_{\mathbb{R}^+}u)(x)$ and the equations

$$\mathcal{D}^l\mathcal{S}^{(m)}_{\mathbb{R}^+}u(0) = 0 \ \text{ for } \ k = 0,1,\dots,m \ \text{ and } \ l = 0,1,\dots,m-1\,. \tag{2.18}$$

The boundedness in (2.11) results from (2.15) and (2.19).

If $v \in \widetilde{W}^m_2(\mathbb{R}^+, x^\alpha)$ then $x^{-m}v \in L_2(\mathbb{R}^+, x^\alpha)$ and $\mathcal{S}_{\mathbb{R}^+}(y^{-m}v) \in L_2(\mathbb{R}^+, x^\alpha)$.

If we apply (b), with the choice $\alpha' = k = m = 0$, we get

$$\begin{aligned}
\mathcal{M}(\mathcal{S}^{(m)}_{\mathbb{R}^+}v)(\cdot) &= \mathcal{M}(\mathcal{S}_{\mathbb{R}^+}y^{-m}v)(\cdot + m) \in L_2(\mathbb{R}_{\alpha-m}) \ \text{ and}\\
\mathcal{M}(\mathcal{S}^{(m)}_{\mathbb{R}^+}v)(z) &= \coth\pi i(z+m)\mathcal{M}(y^{-n}v)(z+m)\\
&= \coth\pi iz\mathcal{M}v(z) \in L_2(\mathbb{R}_{\alpha-n}, (1+|z|)^m)\,,
\end{aligned}$$

since $\coth\pi iz = \coth\pi i(z-m)$ is bounded for $z \in \mathbb{R}_{\alpha-m}$ and $\mathcal{M}\mathcal{S}_{\mathbb{R}^+}v(z) = \coth\pi iz\mathcal{M}v(z)$. □

Lemma 2.3 *Let $\nu_0 \in \mathbb{R}$ and*

$$\begin{aligned}
M_1(r) &:= \int_0^\infty \left|x^{\nu_0-\frac{1}{2}}(i+\ln x)r(x)\right|^2 dx < \infty,\\
M_2(r) &:= \int_0^\infty \left|x^{\nu_0+\frac{1}{2}}\ln x\mathcal{D}r(x)\right|^2 dx < \infty\,;
\end{aligned} \tag{2.19}$$

then $\mathcal{M}r(\nu - i\cdot) \in V_1(\mathbb{R})$, where $V_1(\mathbb{R})$ stands for the space of functions with finite total variation on the entire real line $\mathbb{R}$.

If, in particular, conditions (0.3) *on $r(x)$ hold, then* (2.19) *holds as well and $\mathcal{M}r(\nu - i\cdot) \in V_1(\mathbb{R})$.*

Proof. Since

$$\mathcal{M}r(\omega) := \int_0^\infty t^\omega r(t)\frac{dt}{t}\,, \qquad \omega = \nu_0 - i\lambda\,,$$

(see (1.2)), we have

$$\begin{aligned}
\left(\frac{1}{2}-i\lambda\right) \quad \mathcal{D}_\lambda \quad & \mathcal{M}r(\nu_0-i\lambda) - i\left(\frac{1}{2}-i\lambda\right)\int\limits_0^\infty x^{\frac{1}{2}-i\lambda-1}\left[x^{\nu_0-\frac{1}{2}}r(x)\ln x\right]dx \\
= \quad & -i\int\limits_0^\infty \mathcal{D}_x\left(x^{\frac{1}{2}-i\lambda}\right)\left[x^{\nu_0-\frac{1}{2}}r(x)\ln x\right]dx = i\int\limits_0^\infty x^{\frac{1}{2}-i\lambda}\mathcal{D}_x\left[x^{\nu_0-\frac{1}{2}}r(x)\ln x\right]dx \\
= \quad & i\int\limits_0^\infty x^{\frac{1}{2}-i\lambda}\left\{x^{\nu_0-\frac{1}{2}}\left[\left(\nu_0-\frac{1}{2}\right)\ln x+1\right]r(x)+x^{\nu_0+\frac{1}{2}}\ln x\mathcal{D}_x r(x)\right\}\frac{dx}{x} =: g(\lambda).
\end{aligned}$$

Then $g \in L_2(\mathbb{R})$ due to the Parseval's equality in Lemma 2.2(a) and to condition (2.19). Therefore

$$\mathcal{D}_\lambda\mathcal{M}r(\nu-i\cdot) = \left(\frac{1}{2}-i\cdot\right)^{-1}g(\lambda)\,, \quad g\in L_2(\mathbb{R})$$

and, by Schwartz's inequality,

$$\|\mathcal{D}_\lambda\mathcal{M}r(\nu-i\cdot\|_1 \le \|\left(\frac{1}{2}-i\cdot\right)^{-1}\|_2\|g\|_2 \le \infty.$$

Since $V_1(\mathcal{M}r) \le \|\mathcal{D}_\lambda\mathcal{M}r\|_1 < \infty$ we have proved the claimed inclusion $\mathcal{M}r(\nu-i\cdot) \in V_1(\mathbb{R})$. $\square$

2.3 Mellin convolution operators

Let us start with the boundedness of operator $\mathcal{A}$ in (0.1).

Theorem 2.4 *Let conditions* (0.2)–(0.4) *hold with*

$$\nu_0 = \frac{1}{p}+\alpha-m, \quad m\in N_0\,. \tag{2.20}$$

If $c_0, c_1 \in C^m(\Im)$, $x^j(1-x)^j\mathcal{D}^jk_0 \in C^m(\Im\times\Im)$, $k=0,1,\cdots,m$ *(see* (0.2)*), then the operator*

$$\mathcal{A}: \widetilde{W}^m_{p,\alpha,\beta}(\Im) \to W^m_{p,\alpha,\beta}(\Im)\,, \tag{2.21}$$

defined in (0.1)*, is bounded.*

Proof. Since $\mathcal{D}^k\mathcal{S}_\Im u = \mathcal{S}_\Im\mathcal{D}^k u$, where

$$S_\Im u(x) := \frac{1}{\pi i}\int\limits_0^1\frac{u(y)dy}{y-x}\,, \qquad x\in\Im \tag{2.22}$$

for $u\in\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ (see (2.17) and (1.11) and since $\mathcal{S}_\Im$ is a bounded operator in $L_{p,\alpha,\beta}(\Im)$, we have

$$\|\mathcal{S}_\Im u\|^{(m)}_{p,\alpha,\beta} \le \|\mathcal{S}_\Im\|_{p,\alpha,\beta}\|u\|^{(m)}_{p,\alpha,\beta}\,.$$

Thus, we need to consider the operator $\mathcal{R}_{\Im} := \mathcal{R}_{\Im,o}$ only, where

$$\mathcal{R}_{\Im,k}u(x) := \int_0^1 \left(\frac{y}{x}\right)^k r\left(\frac{x}{y}\right) u(y)\frac{dy}{y} \quad \text{for } k = 0, 1, \ldots, m\,. \tag{2.23}$$

Since with (2.13)

$$\mathcal{D}^k \mathcal{R}_{\Im} u = \mathcal{R}_{\Im,k}\mathcal{D}^k u \quad \text{for } u \in \widetilde{W}^m_{p,\alpha,\beta}(\Im) \tag{2.24}$$

it suffices to show that the $\mathcal{R}_{\Im,k}$ are bounded operators in $L_{p,\alpha,\beta}(\Im)$. For this purpose, the operator

$$\mathcal{Z}_\alpha u(\xi) := e^{-(\frac{1}{p}+\alpha)\xi} u(e^{-\xi})$$

can be applied which defines the isomorphism

$$\mathcal{Z}_\alpha : L_{p,\alpha,\beta}(\Im) \to L_p(\mathbb{R}^+, (1-e^{-\xi})^\beta) = L_p(\mathbb{R}^+, \xi^\beta(1+\xi)^{-\beta})$$

(the spaces coincide since the weights on the right–hand side are equivalent); $\mathcal{Z}_\alpha$ has the inverse

$$\mathcal{Z}^{-1}_\alpha v(x) = x^{-(\frac{1}{p}+\alpha)} v(-\ln x)\,.$$

The transformed operator

$$\mathcal{Z}_\alpha \mathcal{R}_{\Im,k} \mathcal{Z}^{-1}_\alpha v(\xi) = \int_0^\infty g_k(\xi-\eta)v(\eta)d\eta \qquad \text{with} \qquad g_k(\xi) = r(e^{-\xi})e^{(k-\frac{1}{p}-\alpha)\xi} \tag{2.25}$$

is a Fourier convolution whose boundedness in the space $L_p\left(\mathbb{R}^+, \xi^\beta(1+\xi)^{-\beta}\right)$ follows, if the Fourier transformed kernel

$$\begin{aligned}\mathcal{F}g_k(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda\xi} g_k(\xi)d\xi = \int_{-\infty}^{\infty} e^{i\lambda\xi+(k-\frac{1}{p}-\alpha)\xi} r(e^{-\xi})d\xi \\ &= -\int_0^\infty x^{\frac{1}{p}+\alpha-k-i\lambda} r(x)\frac{dx}{x} = -\mathcal{M}r(\frac{1}{p}+\alpha-k-i\lambda)\end{aligned}$$

(see ((1.2)) for $\mathcal{M}$) has a bounded variation $\mathcal{F}g_k \in V_1(\mathbb{R})$(see [14, Theorem 1] and [54]). This is guaranteed by Theorem 2.4 since conditions (0.3) and (2.20) hold. $\square$

Remark 2.5 *In Theorem* 2.4 *we have proved more than claimed: if conditions* (0.2), (0.4), (2.20) *hold and the Mellin transfom* $\mathcal{M}r(\nu - i\lambda)$ *has a bounded variation on* $\mathbb{R}$ *(i.e* $\mathcal{M}r(\nu - i\lambda) \in V_1(\mathbb{R})$*), then operator* (0.1) *is bounded in* $L_{p,\alpha,\beta}(\Im)$ *provided* $c_0, c_1 \in C^m(\Im),\ \ k_0 \in C^m(\Im \times \Im)$.

Lemma 2.6 *If* (0.4) *and* (2.19) *hold, then*

$$\|\mathcal{R}u\|_{p,\alpha,\beta} \le M\left[M_1(r) + M_2(r)\right]\|u\|_{p,\alpha,\beta} \tag{2.26}$$

(see (2.10) *for* $\mathcal{R}u$*) with a constant* M *independent of* $u \in L_{p,\alpha,\beta}(\Im)$.

Proof. Let $\mathcal{Z}_\alpha, g_k$ be as in (2.23)– (2.25) and recall, that

$$\|\mathcal{Z}_\alpha \mathcal{R}_\Im \mathcal{Z}_\alpha^{-1} u\|_{p,\alpha} \leq M\left(\|\mathcal{F}g_0 \,|\, L_\infty(\mathbb{R})\| + V_1(\mathcal{F}g_1)\right)$$

(see [14, 55]); since

$$\begin{aligned} \|\mathcal{F}g_0 \,|\, L_\infty(\mathbb{R})\| = \sup\{|\mathcal{F}g_0(\lambda)| : \lambda \in \mathcal{R}\} &= \sup\left\{\left|\mathcal{M}r(\frac{1}{p} + \alpha - i\lambda)\right| : \lambda \in \mathbb{R}\right\} \\ &\leq \|t^{-1}(i + \ln t)^{-3}\|_2 M_1(r) \leq M_0 M_1(r)\,. \end{aligned}$$

we obtain (2.26). □

Lemma 2.7 *Let $1 < p < \infty$, $0 < \alpha + 1/p < 1$ and γ_0, γ_1 are such that*

$$0 < \alpha + 1/p + \gamma_0, \quad \alpha + 1/p + \gamma_0 + \gamma_1 < 0\,. \tag{2.27}$$

If a function $g(x, y)$ has the estimate

$$|g(x, y)| \leq M\left(\frac{x}{y}\right)^{\gamma_0}\left(1 + \frac{x}{y}\right)^{\gamma_1}, \quad x, y \in \mathbb{R}\,, \quad 0 < M < \infty\,, \tag{2.28}$$

then the operator

$$G\varphi(x) := \int_0^\infty g(x, y)\varphi(y)\frac{dy}{y}\,, \qquad G : L_{p,\alpha}(\mathbb{R}^+) \to L_{p,\alpha}(\mathbb{R}^+) \tag{2.29}$$

is bounded.

Proof. We shall prove that the operator

$$G_\alpha\varphi(x) := \int_0^\infty \left(\frac{x}{y}\right)^\alpha g(x, y)\varphi(y)\frac{dy}{y}$$

is bounded in $L_p(\mathbb{R}^+)$, which is, obviously, the same. Let us choose $\mu = 1/pp'$, $p' = p/(p-1)$; applying the Hölder inequality we proceed as follows

$$\begin{aligned} \|G_\alpha\varphi\|_p &= \left(\int_0^\infty \left|\int_0^\infty \left(\frac{x}{y}\right)^\alpha g(x, y)\varphi(y)\frac{dy}{y}\right|^p dx\right)^{\frac{1}{p}} \\ &\leq \left[\int_0^\infty \left(\int_0^\infty \left|\left(\frac{x}{y}\right)^\alpha g(x, y)\frac{1}{y}\right|^{\frac{1}{p'} + \frac{1}{p}} |\varphi(y)| dy\right)^p dx\right]^{\frac{1}{p}} \\ &\leq \left[\int_0^\infty \left(\int_0^\infty \left(\frac{x}{y}\right)^{\alpha + \mu p'} |g(x, y)|\frac{dy}{y}\right)^{\frac{p}{p'}} \int_0^\infty \left(\frac{x}{y}\right)^{\alpha - \mu p} |g(x, y)||\varphi(y)|^p \frac{dy}{y} dx\right]^{\frac{1}{p}} \end{aligned}$$

$$\leq \quad M\left[\int_0^\infty\left(\int_0^\infty\left(\frac{x}{y}\right)^{\alpha+\frac{1}{p}+\gamma_0}\left(1+\frac{x}{y}\right)^{\gamma_1}\frac{dy}{y}\right)^{\frac{p}{p'}}\times\right.$$

$$\left.\times\int_0^\infty\left(\frac{x}{y}\right)^{\alpha-\frac{1}{p'}+\gamma_0}\left(1+\frac{x}{y}\right)^{\gamma_1}|\varphi(y)|^p\frac{dy}{y}dx\right]^{\frac{1}{p}}$$

$$\leq \quad M\left[\int_0^\infty|\varphi(y)|^p dy\left(\int_0^\infty t^{\alpha+\frac{1}{p}+\gamma_0-1}\left(1+t\right)^{\gamma_1}dt\right)^{\frac{p}{p'}}\int_0^\infty t^{\alpha+\frac{1}{p}+\gamma_0-1}(1+t)^{\gamma_1}dt\right]^{\frac{1}{p}}$$

$$\leq M\int_0^\infty|\varphi(y)|^p dy\int_0^\infty t^{\alpha+\frac{1}{p}+\gamma_0-1}\left(1+t\right)^{\gamma_1}dt \leq M_1\|\varphi\|_p$$

(see condition (2.27)). □

Lemma 2.8 *Let* (2.26) *hold and*

$$|r(x)| \leq Mx^{\gamma_0}(1+x)^{\gamma_1} \tag{2.30}$$

(*cf.* (0.3),(0.4)). *Then the discretized Mellin convolution operator*

$$G_r := \left\|r\left(\frac{j}{k}\right)\frac{1}{k}\right\|_{j,k\in\mathcal{N}} : \ell_{p,\alpha}(\mathcal{N}) \to \ell_{p,\alpha}(\mathcal{N}) \tag{2.31}$$

is bounded.

Proof. Let $\xi = \{\xi_j\}_{j=1}^\infty \in \ell_{p,\alpha}(\mathcal{N})$ and define

$$[S(\xi)]^p := \sum_{j=1}^\infty\left|\sum_{k=1}^\infty\left(\frac{j}{k}\right)^\alpha r\left(\frac{j}{k}\right)\frac{k^\alpha\xi_k}{x}\right|^p.$$

The lemma would be proved if there is a constant M_0, independent of ξ, such that

$$[S(\xi)]^p \leq M_0^p\|\xi\|_{p,\alpha}^p = M_0^p\sum_{k=1}^\infty|k^\alpha\xi_k|^p.$$

From (2.27) we see that $\gamma_1 < 0$. Choose any $0 < \varepsilon < \min\{-\gamma-1, \alpha+1/p+\gamma_0\}$ (see (2.27)). Now

$$\begin{aligned}
[S(\xi)]^p \quad &\leq \quad M^p\sum_{j=1}^\infty\left[\sum_{k=1}^\infty\left(\frac{j}{k}\right)^{\alpha+\gamma_0}\left(1+\frac{j}{k}\right)^{\gamma_1}\frac{k^\alpha|\xi_k|}{k}\right]^p \\
&= \quad M^p\sum_{j=1}^\infty j^{p(\alpha+\gamma_0)}\left(\sum_{k=1}^\infty\frac{(j+k)^{\gamma_1+2\varepsilon}}{k^{\alpha+\gamma_0+\gamma_1+1+\varepsilon}}-\frac{k^{\alpha+\varepsilon}|\xi_k|}{(j+k)^{2\varepsilon}}\right)^p.
\end{aligned}$$

By Hölders inequality we have

$$[S(\xi)]^p \leq M^p\sum_{j=1}^\infty j^{p(\alpha+\gamma_0)}\left(\sum_{k=1}^\infty\frac{(j+k)^{(\gamma_1+2\varepsilon)p'}}{k^{(\alpha+\gamma_0+\gamma_1+1+\varepsilon)p'}}\right)^{p/p'}\times\sum_{k=1}^\infty\frac{k^{p(\alpha+\varepsilon)}|\xi_k|^p}{(j+k)^{2p\varepsilon}}. \tag{2.32}$$

Obviously,

$$S_1(j) := \sum_{k=1}^{\infty} \frac{(j+k)^{(\gamma_1+2\varepsilon)p'}}{k^{(\alpha+\gamma_0+\gamma_1+1+\varepsilon)p'}} \leq \int_{-\infty}^{\infty} \frac{(j+x)^{(\gamma_1+2\varepsilon)p'}}{x^{(\alpha+\gamma_0+\gamma_1+1+\varepsilon)p'}} dx < \infty$$

provided

$$-(\alpha+\gamma_0+\gamma_1+\varepsilon)p' > -1 \quad \text{and} \quad -(\alpha+\gamma_0+\gamma_1-\varepsilon)p' < -1. \tag{2.33}$$

These inequalities can be equivalently written as follows

$$\alpha+\gamma_0+\gamma_1+\frac{1}{p} < -\varepsilon < 0 \quad \text{and} \quad 0 < \varepsilon < \alpha+\gamma_0+\frac{1}{p}$$

and (2.33) follow from (2.27) and choice of ε.

By inserting $x = jt$ we find: there exists a constant A_1 such that

$$S_1(j) \leq A_1^{p'} j^{-(\alpha+\gamma_0+1-\varepsilon)p'+1}. \tag{2.34}$$

From (2.32) and (2.34),

$$\begin{aligned} [S(\xi)]^p &\leq (MA_1)^p \sum_{j=1}^{\infty} j^{p\varepsilon-1} \sum_{k=1}^{\infty} \frac{k^{p(\alpha+\varepsilon)}|\xi_k|^p}{(j+k)^{2p\varepsilon}} \\ &= (MA_1)^p \sum_{k=1}^{\infty} k^{p(\alpha+\varepsilon)}|\xi_k|^p \sum_{j=1}^{\infty} \frac{j^{p\varepsilon-1}}{(j+k)^{2p\varepsilon}}. \end{aligned}$$

Define

$$S_2(k) := \sum_{j=1}^{\infty} j^{p\varepsilon-1}(j+k)^{-2p\varepsilon}.$$

Arguing as for $S_1(j)$ we have

$$S_2(k) \leq \int_{p}^{\infty} x^{p\varepsilon-1}(k+x)^{-2p\varepsilon} dx$$

and since $p\varepsilon - 1 > -1$ and $-1 - p\varepsilon < -1$ the integral is finite. By inserting $x = k\tau$ we find: there exists a constant A_2 such that

$$S_2(k) \leq A_2^p k^{-p\varepsilon}. \tag{2.35}$$

Inequality (2.35) yields

$$[S(\xi)]^p \leq (MA_1A_2)^p \sum_{k=1}^{\infty} (k^\alpha |\xi_k|)^p,$$

so that (2.31) follows at once with $M_0 = MA_1A_2$ and the lemma is proved. □

Remark 2.9 *It is proved in [46, 47, 51] and in [27, Section 2.47] that the discretized Mellin convolution operator* (1.30) *belongs to the Banach algebra generated by Töplitz operators* (1.31) *in the space* $\ell_{p,\alpha}(\mathcal{N})$. *We have given the proof of boundedness in Lemma 2.8 because we have not found a relevant reference for this assertion.*

Let us consider the following operators

$$
\begin{aligned}
\widetilde{S}_{\Im,\theta,\alpha,\beta}\varphi(x) &:= \frac{1}{\pi i}\int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)\alpha}\left(\frac{1-x}{1-y}\right)^{(\theta-1)\beta}\left[\frac{\sigma_\theta'(x)}{\sigma_\theta'(y)}\right]^{\frac{1}{p}}\frac{\sigma_\theta'(y)\varphi(y)dy}{\sigma_\theta(y)-\sigma_\theta(x)},\\
S^0_{\mathbb{R}^+,\theta,\gamma}\varphi(x) &:= \frac{1}{\pi i}\int_0^\infty \left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\gamma)}\frac{\theta y^{\theta-1}\varphi(y)dy}{y^\theta-x^\theta},\\
\widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta}\varphi(x) &:= \frac{1}{\pi i}\int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)\alpha}\left(\frac{1-x}{1-y}\right)^{(\theta-1)\beta}\left[\frac{\sigma_\theta'(x)}{\sigma_\theta'(y)}\right]^{\frac{1}{r}} r\left(\frac{\sigma_\theta(x)}{\sigma_\theta(y)}\right)\times\frac{\sigma_\theta'(y)}{\sigma_\theta(y)}\varphi(y)dy,\\
\mathcal{R}^0_{\mathbb{R}^+,\theta,\gamma}\varphi(x) &:= \int_0^\infty \left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\gamma)} r\left(\left(\frac{x}{y}\right)^\theta\right)\varphi(y)\frac{dy}{y};
\end{aligned}
\tag{2.36}
$$

we shall use $v_0\in C^\infty(\Im)$ to denote the cut-off function defined in (2.7) and $v_t\in C^\infty(\Im)$, $0<t<1$, a similar cut-off function which is equal 1 in some neighbourhood of $t\in(0,1)$ and vanishes in some neighbourhood of the endpoints $0,1\in\Im$.

Theorem 2.10 *Let conditions* (0.3),(0.4) *hold. Then operators*

$$\widetilde{S}_{\Im,\theta,\alpha,\beta},\ \widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta} : L_{p,\alpha,\beta}(\Im)\longrightarrow L_{p,\alpha,\beta}(\Im), \tag{2.37}$$

$$S^0_{\mathbb{R}^+,\theta,\gamma},\ \mathcal{R}^0_{\mathbb{R}^+,\theta,\gamma} : L_{p,\alpha}(\mathbb{R}^+)\longrightarrow L_{p,\alpha}(\mathbb{R}^+), \tag{2.38}$$

are bounded and operators

$$
\begin{aligned}
T_0 &:= v_0(\widetilde{S}_{\Im,\theta,\alpha,\beta}-S^0_{\Im,\theta,\alpha})|_{L_{p,\alpha,\beta}(\Im)},\\
T_1 &:= v_0\mathcal{I}(\widetilde{S}_{\Im,\theta,\alpha,\beta}-S^0_{\Im,\theta,\beta})\mathcal{I}|_{L_{p,\alpha,\beta}(\Im)},\\
T_2 &:= v_t(\widetilde{S}_{\Im,\theta,\alpha,\beta}-S_{\mathbb{R}})|_{L_{p,\alpha,\beta}(\Im)},\\
T_3 &:= a\mathcal{R}_{\mathbb{R}^+,\theta,\alpha}|_{L_{p,\alpha,\beta}(\Im)},\\
T_4 &:= (\widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta}-\theta\mathcal{R}_{\mathbb{R}^+,\theta,\alpha}|L_{p,\alpha,\beta(\Im)}
\end{aligned}
\tag{2.39}
$$

are all compact in $L_{p,\alpha,\beta}(\Im)$ *provided at* $a\in C(\Im)$ *and* $a(0)=0$ *(see* (2.8) *for* $\mathcal{I}$*).*

Proof. Clearly

$$\widetilde{S}_{\Im,\theta,\alpha,\beta}=V_{\theta,\alpha,\beta}S_\Im V^{-1}_{\theta,\alpha,\beta},\quad \mathcal{R}_{\Im,\theta,\alpha,\beta}=V_{\theta,\alpha,\beta}\mathcal{R}_\Im V^{-1}_{\theta,\alpha,\beta} \tag{2.40}$$

(see (1.7), (1.8)). Since operators $S_\Im,\mathcal{R}_\Im,V^{\pm}_{\theta,\alpha,\beta}$ are bounded in $L_{p,\alpha,\beta}$ (see (1.9) and Theorem 2.4), boundedness of operators (2.37) is evident.

Similarly,

$$\begin{array}{l} S^0_{\mathbb{R}^+,\theta,\gamma} = V_{\theta,\gamma} S_{\mathbb{R}^+} V^{-1}_{\theta,\gamma}, \qquad \gamma = \alpha, \beta \\ \mathcal{R}^0_{\mathbb{R}^+,\theta,\gamma} = V_{\theta,\gamma} \mathcal{R}_{\mathbb{R}^+} V^{-1}_{\theta,\gamma}, \end{array} \tag{2.41}$$

where

$$V_{\theta,\gamma}\varphi(x) := x^{(\theta-1)(\frac{1}{p}+\gamma)}\varphi(x^\theta), \qquad V^{-1}_{\theta,\gamma} = V_{\frac{1}{\theta},\gamma} \tag{2.42}$$

and

$$V_{\theta,\gamma}, V^{-1}_{\theta,\gamma} : L_{p,\gamma}(\mathbb{R}^+) \longrightarrow L_{p,\gamma}(\mathbb{R}^+) \tag{2.43}$$

are automorphisms. Operators $S_{\mathbb{R}^+}$ and $\mathcal{R}_{\mathbb{R}^+}$ are bounded in $L_{p,\gamma}(\mathbb{R}^+)$ ($\gamma = \alpha, \beta$; see [5, 12, 24] and Theorem 2.4) and boundedness of operators (2.38) is also evident.

Since

$$\sigma'_\theta(x) = \frac{\theta x^{\theta-1}(1-x)^{\theta-1}}{[x^\theta + (1-x)^\theta]^2} = x^{\theta-1}(1-x)^{\theta-1} g_\theta(x), \quad g_\theta \in C^\infty(\Im), \tag{2.44}$$

$$|\sigma_\theta(x) - \sigma_\theta(y)| = |\sigma'_\theta(\mu_0 x + (1-\mu_0)y)||x-y| \geq C_1|x+y|^{\theta-1}|x-y|, \quad 0 < c_1 < \infty,$$

the operator

$$T_5\varphi(x) := v_0(x)[S_{\Im,\theta,\alpha,\beta} - \widetilde{S}^0_{\mathbb{R}^+,\theta,\alpha}]|_{L_{p,\alpha,\beta}(\Im)}\varphi(x) = \int_0^1 \tilde{g}(x,y)\varphi(y)dy,$$

$$\widetilde{S}^0_{\Im,\theta,\alpha}\varphi(x) := \frac{1}{\pi i}\int_0^1 \left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)} \frac{\sigma'_\theta(y)\varphi(y)dy}{\sigma_\theta(y) - \sigma_\theta(x)}$$

has a bounded kernel:

$$\begin{aligned} |\tilde{g}(x,y)| \;&:=\; \left|\frac{v_0(x)}{\pi i}\left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)} \frac{\sigma'_\theta(y)}{\sigma_\theta(y)-\sigma_\theta(x)}\left[\left(\frac{1-x}{1-y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)}\frac{g_\theta(x)}{g_\theta(y)} - 1\right]\right| \\ &\leq\; C_2\frac{x^{(\theta-1)(\frac{1}{p}+\alpha)}y^{(\theta-1)(\frac{1}{p}+\alpha)}(1-y)^{(\theta-1)(1-\frac{1}{p}-\alpha)}}{|x+y|^{\theta-1}} \leq C_3 < \infty \end{aligned}$$

(recall that $0 < \frac{1}{p} + \alpha < 1$). Therefore if we can prove that the operator

$$T_6\varphi = v_0(S^0_{\Im,\theta,\alpha} - \widetilde{S}^0_{\Im,\theta,\alpha})\varphi$$

has a bounded kernel

$$\begin{aligned} g_6(x,y) \;&=\; \left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)}\frac{v_0(x)}{\pi i}\left[\frac{\sigma'_\theta(y)}{\sigma_\theta(y)-\sigma_\theta(x)} - \frac{\theta y^{\theta-1}}{y^\theta - x^\theta}\right] = \\ &=\; \frac{v_0(x)}{\pi i}\left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)}\frac{\sigma'_\theta(y)}{\sigma_\theta(y)-\sigma_\theta(x)} g^1_6(x,y), \\ &\qquad g^1_6(x,y) := 1 - \left[\frac{\theta y^{\theta-1}}{\sigma_\theta(y)}\frac{\sigma_\theta(x)-\sigma_\theta(y)}{y^\theta - x^\theta},\right] \end{aligned}$$

compactness of the operator T_0 in (2.39) will become evident.

We have $\lim\limits_{y\to x} g_6^1(x,y)=0$ and

$$\mathcal{D}_x g_6^1(x,y)=\left[\frac{\theta(\theta-1)y^{\theta-2}}{\sigma_\theta'(y)}-\frac{\theta\sigma_\theta''(y)y y^{\theta-1}}{[\sigma_\theta'(y)]^2}\right]\frac{\sigma_\theta(y)-\sigma_\theta(x)}{y^\theta-x^\theta}$$
$$+\frac{\theta y^{\theta-1}}{\sigma_\theta'(y)}\left[\frac{\sigma_\theta'(y)}{y^\theta-x^\theta}-\frac{[\sigma_\theta(y)-\sigma_\theta(x)]\theta y^{\theta-1}}{(y^\theta-x^\theta)^2}\right]$$
$$=\left[\frac{\theta-1}{y}+0(1)-\frac{\theta-1}{y}+0(1)\right]0(1)+0(1)\left[\frac{1}{y}+0(1)-\frac{1}{y}-0(1)\right]=0(1);$$

therefore; $g_6^1(x,y)\le C_4|x-y|$ and we obtain

$$|g_6^1(x,y)|\le\frac{C_4}{C_1}\frac{x^{(\theta-1)(\frac{1}{p}+\alpha)}y^{(\theta-1)(1-\frac{1}{p}-\alpha)}(1-y)^{\theta-1}}{|(x+y)^{\theta-1}|}\le\frac{C_4}{C_1}<\infty.$$

A similar estimates can be applied to the kernel of operator T_1 in (2.39).

Now let us consider the operator T_2. If

$$b_2(x):=x^{(\theta-1)\alpha}(1-x)^\beta[\sigma_\theta'(x)]^{\frac{1}{p}},$$

then $v_t b_2\in C^\infty(\Im)$ and

$$|v_t(x)b_2(x)-v_t(y)b_2(y)|\le C_5|x-y|\tilde{v}_t\left(\frac{x+y}{2}\right), \tag{2.45}$$

where $\tilde{v}_t(x)$ also vanishes in some neighbourhoods of $0,1\in\Im$. Since

$$\begin{aligned}
T_2\varphi(x) &= \int_0^1[g_2^1(x,y)+g_2^2(x,y)]\varphi(y)dy,\\
g_2^1(x,y) &= \frac{1}{\pi i}\frac{v_t(x)b_2(x)-v_t(y)b_2(y)}{b_2(y)}\frac{\sigma_\theta'(y)}{\sigma_\theta(y)-\sigma_\theta(x)},\\
g_2^2(x,y) &= \frac{v_t(y)}{\pi i}\left[\frac{\sigma_\theta'(y)}{\sigma_\theta(y)-\sigma_\theta(x)}-\frac{1}{y-x}+\frac{1}{y-x}-\frac{\theta y^{\theta-1}}{y^\theta-x^\theta}\right],
\end{aligned}$$

we get, due to (2.44), (2.45),

$$|g_2^1(x,y)|\le\frac{C_5}{C_1}\frac{y^{(\theta-1)}(1-y)^{\theta-1}\tilde{v}_t\left(\frac{x+y}{2}\right)}{\pi(x+y)^{\theta-1}y^{(\theta-1)(\frac{1}{p}+\alpha)}(1-y)^{(\theta-1)(\frac{1}{p}+\beta}}\le C_6,\ \ |g_2^2(x,y)|\le C_7,$$

because,

$$\frac{|v_t(y)|}{\pi}\left|\frac{\sigma_\theta'(y)}{\sigma_\theta(y)-\sigma_\theta(x)}-\frac{1}{y-x}\right|=\frac{|v_t(y)\int_x^y[\sigma_\theta'(y)-\sigma_\theta'(t)]dt|}{|\sigma_\theta(y)-\sigma_\theta(x)||y-x|}\le C_8\frac{\int_x^y(y-t)dt}{(y-x)^2}\le\frac{1}{2}C_8$$

and, similarly

$$\frac{|v_t(y)|}{\pi}\left|\frac{1}{y-x}-\frac{\theta y^{\theta-1}}{y^\theta-x^\theta}\right]\leq C_9.$$

Thus, T_2 in (2.39) has a bounded kernel.

If $a(x)=0$ in some neighbourhood of $0\in\Im$ and $r(x)=0$ for sufficiently large $x>N$, then compactness of T_3 in (2.39) becomes evident, since the operator has a bounded kernel. If $a(x)=0$ only for $x=0$ and $r(x)\neq 0$ for a large $x>N$, we can find approximations $\|a-a_\varepsilon|C(\Im)\|\to 0$, $M_1(r-r_\varepsilon)\to 0$, $M_2(r_\varepsilon)\leq M_2(r)$ (see (2.19)) where $a_\varepsilon(x)$ and $r_\varepsilon(y)$ vanish for $|x|<\varepsilon$, $|x-1|<\varepsilon$, $|y|>\frac{1}{\varepsilon}$. Due to Lemma 2.6 operators $T_3^{(\varepsilon)}=a_\varepsilon\mathcal{R}^{(\varepsilon)}_{\Im,\theta,\alpha}$ (with the kernel r_ε) approximate T_3 in norm and are compact; therefore T_3 is compact.

Due to a similar property for operator $\widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta}$ we can easily get that operator $\widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta}\to\widetilde{\mathcal{R}}_{\Im,\theta,\alpha}$, where

$$\widetilde{\mathcal{R}}_{\Im,\theta,\alpha}\varphi(x)=\int\limits_0^1\left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\alpha)}r\left(\frac{\sigma_\theta(x)}{\sigma_\theta(y)}\right)\varphi(y)\frac{dy}{y}, \tag{2.46}$$

is compact. Thus, it remains to show that the operator

$$T_4':=(\widetilde{\mathcal{R}}_{\Im,\theta\alpha}-\widetilde{\mathcal{R}}_{\mathbb{R}^+,\theta\alpha})|_{L_{p,\alpha,\beta(\Im)}} \tag{2.47}$$

is compact in $L_{p,\alpha,\beta}(\Im)$.

If we recall (0.6) we find

$$\frac{\sigma_\theta(x)}{\sigma_\theta(y)}=\left(\frac{x}{y}\right)^\theta\frac{h_\theta(y)}{h_\theta(x)}=\left(\frac{x}{y}\right)^\theta\left[1+(y-x)\frac{h'_\theta(\mu_0x+(1-\mu_0)y)}{h_\theta(x)}\right],$$
$$h_\theta(x):=x^\theta+(1-x)^\theta\neq 0,\quad h_\theta(x)\in C^\infty(\Im),\quad x,y\in\Im$$

for some $0<\mu_0<1$. By the same formula for a remainder we get

$$r\left(\frac{\sigma_\theta(x)}{\sigma_\theta(y)}\right)-r\left(\left(\frac{x}{y}\right)^\theta\right)=(y-x)g_0(x,y),$$
$$g_0(x,y)=\left(\frac{x}{y}\right)^\theta\frac{h'_\theta(\mu_0x+(1-\mu_0)y)}{h_\theta(x)}\times$$
$$\times r'\left(\left(\frac{x}{y}\right)^\theta\left[\mu_1+(1-\mu_1)(y-x)\frac{h'_\theta(\mu_0x+(1-\mu_0)y)}{h_\theta(x)}\right]\right);$$

due to (0.3) $g_0(x,y)$ has the following estimates

$$|g_0(x,y)|\leq M_2\left(\frac{x}{y}\right)^\theta\left(\frac{x}{y}\right)^{(\gamma_0-1)}\left(1+\left(\frac{x}{y}\right)^\theta\right)^{\gamma_1}=M_2\left(\frac{x}{y}\right)^{\gamma_0}\left(1+\left(\frac{x}{y}\right)^\theta\right)^{\gamma_1}. \tag{2.48}$$

The operator

$$V_{1/\theta,\alpha}T_4'V_{\theta,\alpha}\varphi(x)=\int\limits_0^1(y^{\frac{1}{\theta}}-x^{\frac{1}{\theta}}g_0(x^{\frac{1}{\theta}},y^{\frac{1}{\theta}})\varphi(y)\tfrac{dy}{y}=$$
$$(Gy^{\frac{1}{\theta}}\varphi)(x)-(x^{\frac{1}{\theta}}\varphi)(x),$$

where G, $V_{\theta,\alpha}$ are default in (2.29) and (2.42) and $g(x,y)=g_0(x^{\frac{1}{\theta}},y^{\frac{1}{\theta}})$, is equivalent to T_4' in $L_{p,\alpha,\beta}(\Im)$ space (see (2.43)) and the kernel function $g(x,y)$ has estimates (2.28) (cf. (2.48)). Therefore G is bounded in $L_{p,\alpha,\beta}(\Im)$ (see Lemma 2.7). Since $x^{\frac{1}{\theta}}$ vanishes at 0, the operators $Gy^{\frac{1}{\theta}}I$ and $x^{\frac{1}{\theta}}G$ are compact in $L_{p,\alpha,\beta}(\Im)$ (see a similar proof for T_3 above). Thus, T_4 is compact in $L_{p,\alpha,\beta}(\Im)$. □

Theorem 2.11 *Let conditions* (0.3), (0.4) *hold and*

$$\theta(\gamma_0+\frac{1}{p}+\alpha)-m>0\ \text{ for some integer }\ m\in\mathcal{N}. \tag{2.49}$$

Then operators $\widetilde{\mathcal{R}}_{\Im,\theta,\alpha\beta}$ and $\mathcal{R}^0_{\mathbb{R}^+,\theta,\alpha}$ in (2.36) *are bounded in $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ and in $\widetilde{W}^m_{p,\alpha}(\mathbb{R}^+)$, respectively and operators T_3 and T_4 in* (2.39) *are compact in $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ provided $a\in C^m(\Im)$, $a(0)=0$.*

Proof Boundedness follows from Lemma 2.7 and equality (2.37) since

$$\mathcal{D}^k\mathcal{R}^0_{\mathbb{R}^+,\theta,\gamma}\varphi(x)\ =\ \int\limits_0^\infty\left(\frac{x}{y}\right)^{(\theta-1)(\frac{1}{p}+\gamma)-k}r\left(\left(\frac{x}{y}\right)^\theta\right)\varphi(y)\frac{dy}{y}, \tag{2.50}$$
$$0\le k\le m,\quad \varphi\in\widetilde{W}^m_{p,\alpha}(\mathbb{R}^+)$$

(see (2.13)) and the kernel of the operator $\mathcal{D}^k\mathcal{R}^0_{\mathbb{R}^+,\theta,\gamma}$ satisfies conditions of Lemma 2.7 (see (2.49)). Similarly with the operator $\widetilde{\mathcal{R}}_{\Im,\theta,\alpha,\beta}$.

Compactness of T_3 and T_4 can be proved directly as in the foregoing Theorem 2.11. The assertion can also be derived by an indirect method: it can be proved that operators $\tilde{\mathcal{R}}_{\Im,\theta,\alpha\beta}$, $\mathcal{R}^0_{\mathbb{R}^+,\theta,\alpha}$ are bounded in the Bessel potential spaces $\widetilde{H}^{m+\varepsilon}_{p,\alpha,\beta}(\Im)$ and in $\tilde{H}^{m+\varepsilon}_{p,\alpha}(\mathbb{R}^+)$ respectively for some $\varepsilon>0$ (see [16, 61]); then, by interpolation theorem for compact operators (see [61]) we get T_3, T_4 are compact in $\widetilde{H}^m_{p,\alpha,\beta}(\Im)=\widetilde{W}^m_{p,\alpha,\beta}(\Im)$. □

Remark 2.12 *The operators $\widetilde{S}_{\Im,\theta,\alpha,\beta}$ and $S^0_{\mathbb{R}^+,\theta,\gamma}$ are unbounded, in general, in Sobolev spaces $\widetilde{W}^m_{p,\alpha,\beta}(\Im)$ and in $W^m_p(\mathbb{R}^+,x^\gamma(1+x)^\nu)$ for any $\nu\in\mathbb{R}$, respectively, if $m\ge1$ and θ is large.*

We should explain this on the example of the operator $S^0_{\mathbb{R}^+,\alpha}$: this operator is bounded in $L_{p,\alpha}(\mathbb{R}^+)$ if and only if

$$S^\theta_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int\limits_0^\infty\frac{\theta y^{\theta-1}}{y^\theta-x^\theta}\varphi(y)dy \tag{2.51}$$

is bounded in $L_{p,\mu}(\mathbb{R}^+) = L_p(\mathbb{R}^+, x^\mu(1+x)^\nu)$ with $\mu = \alpha + (\theta-1)(\frac{1}{p}+\alpha)$, i.e. iff

$$-\frac{1}{p} < \mu < \theta - \frac{1}{p}, \qquad -\frac{1}{p} < \mu + \nu < \theta - \frac{1}{p}, \tag{2.52}$$

since $0 < \frac{1}{p} + \alpha < 1$ (see [24, 33] for $\theta = 1$; the case $\theta \neq 1$ is reduced to $\theta = 1$ as in (2.41), (2.43)).

Integrating by parts we find that

$$\begin{aligned} \mathcal{D}_x S^\theta_{\mathbb{R}^+}\varphi(x) &= \tfrac{1}{\pi i}\mathcal{D}_x \int_0^\infty \mathcal{D}_y \ln(y^\theta - x^\theta)\varphi(y)dy = \\ &= \tfrac{1}{\pi i}\int_0^\infty \left(\tfrac{x}{y}\right)^{\theta-1} \tfrac{\theta y^{\theta-1}}{y^\theta - x^\theta}\mathcal{D}_y\varphi(y)dy, \quad \varphi \in \widetilde{W}^m_p(\mathbb{R}^+, x^\mu(1+x)^\nu). \end{aligned}$$

Thus, $S^\theta_{\mathbb{R}^+}$ is bounde in $\widetilde{W}^m_p(\mathbb{R}^+, x^\mu(1+x)^\nu)$ iff $S^\theta_{\mathbb{R}^+}$ is bounded in $L_p(\mathbb{R}^+, x^{\mu+m(\theta-1)}(1+x)^\nu)$ (cf. (2.52)).

3 Preliminaries

3.1 Splines

Let us recall here more information about splines, defined in subsection 1.2.

If we denote

$$x^m_+ := \begin{cases} x^m, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{3.1}$$

we can prove that $\Phi_m(x)$, defined in (1.16), can be written as

$$\Phi_m(x) = \frac{1}{m!}\sum_{k=0}^{m+1}(-1^k)\begin{pmatrix} m+1 \\ k \end{pmatrix}\left(x + \frac{m+1}{2} - k\right)^m_+ \tag{3.2}$$

for all $m \in \mathcal{N}$, moreover, for $m \geq 2$ there holds the following recurrence formula

$$\Phi_m(x) = \frac{1}{m}\left(x + \frac{m+1}{2}\right)\Phi_{m-1}\left(x + \frac{1}{2}\right) - \frac{1}{m}\left(x - \frac{m+1}{2}\right)\Phi_{m-1}\left(x - \frac{1}{2}\right). \tag{3.3}$$

For the cases $m = 1, 2$ we have

$$\begin{gathered} \Phi_1(x) = (1+x)\chi_{[-1,0)}(x) + (1-x)\chi_{[0,1]}(x), \\ \Phi_2(x) = \left(\frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8}\right)\chi_{[-3/2,-1/2]}(x) + \left(\frac{3}{4} - x^2\right)\chi_{[-1/2,1/2]}(x) + \\ + \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8}\right)\chi_{[1/2,3/2]}(x) \end{gathered} \tag{3.4}$$

Since

$$\mathcal{D}\Phi_{m,j}^{(n)}(x) = \mathcal{D}\int\limits_{-\infty}^{\infty} \chi_{[-1/2,1/2]}(me - j - y)\Phi_{m-1}(y)dy =$$

$$= \mathcal{D}_n \int\limits_{nx-j-\frac{1}{2}}^{nx-j+\frac{1}{2}} \Phi_{m-1}(y)dy = n\left[\Phi_{m-1,j}^{(n)}\left(x + \frac{1}{2n}\right) - \right.$$

$$\left. -\Phi_{m-1,j+1}^{(n)}\left(x + \frac{1}{2n}\right)\right] = -n\Delta_j\Phi_{m-1,j}\left(x + \frac{1}{2n}\right),$$

$$\Delta_j\xi_{jk} := \xi_{j+1,k} - \xi_{jk},$$

we easily find

$$\mathcal{D}^\ell\Phi_{mj}^{(n)}(x) = (-n)^\ell\Delta_j^\ell\Phi_{m-\ell,j}^{(n)}\left(x + \frac{\ell}{2n}\right), \quad \ell \le m, \tag{3.5}$$

where

$$\Delta_j\xi_{jk} := \sum_{i=0}^{\ell}(-1)^{\ell-1}\left(\begin{array}{c} \ell \\ i \end{array}\right)\xi_{j+1,k}. \tag{3.6}$$

Therefore,

$$\begin{aligned} \mathcal{D}^m\Phi_{mj}^{(n)}(x) &= (-n)^m\sum_{i=0}^{m}(-1)^{m-i}\left(\begin{array}{c} m \\ i \end{array}\right)\Phi_{0,j+1}^{(n)}\left(x + \frac{m}{2n}\right) = \\ &= n^m\sum_{i=0}^{m}(-1)^i\left(\begin{array}{c} m \\ i \end{array}\right)\chi_{\left[\frac{2j+2i-m-1}{2n},\frac{2j+2i-m+1}{2n}\right]}(x). \end{aligned} \tag{3.7}$$

Due to the definition of $\Phi_m(x)$ (see (1.16))

$$\int\limits_{-\infty}^{\infty}\Phi_m(x)dx = \left[\int\limits_{-\infty}^{\infty}\Phi_0(x)dx\right]^{m+1} = 1; \tag{3.8}$$

as a consequence we derive

$$\sum_{j=-\infty}^{\infty}\Phi_m(j + x) \equiv 1, \quad x \in \mathbb{R}. \tag{3.9}$$

In fact,

$$\sum_{j=-\infty}^{\infty}\Phi_m(j + x) = \sum_{j=-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\Phi_{m-1}(j + x - y)\Phi_0(y)dy =$$

$$= \sum_{j=-\infty}^{\infty}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\Phi_{m-1}(j + x - y)dy = \int\limits_{-\infty}^{\infty}\Phi_{m-1}(x - y)dy =$$

$$= \int\limits_{-\infty}^{\infty}\Phi_{m-1}(y)dy = 1.$$

Now if $\langle\cdot,\cdot\rangle$ denotes the scalar product

$$\langle\varphi,\psi\rangle := \int_{-\infty}^{\infty} \overline{\varphi(y)}\psi(y)dy, \tag{3.10}$$

then

$$\langle\Phi_k,\Phi_m\rangle = \int_{-\infty}^{\infty} \overline{\Phi_k(y)}\Phi_m(y)dy = \int_{-\infty}^{\infty} \Phi_k(-y)\Phi_m(y)dy = \Phi_{k+m+1}(0), \tag{3.11}$$

since (see (1.16))

$$\Phi_m(x) \equiv \Phi_m(-x). \tag{3.12}$$

Similarly

$$\langle\Phi_k(\cdot \pm j),\Phi_m\rangle = \Phi_{k+m+1}(\pm j) = \Phi_{k+m+1}(j). \tag{3.13}$$

It is convenient to define the Fourier transform $\mathcal{F}$ (and its inverse $\mathcal{F}^{-1}$) as follows

$$\mathcal{F}^{\pm 1}\varphi(n) := \int_{-\infty}^{\infty} \ell^{-2\pi nyi}\varphi(y)dy, \quad \mathcal{F}^{-1}\psi(y) := \int_{-\infty}^{\infty} \ell^{2\pi nyi}\psi(n)dn, \quad n \in \mathbb{R}. \tag{3.14}$$

Due to (1.16) we have

$$\begin{aligned}\mathcal{F}^{\pm 1}\Phi_m(n) &= \int_{-\infty}^{\infty} \ell^{\mp 2\pi nyi}\Phi_m(y)dy = \left[\int_{-\infty}^{\infty} \ell^{\mp 2\pi nyi}\Phi_0(y)dy\right]^{m+1} = \\ &= \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \ell^{\mp 2\pi nyi}dy\right]^{m+1} = \left[\frac{\sin\pi n}{\pi n}\right]^{m+1},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Phi_m(x) = \mathcal{F}^{-1}\mathcal{F}\Phi_m(x) &= \int_{-\infty}^{\infty} \ell^{\mp 2\pi nyi}\left[\frac{\sin\pi n}{\pi n}\right]^{m+1} dn = \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \ell^{\mp 2\pi nyi}\left[\frac{\sin\pi n}{\pi n}\right]^{m+1} dn.\end{aligned} \tag{3.16}$$

3.2 Projections on spline spaces.

Together with $S_k^{(n)}(\Im)$ (see (1.18)) we should consider the splane spaces $S_k^{(n)}(\mathbb{R})$, generated by all functions $\Phi_{k,j}^{(n)}(x)$, $j = 0,\pm 1,\ldots$, defined in (1.18)

$$S_k^{(n)}(\mathbb{R}) := \{\ \text{Span}\,\{\Phi_{k,j}^{(n)} : j = 0,\pm 1,\ldots\}. \tag{3.17}$$

For a Banach space $X(\mathbb{R})$, which contains $S_k^{(n)}(\mathbb{R})$ (i.e. $S_k^{(n)}(\mathbb{R}) \subset X(\mathbb{R})$) by $S_k^{(n)}(X(\mathbb{R}))$ is denoted the closure of $S_k^{(n)}(\mathbb{R})$ in $X(\mathbb{R})$. If, for example,

$$W_{p,\alpha}^m(\mathbb{R}) := W_p^m(\mathbb{R},|x|^\alpha) := \{\varphi \in L_p(\mathbb{R},x^\alpha) : \mathcal{D}^k\varphi \in L_p(\Im,x^\alpha)\}, \quad m = 0,1,\ldots \tag{3.18}$$

then $S_k^{(n)}(\mathbb{R}) \subset W_{p,\alpha}^m(\mathbb{R})$ provided $k \geq m$ and (0.4) holds for $\alpha \in \mathbb{R}$; $S_m^{(n)}(\mathbb{R})(W_{p,\alpha}^m(\mathbb{R}))$ denotes the closure of the spline space (3.17) with respect to the norm in $W_{p,\alpha}^m(\mathbb{R})$:

$$\|\varphi|W_{p,\alpha}^m(\mathbb{R})\| = \sum_{k=0}^m \left(\int_{-\infty}^{\infty} |x^\alpha \mathcal{D}^k \varphi(x)|^p dx \right)^{1/p}$$

(cf. (1.11)).

Let us consider the operators

$$\tilde{p}_k^{(n)}\varphi(x) := E_k^{(n)}\mathcal{D}^{(n)}\varphi(x) = \sum_{j=-\infty}^{\infty} \varphi\left(\frac{j}{n}\right)\Phi_{kj}^{(n)}(x),$$

$$\mathcal{D}^{(n)}\varphi := \left\{\varphi\left(\frac{j}{n}\right)\right\}_{j=-\infty}^{\infty}, \quad \varphi \in W_{p,\alpha}^m(\Im), \quad m \geq 1, \tag{3.19}$$

$$E_k^{(n)}\xi(x) := \sum_{j=-\infty}^{\infty} \xi_j \Phi_{mj}^{(n)}(x), \quad \xi = \{\xi_j\}_{j=-\infty}^{\infty}.$$

Operators $\tilde{p}_k^{(n)}$ are correctly defined, as will be proved in the next theorem. Let us notice here that since $\Phi_{1j}^{(n)}\left(\frac{j}{n}\right) = \Phi_1(0) = 1$, $\Phi_{1j}^{(n)}\left(\frac{\ell}{n}\right) = 0$ if $\ell \neq j$, we get (see Remark 3.8 below)

$$\left(\tilde{p}_1^{(n)}\right)^2\varphi = \varphi \ \text{ while } \left(\tilde{p}_1^{(n)}\right)^2\varphi \neq \varphi \ \text{ if } k > 1 \ \text{ for all } \varphi \in W_{p,\alpha}^m(\Im). \tag{3.20}$$

Thus operator $\tilde{p}_k^{(n)}$, projecting the space $W_{p,\alpha}^m(\mathbb{R})$ onto $S_m^{(n)}(W_{p,\alpha}^m(\mathbb{R}))^k$, are not projectors unless $k = 1$. Projectors $p_k^{(n)}$ will be constructed in subsection 3.4, following [27, Sect.2.7]

Theorem 3.1 *The operators*

$$\tilde{p}_k^{(n)} : W_{p,\alpha}^m(\mathbb{R}) \longrightarrow S_k^{(n)}(W_{p,\alpha}^m(\mathbb{R})), \quad k \geq m \geq 1 \tag{3.21}$$

are correctly defined, bounded and converge strongly to the identity:

$$\lim_{n\to\infty} \tilde{p}_k^{(n)}\varphi = \varphi \qquad \text{for all} \qquad \varphi \in W_{p,\alpha}^m(\mathbb{R}). \tag{3.22}$$

Proof. From the definitions of $\Phi_{kj}^{(n)}$ and $\tilde{p}_k^{(n)}$ in (1.16) and (3.19) we easily find that

$$\tilde{p}_k^{(n)}\varphi(x) = n\Phi_{k-m-1,0}^{(n)} * \tilde{p}_m^{(n)}\varphi(x) = \int_{-\infty}^{\infty} \Phi_{k-m-1}(n(x-y))\tilde{p}_k^{(n)}\varphi(y)dy. \tag{3.23}$$

Let us check first that the convolution operators

$$\mathcal{K}_\ell^{(n)}\varphi(x) := n\Phi_{\ell,0}^{(n)} * \varphi(x) = n\int_{-\infty}^{\infty} \Phi_{\ell,0}^{(n)}(x-y)\varphi(y)dy, \quad \ell \geq 0, \tag{3.24}$$

are bounded in $W_{p,\alpha}(\mathbb{R})$. For $\ell \geq 1$ this follows from the Stechkin's theorem (see [5, 12]) since the symbol of the convolution operator

$$k_\ell(\theta) n \mathcal{F}\Phi^{(n)}_{\ell,0}(\theta) = \mathcal{F}\Phi_\ell\left(\frac{\theta}{n}\right) = \left[\frac{n \sin \pi \frac{\theta}{n}}{\pi\theta}\right]^{\ell+1}$$

(see (3.15)) is of bounded variation $k_\ell \in V_1(\mathbb{R})$. For $\ell = 0$ this argument fails and we give the direct proof in the general case $\ell \geq 0$.

We have

$$\|\mathcal{D}^s \mathcal{K}_\ell \varphi | L_{p,\alpha}(\mathbb{R})\| = \left(\int\limits_{-\infty}^{\infty} t^\alpha n \int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}} \Phi_\ell(t-\tau)|\mathcal{D}^s\varphi(\tau)d\tau|^p dt\right)^{1/p} \leq$$

$$\leq \left(\int\limits_{-\infty}^{\infty} |t|^{\alpha p} n^p \int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}} |\tau|^{\alpha p}|\mathcal{D}^s\varphi(\tau)d\tau|^p d\tau \left(\int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}} |\tau|^{-\alpha p'} d\tau\right)^{p/p'} dt\right)^{1/p},$$

$$s = 0, 1, \dots, n, \quad p' := \frac{p}{p-1},$$

since $|\Phi^{(n)}_{\ell,0}(t)| \leq 1$ and we applied the Hölder's inequality. Applying the mean value theorem we easily prove that

$$t^{\alpha p}\left(\int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}} |\tau|^{-\alpha p'} d\tau\right)^{p/p'} \leq c_0 n^{-p/p'},$$

where C_0 is independent of t (but depends on p and ℓ). Therefore

$$\|\mathcal{D}^s \mathcal{K}_\ell \varphi | L_{p,\alpha}(\mathbb{R})\| c_0 n^{1-\frac{1}{p}} \left(\int\limits_{-\infty}^{\infty} \int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}} |\tau|^{-\alpha p'}|\varphi(\tau)|^p d\tau dt\right)^{1/p} =$$

$$= c_0 n^{1-\frac{1}{p'}} \left(\int\limits_{-\infty}^{\infty} |\tau|^\alpha |\mathcal{D}^s\varphi(\tau)|^p d\tau \int\limits_{t-\frac{\ell+1}{n}}^{t+\frac{\ell+1}{n}}\right)^{1/p} = c_1\|\mathcal{D}^s\varphi | L_{p,\alpha}(\mathbb{R})\|,$$

where $c_1 = (\ell+1)^{1/p} c_0$ and the boundedness $\mathcal{K}_\ell : W^m_{p,\alpha}(\mathbb{R}) \to W^m_{p,\alpha}(\mathbb{R})$ follows.

Since

$$|\Phi^{(n)}_{\ell,0}| \leq 1, \quad \mathbf{supp}\, \Phi^{(n)}_{\ell,0}\left[-\frac{\ell+1}{2n}, \frac{\ell+1}{2n}\right] \quad n\int\limits_{-\infty}^{\infty} \Phi^{(n)}_{\ell,0}(t)dt = 0,$$

(see (3.17),(3.18) and (3.8)) we can apply the well-known convergence result (see [][29])

$$\lim_{n\to\infty} \mathcal{K}^{(n)}_\ell \varphi = \lim_{n\to\infty} n\Phi^{(n)}_{\ell,0}\varphi * \varphi = \varphi, \quad \varphi \in W^m_{p,\alpha}(\mathbb{R}). \tag{3.25}$$

From the representation (3.22), from the boundedness of $\mathcal{K}_\ell^{(n)}$ and from (3.25) it follows that we can suppose $k=m$.

Applying (3.5) we get

$$\begin{aligned}\mathcal{D}^\ell \tilde{p}_m^{(n)}\varphi(x) &= (-n)^\ell \sum_{j=-\infty}^{\infty} \varphi\left(\frac{j}{n}\right)\Delta_j^\ell \Phi_{m-\ell,j}^{(n)}\left(x+\frac{\ell}{2n}\right) = \\ &= n^\ell \sum_{j=-\infty}^{\infty} \Delta_j^\ell\varphi\left(\frac{j}{n}\right)\Phi_{m-\ell,j}^{(n)}(x), \qquad \ell=1,\dots,m,\end{aligned} \tag{3.26}$$

where (see (3.6)) $\varphi\in C_0^\infty(\mathbb{R})$ and

$$\begin{aligned}&\Delta_j^\ell\varphi\left(\frac{j}{n}\right) := \sum_{i=0}^{\ell}(-1)^\ell \binom{\ell}{i}\varphi\left(\frac{j+1}{n}\right) = \int_{\Im_{\ell,j}^{(n)}} \partial^\ell\varphi(t_1+\dots+t_\ell)dt, \\ &dt := dt_1,\dots,dt_\ell, \quad \Im_{\ell,j}^{(n)} := \left[\frac{j}{n},\frac{j+1}{n}\right]\times\left[0,\frac{1}{n}\right]\times\dots\times\left[0,\frac{1}{n}\right]\times(\ell-1).\end{aligned} \tag{3.27}$$

We proceed as follows

$$\begin{aligned}&\|\mathcal{D}^\ell\tilde{p}_m^{(n)}\varphi|L_{p,\alpha}(\mathbb{R})\| = \left(\int_{-\infty}^{\infty}\left|x^\alpha n^\ell\sum_{j=-\infty}^{\infty}\Delta_j^\ell\varphi\left(\frac{j+1}{n}\right)\Phi_{m-\ell,j}^{(n)}(x)\right|^p dx\right)^{1/p} \le \\ &\le c_0\Big(\sum_{j=-\infty}^{\infty} n^{1-1/p'-\alpha}\int_{\frac{j}{n}}^{\frac{j+1}{n}}|x^\alpha\mathcal{D}^\ell\varphi(t)|^p dt\int_{j-\frac{m-\ell}{2n}}^{j+\frac{m-\ell}{2n}}|x|^{\alpha p}dx\Big)^{1/p}.\end{aligned} \tag{3.28}$$

Since, due to (3.27),

$$\begin{aligned}\left|\Delta_j^\ell\varphi\left(\frac{j+1}{n}\right)\right| &\le n^{1-\ell}\int_{\frac{j}{n}}^{\frac{j+1}{n}}|\partial\varphi(t)|dt \le n^{1-\ell}\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t^\alpha\partial^\alpha\varphi(t)|^p dt\right)^{1/p}\times \\ &\times\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{-\alpha p'}dt\right)^{1/p'} \le c_0 n^{1-1/p'-\alpha}\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t^\alpha\partial^\alpha\varphi(t)|^p dt\right)^{1/p}\end{aligned}$$

and we apply the first of the following two inequalities

$$\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{-\alpha p'}dt\right)^{1/p'} \le c_0 n^{-1/p'-\alpha}, \qquad \left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{\alpha p}dt\right)^{1/p} \le C_0' n^{-1/p'+\alpha} \tag{3.29}$$

which can be easily obtained with the help of the mean value theorem (constants c_0, c_0' depend on p and ℓ, but not on n).

Applying the second inequality in (3.29), we get from (3.28) the following

$$\begin{aligned}\|\mathcal{D}^\ell \tilde{p}_m^{(n)}\varphi|L_{p,\alpha}(\mathrm{I\!R})\| \le c_0 \sum_{j=-\infty}^{\infty}\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} |k^\alpha \mathcal{D}^\ell\varphi(t)|^p dt\right)^{1/p} &= \\ = c_0\ell\left(\int_{-\infty}^{\infty}|x^\alpha\mathcal{D}^\ell\varphi(t)|^p dt\right)^{1/p} &= \\ = c_0\ell\|\mathcal{D}^\ell\varphi|L_{p,\alpha}(\mathrm{I\!R})\|.\end{aligned} \tag{3.30}$$

For the case $\ell = 0$ we apply the following obvious equality

$$\varphi\left(\frac{j}{n}\right) = n\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left[\varphi'(t)\left(t-\frac{j+1}{n}\right)+\varphi(t)\right]dt,$$

and the second inequality in (3.29); we get

$$\left|\varphi\left(\frac{j}{n}\right)\right|^p \le n^p\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}(|\varphi'(t)|+|\varphi(t)|)dt\right)^p \le n^p\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{\alpha p}(|\varphi'(t)|^p+|\varphi(t)|^p\right)dt \times$$

$$\times\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{-\alpha p'}dt\right)^{p/p'} \le c_1 n j^{-\alpha p'}\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{\alpha p}(|\varphi'(t)|^p+|\varphi(t)|^p)dt.$$

z

Further we proceed as follows (applying the second inequality in (3.29):

$$\begin{aligned}\|\tilde{p}_m^{(n)}\varphi|L_{p,\alpha}(\mathrm{I\!R})\| = \left(\int_{-\infty}^{\infty}|x^\alpha\varphi\left(\frac{j}{n}\right)\sum_{-\infty}^{\infty}\varphi\left(\frac{j}{n}\right)\Phi_{m,j}^{(n)}(x)|^p dx\right)^{1/p} &\le \\ \le c_1 n^{1/p}\sum_{j=-\infty}^{\infty}(j^{-\alpha p'}\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{\alpha p}(|\varphi'(t)|^p+|\varphi(t)|^p)\int_{j-\frac{1}{2n}}^{j+\frac{1}{2n}}|x|^{\alpha p}dx)^{1/p} &\le \\ \le c_2\sum_{j=-\infty}^{\infty}\int_{\frac{j}{n}}^{\frac{j+1}{n}}|t|^{\alpha p}(|\varphi'(t)|^p+|\varphi(t)|^p)dt)^{1/p} &\le \\ \le c_2(\|\mathcal{D}\varphi\|_{p,\alpha}+\|\varphi\|_{p,\alpha}).\end{aligned}$$

The obtained inequality together with (3.30) yield the boundedness (3.21) for $k = m$, since the lineal $c_0^\infty(\mathrm{I\!R})$ is dense in $W_{p,\alpha}^m(\mathrm{I\!R})$.

To prove (3.22) (for $k = m$) again it is sufficient to take $\varphi \in c_0^\infty(\mathrm{I\!R})$. Due to (3.9) we find

$$\sum_{j=-\infty}^{\infty}\Phi_{\ell,j}^{(n)}(x) = \sum_{j=-\infty}^{\infty}\Phi_\ell(nx-j) = \sum_{j=-\infty}^{\infty}\Phi_\ell(nx+j) \equiv 1$$

and, applying (3.26),(3.27),(3.29) we proceed as follows

$$\|\mathcal{D}^\ell[\varphi-\tilde{p}^{(n)}_m\varphi]|L_{p,\alpha}(\mathbb{R})\| = \|\sum_{j=-\infty}^{\infty}[\mathcal{D}^\ell\varphi - n^\ell\Delta^\ell_j\varphi\left(\frac{j}{n}\right)]\Phi^{(n)}_{m-\ell,j}||L_{p,\alpha}(\mathbb{R})\| =$$

$$= \left(\int_{-\infty}^{\infty}\left|\sum_{j=-\infty}^{\infty} n^\ell x^\ell \int_{\Im^{(n)}_{\ell,j}}[\mathcal{D}^\ell\varphi(x)-\mathcal{D}^\ell\varphi(t)]dt\Phi^{(n)}_{m-\ell,j}(x)\right|^p dx\right)^{1/p} \leq$$

$$\leq \sum_{j=-\infty}^{\infty} n^\ell\left(\int_{\Im^{(n)}_{\ell,j}}|t_1|^{\alpha p}|\mathcal{D}^\ell\varphi(x)-\mathcal{D}^\ell\varphi(t)|^p dt\right)^{1/p}\left(\int_{\Im^{(n)}_{\ell,j}}|t|^{-\alpha p'}dt\right)^{1/p'} \times$$

$$\times\int_{j-\frac{m-\ell}{2n}}^{j+\frac{m-\ell}{2n}}|x|^{\alpha p}dx)^{1/p} \leq c_3\varepsilon_{\ell,n},$$

where

$$\varepsilon_{\ell,n} := \sup_j[n^\ell\int_{\Im^{(n)}_{\ell,j}|t|^{\alpha p}}|\mathcal{D}^\ell\varphi(x)-\mathcal{D}^\ell\varphi(t)|^p dt]^{1/p}\to 0 \quad as \quad n\to\infty$$

due to the continuity of $\mathcal{D}^\ell\varphi(x)$ (we remind that $\varphi\in c_0^\infty(\mathbb{R})$); for a general $\varphi\in W^m_{p,\alpha}(\mathbb{R})$ the convergence (3.22) follows since $c_0^\infty(\mathbb{R})$ is a dense subset of $W^m_{p,\alpha}(\mathbb{R})$. □

3.3 De Boor's estimates.

By analogy with (1.21)

$$\ell_{p,\alpha}(\mathcal{Z}) := \{\xi=\{\xi_j\}_{j=-\infty}^{\infty} : \|\xi|\ell_{p,\alpha}(\mathcal{Z})\| = \left(\sum_{j=-\infty}^{\infty}(1+|j|)^{\alpha p}|\xi_j|^p\right)^{1/p} < \infty\}. \tag{3.31}$$

For the norm we should use also the notation $\|\xi\|_{p,\alpha} = \|\xi|\ell_{p,\alpha}(\mathcal{Z})\|$.

Theorem 3.2 *Let $k\geq m$ and (0.4) hold for α and p. The operator*

$$E^{(n)}_m : \ell_{p,\alpha}(\mathcal{Z}) \to S^{(n)}_m(W^k_{p,\alpha}(\mathbb{R})) \tag{3.32}$$

(see (3.19)) is an isomorphism of Banach spaces and the following estimates hold

$$\begin{aligned} c_0^{-1}n^{-\frac{1}{p}-\alpha}\|\xi\|^{(n)}_{p,\alpha} &\leq \|E^{(n)}_m\xi|W^k_{p,\alpha}(\mathbb{R})\| \leq \\ \leq c_0 n^{-\frac{1}{p}-\alpha}\|\xi\|^{(n)}_{p,\alpha} &\leq c_0 n^{-\frac{1}{p}-\alpha}\frac{(2n)^{m+1}-1}{2n-1}\|\xi\|_{p,\alpha}, \end{aligned} \tag{3.33}$$

where

$$\|\xi\|^{(n)}_{p,\alpha} := \sum_{k=0}^{n} n^\ell\|\Delta^\ell\xi\|_{p,\alpha}, \quad \Delta^\ell\xi := \left\{\sum_{i=0}^{\ell}(-1)^{\ell-i}\left(\begin{array}{c}\ell\\ i\end{array}\right)\xi_{j+1}\right\}_{j=-\infty}^{\infty}. \tag{3.34}$$

Proof. Similarly to (3.23) and (3.26) we obtain

$$\begin{aligned}&E_m^{(n)}\xi(x)=n(\Phi_{\ell-1,0}^{(n)}*E_{m-\ell}^{(n)}\xi)(x),\\ &\mathcal{D}^\ell E_m^{(n)}\xi(x)=n(\mathcal{D}^\ell\Phi_{\ell-1,0}^{(n)}*E_{m-\ell}^{(n)}\xi)(x)=\\ &n^\ell\sum_{j=-\infty}^{\infty}(\Delta^\ell\xi)_j\Phi_{m-\ell,j}^{(n)}(x)=n^\ell E_{m-\ell}^{(n)}\Delta^\ell\xi(x),\quad \ell=1,\dots,m.\end{aligned}\tag{3.35}$$

By means of the mean value theorem we can derive the following analogue of formulae (3.29):

$$\left(\int\limits_{j-\frac{m-\ell}{n}}^{j+\frac{m-\ell}{n}}|x|^{\alpha p}dx\right)^{1/p}\le c_0(1+|j|)^{\alpha p}n^{-\frac{1}{p}-\alpha p},\tag{3.36}$$

where $c_0=$ const is independent of $n=1,2,\dots$.

Applying (3.35) and (3.36) we proceed similarly to (3.28), (3.30):

$$\begin{aligned}\|E_m^{(n)}\xi|W_{p,\alpha}^k(\mathbb{R})\| &= \sum_{\ell=0}^{k}\|\mathcal{D}^\ell E_m^{(n)}\xi|W_{p,\alpha}^k(\mathbb{R})\ |L_{p,\alpha}(\mathbb{R})\|=\\ &= \sum_{\ell=0}^{k}n^\ell\left(\int\limits_{j=-\infty}^{\infty}|\sum_{j=-\infty}^{\infty}\Phi_{m-\ell,j}^{(n)}(x)(\Delta^\ell\xi|^p|x|^{\alpha p}dx\right)^{1/p}\le\\ &\le \sum_{\ell=0}^{k}n^\ell(\sum_{j=-\infty}^{\infty}|\Delta^\ell\xi)_j|^p\int\limits_{j-\frac{m-\ell}{2n}}^{j+\frac{m-\ell}{2n}}|x|^{\alpha p}dx)^{1/p}\le\\ &\le c_0\sum_{\ell=0}^{k}n^\ell\left(\int\limits_{j=-\infty}^{\infty}(1+|j|)^{\alpha p}n^{-\alpha p-1}|\Delta^\ell\xi)_j|^p\right)^{1/p}=\\ &= c_0\sum_{\ell=0}^{k}n^\ell-\frac{1}{p}-\alpha\|\Delta^\ell\xi\|_{p,\alpha}=c_0n^{-\frac{1}{p}-\alpha}\|\xi\|_{p,\alpha}^{(n)}\le\\ &\le c_0n^{-\frac{1}{p}-\alpha}\frac{(2n)^{m+1}-1}{2n-1}\|\xi\|_{p,\alpha},\end{aligned}$$

since $|\Phi_{m-\ell,j}^{(n)}\le 1$ and this function is supported on the interval $\left[j-\frac{m-\ell}{2n},j+\frac{m-\ell}{2n}\right]$ (see (1.17) and (3.18)).

The right inequalities in (3.33) are proved.

To prove the left inequalities (i.e. the inverse estimates) we recall the one proved in [27, Theorem 2.6]: the inequality

$$\|E_m^{(n)}n|L_{p,\alpha}(\mathbb{R})\|\le c_1n^{-\frac{1}{p}-\alpha}\|n\in\ell_{p,\alpha}(\mathcal{Z})\|,\quad c_2=const,\quad n\in\ell_{p,\alpha}(\mathcal{Z})\tag{3.37}$$

holds provided the condition

$$\lambda^{\Phi_m,\Phi_m}(n):=\sum_{j=-\infty}^{\infty}\left(\int\limits_{\mathbb{R}}\Phi_m(t+j)\bar{\Phi_m}(t)\right)\ell^{ink}\ne 0\tag{3.38}$$

is fulfilled for all $0 \le n \le 2\pi$. Due to the formula 2.7.6(4) in [27].

$$\lambda^{\Phi_m,\Phi_m}(n) = \sum_{j=-\infty}^{\infty} \left|\mathcal{F}\Phi_m\left(\frac{n}{2\pi}+j\right)\right|^2 \neq 0 \quad (0 \le n \le 2\pi)$$

(cf. (3.14),(3.15)) and (3.38) holds.

We proceed with the help of (3.35) and (3.37):

$$\|E_m^{(n)}\xi|W_{p,\alpha}^k(\mathbb{R})\| = \sum_{\ell=0}^{k} \|\mathcal{D}^\ell E_m^{(n)}\xi|L_{p,\alpha}(\mathbb{R})\| = \sum_{\ell=0}^{k} \|E_{m-\ell}^{(n)}\Delta^\ell\xi\|_{p,\alpha} \ge$$
$$\ge c_1 \sum_{\ell=0}^{k} n^{\ell-\frac{1}{p}-\alpha}\|\Delta^\ell|\ell_{p,\alpha}(\mathcal{Z})\| = c_2 n^{\frac{1}{p}-\alpha}\|\xi\|_{p,\alpha} \ge\ge c_2 n^{-\frac{1}{p}-\alpha}\|\xi\|_{p,\alpha}.$$

□

Corollary 3.3 *The spline spaces $S_m^{(n)}(W_{p,\alpha}^k(\mathbb{R}))$ are independent of $k = 1,2,\ldots,m$ and can be described as follows:*

$$S_m^{(n)}(W_{p,\alpha}^k(\mathbb{R})) = \{\varphi \in W_{p,\alpha}^k(\mathbb{R}) : \varphi = \sum_{j=-\infty}^{\infty} \xi_j \Phi_{m,j}^{(n)}, \xi = \{\xi_j\}_{-\infty}^{\infty} \in \ell_{p,\alpha}(\mathcal{Z})\}.$$

Remark 3.4 *C.de Bohr was first who proved estimates like* (3.33) *(cf. [4]). In [27, Sect. 2.12.2.] the estimates* (3.33) *are proved for* $k = 0$ *and in our proof we applied this result.*

3.4 Spline projections and quasiprojections.

Based on the operators $\tilde{l}_m^{(n)}$ (see (3.19)) and following [27, Sect. 2.7] we should describe spline projections (see Theorem (3.7))

$$\begin{gathered} p_m^{(n)} = W_{p,\alpha}^k(\mathbb{R}) \to S_m^{(n)}(W_{p,\alpha}^k(\mathbb{R})), \quad 1 \le k \le m, \\ (p_m^{(n)})^2 = p_m^{(n)}, \quad p_m^{(n)}\varphi\left(\frac{j}{n}\right) = \varphi\left(\frac{j}{n}\right), \quad j = 0, \pm 1, \ldots. \end{gathered} \tag{3.39}$$

For this we shall prove the following.

Lemma 3.5 *The operator*

$$\mathcal{D}^{(n)} : W_{p,\alpha}^K(\mathbb{R}) \to \ell_{p,\alpha}(\mathcal{Z}), \quad 1 \le 1, \quad 1 < p < \infty, \quad 0 < \frac{1}{p} + \alpha < 1 \tag{3.40}$$

(see (3.19) *for* $\mathcal{D}^{(n)}$*) is bounded and*

$$\|\mathcal{D}^{(n)}\varphi|\ell_{p,\alpha}(\mathcal{Z})\| \le c' n^{\frac{1}{p}+\alpha}\|\varphi|W_{p,\alpha}^k(\mathbb{R})\|. \tag{3.41}$$

Proof Since $\tilde{p}_m^{(n)} = E_m^{(n)}\mathcal{D}^{(n)}$ is bounded un $W^k_{p,\alpha}(\mathbb{R})$ (see Theorem 3.1) and $E_m^{(n)}$ is invertible with the norm estimate

$$\|(E_m^{(n)})^{-1}\| \leq c_o n^{\frac{1}{p}+\alpha}$$

(see (3.32) and (3.33)), we get the boundedness (3.40) and estimates (3.41) since $\mathcal{D}^{(n)} = (E_m^{(n)})^{-1}P_m^{(n)}$. $\square$.

It is an easy exercise to find out that the operator

$$\mathcal{D}^{(n)}E_m^{(n)} = \|\Phi_m(j-\ell)\|_{j,\ell\in\mathcal{Z}} =: Tg_m^0 \tag{3.42}$$

is a Töplitz operator (see [27, Propos. 2.14 and 2.18]) with the symbol, defined in (1.24) Due to the property (3.9)

$$g_m(0) = g_m(2\pi) = 1, \quad g_m \in C^\infty([0,2\pi]). \tag{3.43}$$

From (3.4) we easily find

$$g_0(n) \equiv g_1(n) \equiv 1, \quad g_2(n) = \frac{3+\cos n}{4}. \tag{3.44}$$

Lemma 3.6 *The condition* (1.20) *is implied hold, by the condition* (1.21).

Proof. From the formula [27, 2.7.6(4)] we get

$$\begin{aligned} g_m(n) &= \lambda^{\Phi_m,\delta_0^{(1)}}\left(\frac{n}{2\pi}\right) = \sum_{j=-\infty}^{\infty} \mathcal{F}\Phi_m\left(\frac{n}{2\pi}+j\right)\mathcal{F}\delta_j^{(1)}(n) = \\ &= \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}\Phi_m\left(\frac{n}{2\pi}+j\right) = \sum_{j=-\infty}^{\infty}\left[\frac{\sin\pi\left(\frac{n}{2\pi}+j\right)}{\frac{n}{2}+\pi j}\right]^{m+1} = \\ &= \begin{cases} \left(\frac{\sin\frac{n}{2}}{\pi}\right)\sum\limits_{j=-\infty}^{\infty}\frac{(-1)^{j(m+1)}}{\left(\frac{n}{2\pi}+j\right)^{m+1}}, & 0\leq n<\pi, \\ \left(\dfrac{\sin\left(\pi-\frac{n}{2}\right)}{\pi}\right)\sum\limits_{j=-\infty}^{\infty}\dfrac{(-1)^{j(m+1)}}{\left(\frac{n}{2}+j\right)^{m+1}}, & \pi<n\leq 2\pi, \end{cases} \end{aligned} \tag{3.45}$$

where $\mathcal{F}$ is the Fourier transform (see (3.14), (3.15)). Obviuosly. (1.20) holds provided $m = 2\ell - 1$ is add.

It is known, that

$$\Phi_m(0) > \frac{1}{2} \quad \text{if} \quad m = 0,1,\ldots,7, \tag{3.46}$$

while $\Phi_8(0) < \frac{1}{2}$ (see [26]); therefore

$$\begin{aligned} |g_m(n)| &= \left|\sum_{j=-\infty}^{\infty}\Phi_m(j)\ell^{inj}\right| = \left|\Phi_m(n) + 2\sum_{j=1}^{\infty}\Phi_m(j)\cos nj\right| = \\ &= \left|\sum_{j=-\infty}^{\infty}\Phi_m(j) - 2\sum_{j=1}^{\infty}\Phi_m(j)(1-\cos nj)\right| = \left|1-2\sum_{j=1}^{\infty}\phi_m(j)(1-\cos nj)\right| \geq \\ &\geq |1-2\Phi_m(0)| > 0, \quad 0\leq n\leq 2\pi, \quad m=1,2,\ldots,7, \end{aligned}$$

since $\sum\limits_{j=-\infty}^{\infty} \Phi_m(j) = 1$ (see (3.9)).

Further we shall suppose condition (1.20) fulfilled. Then the operator $T^0_{g_m^{-1}}$ is bounded in $\ell_{p,\alpha}(\mathcal{Z})$ (see [5, 12]) since $g_m^{-1} \in V_1(\mathbb{R})$–has a bounded total variation on $\mathbb{R}$.

Theorem 3.7 *The operator*

$$p_m^{(n)} := E_m^{(n)} T^0_{g_m^{-1}} \mathcal{D}^{(n)} \tag{3.47}$$

has all properties listed in (3.39) *and converges strongly to the identity operator as* $n \to \infty$

$$\lim_{n\to\infty} p_m^{(n)} \varphi = \varphi \quad \text{for all} \quad \varphi \in W^k_{p,\alpha}(\Im), \quad 1 \le k \le m. \tag{3.48}$$

Proof. (see [27, Sect. 2.6.3, 2.7.4]. Boundedness of $p_m^{(n)}$ follows from Theorem 3.2 and Lemma 3.5.

$p_m^{(n)}$ is a projection since

$$(p_m^{(n)})^2 = E_m^{(n)} T^0_{g_m^{-1}} \mathcal{D}^{(n)} E_m^{(n)} T^0_{g_m^{-1}} \mathcal{D}^{(n)} = E_m^{(n)} T^0_{g_m^{-1}} \mathcal{D}^{(n)} T^0_{g_m} T^0_{g_m^{-1}} \mathcal{D}^{(n)} = p_m^{(n)}$$

(see (3.42)) and

$$\mathcal{D}^{(n)} p_m^{(n)} \varphi = \mathcal{D}^{(n)} E_m^{(n)} T^0_{g_m^{-1}} \mathcal{D}^{(n)} \varphi = T^0_{g_m} T^0_{g_m^{-1}} \mathcal{D}^{(n)} \varphi = \mathcal{D}^{(n)} \varphi,$$

which proves the last equality in (3.39).

From (3.47) and (3.48) we find that

$$p_m^{(n)} = E_m^{(n)} T^0_{g_m^{-1}} (E_m^{(n)})^{-1} \widetilde{p}_m^{(n)};$$

since $\widetilde{p}_m^{(n)}$ are uniformly bounded, due to (3.22) and the Banach–Steinhaus theorem, invoking (3.33) we conclude the same for $p_m^{(n)}$:

$$\|p_m^{(n)} \varphi | W^k_{p,\alpha}(\mathbb{R})\| \le M \|\varphi | W^k_{p,\alpha}(\mathbb{R})\| \tag{3.49}$$

with a constant M independent of n and $\varphi \in W^k_{p,\alpha}(\mathbb{R})$.

To prove (3.48) we proceed as follows

$$\lim_{n\to\infty} \|p_m^{(n)} \varphi - \varphi | W^k_{p,\alpha}(\mathbb{R})\| \le \lim_{n\to\infty} (\|p_m^{(n)} (\varphi - \widetilde{p}_m^{(n)} \varphi) | W^k_{p,\alpha}(\mathbb{R})\| +$$
$$\|\widetilde{p}_m^{(n)} \varphi - \varphi | W^k_{p,\alpha}(\mathbb{R})\|) \le \lim_{n\to\infty} (\|p_m^{(n)}\|^{(k)}_{p,\alpha} + 1) \|\widetilde{p}_m^{(n)} \varphi - \varphi | W^k_{p,\alpha}(\mathbb{R})\| = 0$$

(cf. (3.49)) since $\widetilde{p}_m^{(n)} \varphi \in S_m^{(n)}(\mathbb{R})$ and, therefore, $p_m^{(n)} \widetilde{p}_m^{(n)} \varphi = \widetilde{p}_m^{(n)} \varphi$. $\square$.

Remark 3.8 *Since* $g_1(\theta) \equiv 1$ (*see* (3.44)) *we find that* $p_1^{(n)} = \widetilde{p}_1^{(n)}$.

Let $\mathbb{R}^+ = [0, \infty\}$ and

$$\begin{aligned} \widetilde{W}^k_{p,\alpha}(\mathbb{R}^+) = \{\varphi \in W^k_{p,\alpha}(\mathbb{R}) : \mathbf{supp}\,\varphi \subset \mathbb{R}^+\}, \\ S^{(n)}_m(\mathbb{R}^+) = \mathbf{Span}\{\Phi^{(n)}_{mj}\}^\infty_{j=\left[\frac{m}{2}\right]+1}; \end{aligned} \tag{3.50}$$

if, as usual, $S^{(n)}_m(\widetilde{W}^k_{p,\alpha}(\mathbb{R}^+))$ denotes the subspase of $W^k_{p,\alpha}(\mathbb{R}^+)$ generated by splines $S^{(n)}_m(\mathbb{R}^+)$ (see subsection 3.4), we can define the spline projector

$$P^k_{\mathbb{R}^+,m} : \widetilde{W}^k_{p,\alpha}(\mathbb{R}^+) \to S^{(n)}_m(\widetilde{W}^k_{p,\alpha}(\mathbb{R}^+)) \tag{3.51}$$

as follows: let

$$\begin{aligned} \widetilde{E}^{(n)}_m \xi(x) := \sum_{j=1}^{\infty} \xi_j \Phi^{(n)}_{mj}\left(x - \frac{1}{n}\left[\frac{m}{2}\right]\right), \\ \overset{\wedge}{\mathcal{D}}^{(n)} \varphi := \left\{\varphi\left(\frac{1}{n}\right)\right\}^\infty_{j=1}; \end{aligned} \tag{3.52}$$

then

$$\begin{aligned} \overset{\wedge}{\mathcal{D}}^{(n)} \widetilde{E}^{(n)}_m &= \left\|\Phi_m\left(j - \ell - \frac{1}{n}\left[\frac{m}{2}\right]\right)\right\|^\infty_{j,\ell=1} = T_{\widetilde{g}_m}, \\ \widetilde{g}_m(\theta) &= \ell^{-\left[\frac{m}{2}\right]\theta i} g_m(\theta); \end{aligned}$$

the Töplit operator is right invertible unless $m = 1$ (then it is invertible), since $\operatorname{ind} \widetilde{g}_m = -\left[\frac{m}{2}\right]$ (see [5, 12, 25]). The operator $P^{(n)}_{\mathbb{R}^+,m} = \widetilde{E}^{(n)}_m (T_{\widetilde{g}_m})^{-1} \overset{\wedge}{\mathcal{D}}^{(n)}$ is a projector (3.51), but it does not suit our purposes $P^{(n)}_{\mathbb{R}^+,m}\varphi\left(\frac{j}{n}\right) \neq \varphi\left(\frac{j}{n}\right)$. We would prefer other operator – quasiprojector, which is defined as follows

$$\hat{P}^{(n)}_{\mathbb{R}^+,m}\varphi = \hat{E}^{(n)}_m T^{-1}_{g_m} \overset{\wedge}{\mathcal{D}}^{(n)} : W^k_{p,\alpha}(\mathbb{R}^+) \to S^{(n)}_m(W^k_{p,\alpha}(\mathbb{R}^+)), \quad 1 \le k \le m < \infty, \tag{3.53}$$

where $\overset{\wedge}{\mathcal{D}}^{(n)}$ is defined in (3.52) and

$$\hat{E}^{(n)}_m \xi(x) := \sum_{j=\left[\frac{m}{2}\right]+1}^{\infty} \xi_j \Phi^{(n)}_{mj}(x).$$

The inverse $(T_{g_m})^{-1}$ to the Töplitz operator $T_{g_m} = \|\Phi_m(j-\ell)\|^\infty_{j,\ell=1}$ exists (see [5, 12, 25]) since we suppose (1.28) fulfilled, which implies (1.27) (see (3.6)) and, additionally, $\operatorname{ind} g_m = 0$ since $g_m(\theta)$ is real valued.

$\hat{P}^{(n)}_{\mathbb{R}^+,m}$ has the following properties

$$\begin{gathered} (\hat{P}^{(n)}_{\mathbb{R}^+,m})^2 = \hat{E}^{(n)}_m (T_{g_m})^{-1} \overset{\wedge}{\mathcal{D}}^{(n)} \hat{E}^{(n)}_m T^{-1}_{g_m} \overset{\wedge}{\mathcal{D}}^{(n)} = \hat{E}^{(n)}_m T^{-1}_{g_m} R_m T^{-1}_{g_m} \overset{\wedge}{\mathcal{D}}^{(n)} = \hat{P}^{(n)}_{\mathbb{R}^+,m} + K^{(n)}_m, \\ \hat{P}^{(n)}_{\mathbb{R}^+,m}\varphi\left(\frac{j}{n}\right) = \varphi\left(\frac{j}{n}\right), \quad j = \left[\frac{m}{2}\right] + 1, \ldots, \end{gathered} \tag{3.54}$$

where $K_m^{(n)} = \hat{E}_m^{(n)} (I - R_m)\overset{\wedge}{\mathcal{D}}^{(n)}$ is a finite ($\left[\frac{m}{2}\right]$–dimensional) operator since

$$R_m\xi := \{0, \dots, 0, \xi_{\left[\frac{m}{2}\right]+1}, \dots\}$$

and $I - R_m$ is $\left[\frac{m}{2}\right]$-dimensional projector. Another example of the quasiprojector is the operator $\hat{P}_{\Im,m}^{(n)}$, defined in (1.19).

Operators $\hat{P}_{\mathbb{R}^+,m}^{(n)}$ and $\hat{P}_{\Im,m}^{(n)}$ are both restrictions of the projector $P_m^{(n)}$ (see (3.47)) to the half–axes $\mathbb{R}^+$ and to the interval $\Im = [0, 1]$, respectively.

$H_p^s(\mathbb{R})$ and $\mathcal{Z}^t(\mathbb{R})$ ($1 < p < \infty$, $s \in \mathbb{R}$, $0 < t < \infty$) denote the Bessel and Zygmound spaces (see definitions in [29, 60, 61]); $H_p^m(\mathbb{R}) = W_p^m(\mathbb{R})$ for integer $m = 1, 2, \dots$ and $\mathcal{Z}^t(\mathbb{R}) = C^+(\mathbb{R})$ (the Hölder spaces) for non–integer $t \in \mathbb{R}^+ \backslash \{0, 1, \dots\}$.

Remark 3.9 *In [21, 23] it is proved that:*

(i) *if* $-\infty < s \le \nu \le m + 1$ *and* $s < m + 1/2$, *then there exists a constant* C, *independent of* $n = 1, 2, \dots,$ *such that*

$$\mathbf{inf}\{\|(u - v)|H_p^s(\mathbb{R})\| : \nu \in S_m^{(n)}(\mathbb{R})\} \le Cn^{s-\nu}\|u|H_p^\nu(\mathbb{R})\|$$

for any $u \in H_p^\nu(\mathbb{R})$*;*

(ii) *if* $0 < s < \nu \le m$, *there exists a constant* C, *independent of* $n = 1, 2, \dots,$ *such that*

$$\mathbf{inf}\{\|(u - v)|\mathcal{Z}^s(\mathbb{R})\| : \nu \in S_m^{(n)}(\mathbb{R})\} \le Cn^{s-\nu}\|u|\mathcal{Z}^\nu(\mathbb{R})\|$$

for any $u \in \mathcal{Z}^\nu(\mathbb{R})$.

Better approximation can be gained with non–uniform meshes. If $\Delta_n = \{x_1, \dots, x_n\} \subset \Im = [0, 1]$ are such that $|x_j - x_{j-1}| \sim \frac{1}{n}\left(\frac{j}{n}\right)^q$ $\quad(q \ge 1)$, then we write $\Delta_n \in \mathcal{M}_q$ (see [21]). Let

$$L_\mu^{p,m}(\Im) := \{\varphi \in L_p(\Im) : x^{k-\mu}\mathcal{D}^k\varphi \in L_p(\Im), \quad k = 1, \dots, m\}.$$

Lemma 3.10 *If* $u \in L_\mu^{p,m}(\Im)$, $\mu < \nu$, $\Delta_n \in \mathcal{M}_q$, $q \ge m(\rho - \mu)$, *then*

$$\|(I - Q_m^{(n)})u|L_\mu^p(\Im)\| \le Cn^{-m}\|u|L_\rho^{p,m}(\Im)\|,$$
$$\|(I - Q_m^{(n)})u|L^p(\Im)\| \le Cn^{-\ell}\|u|W_p^m(\Im)\|,$$

where $Q_m^{(n)}$ *are the orthgonal projectots* $Q_m^{(n)} : L_2(\Im) \to S_m^{(n)}(L_2(\Im))$ *(see below).*

We conclude this subsection by several remarks on orthogonal projectors.

The projector $P_m^{(n)}$ in (3.47) is not orthogonal, i.e. a self adjoint one (in $L_2(\mathbb{R})$-space). To construct the orthogonal projector we should consider the operators

$$\mathcal{D}_m^{(n)}\varphi := \left\{ n\int_{-\infty}^{\infty} \Phi_{mj}^{(n)}(y)\varphi(y)dy \right\}_{j=-\infty}^{\infty},$$

which are bounded

$$\mathcal{D}_m^{(n)} : W_{p,\alpha}^k(\mathbb{R}) \to \ell_{p,\alpha}(\mathcal{Z})$$

(as $\mathcal{D}_m^{(n)}$ in (3.40)). Then

$$\begin{aligned}
\mathcal{D}_m^{(n)} E_m^{(n)} \xi &= \left\{ \sum_{\ell=-\infty}^{\infty} \xi_\ell \int_{-\infty}^{\infty} \Phi_m(y-(j-\ell))\Phi_m(y)dy \right\}_{j=-\infty}^{\infty} = \\
&= \left\{ \sum_{J=-\infty}^{\infty} \xi_\ell \Phi_{2m+1}(j-\ell) \right\}_{j=-\infty}^{\infty} = T_{g_{2m+1}}^0 \xi
\end{aligned}$$

(see (3.13)). Due to Lemma (3.6) $T_{g_{2m+1}}^0$ is invertible for all $m = 1, 2, \ldots$. Therefore the operator

$$Q_m^{(n)} = E_m^{(n)} T^0_{g^{-1}_{2m+1}} \mathcal{D}_m^{(n)}$$

is a projector

$$\begin{aligned}
Q_m^{(n)} &: W_{p,\alpha}^k(\mathbb{R}) \to S_m^{(n)}(W_{p,\alpha}^k(\mathbb{R})), \quad 1 \le k \le m \\
(Q_m^{(n)})^2 &= Q_m^{(n)}
\end{aligned}$$

and is self-adjoint $(Q_m^{(n)})^* = Q_m^{(n)}$, since

$$\begin{aligned}
(\mathcal{D}_m^{(n)})^* &= E_m^{(n)}, \quad (E_m^{(n)})^* = \mathcal{D}_m^{(n)}, \\
(T_{g_{2m+1}}^0)^* &= T_{\bar{g}_{2m+1}}^0 = T_{g_{2m+1}}^0.
\end{aligned}$$

For the space $\widetilde{W}_{p,\alpha}^k(\mathbb{R}^+)$ we can define a quasiprojector

$$\hat{Q}{}_m^{(n)} = \hat{E}{}_m^{(n)} (T_{g_{2m+1}})^{-1} \overset{\wedge}{\mathcal{D}}{}_m^{(n)} : \widetilde{W}_{p,\alpha}^m(\mathbb{R}^+) \to S_m^{(n)}(\widetilde{W}_{p,\alpha}^m(\mathbb{R}^+))$$

where $\overset{\wedge}{\mathcal{D}}{}_m^{(n)}$ is the restriction of $\mathcal{D}_m^{(n)}$ to $\mathbb{R}^+$:

$$\overset{\wedge}{\mathcal{D}}{}_m^{(n)} \psi := \left\{ \underbrace{0, \ldots 0}_{\left[\frac{m}{2}\right]-times}, n \int_0^{\infty} \Phi^{(n)}_{m,\left[\frac{m}{2}\right]+1}(y)\psi(y)dy, \ldots \right\}, \quad \psi \in \widetilde{W}_{p,\alpha}^m(\mathbb{R}^+);$$

$T_{g_{2m+1}} = \|\Phi_{2m+1}(j-\ell)\|_{j,\ell=1}^{\infty}$ is a Töplitz matrix and is invertible in $\ell_{p,\alpha}(\mathcal{N})$ (see (3.53). Since

$$\overset{\wedge}{\mathcal{D}}{}_m^{(n)} \hat{E}{}_m^{(n)} = T_{g_{2m+1}} R_m$$

(see (3.54) for R_m), we easily find that

$$\begin{aligned}
(\hat{Q}{}_m^{(n)})^2 &= Q_m^{(n)} + \widetilde{K}_m^{(n)}, \\
\widetilde{K}_m^{(n)} &:= \hat{E}{}_m^{(n)} T_{g_{2m+1}}^{-1} (I - R_m) T g_{2m+1} R_m T_{g_{2m+1}}^{-1} \overset{\wedge}{\mathcal{D}}{}_m^{(n)} + \\
&\quad + \hat{E}{}_m^{(n)} (I - R_m) T_{g_{2m+1}}^{-1} \overset{\wedge}{\mathcal{D}}{}_m^{(n)},
\end{aligned}$$

where $\widetilde{K}_m^{(n)}$ is a finite dimensional operator.

If we define

$$\widetilde{\mathcal{D}}_m^{(n)}\varphi := \left\{ n\int\limits_0^\infty \Phi_{mj}^{(n)}\left(y - \frac{1}{n}\left[\frac{m}{2}\right]\right)\varphi(y)dy \right\}_{j=1}^\infty$$

we can prove that $\widetilde{\mathcal{D}}_m^{(n)}\widetilde{E}_m^{(n)} = T_{\widetilde{g}_{2m+1}}$ (see (3.52)) where $T_{\widetilde{g}_{2m+1}}$ is a right invertible Töplitz matrix (operator); the operator $\widetilde{Q}_m^{(n)} = \widetilde{E}_m^{(n)} T_{\widetilde{g}_{2m+1}}^{-1} \widetilde{\mathcal{D}}_m^{(n)}$ would be a projector, but non self–adjoint, because $\overline{\widetilde{g}_{2m+1}} \neq \widetilde{g}_{2m+1}$.

Again, for $m = 1$ the operatorrs $\widetilde{Q}_1^{(n)} = Q_1^{(n)}$ coinside and represent orthogonal projectors.

3.5 Banach algebra technique in approximation methods.

In the present subsection we recall some results from [41] which would be applied to prove stability results claimed in Theorems 1.5 and 1.8.

The algebraic approach to the proof of stability was first suggested by A. Kozak in [35] and developed further in [56]. We stick here to the sheme suggested in [41].

Let X be a Banach space and $X_n \subset X$–subspaces with projections $P_n : X \to X_n$ converging to the identity strongly: $P_n\varphi \to \varphi$ as $n \to \infty$ for all $\varphi \in X$.

$\Omega_b(X)$ will denote the Banach algebra of sequences $\mathcal{A} := \{A_n\}_{n\in\mathcal{N}}$ of uniformly bounded linear operators $A_n : X_n \to X_n$, with the pointwise composition as multiplication

$$\{A_n\}\{B_n\}_n = \{A_nB_n\}_n$$

and endowed with the uniform norm

$$\|\{A_n\}_n\| := \sup_n \|A_n\|.$$

Let $\Omega_0(X)$ denote the ideal of sequences $\{T_n\}_n$ converging to 0 in norm

$$\Omega_0(\mathcal{L}) = \{\{T_n\} \in \Omega_b(X) : \lim_n \|T_n\| = 0\}. \tag{3.55}$$

We consider two further ideals

$$\begin{array}{lll} \Omega_c(X) := \{\{B_n + P_nT|_{X_n}\}_n : \{B_n\}_n \in \Omega_0(X), & T \quad \text{is compact in} \quad X\}, \\ \Omega_s(X) := \{\{\mathcal{D}_n\}_n \in \Omega_b(X) : \lim\limits_n \mathcal{D}_n\varphi = 0 & \text{for all} \quad \varphi \in X\}. \end{array} \tag{3.56}$$

A sequence $\{A_n\} \in \Omega_b(X)$ is called stable if

(a) it converges strongly to some bounded operator in ??: $\lim_n A_n\varphi = A\varphi$ for all $\varphi \in X$;

(b) An are invertible for all sufficiently large $n \geq N_0$ and the inverses are uniformly bounded $\sup\limits_n \|A_n^{-1}\| \leq M < \infty$.

It is easy to ascertain thet stability is equivalent the invertibility of the quotient class $\{A_n\}_n^0$ corresponding to the sequence $\{A_n\}_n$ in the quotient algebra $\Omega_b(X)/\Omega_0(X)$.

The next theorem was proved in [56]. The short and clegant proof exposed here is suggested in [41].

Theorem 3.11 *A sequence $\{A_n\}_n \in \Omega_b(X)$ is stable if and only if:*

(i) *the limit operator $A_n\varphi = \lim_n A_n\varphi$ is invertible;*

(ii) *the quotient class $\{A_n\}_n$ is invertible in the quotient algebra $\Omega_b(X)/\Omega_0(X)$.*

Proof. It is easy to check that

$$\Omega_c(X) \cap \Omega_s(X) = \Omega_0(X). \tag{3.57}$$

The quotient class $\{A_n\}_n^0 \in \Omega_b(X)/\Omega_0(X)$ is invertible if and only if the quotient classes $\{A_n\}_n^c \in \Omega_b(X)/\Omega_c(X)$ and $\{A_n\}_n^s \in \Omega_b(X)/\Omega_s(X)$ are both invertible. In fact, if $\{A_n\}_n^c$ and $\{A_n\}_n^s$ are invertible, there exist $\{C_n\}_n$, $\{S_n\}_n \in \Omega_b(X)$ such that

$$\begin{aligned} \{C_n\}_n\{A_n\}_n &= \{I + T_n\}_n, \quad \{T_n\}_n \in \Omega_c(X), \\ \{S_n\}_n\{A_n\}_n &= \{I + K_n\}_n, \quad \{K_n\}_n \in \Omega_s(X), \end{aligned}$$

where I is the identity operator. If $B_n = C_n + S_n - C_nAS_n$, then

$$\{B_n\}_n\{A_n\}_n = \{I - T_nK_n\}_n$$

where, due to (3.57), $\{T_nK_n\}_n \in \Omega_0(X)$; similarly is proved that $\{A_n\}_n\{B_n\}_n = \{I - K_nT_n\}_n$ and, therefore, $\{B_n\}_n^0$ is the inverse to $\{B_n\}_n^0$ in $\Omega_s(X)/\Omega_0(X)$.

The inverse assertion is trivial, since $\Omega_0(\mathcal{L})$ is contained in both ideals $\Omega_c(X)$ and $\Omega_s(X)$.

To conclude the proof it remains to note that invertibility in the quotient algebra $\Omega_b/\Omega_s(X)$ is equivalent to the invertibility of the limit operator A in X. □

After "algebraizing" an approximation scheme in Theorem (3.11) we should apply a localization principle from [24] (see Section 4). The concluding propositions in this section will be applied to implement the local principle.

Let

$$\begin{aligned} \dot{C}(\mathbb{R}) &:= \{b \in C(\mathbb{R}) : b(-\infty) = b(\infty)\}, \\ C^k(\mathbb{R}) &:= \{b \in \dot{C}(\mathbb{R}) : \mathcal{D}^\ell b \in \dot{C}(\mathbb{R}), \qquad \ell = 1, \dots, k\}, \end{aligned} \tag{3.58}$$

and the norm of an operator A in $W^k_{p,\alpha}(\mathbb{R})$ is denoted by $\|A\|^{(k)}_{p,\alpha}$.

For a polinomial

$$\lambda_0^k(n) := (1 + 2\pi\eta)^k - \sum_{\ell=0}^{k} \begin{pmatrix} k \\ \ell \end{pmatrix} (2\pi i\eta)^\ell$$

the operator

$$\begin{aligned} &\Lambda^k\varphi := \mathcal{F}^{-1}\lambda_0^k\mathcal{F}\varphi = \sum_{\ell=0}^{k} \begin{pmatrix} k \\ \ell \end{pmatrix} \mathcal{D}^\ell\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}) \\ &\Lambda^k : W^k_{p,\alpha}(\mathbb{R}) \to L_{p,\alpha}(\mathbb{R}) \end{aligned} \tag{3.59}$$

is, obviously, bounded (see (3.14) for the Fourier transforms $\mathcal{F}^{\pm 1}$). It is less obvious but also well–known that the inverse operator to Λ^k exists and

$$\Lambda^{-k}\varphi = \mathcal{F}^{-1}\lambda_0^{-k}\mathcal{F}\varphi, \quad \Lambda^{-k}\Lambda^k\varphi = \Lambda^k\Lambda^{-k}\varphi = \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}) \tag{3.60}$$

(see [12, 16, 29, 61] for the Bessel potential operators).

Lemma 3.12 *Let $a \in V_1(\mathbb{R}) \cap C(\dot{\mathbb{R}})$, $b \in C^k(\dot{\mathbb{R}})$ and $1 < p < \infty$, $0 < \frac{1}{p} + \alpha > 1$, $k, m \in \mathcal{N}$, $1 \le k \le m$. Then*

$$B_m^{(n)}(a)bI - bB_m^{(n)}(a) = \sum_{\ell=0}^{k} \hat{B}{}_j^{(n)}(a)T_j + R_n, \tag{3.61}$$

where

$$B_j^{(n)}(a) := E_j^{(n)}T_a^0\mathcal{D}^{(n)}, \quad \hat{B}{}_j^{(n)}(a) := \Lambda^{-k}B_j^{(n)}(a)\Lambda^k, \quad \lim_{n\to\infty}\|R_n\|_{p,\alpha}^{(k)} = 0, \tag{3.62}$$

and [2] *$T_0, \ldots, \mathcal{T}_k$ are some compact operators in $W_{p,\alpha}^k(\mathbb{R})$.*

Proof. For $k = 0$ the claim was proved in [27, Sect. 2.7.5] in the following stronger form:

$$\text{if } R_{n,\ell}^1 := B_\ell^{(n)}(a)bI - bB_\ell^{(n)}(a), \quad \text{then } \lim_{n\to\infty}\|R_{n,\ell}^1\|_{p,\alpha} = 0. \tag{3.63}$$

Since the operator (3.59) and the inverse

$$\Lambda^{-k} : L_{p,\alpha}(\mathbb{R}) \to W_{p,\alpha}^{(k)}(\mathbb{R}) \tag{3.64}$$

are isomorphisms, we have to prove that

$$A_m^{(n)} := \Lambda^k[B_m^{(n)}(a)bI - bB_m^{(n)}(a)]\Lambda^{-k} = \sum_{j=0}^{k} B_j^{(n)}T_j' + R_n^2, \quad \lim_{n\to\infty}\|R_n^2\|_{p,\alpha} = 0, \tag{3.65}$$

where $T_0', \ldots, T_k'$ are compact in $L_{p,\alpha}(\mathbb{R})$.

In (3.35) we already proved that

$$\mathcal{D}^\ell E_m^{(n)}\xi = n^\ell E_{m-\ell}^{(n)}\Delta^\ell\xi, \quad l \le m;$$

it can be easily verified that

$$\Delta^\ell T_a^0\xi = T_a^0\Delta^\ell\xi, \quad n^\ell\Delta^\ell\mathcal{D}^{(n)}\varphi = \mathcal{D}^{(n)}\Delta_{\frac{1}{n}}^\ell\varphi,$$

[2] Note that if $\hat{B}{}_j^{(n)}$ i.e. $B_j^{(n)}$ converge to some bounded operators $\hat{B}_j$ strongly as $n \to \infty$, then $\sum\limits_{\ell=0}^{k} \hat{B}{}_j^{(n)}(a)T_j = T + \widetilde{R}_n$, $\|\mathbb{R}_n\|_{p,\alpha}^{(k)} \to 0$ (see Lemma (3.13) below)

where $\Delta_{\frac{1}{n}}\varphi(x) := n\left[\varphi\left(x+\frac{1}{n}\right)-\varphi(x)\right] = \mathcal{D}\varphi(x)+R^3_n\varphi(x)$, $\|R^3_n\varphi\|_{p,\alpha} \leq (2n)^{-1}\|\varphi\|^{(2)}_{p,\alpha}$, for all $\varphi \in W^2p,\alpha(\mathbb{R})$. These formulas result into the following:

$$\begin{aligned}&\mathcal{D}^\ell B^{(n)}_m(a)\Lambda^{-k} = B^{(n)}_{m-\ell}(a)\Delta^\ell_{\frac{1}{n}}\Lambda^{-k} = B^{(n)}_{m-\ell}(a)\mathcal{D}^\ell\Lambda^{-k} + R^4_{m,\ell},\\ &\|R^4_{n,\ell}\psi\|_{p,\alpha} \leq M_1 n^{-1}\|\psi\|^{(k)}_{p,\alpha} \quad \text{for all} \quad l \leq k-1, \quad \text{and all} \quad \psi \in W^k_{p,\alpha}(\mathbb{R}).\end{aligned} \tag{3.66}$$

Since $\mathcal{D}^\ell\Lambda^k = W^0_{g_{\ell,k}} := \mathcal{F}^{-1}g_{\ell,k}\mathcal{F}$ is a convolution operator (see [12, 14]) with the symbol

$$g_{\ell,k}(\theta) = \frac{(2\pi\theta i)^\ell}{(1+2\pi\theta i)^k},$$

operators

$$T_{i,\ell} : \mathcal{D}^i b\mathcal{D}^\ell\Lambda^{-k} \quad \text{are compact in} \quad L_{p,\alpha}(\mathbb{R}) \quad \text{provided} \quad l \leq k-1,\, 0 \leq i \leq k. \tag{3.67}$$

(3.67) follows from [14, Theorem] since $\partial^i b(\pm\infty) = g_{\ell,k}(\pm\infty) = 0$.

From (3.59), (3.65) and (3.66) we find that

$$\begin{aligned}A^{(n)}_m &= \sum_{\ell=0}^{k}\mathcal{D}^\ell[B^{(n)}_m bI - bB^{(n)}_m(a)]\Lambda^{-k} =\\ &= \sum_{\ell=0}^{k}\sum_{\substack{i=0\\ i\neq k}}^{\ell}\begin{pmatrix}k\\ \ell\end{pmatrix}\begin{pmatrix}\ell\\ i\end{pmatrix}[B^{(n)}_{m-\ell}(a)(\mathcal{D}^{\ell-i}b)-\\ &-(\mathcal{D}^{\ell-i}b)B^{(n)}_{m-\ell}(a)]\mathcal{D}^i\Lambda^{-k}\varphi + R^5_n\varphi + R^6_n\varphi,\\ &R^6_n\varphi := [B^{(n)}_{m-k}(a)bI - bB^{(n)}_{m-k}(a)]\Delta^k_{\frac{1}{n}}\Lambda^{-k}\varphi\end{aligned} \tag{3.68}$$

with $\lim\limits_{n\to\infty}\|R^5_n\|_{p,\alpha} = 0$. Since

$$\lim_{n\to\infty}\Delta^k_{\frac{1}{n}}\Lambda^{-k}\varphi = 0 \quad \text{for all} \quad \varphi \in L_{p,\alpha}(\mathbb{R}),$$

due to (3.63) the last summand in (3.68) converges to 0 uniformly

$$\lim_{n\to\infty}\|R^6_n\|_{p,\alpha} = 0. \tag{3.69}$$

Due to (3.63), (3.68), (3.69) can proceed in (3.68) as follows

$$A^{(n)}_m\sum_{\ell=0}^{k}\sum_{\substack{i=0\\ i\neq k}}^{\ell}\begin{pmatrix}k\\ \ell\end{pmatrix}\begin{pmatrix}\ell\\ i\end{pmatrix}[B^{(n)}_{m-\ell}(a) - B^{(n)}_{m-i}(a)](\mathcal{D}^{\ell-i}b)\mathcal{D}^i\Lambda^{-k} + R^2_n = \sum_{j=0}^{k}B^{(n)}_j(a)T'_j + R^2_n,$$

where $\lim\limits_{n\to\infty}\|R^2_n\|_{p,\alpha} = 0$ and T'_j are linear combinations of operators $(\mathcal{D}^{\ell-i}b)\mathcal{D}^i\Lambda^{-k}$ for $0 \leq i \leq \ell \leq k$, $i < k$ and thus are compact due to (3.67). $\square$

Lemma 3.13 *Let $b \in \dot{C}^k(\mathbb{R})$ and $1 < p < \infty$, $0 < \frac{1}{p} + \alpha > 1$, $k, m \in \mathcal{N}$, $1 \leq k \leq m$. Then the commutator*

$$[P_m^{(n)}, bI] = T + R_n, \quad [A, B] := AB - BA \tag{3.70}$$

where $P_m^{(n)}$ are projections (see (3.47)),

$$\lim_{n\to\infty} \|R_n\|_{p,\alpha}^k = 0,$$

and T is a compact operator in $W_{p,\alpha}^k(\mathbb{R})$.

Proof. From (3.47) and (3.68) we find that $P_j^{(n)} = B_j^{(n)}(g_j^{-1})$ and from (3.66) we get

$$B_j^{(n)}(g_m^{-1})\Delta_{\frac{1}{n}}^{m-j}\psi = E_j^{(n)}T_{g_m^{-1}}^0\mathcal{D}^{(n)}\Delta_{\frac{1}{n}}^{m-j}\psi = \mathcal{D}^{m-j}P_m^{(n)}\psi, \quad \psi \in W_{p,\alpha}^{-m-j}(\mathbb{R}).$$

Therefore from (3.48) we derive

$$\lim_{n\to\infty} \hat{B}{}_j^{(n)}(g_m^{-1})\varphi = \lim_{n\to\infty} \Lambda^{-k}B_j^{(n)}(g_m^{-1})\Lambda^{-k}\varphi = \Lambda^{-k}\Lambda^k\varphi \quad \text{for all} \quad \varphi \in W_{p,\alpha}^k(\mathbb{R})$$

(see (3.62)) and

$$\lim_{n\to\infty} \|\hat{B}{}_j^{(n)}T' - T'\|_{p,\alpha}^{(k)} = 0 \tag{3.71}$$

for any compact operator T' in $W_{p,\alpha}^k(\mathbb{R})$. In (3.71) we applied a well–known assertion: if operators B converge strongly in a Banach space X and K is compact, then B_nK converges in norm (see e.g. [36, Theorem 10.6]). (3.70) follows from (3.61) and (3.71). □

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