THE GREEN FORMULA AND LAYER POTENTIALS

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The explicit form of all possible variants of the Green formula is described for a boundary value problem when the "basic" operator is an arbitrary partial differential operator with variable matrix coefficients and the "boundary" operators are quasi–normal with vector–coefficients. If the system possesses a fundamental solution, a representation formula for the solution is derived and boundedness properties of the relevant layer potentials, mapping function spaces on the boundary (Bessel potential, Besov, Zygmund spaces) into appropriate weighted function spaces on the domain are established. We conclude by discussing some closely related topics: traces of functions from weighted spaces, traces of potential–type functions, Plemelji formulae, Calderón projections, and minimal smoothness requirements for the surface and coefficients.

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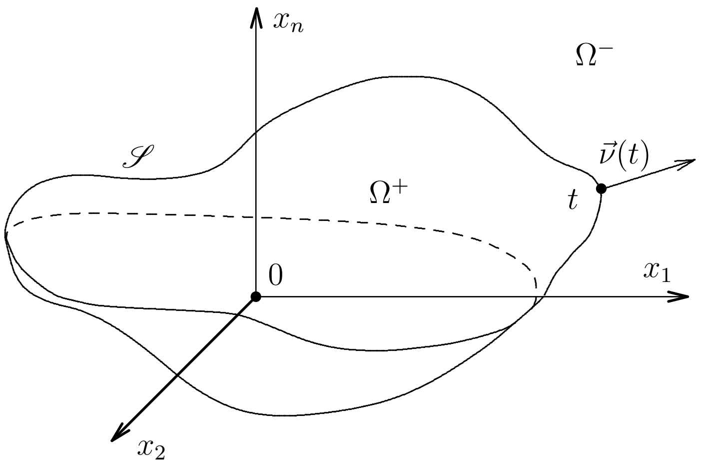
Introduction

Let $\Omega^+ \subset \mathbb{R}^n$ be a domain with a smooth boundary $\partial\Omega^+ = \mathscr{S}$, $\Omega^- := \mathbb{R}^n \setminus \overline{\Omega^+}$ and let $\vec{\nu}(t) = (\nu_1(t), \dots, \nu_n(t))$, $t \in \mathscr{S}$ be the outward unit normal vector (see Fig.1).



Let $\gamma_{\mathscr{S}}^{\pm}$ denote the trace operators on the boundary:

$$\gamma_{\mathscr{S}}^{\pm} u(t) := \lim_{\substack{x \to t \\ x \in \Omega^{\pm}, t \in \mathscr{S}}} u(x) \,.$$

We consider a boundary value problem of the form

$$\begin{cases} \mathbf{A}(x, D_x) u(x) = f(x), & x \in \Omega^{\pm}, \\ \gamma_{\mathscr{S}}^{\pm} \mathbf{b}_j u(t) = g_j(t), & j = 0, \dots, \omega - 1, \;\; t \in \mathscr{S}\,, \quad \omega \leq m\,, \end{cases} \tag{0.1}$$

with a partial differential operator (we call it a “basic” operator) with $N \times N$ matrix coefficients

$$\mathbf{A}(x, D_x) := \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \,, \qquad a_\alpha \in C^\infty(\overline{\Omega}^{\pm}, \mathbb{C}^{N \times N}) \tag{0.2}$$

and with a quasi–normal system of “boundary” operators

$$\mathbf{b}_j(t, D_t) = \sum_{|\alpha| \leq m_j} b_{j\alpha}(t) \partial_t^\alpha \,, \quad b_{j\alpha} \in C^\infty(\mathscr{S}, \mathbb{C}^N)\,, \quad m_j \leq m - 1\,, \quad j = 0, \dots, \omega - 1$$

with vector–row coefficients of length N. Extending arbitrarily the “boundary” operator system $\{\mathbf{b}_j\}_{j=0}^{\omega-1}$ to a Dirichlet system $\{\mathbf{b}_j\}_{j=0}^{mN-1}$, it is possible to find then unique system of "boundary" differential operators $\{\mathbf{c}_j\}_{j=0}^{mN-1}$ such that the Green formula

$$\int\limits_{\Omega^+}((\mathbf{A}u)^\top\overline{v} - u^\top\overline{\mathbf{A}^*v})dy = \pm\sum_{j=0}^{mN-1}\oint\limits_{\mathscr{S}}\mathbf{b}_j u\overline{\mathbf{c}_j v}d_\tau\mathscr{S} \tag{0.3}$$

holds (see Theorem 1.6) with the formally adjoint operator $\mathbf{A}^*$ to (0.1). The system $\{\mathbf{c}_j\}_{j=0}^{mN-1}$ is a Dirichlet system if and only if the "basic" operator $\mathbf{A}(x, D_x)$ is normal.

Moreover, if the "basic" operator is normal, it is possible to prescribe parts of both systems $\{\mathbf{b}_j\}_{j=0}^{kN-1}$ and $\{\mathbf{c}_j\}_{j=kN}^{(m-k)N-1}$, if they are both Dirichlet systems, and find missing parts in a unique way such that the Green formula (0.3) holds.

For a formally self–adjoint operator of even order $m = 2\ell$ a simplified Green formula is proved separately (see Theorem (1.7)).

The Green formula (0.3) was proved in [Ta1, Ta2] for a rectangular system of "basic" operators with $\ell \times k$ matrix coefficients with an injective principal symbol (see [LM1, Ch.2, Theorem 6.1] for scalar elliptic operators (i.e. $N = 1$) and [Ro1, RS2] for elliptic Agmon–Douglis–Nirenberg systems; see also the survey [Ag1, § 4]). All the investigations mentioned in [LM1, Ro1, RS2, Ta1, Ta2] are based on local diffeomorphisms which replace the domains $\Omega^\pm$ by the half–space $\mathbb{R}^n_+$. The present approach is direct and relies on the partial integration formulae (1.23)–(1.24), which follow from the Gauss divergence formula and the Stokes formula for differential forms. Other important ingredients are the special Green formula with the normal derivatives $\mathbf{B}_j = \partial^j_\nu$ as "boundary" operators (see Theorem 1.10; for similar formulae, the interested reader may also consult [CP1, CW1, Di1, Se1]) and Lemma 4.7 (see [LM1, Ch. 2, Lemma 2.1] and [RS2, (11)] for the scalar case).

Moreover, the approach is constructive and allows us to write the "boundary" differential operators $\{\mathbf{c}_j(x, D_x)\}_{j=0}^{mN-1}$ in explicit form (see Theorem 1.11) provided the "boundary" operators $\{\mathbf{b}_j(x, D_x)\}_{j=0}^{mN-1}$ are fixed. The algorithm is purely algebraic and involves only the coefficients of the differential operators $\mathbf{A}(x, D_x)$ and $\mathbf{B}_j(x, D_x)$.

Let us note that explicit formulae were previously known only for the symbols of the operators $\mathbf{c}_j(x, D_x)$, $j = 0, \dots, mN - 1$ (cf. [Ta1, § 8.33]).

In order to demonstrate an essential application of the Green formula, let us assume that $\mathbf{A}(x, D_x)$ has a two–sided inverse on the entire space $\mathbb{R}^n$

$$\mathbf{A}(x, D_x)\mathbf{F}_\mathbf{A}(x, D_x) = \mathbf{I}, \quad \mathbf{F}_\mathbf{A}(x, D_x)\mathbf{A}(x, D_x) = I\,,$$

i.e. the operator has a fundamental solution. As is well know, this entails that $\mathbf{A}(x, D_x)$ is elliptic and (for $n > 2$) it has even order $m = \operatorname{ord}\mathbf{A} = 2\ell$. We "insert" the distributional Schwartz kernel $v_{\varepsilon,x}(y) = \chi_\varepsilon(x - y)\mathscr{K}_\mathbf{A}(x, y)$ of the fundamental solution $\mathbf{F}_\mathbf{A}(x, D_x)$, suitably truncated near the diagonal set $x = y$ into the Green formula (0.3). Making $\varepsilon \to 0$ yields a representation of the solution $u(x)$ to the elliptic equation $\mathbf{A}(x, D_x)u(x) = f(x)$ in the domain $\Omega^\pm$

$$\chi_{\Omega^\pm}(x)u(x) = \mathbf{N}_{\Omega^\pm}f(x) \pm \sum_{j=0}^{2\ell-1}\mathbf{V}_j\gamma^\pm_\mathscr{S}\mathbf{B}_j u(x)\,, \tag{0.4}$$

where $\chi_{\Omega^\pm}$ stands for the characteristic function of $\Omega^\pm \subset \mathbb{R}^n$. The operators

$$\begin{aligned}\mathbf{N}_{\Omega^\pm}\varphi(x) &:= \int_{\Omega^\pm}\left[\overline{\mathscr{K}_{\mathbf{A}^*}(y, x)}\right]^\top\varphi(y)dy = \int_{\Omega^\pm}\mathscr{K}_\mathbf{A}(x, y)\varphi(y)dy\,,\\ \mathbf{V}_j\psi(x) &:= \oint_\mathscr{S}\left[\overline{\mathbf{C}_j(\tau, D_\tau)\mathscr{K}^\top_\mathbf{A}(x, \tau)}\right]^\top\varphi(\tau)\;d_\tau\mathscr{S}\,, \quad j = 0, \dots, 2\ell\end{aligned} \tag{0.5}$$

are the volume (NEWTON) and the layer potentials, respectively (see (3.3)–(3.7)).

The layer potentials $\mathbf{V}_0, \ldots, \mathbf{V}_{2\ell-1}$ extend functions defined on the boundary into the domain and their continuity properties have essential applications in many investigations. A partial list includes potential theoretic methods (see [CW1, DNS1, Gu1, KGBB1, Lo1, MMT1, Se1] etc.), a priori estimates of solutions of BVPs (see [CW1, DNS1, DN1, Gr3, LM1] etc. and Corollary 3.4), full asymptotic expansions of solutions to crack–type and mixed BVPs for elliptic partial differential equations (see [CD2]).

As a particular case of Theorem 3.2 we can formulate the following (see §1 for the definition of the BESSEL potential $\mathbb{H}^s_{p,loc}(\overline{\Omega^\pm})$, BESOV $\mathbb{B}^s_{p,p}(\mathscr{S})$ and other spaces).

Theorem 0.1 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $\mu_j = \operatorname{ord}\mathbf{C}_{\rm j} < \operatorname{ord}\mathbf{A} = 2\ell$. The layer potentials*

$$\mathbf{V}_j \ : \ \mathbb{X}^s_p(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2\ell-j+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) , \qquad j = 0, \ldots, 2\ell-1 \tag{0.6}$$

are all continuous when either $\mathbb{X}^s_p(\mathscr{S}) = \mathbb{H}^s_p(\mathscr{S})$ or $\mathbb{X}^s_p(\mathscr{S}) = \mathbb{B}^s_{p,p}(\mathscr{S})$.

Theorem 0.1 is proved with the help of Lemma 4.8, which has independent interest. It allows the representation of the layer potentials in (0.5) in the form of volume potentials, i.e. pseudodifferential operators (PsDOs). For the sake of this introduction, below we record a slightly particular case of this lemma.

Lemma 0.2 *Let $s > 0$, $s \notin \mathbb{N}$, $k = 0, 1, \ldots$, $1 < p < \infty$, $1 \le q \le \infty$. Let $\mathbf{A}(x, D_x)$ in (0.2) be a normal operator $\det \mathscr{A}({\rm t}, \vec{\nu}({\rm t})) \neq 0$ for all $t \in \mathscr{S}$ and have order $\operatorname{ord}\mathbf{A} = {\rm m}$.*

For a DIRICHLET system $\{\mathbf{B_j}\}^{m-1}_{j=0}$ of "boundary" differential operators of order $m-1$ with C^∞–smooth $N \times N$ matrix coefficients there exists a continuous linear operator

$$\mathscr{P} : \ \bigotimes_{j=0}^{m-1} \ \mathbb{B}^{s+m-1-j}_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^{s+m-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{0.7}$$

such that

$$\gamma^+_{\mathscr{S}}\mathbf{B}_j\mathscr{P}\Phi = \varphi_j, \quad \mathbf{A}\mathscr{P}\Phi \in \widetilde{\mathbb{H}}^{s-1+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \tag{0.8}$$

for $j = 0, \ldots, m-1$ and arbitrary $\Phi = (\varphi_0, \ldots, \varphi_{m-1}) \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s+m-1-j}_{p,p}(\mathscr{S})$.

A similar assertion is proved in [LM1, Ch.2, Theorem 6.1] for the scalar case; see also [LM1, Ch.2, Lemmata 2.1 and 2.2] and [Hr2, Theorem 1.2.6]. Our proof is carried out for matrix–valued operators, it is more transparent and the spaces involved are more general (we consider weighted spaces $\mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm})$ as well).

Theorem 0.1 can be derived from the results on PsDOs with the transmission property (see [Bo1, Gr1, Gr2, Jo2, RS1] and the survey [BS1, Theorems 2.17, 2.21]). The approach suggested here is different, works for weighted spaces and seems to be simpler. It has consequences which are perhaps difficult to obtain by the methods suggested earlier (see, e.g., §§ 6.3–6.5 below).

In § 1.1 we discuss the GREEN formula (0.3) and related topics. Namely, we recall the definitions of normal operators, DIRICHLET systems of operators and formal adjoint BVPs (see [LM1]), and we also introduce systems of quasi–normal operators. BVPs with quasi–normal "boundary" operators include mixed–type problems arising in elasticity, the diffraction of electromagnetic waves and many other problems in mathematical physics. Our main results are Theorems 1.6 and 1.7, dealing with GREEN's formula. The proofs are deferred to §§ 5.1, 5.2. In § 1.2 we set the stage by discussing several prerequisites, such as the GÜNTER and the STOKES tangential derivatives, and various partial integration formulae for a domain and on the surface, respectively, based on the GAUSS divergence formula and the STOKES formula for differential forms (see Lemma 1.8). A particular GREEN formula is proved in Theorem 1.10 for arbitrary "basic" operators when the "boundary" operators are given by the normal derivatives $\mathbf{B}_j = \partial^j_{\overline{\nu}}$. In Theorem1.11 we find the explicit record for the "boundary" operators $\{\mathbf{c}_j(x, D_x)\}_{j=0}^{mN-1}$ in the GREEN formula (0.3) when the extended DIRICHLET system $\{\mathbf{b}_j(x, D_x)\}_{j=0}^{mN-1}$ is fixed.

In § 2 we define the BESSEL potential $\mathbb{H}^{r,k}_{p,loc}(\overline{\Omega^\pm})$, the BESOV $\mathbb{B}^{r,k}_{p,q,loc}(\overline{\Omega^\pm})$ and the ZYGMUND $\mathbb{Z}^{r,k}(\overline{\Omega^\pm})$ spaces with weights .

In § 3, relying on the GREEN formula for an elliptic differential equation (provided the "basic" operator has a fundamental solution), a representation formula for a solution is derived. The result on the continuity of the layer potentials, intervening in the aforementioned representation formula of solutions, as well as of more general potential–type operators is also formulated (cf. Theorem 0.1).

More concretely, we prove the continuity of layer potentials from the boundaryBESSEL potential $\mathbb{H}^s_p(\mathscr{S})$ and BESOV spaces $\mathbb{B}^s_{p,p}(\mathscr{S})$, $\mathbb{B}^s_{p,q}(\mathscr{S})$ (including ZYGMUND spaces $\mathbb{Z}^s(\mathscr{S})$ $= \mathbb{B}^s_{\infty,\infty}(\mathscr{S})$) into appropriate weighted BESSEL potential spaces $\mathbb{H}^{r,k}_{p,loc}(\overline{\Omega^\pm})$ as well as BESOV spaces $\mathbb{B}^{r,k}_{p,q,loc}(\overline{\Omega^\pm})$, defined both in the exterior Ω^- and the interior Ω^+ of the surface $\mathscr{S}$ (see Theorem 3.2). In the last part of this section, *a priori* estimates for solutions of the BVP (0.1) are obtained when the "basic" operator is hypoelliptic (see Corollary 3.4 and Remark 3.5).

In § 4 a basic auxiliary result, i.e. Lemma 4.8, is proved. This lemma plays a crucial role in the proof of Theorem 3.2 in § 5.3.

In § 5 we present the proofs of Theorems 1.6, 1.7 and 3.2.

In § 6.1 we prove that the generalized layer potentials, involving integral operators with supersingular kernels on the boundary surface, have well defined traces on the boundary of the domain, when interpreted as classical PsDOs. Such interpretation of supersingular integral operators is necessary because they encounter in many problems of mathematical physics (e.g. derivatives of the double layer potential for a second order differential operator) and does not exist in usual sense.

In § 6.2 we extend the trace theorem (see also Theorem 4.6) and the basic Lemma 4.8 to functions in weighted spaces.

In § 6.3 we prove a theorem on the CALDERÓN projections, related to the GREEN formula (0.3) and the corresponding layer potentials (0.5). Its essence is that the operators $\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}} := \pm\gamma^\pm_{\mathscr{S}}\mathbf{B}_j\mathbf{V}_j$ for $j = 0, \ldots, 2\ell - 1$ are proved to be projections $(\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}})^2 = \mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}$, $\mathbf{P}^-_{\mathbf{A},\mathbf{j}} + \mathbf{P}^+_{\mathbf{A},\mathbf{j}} = I$ in the spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\mathbb{B}^s_{p,q}(\mathscr{S})$.

In § 6.4 we establish the PLEMELJI formulae (the jump relations) for the layer potentials.

In § 6.5 we indicate how to substantially weaken the smoothness assumptions on the boundary $\mathscr{S} = \partial\Omega^\pm$ of the domain and on coefficients of the differential operators. Such results are important especially in the context of the recent progress in the theory of BVPs for differential equations in domains with LIPSCHITZ boundaries. These investigations are based on results for layer potentials on LIPSCHITZ surfaces (see [Ke1,Ke2,MMP1,MMT1,MT1] and the literature cited therein). Most recent and general results in this direction are obtained in [MT2].

Most of the above–mentioned results on the GREEN formula, layer potentials, the PLEMELJI formulae under minimal restrictions on the boundary manifold and coefficients are known for second order equations (see [MMT1, MT1, MT2] for recent results). Less is known for higher order equations (see [CP1, CW1, Di1, Gr1, LM1, Ro1, Se1]). CALDERÓN projections have been investigated in [Se1] (see also [CP1, CW1, Gr1, Di1]).

Acknowledgements: The author thanks E.Shargorodsky who suggested the first

version of Theorem 2.2 and Lemma 4.8, as well as the results of § 6.4. E.Shargorodsky, M.Mitrea and F.-O.Speck also had made many valuable remarks while reading the manuscript.

The investigation was supported by grants:
a) INTAS 96 – 0876;
b) Academy of Sciences of Georgia 1.2.97.

1 The Green formula and boundary value problems

1.1. The Green formula for quasi–normal BVPs. Let Ω^+, $\partial\Omega = \mathscr{S}$ and $\vec{\nu}(t)$ be the same as in the Introduction [1)] and consider a partial differential operator with $N \times N$ matrix coefficients

$$\mathbf{A}(x, D_x) := \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha, \quad a_\alpha \in C^\infty(\overline{\Omega}^\pm, \mathbb{C}^{N\times N}). \tag{1.1}$$

The operator

$$\mathbf{A}^*(x, D_x) = \sum_{|\alpha|\leq m} (-1)^\alpha \partial_x^\alpha \overline{[a_\alpha(x)]}^\top I\,, \tag{1.2}$$

where $\mathscr{B}^\top$ denotes the transposed matrix to $\mathscr{B}$, is the formal adjoint to $\mathbf{A}(x, D_x)$ with respect to the sesquilinear form

$$(u, v) := \int\limits_{\Omega^\pm} [u(y)]^\top \overline{v(y)} dy.$$

Definition 1.1 (see [LM1, Ch.2, § 1.4]). *The operator* $\mathbf{A}(x, D_x)$ *in* (1.1) *is called* **normal** *on* $\mathscr{S}$ *if*

$$\inf |\det \mathscr{A}_0(t, \vec{\nu}(t))| \neq 0, \quad t \in \mathscr{S}\,, \quad |\xi| = 1\,, \tag{1.3}$$

where $\mathscr{A}_0(x, \xi)$ *denotes the* **homogeneous principal** *symbol of* $\mathbf{A}$

$$\mathscr{A}_0(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x)(-i\xi)^\alpha, \quad x \in \overline{\Omega^\pm}\,, \quad \xi \in \mathbb{R}^n. \tag{1.4}$$

The condition (1.3) means that the surface $\mathscr{S}$ is not characteristic for the operator $\mathbf{A}(x, D_x)$. Normal operators contain, as a subclass, elliptic operators on the surface

$$\inf |\det \mathscr{A}_0(\mathrm{t}, \xi)| \neq 0 \quad \text{for all} \quad \mathrm{t} \in \mathscr{S}, \quad \xi \in \mathrm{S}^{\mathrm{n}-1}, \tag{1.5}$$

where $S^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ is the unit sphere in $\mathbb{R}^n$; these two definitions coincide for operators with constant coefficients since the unit normal vector $\vec{\nu}(t)$ runs through the entire unit sphere if t ranges over the closed smooth surface $\mathscr{S}$. In fact, the surface $\mathscr{S} = \partial\Omega^+$ is the boundary of the domain Ω^+ and thus any connected part of this boundary can be continuously deformed to the unit sphere. If we suppose that the unit normal, while ranging over the surface $\mathscr{S}$, leaves some (obviously open) domain on the unit sphere uncovered, we end up with a contradiction.

Let us consider a BVP with mixed conditions

$$\begin{cases} \mathbf{A}(x, D_x)u(x) = f(x), & x \in \Omega^\pm, \\ \gamma_{\mathscr{S}}^\pm \mathbf{b}_j u(t) = g_j(t), & j = 0, \dots, \omega - 1, \;\; t \in \mathscr{S}\,, \quad \omega \leq mN\,, \end{cases} \tag{1.6}$$

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1) Optimal smoothness constraints on $\partial\Omega = \mathscr{S}$ will be discussed later on in § 6.5.

where $\mathbf{A}(x, D_x)$ is the "basic" operator, defined in (1.1) and

$$\mathbf{b}_j(t, D_t) = \sum_{|\alpha| \le m_j} b_{j\alpha}(t) \partial_t^\alpha , \qquad b_{j\alpha} \in C^\infty(\mathscr{S}, \mathbb{C}^N) \tag{1.7}$$

are "boundary" differential operators with vector–row coefficients of length N and $\operatorname{ord} \mathbf{b}_{\rm j} = {\rm m_j} \le {\rm m} - 1$.

Together with (1.6) we will consider the boundary value problem with the formal adjoint "basic" operator

$$\begin{cases} \mathbf{A}^*(x, D_x) v(x) = d(x), & x \in \Omega^\pm, \\ \gamma_{\mathscr{S}}^\pm \mathbf{c}_{mN-j-1} v(t) = h_j(t), \quad j = 0, \ldots, \omega^* - 1, & t \in \mathscr{S} \end{cases} \tag{1.8}$$

(see (1.2)); here $\omega^* \le mN$, $\operatorname{ord} \mathbf{c}_{\rm j} = \mu_{\rm j} \le {\rm m} - 1$, and $\mathbf{c}_j(t, D_t)$ are some "boundary" differential operators

$$\mathbf{c}_j(t, D_t) = \sum_{|\alpha| \le \mu_j} c_{j,\alpha}(t) \partial_t^\alpha , \qquad c_{j,\alpha} \in C^\infty(\mathscr{S}, \mathbb{C}^N) \tag{1.9}$$

with vector–row coefficients of length N.

A particular case of BVP (1.6) is the following

$$\begin{cases} \mathbf{A}(x, D_x) u(x) = f(x), & x \in \Omega^\pm, \\ \gamma_{\mathscr{S}}^\pm \mathbf{B}_j u(t) = G_j(t), \qquad j = 0, \ldots, \ell - 1, & t \in \mathscr{S}, \end{cases} \tag{1.10}$$

where

$$\mathbf{B}_j(t, D_t) = \sum_{|\alpha| \le m_j} b_{j\alpha}(t) \partial_t^\alpha , \qquad b_{j\alpha} \in C^\infty(\mathscr{S}, \mathbb{C}^{N \times N})$$

are "boundary" operators with $N \times N$ matrix coefficients and $\operatorname{ord} \mathbf{B}_j = m_j \le m - 1$. The formal adjoint BVP of (1.10) can be written in the form

$$\begin{cases} \mathbf{A}^*(x, D_x) v(x) = d(x), & x \in \Omega^\pm, \\ \gamma_{\mathscr{S}}^\pm \mathbf{C}_{m-j-1} v(t) = H_j(t), \quad j = 0, \ldots, \ell^* - 1, & t \in \mathscr{S} \end{cases} \tag{1.11}$$

(see (1.2)), where $\ell^* \le m$ and $\mathbf{C}_j(t, D_t)$ are some "boundary" differential operators

$$\mathbf{C}_j(t, D_t) = \sum_{|\alpha| \le \mu_j} c_{j,\alpha}(t) \partial_t^\alpha , \qquad c_{j,\alpha} \in C^\infty(\mathscr{S}, \mathbb{C}^{N \times N})$$

with $\operatorname{ord} \mathbf{C}_{\rm j} = \mu_{\rm j} \le {\rm m} - 1$.

The BVPs (1.10) are encountered, e.g., in elasticity, when the displacement or the stress fields are prescribed (these BVPs are denoted there by $I^\pm$ and by $II^\pm$, respectively). The BVPs (1.6) also cover the mixed problems of elasticity when the normal component of the displacement and both tangent components of the stress fields ($III^\pm$ BVP) or the normal component of the stress and both tangent components of the displacement fields ($IV^\pm$ BVP) are prescribed (see [KGBB1, §§ 1.8–1.10]).

Definition 1.2 *A system* $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$ *of differential operators with matrix* $N\times N$ *coefficients is called a* DIRICHLET *system of order* k *if all participating operators are normal on* $\mathscr{S}$ *(see Definition* 1.1*) and, after renumbering,* $\operatorname{ord}\mathbf{B}_{\rm j} = {\rm j}$, $j=0,1,\dots,k-1$.

A system of differential operators $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}$ *with row–vector coefficients of length* N *is said to be a* DIRICHLET *system of order* k *if*

$$\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1} = \mathscr{H}_0\,\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$$

where $\{\mathbf{b}_j(t,D_t)\}_{j=\omega}^{kN-1}$ *is a* DIRICHLET *system and* $\mathscr{H}_0$ *is a constant* $kN\times kN$ *matrix, interchanging rows.*

Definition 1.3 *A system* $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$ *is said to be* **quasi–normal system** *if:*

i. the principal homogeneous symbols $\mathbf{b}_{j,0}(t,\vec{\nu}(t))$, $j=0,\dots,\omega-1$ *evaluated at the normal vectors* $\xi=\vec{\nu}(t)$ *are linearly independent vector–rows for all* $t\in\mathscr{S}$ *on the boundary;*

ii. operators $\mathbf{b}_0(t,D_t),\dots,\mathbf{b}_{\omega-1}(t,D_t)$ *with equal order are, at most,* N.

Lemma 1.4 *For any arbitrary quasi–normal system of operators* $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$, $\operatorname{ord}\mathbf{b}_{\rm j}\le {\rm m}-1$, *there exists a non–unique extension up to a* DIRICHLET *system*

$$\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1} = \mathscr{H}_0\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$$

of order m *with some constant* $mN\times mN$ *matrix* $\mathscr{H}_0$.

Proof. Let us select among the "boundary" row–operators $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$ those with equal orders m_j and add to the selected rows new rows of differential operators of the same order in such a way that the resulting $N\times N$ matrix–operator $\mathbf{B}_j(t,D_t)$ will have linearly independent rows in the principal homogeneous symbol $\mathscr{B}_{j,0}(t,\vec{\nu}(t))$, i.e. will be normal. Next we extend the system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{\ell}$ up to a DIRICHLET system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$ of order m by adding normal operators with missing orders (say, $\partial^{m_k}_{\vec{\nu}(t)}$, $k=\ell+1,\dots,m-1$).

As the concluding step we rearrange rows in the extended system $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$ with the help of some matrix $\mathscr{H}_0$ which has entries 0 and 1 to get a DIRICHLET system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$. ■

Definition 1.5 (1.8) *is called* **formally adjoint** *to BVP* (1.6) *if there exist two systems of "boundary" differential operators*

$$\mathbf{b}_j(t,D_t)=\sum_{|\alpha|\le m_j} b_{j\alpha}(t)\partial_t^\alpha,\qquad \mathbf{c}_k(t,D_t)=\sum_{|\alpha|\le \mu_k} c_{k\alpha}(t)\partial_t^\alpha\,,$$

$$b_{j\alpha},c_{k\alpha}\in C^\infty(\mathscr{S},\mathbb{C}^N),\quad j,k=0,\dots,mN-1\,,$$

which are extensions of systems $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{\omega-1}$ *and* $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{\omega^*-1}$, *respectively, such that the* GREEN *formula*

$$\int_{\Omega^\pm}((\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}^*v})dy=\pm\sum_{j=0}^{mN-1}\oint_{\mathscr{S}}\mathbf{b}_ju\overline{\mathbf{c}_jv}d_\tau\mathscr{S} \tag{1.12}$$

holds[2] *with* $u,v\in C^\infty(\overline{\Omega^\pm},\mathbb{C}^N)$.

[2]The integral $\oint_{\mathscr{S}}$ is used to underline that integration is performed over the closed surface $\mathscr{S}$.

For BVP (1.10) and its formal adjoint (1.11) the GREEN formula (1.12) takes the form

$$\int_{\Omega^\pm} ((\mathbf{A}u)^\top \overline{v} - u^\top \overline{\mathbf{A}^* v})dy = \pm \sum_{j=0}^{m-1} \oint_{\mathscr{S}} (\mathbf{B}_j u)^\top \overline{\mathbf{C}_j v} d_\tau \mathscr{S}\,, \tag{1.13}$$

where the "boundary" differential operators $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$ and $\{\mathbf{C}_j(t,D_t)\}_{j=0}^{m-1}$ have $N\times N$ matrix coefficients. If (1.8) is formally adjoint to BVP (1.8), then [3)]

$$m_j + \mu_j = m-1, \qquad j = 0,\ldots .\omega - 1. \tag{1.14}$$

Since the DIRICHLET systems participating in the GREEN formulae (1.12) might differ from the DIRICHLET systems in (1.13) only by some rearrangement of rows (cf. (5.1)), we will address most frequent more convenient formula (1.13).

Theorem 1.6 *If either $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$ or $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ is a fixed* DIRICHLET *system of "boundary" operators, then the* GREEN *formula* (1.12) *holds, then the related system (respectively, $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ or $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$) is unique and BVP* (1.8) *is formally adjoint to* (1.6).

The related system $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ (the system $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$, respectively) is a DIRICHLET *system if and only if the "basic" operator $\mathbf{A}(x,D_x)$ is normal.*

If the "basic" operator $\mathbf{A}(x,D_x)$ is normal, $\omega = kN$, $\omega^* = (m-k)N$, the related systems $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}$, $\{\mathbf{c}_{mN-j-1}(t,D_t)\}_{j=0}^{(m-k)N-1}$ are fixed and one of them is quasi–normal, then the GREEN *formula* (1.12) *holds if and only if both of them are* DIRICHLET *systems* $\mathrm{ord}\,\mathbf{b}_\mathrm{j} = \mathrm{ord}\,\mathbf{c}_{\mathrm{mN-j-1}} = \mathrm{j}$ *(of order k and $m-k$, respectively). The extended systems $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{mN-1}$ and $\{\mathbf{c}_j(t,D_t)\}_{j=0}^{mN-1}$ in* (1.12) *are then* DIRICHLET *systems (of order m) and are unique.*

The proof is deferred to § 5.1. The first part of the Theorem for scalar elliptic operators has been proved earlier (see [LM1, Ch. 2, Theorem 2.1]) and for elliptic AGMON–DOUGLIS–NIRENBERG systems–in [Ro1, RS1]. The most general case, to our best knowledge, is considered in [Ta1, Ta2], where the "basic" and "boundary" operators have "rectangular" $k\times\ell$ matrix coefficients and the "basic" operator has an injective principal symbol.

It is well–known that if $\mathbf{A}(x,D_x)$ is scalar ($N=1$), is elliptic and has real valued coefficients (or complex valued coefficients and $n>2$) than it is proper elliptic and has even order $\mathrm{ord}\,\mathbf{A}(\mathrm{x},\mathrm{D_x}) = \mathrm{m} = 2\ell$ (see [LM1, Ch.2, §§ 1.1]). Although for the non–scalar case ($N=2,3,\ldots$) matters are different (see § 6.6), many elliptic systems arising in applications (to, e.g., elasticity, thermo elasticity, hydrodynamics) have even order. Let us consider some simplification of GREEN's formula for such systems, especially when the system is formally self–adjoint.

Assume that the operator in (1.1) has even order $m=2\ell$. Then, it can be represented in the form

$$A(x,D_x) = \sum_{|\alpha|,|\beta|\le\ell} (-1)^{|\alpha|}\partial_x^\alpha a_{\alpha,\beta}(x)\partial_x^\beta, \quad a_{\alpha,\beta}\in \mathbf{C}^\infty(\overline{\Omega^\pm},\mathbb{C}^{N\times N}) \tag{1.15}$$

(the representation is not unique) and with it one associates the following sesquilinear form

$$\mathscr{A}(u,v) := \int_{\Omega^\pm} \sum_{|\alpha|,|\beta|\le\ell} \left[a_{\alpha,\beta}(y)\partial_y^\beta u(y)\right]^\top \overline{\partial_y^\alpha v(y)}dy, \quad u,v\in C_0^\infty(\overline{\Omega^\pm},\mathbb{C}^N)\,. \tag{1.16}$$

[3)] (1.14) follows, e.g., from the formulae (1.38) for "boundary" operators $\{\mathbf{C}_j(t,D_t)\}_{j=0}^{m-1}$.

Theorem 1.7 *For an arbitrary "basic" differential operator* (1.15) *of even order* 2ℓ *and an arbitrary* Dirichlet *system* $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{\ell-1}$ *of order* ℓ *of "boundary" differential operators with matrix* $N\times N$ *coefficients there exists a system* $\{\mathbf{C}_j(t,D_t)\}_{j=0}^{\ell-1}$ *of "boundary" operators with* $\operatorname{ord}\mathbf{B}_{\rm j}+\operatorname{ord}\mathbf{C}_{\rm j}=2\ell-1$ *such that*

$$\mathscr{A}(u,v)=\int\limits_{\Omega^\pm}(\mathbf{A}u)^\top\overline{v}dy\pm\sum_{j=0}^{\ell-1}\int\limits_{\mathscr{S}}(\mathbf{C}_ju)^\top\overline{\mathbf{B}_jv}d_\tau\mathscr{S}\,,\quad u,v\in C_0^\infty(\overline{\Omega^\pm},\mathbb{C}^N)\,. \tag{1.17}$$

If $\mathbf{A}$ *is formally self–adjoint,* $\mathbf{A}=\mathbf{A}^*$*, we get the following simplified* Green *formula*

$$\int\limits_{\Omega^\pm}\left[(\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}v}\right]dy=\pm\sum_{j=0}^{\ell-1}\int\limits_{\mathscr{S}}\Big[(\mathbf{C}_ju)^\top\overline{\mathbf{B}_jv}-(\mathbf{B}_ju)^\top\overline{\mathbf{C}_jv}\Big]d_\tau\mathscr{S}. \tag{1.18}$$

The proof is deferred to § 5.2. A slightly different proof in the scalar case, $N=1$, and for elliptic operators can be found in [LM1, Ch. 2, § 2.4].

1.2. Partial integration and the special Green formula. Let us consider "extended" normal derivatives

$$\partial_{\vec\nu(x)}:=\vec\nu(x)\cdot\nabla=\sum_{k=1}^n\nu_k(x)\partial_k,,\quad\nabla:=(\partial_1,\dots,\partial_n)\,,\quad x\in\mathbb{R}^n\,,\quad j=0,1,\dots\,, \tag{1.19}$$

where $\vec\nu(x)=(\nu_1(x),\dots,\nu_n(x))$, $x\in\mathbb{R}^n$ is some C^∞–smooth vector field which coincides with the unit normal vector field on $\mathscr{S}$ and stabilises to the unit normal vector to the coordinate hyperplane $x_n=0$ in a neighbourhood of infinity: $\vec\nu(x)\equiv\vec\nu^{(n)}:=(0,\dots,0,1)$ when $|x|>R$ for a sufficiently large R.

A first order linear differential operator

$$\vec h(x)\cdot\nabla:=\sum_{k=1}^n h_k(x)\partial_k\,,\quad\vec h(x)=(h_1(x),\dots,h_n(x)) \tag{1.20}$$

can be applied to arbitrary function $\varphi\in C^1(\mathscr{S})$ defined only on the surface $\mathscr{S}$ if the operator $\vec h(x)\cdot\nabla$ is tangent, i.e., the directing vector $\vec h(x)$ is tangent to $\mathscr{S}$:

$$\forall t\in\mathscr{S}\,,\quad\vec\nu(t)\cdot\vec h(t)\equiv 0\,.$$

In fact, then we can write

$$\vec h(t)\cdot\nabla\varphi(t):=\lim_{\lambda\to0}\frac{\varphi(t+\lambda\vec h_{\mathscr{S}}(t))}{\lambda}\,,\quad\varphi\in C^1(\mathscr{S})\,,$$

where $\lambda\vec h_{\mathscr{S}}(t)$ is the projection of the tangent vector $\lambda\vec h(t)$ onto the surface $\mathscr{S}$ (the projection is correctly defined for small $|\lambda|<\varepsilon$).

The following two classes of tangent operators are of special interest for us the Günter $\mathscr{D}_j$ and the Stokes $\mathscr{M}_{j,k}$ derivatives:

$$\begin{aligned}&\mathscr{D}_x:=(\mathscr{D}_1,\dots,\mathscr{D}_n)\,,\quad\mathscr{D}_j:=\partial_j-\nu_j(x)\partial_{\vec\nu(x)}=\vec d_j\cdot\nabla\,,\\&\mathscr{M}_x:=[\mathscr{M}_{j,k}]_{n\times n}\,,\quad\mathscr{M}_{j,k}:=\nu_j(x)\partial_k-\nu_k(x)\partial_j=\vec m_{j,k}\cdot\nabla\,.\end{aligned} \tag{1.21}$$

It is easy to ascertain that the corresponding directing vectors are tangent to $\mathscr{S}$:

$$\vec{\nu}(t)\cdot\vec{d}_j(t)\equiv\vec{\nu}(t)\cdot\vec{m}_{j,k}(t)\equiv 0,\quad t\in\mathscr{S}\,.$$

Only $n-1$ out of n derivatives $\mathscr{D}_1,\ldots,\mathscr{D}_n$ and out of n^2 derivatives $\mathscr{M}_{1,1},\ldots,\mathscr{M}_{n,n}$ are linearly independent and the following relations are valid:

$$\begin{gathered}\mathscr{D}_j:=-\sum_{k=1}^{n}\nu_k\mathscr{M}_{j,k}\,,\quad \sum_{k=1}^{n}\nu_k\mathscr{D}_k=0\,,\\ \mathscr{M}_{j,k}=\nu_j\mathscr{D}_k-\nu_k\mathscr{D}_j\,,\quad \mathscr{M}_{j,j}=0\,,\quad \mathscr{M}_{j,k}=-\mathscr{M}_{k,j}\,.\end{gathered}\tag{1.22}$$

The tangent derivatives $\mathscr{D}_j$ were introduced in [Gu1, §1.3]), while $\mathscr{M}_{j,k}$ for $n=3$ in [KGBB1, Ch. V]. The derivatives $\mathscr{M}_{j,k}$ are natural entries of the Stokes formula (1.27).

Lemma 1.8 *For a first order differential operator* (1.20) *and for a "tangent" differential operator*

$$\mathbf{G}=\sum_{|\alpha|\le k}g_\alpha(x)\mathscr{D}_x^\alpha=\sum_{|\beta|\le k}g_\beta(x)\mathscr{M}_x^\beta,\qquad x\in\Omega^\pm$$

(see (1.21)–(1.22)*) the following formulae hold:*

$$\int\limits_{\Omega^\pm}[\vec{h}(x)\cdot\nabla]^\top\overline{v(y)}dy=\pm\oint\limits_{\mathscr{S}}\vec{h}(\tau)\cdot\vec{\nu}(\tau)\,[u(\tau)]^\top\,\overline{v(\tau)}d_\tau\mathscr{S}-\int\limits_{\Omega^\pm}[u(y)]^\top\overline{\nabla\cdot\overline{\vec{h}(\tau)}v(y)}dy\,,\tag{1.23}$$

$$\oint\limits_{\mathscr{S}}[\mathbf{G}u(\tau)]^\top\,\overline{v(\tau)}\;d_\tau\mathscr{S}=\oint\limits_{\mathscr{S}}u^\top(\tau)\overline{\mathbf{G}^*_{\mathscr{S}}v(\tau)}\;d_\tau\mathscr{S}\,,\tag{1.24}$$

where

$$\mathbf{G}^*_{\mathscr{S}}=\sum_{|\alpha|\le k}[(\mathscr{D}_x)^*_{\mathscr{S}}]^\alpha\left[\overline{g_\alpha(x)}\right]^\top=\sum_{|\beta|\le k}[(\mathscr{M}_x)^*_{\mathscr{S}}]^\beta\left[\overline{g_\beta(x)}\right]^\top\,,\tag{1.25}$$

$$(\mathscr{D}_j)^*_{\mathscr{S}}u(x)=-\sum_{k=1}^{n}\nu_k\partial_j\nu_ku(x)+\nu_j\partial^*_{\vec{\nu}}u(x)\,,\quad(\mathscr{M}_{j,k})^*_{\mathscr{S}}u(x)=-\mathscr{M}_{j,k}u(x)=\mathscr{M}_{k,j}u(x)\,.$$

In particular,

$$\begin{gathered}\int\limits_{\Omega^\pm}\left[\partial_{\vec{\nu}(y)}u(y)\right]^\top\overline{v(y)}dy=\pm\oint\limits_{\mathscr{S}}[u(\tau)]^\top\,\overline{v(\tau)}d_\tau\mathscr{S}-\int\limits_{\Omega^\pm}[u(y)]^\top\,\overline{\partial^*_{\vec{\nu}(y)}v(y)}dy\,,\\ \int\limits_{\Omega^\pm}[\mathbf{G}u(y)]^\top\,\overline{v(y)}dy=\int\limits_{\Omega^\pm}[u(y)]^\top\,\overline{\mathbf{G}^*v(y)}dy\,,\end{gathered}\tag{1.26}$$

where the (usual) adjoint operator $\mathbf{G}^*$ *is defined as follows (cf*[4]*.* (1.25)*):*

$$\mathbf{G}^*=\sum_{|\alpha|\le k}[\mathscr{D}^*_x]^\alpha\left[\overline{g_\alpha(x)}\right]^\top=\sum_{|\beta|\le k}[\mathscr{M}^*_x]^\beta\left[\overline{g_\beta(x)}\right]^\top\,,$$

---

[4] It is worth to underline that the formally adjoint operators $\mathscr{D}^*_j$, $\mathscr{M}^*_{j,k}$ on the domains $\Omega^\pm$ (see (1.26)) and the "surface" adjoints $(\mathscr{D}_j)^*_{\mathscr{S}}$, $(\mathscr{M}_{j,k})^*_{\mathscr{S}}$ (see (1.25)) differ by a lower order terms $(\mathscr{D}_j)^*_{\mathscr{S}}u=\mathscr{D}^*_ju+h_ju$, $(\mathscr{M}_{j,k})^*_{\mathscr{S}}u=\mathscr{M}^*_{j,k}u+f_{j,k}u$, where h_j and $f_{j,k}$ are functions.

$$\mathscr{D}_j^* u(x) = -\partial_j u(x) - \partial^*_{\vec{\nu}(x)} \nu_j u(x)\,, \quad \mathscr{M}^*_{j,k} u(x) = -\partial_k \nu_j u(x) + \partial_j \nu_k u(x)\,,$$

Proof. Formula (1.23) is a direct consequence of the GAUSS formula on divergence

$$\int\limits_{\Omega^\pm} \partial_k u(y) dy = \pm \oint\limits_{\mathscr{S}} \nu_k(\tau) u(\tau) d_\tau \mathscr{S}\,, \qquad k = 1, 2, \ldots, n$$

(see [Di1], [Si1, 4.13(4)]) which yields

$$\int\limits_{\Omega^\pm} [\partial_k u(y)]^\top \overline{v(y)} dy = \int\limits_{\Omega^\pm} \partial_k [u^\top(y) \overline{v(y)}] dy - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial_k v(y)} dy$$
$$= \pm \oint\limits_{\mathscr{S}} \nu_k(\tau)\, [u(\tau)]^\top\, \overline{v(\tau)} d_\tau \mathscr{S} - \int\limits_{\Omega^\pm} [u(y)]^\top \overline{\partial_k v(y)} dy\,.$$

To prove the first formula in (1.26) we apply (1.23) and note that $\vec{\nu}(t) \cdot \vec{\nu}(t) \equiv 1$. For the second formula in (1.26) it suffices to take a first order tangent operator $\mathbf{G}(D) = \vec{h}(t)\nabla$, apply (1.23) and note that $\vec{h}(t) \cdot \vec{\nu}(t) \equiv 0$.

It suffices to prove formula (1.24) for the generators $\mathscr{D}_j$ and $\mathscr{M}_{j,k}$. To this end we recall the STOKES formula

$$\oint\limits_{\mathscr{S}} (\mathscr{M}_{j,k} u)(\tau) d_\tau \mathscr{S} = \oint\limits_{\mathscr{S}} [\nu_j(\tau)(\partial_k u)(\tau) - \nu_k(\tau)(\partial_j u)(\tau)] d_\tau \mathscr{S} = 0\,, \quad j, k = 1, \ldots, n\,. \tag{1.27}$$

This formula is well–known for $n = 2, 3$ (see, e.g., [Di1, Si1]). In general, for $n = 2, 3, \ldots$, (1.27) follows from another STOKES formula on external differential forms

$$\oint\limits_{\mathscr{S}} d\omega = 0\,, \qquad \text{ord}\ \omega = \dim \mathscr{S} - 1$$

(see [Sc1, (VI.7;3)], [Ca1, Ch. III, § 4.10]). In fact, it is easy to verify that

$$\nu_j d\mathscr{S} = (-1)^{j-1} \bigwedge_{m \neq j} dx_m$$

(see [Sc1, (VI.6;48)] for a detailed proof). With this formula at hand the integrand in (1.27) can be represented as a total differential

$$\mathscr{M}_{j,k} u d\mathscr{S} = (-1)^{j-1} (\partial_k u) \bigwedge_{m \neq j} dx_m - (-1)^{k-1} (\partial_j u) \bigwedge_{m \neq k} dx_m = d \left((-1)^{j+k} u \bigwedge_{m \neq j,k} dx_m \right)$$

for $j > k$ and we get (1.27). Since $\mathscr{M}_{k,j} = -\mathscr{M}_{j,k}$, $\mathscr{M}_{k,k} = 0$ (see (1.22)), (1.27) is proved for all $j, k = 1, \ldots n$.

From (1.27) we derive the following rule of partial integration for the generator $\mathscr{M}_{j,k}$

$$\oint\limits_{\mathscr{S}} \left[\mathscr{M}_{j,k}u(\tau)\right]^\top \overline{v(\tau)} d_\tau\mathscr{S} = \oint\limits_{\mathscr{S}} \mathscr{M}_{j,k}\left[u^\top(\tau)\overline{v(\tau)}\right] d_\tau\mathscr{S} - \oint\limits_{\mathscr{S}} u^\top(\tau)\overline{\mathscr{M}_{j,k}v(\tau)} d_\tau\mathscr{S}$$

$$= \oint\limits_{\mathscr{S}} u^\top(\tau)\overline{(\mathscr{M}_{j,k})^*_{\mathscr{S}} v(\tau)}\, d_\tau\mathscr{S}\,, \qquad j,k = 0,\dots,n$$

where $(\mathscr{M}_{jk})^*_{\mathscr{S}} = -\mathscr{M}_{jk} = \mathscr{M}_{kj}$ and (1.24), (1.25) are proved for the generators $\mathscr{M}_{jk}$. Invoking the relations (1.22) we find

$$(\mathscr{D}_j)^*_{\mathscr{S}} = -\sum_{k=1}^{n} (\mathscr{M}_{j,k})^*_{\mathscr{S}}\nu_k = -\sum_{k=1}^{n} \nu_k\partial_j\nu_k + \nu_j\partial^*_{\vec{\nu}}\,,$$

which yields (1.24) for another generator $\mathscr{D}_j$. ■

Example 1.9 *Let*

$$\mathbf{A}(x, D_x) := \sum_{j,k=0}^{n} a_{j,k}(x)\partial_j\partial_k, \quad a_{j,k} \in C^\infty(\overline{\Omega^\pm}, \mathbb{C}^{N\times N})$$

be an arbitrary second order operator with variable coefficients and consider the DIRICHLET *problem ($Au = f$ in $\Omega^\pm$ and $\gamma^\pm_{\mathscr{S}}u = g$ on $\mathscr{S}$) or the* NEUMANN *problem ($Au = f$ in $\Omega^\pm$ and $\sum\limits_{j,k=0}^{n} a_{j,k}\nu_j\gamma^\pm_{\mathscr{S}}\partial_k u = g$ on $\mathscr{S}$). Applying the partial integration* (1.23), *we arrive at* GREEN*'s formula* (1.13) *with*

$$\mathbf{B}_0(x, D_x) = I\,, \quad \mathbf{B}_1(x, D_x)u(x) = \sum_{j,k=0}^{n} a_{j,k}(x)\nu_j(x)\partial_k u(x)\,,$$

$$\mathbf{C}_0(x, D_x) = I\,, \quad \mathbf{C}_1(x, D_x)u(x) = -\sum_{j,k=0}^{n} \nu_k(x)\partial_j a^*_{j,k}(x)u(x)\,,$$

for the DIRICHLET *problem and with*

$$\mathbf{B}_0(x, D_x)u(x) = \sum_{j,k=0}^{n} a_{j,k}(x)\nu_j(x)\partial_k u(x)\,, \quad \mathbf{B}_1(x, D_x) = I\,,$$

$$\mathbf{C}_0(x, D_x)u(x) = -\sum_{j,k=0}^{n} \nu_k(x)\partial_j a^*_{j,k}(x)u(x)\,, \quad \mathbf{C}_1(x, D_x) = I\,,$$

for the NEUMANN *problem.*

Thus, via partial integration (see (1.23)) we can obtain the special GREEN formula for an arbitrary "basic" operator (not necessarily elliptic; cf. the foregoing Example 1.9). But it is not certain that "boundary" operators in the obtained formula are normal even if the "basic" operator is elliptic. On the other hand, the normality of one of two systems of “boundary” operators is necessary in order to replace them by arbitrary system of “boundary” operators of our choice (see § 5.1). For this reason we derive the special GREEN formula in Theorem 1.10.

The operator $\mathbf{A}(x, D_x)$ in (1.1) can be written in the form

$$\begin{aligned}
\mathbf{A}(x, D_x) &= \mathscr{A}_0(x, \vec{\nu}(x))\partial^m_{\vec{\nu}(x)} + \sum_{j=0}^{m-1} \mathbf{A}_{m-j}(x, \mathscr{D}_x)\partial^j_{\vec{\nu}(x)} \\
&= \mathscr{A}_0(x, \vec{\nu}(x))\partial^m_{\vec{\nu}(x)} + \sum_{j=0}^{m-1} \widetilde{\mathbf{A}}_{m-j}(x, \mathscr{M}_x)\partial^j_{\vec{\nu}(x)} \,, \qquad (1.28)\\
\mathbf{A}_k(x, \mathscr{D}_x) &= \sum_{|\alpha|\le k} a^0_{k,\alpha}(x)\mathscr{D}^\alpha_x = \widetilde{\mathbf{A}}_k(x, \mathscr{M}_x) = \sum_{|\beta|\le k} \widetilde{a}^0_{k,\beta}(x)\mathscr{M}^\beta_x \,, \\
&\mathscr{D}^\alpha_x := \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n \,, \quad \mathscr{M}^\beta_x := \mathscr{M}^{\beta_{1,1}}_{1,1} \cdots \mathscr{M}^{\beta_{n,n}}_{n,n} \,, \\
&\alpha \in \mathbb{N}^n_0 \,, \quad \beta \in \mathbb{N}^{n\times n}_0 \,, \quad x \in \Omega^\pm \,, \quad k = 1, 2, \dots, m \,,
\end{aligned}$$

where $\mathscr{A}_0(x, \xi)$ is the homogeneous principal symbol (see (1.4)) and the derivatives $\partial_{\vec{\nu}(x)}$, $\mathscr{D}_j$, $\mathscr{M}_{j,k}$ are defined in (1.19)–(1.21).

Theorem 1.10 *Let* $\mathbf{A}(x, D_x)$ *be defined as in* (1.1) *and*

$$\mathbf{B}_k(t, D_t) := \partial^k_{\vec{\nu}(t)} \,, \quad \mathbf{C}_k(t, D_t) := \sum_{j=k+1}^{m} (\partial^*_{\vec{\nu}(t)})^{j-k-1} \mathbf{A}^*_{m-j}(t, \mathscr{D}_t) \qquad (1.29)$$

$$= \sum_{j=0}^{m-k-1} (\partial^*_{\vec{\nu}(t)})^j \mathbf{A}^*_{m-j-k-1}(t, \mathscr{D}_t) \,, \quad \partial^*_{\vec{\nu}(t)} u(t) := -\sum_{k=1}^{n} \partial_{t_k} \nu_k(t) u(t) \,.$$

Hence the GREEN *formula* (1.13) *is valid.*

Proof. Applying (1.26) we find the following:

$$\begin{aligned}
\int_{\Omega^\pm} (\mathbf{A}u)^\top \overline{v} dy &= \pm \sum_{k=0}^{m-1} \sum_{j=k+1}^{m} \oint_{\mathscr{S}} \left[\gamma^\pm_{\mathscr{S}} \partial^k_{\vec{\nu}} u\right]^\top \overline{\gamma^\pm_{\mathscr{S}} (\partial^*_{\vec{\nu}})^{j-k-1} \mathbf{A}^*_{m-j} v} d_\tau \mathscr{S} + \int_{\Omega^\pm} u^\top \overline{\mathbf{A}^* v} dy \\
&= \pm \sum_{k=0}^{m-1} \oint_{\mathscr{S}} [\gamma^\pm_{\mathscr{S}} \partial^k_{\vec{\nu}} u]^\top \overline{\gamma^\pm_{\mathscr{S}} \mathbf{C}_k v} d_\tau \mathscr{S} + \int_{\Omega^\pm} u^\top \overline{\mathbf{A}^* v} dy \,.
\end{aligned}$$

The GREEN formula (1.13) for BVPs (1.10), (1.11) with operators (1.29) is proved. ■

BVP (1.10) with the normal “boundary” operators $\mathbf{B}_k = \partial^k_{\vec{\nu}}$, $k = 0, \dots, m-1$, is called the DIRICHLET problem.

The GREEN formulae (1.13) with operators (1.29) can be found in [Se1, (5.3)], [Tv1, Ch.III, (5.41)], [CW1, (1.5)], [CP1, Di1]. This special formula is a crucial component of the proof of Theorem 1.6.

1.3. About “boundary” operators in the Green formula. Next we discuss the problem of finding “boundary” differential operators $\{\mathbf{C}_j(t, D_t)\}^{m-1}_{j=0}$ in the

GREEN formula (1.13) in explicit form, provided the DIRICHLET system $\{\mathbf{B}_j(t, D_t)\}_{j=0}^{m-1}$ is fixed.

Similar formulae hold for the "boundary" differential operators $\{\mathbf{c}_j(t, D_t)\}_{j=0}^{mN-1}$ in the GREEN formula (1.12).

Since $\{\mathbf{B}_j(x, D_x)\}_{j=0}^{m-1}$ is a DIRICHLET system, to simplify the representation formulae hereafter, we suppose (cf. (1.14))

$$\operatorname{ord}\mathbf{B}_{\rm j} = {\rm j}\,, \quad \operatorname{ord}\mathbf{C}_{\rm j} = {\rm m} - 1 - {\rm j}\,, \quad {\rm j} = 0, \dots, {\rm m} - 1\,. \tag{1.30}$$

Let us introduce, for convenience, the following vector–operators of length m:

$$\begin{aligned}
\vec{\mathbf{D}}^{(m)}(x, D_x) &:= \left\{\partial_{\vec{\nu}(x)}^{m-1}, \dots, \partial_{\vec{\nu}(x)}, I, \right\}^{\top}, \\
\vec{\mathbf{B}}^{(m)}(x, D_x) &:= \{\mathbf{B}_0(x, D_x), \dots, \mathbf{B}_{m-1}(x, D_x)\}^{\top}, \\
\vec{\mathbf{C}}^{(m)}(x, D_x) &:= \{\mathbf{C}_0(x, D_x), \dots, \mathbf{C}_{m-1}(x, D_x)\}^{\top}.
\end{aligned} \tag{1.31}$$

When applied to a vector–function they produce longer vector–functions, e.g.,

$$\vec{\mathbf{B}}^{(m)}(x, D_x)u := \{\mathbf{B}_j(x, D_x)u\}_{j=0}^{m-1}\,.$$

Then the GREEN formula (1.13) takes the form

$$\int\limits_{\Omega^{\pm}} ((\mathbf{A}u)^{\top} \cdot \overline{v} - u^{\top} \cdot \overline{\mathbf{A}^{*}v})dy = \pm \oint\limits_{\mathscr{S}} (\vec{\mathbf{B}}^{(m)}u)^{\top} \cdot \overline{\vec{\mathbf{C}}^{(m)}v} d_{\tau}\mathscr{S}\,, \tag{1.32}$$

while the representation (1.28) acquires the form

$$\mathbf{A}(x, D_x) = \left[\vec{\mathbf{A}}^{(m+1)}(x, \mathscr{D}_x)\right]^{\top} \cdot \vec{\mathbf{D}}^{(m+1)}(x, D_x)\,, \tag{1.33}$$

$$\vec{\mathbf{A}}^{(m+1)}(x, D_x) := \{\mathscr{A}_0(x, \vec{\nu}(x)), \mathbf{A}_1(x, \mathscr{D}_x), \dots, \mathbf{A}_m(x, \mathscr{D}_x), \}^{\top}\,,$$

where "$\cdot$" denotes the formal scalar product of vectors. For the DIRICHLET system $\vec{\mathbf{B}}^{(m)}(t, D_t)$ we introduce the $m \times m$ lower–triangular matrix–operator

$$\mathbf{b}^{(m\times m)}(x, \mathscr{D}_x) = \begin{bmatrix} \mathscr{B}_{0,0}(x, \vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{B}_{1,0}(x, \mathscr{D}_x) & \mathscr{B}_{1,0}(x, \vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots, \\ \mathbf{B}_{m-1,0}(x, \mathscr{D}_x) & \mathbf{B}_{m-1,1}(x, \mathscr{D}_x) & \cdots & \mathscr{B}_{m-1,0}(x, \vec{\nu}(x)) \end{bmatrix} \tag{1.34}$$

with the entries $\mathbf{B}_{j,k}(x, \mathscr{D}_x)$ representing "tangent" differential operators of order $j - k$, compiled of matrix coefficients of the representations

$$\mathbf{B}_j(x, D_x) = \mathscr{B}_{j,0}(x, \vec{\nu}(x))\partial_{\vec{\nu}(x)}^{j} + \sum_{k=0}^{j-1} \mathbf{B}_{j,k}(x, \mathscr{D}_x)\partial_{\vec{\nu}(x)}^{k}\,, \tag{1.35}$$

where $\mathscr{B}_{j,0}(x, \xi)$ stands for the principal homogeneous symbol of $\mathbf{B}_j(x, D_x)$ ($j = 0, \dots m - 1$; cf. (1.28)).

Invertible block matrix–operators of type (1.34) will be referred to as admissible operators (cf. [Ag1, § 4]).

Since the entries of the principal diagonal in (1.34) are non–degenerate in the vicinity of $\mathscr{S}$

$$\det \mathscr{B}_{j,0}(x,\vec{\nu}(x)) \neq 0\,, \qquad j=0,\dots,m-1$$

(we remind that the operators $\mathbf{B}_j(t,D_t)$ are normal), $\mathbf{b}^{(m\times m)}(x,\mathscr{D}_x)$ is admissible on $\mathscr{S}$:

$$\left[\mathbf{b}^{(m\times m)}(x,\mathscr{D}_x)\right]^{-1} = \begin{bmatrix} \mathscr{B}^{-1}_{0,0}(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \widetilde{\mathbf{B}}_{1,0}(x,\mathscr{D}_x) & \mathscr{B}^{-1}_{1,0}(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots, \\ \widetilde{\mathbf{B}}_{m-1,0}(x,\mathscr{D}_x) & \widetilde{\mathbf{B}}_{m-1,1}(x,\mathscr{D}_x) & \cdots & \mathscr{B}^{-1}_{m-1,0}(x,\vec{\nu}(x)) \end{bmatrix}, \tag{1.36}$$

$$\widetilde{\mathbf{B}}_{j,k} := -\mathscr{B}^{-1}_{k,0}(x,\vec{\nu}(x))\mathbf{B}_{j,k}(x,\mathscr{D}_x)\mathscr{B}^{-1}_{j,0}(x,\vec{\nu}(x))\,.$$

The set of admissible matrix–operators is an algebra: finite sums, products and even inverses (when meaningful) of admissible matrix–operators are admissible again.

The representations (1.35), in the notation introduced above, can be written in the form

$$\vec{\mathbf{B}}^{(m)}(x,D_x) = \mathbf{b}^{(m\times m)}(x,\mathscr{D}_x)\vec{\mathbf{D}}^{(m)}(x,D_x)\,. \tag{1.37}$$

Theorem 1.11 *Let the* Dirichlet *system* $\vec{\mathbf{B}}^{(m)}(x,D_x)$ *be fixed and suppose that the convention* (1.30) *holds. Then the system* $\vec{\mathbf{C}}^{(m)}(x,D_x)$ *in the* Green *formula* (1.13) (*see* (1.31)) *is found as follows*

$$\vec{\mathbf{C}}^{(m)}(x,D_x) = \left[\left(\mathbf{b}^{(m\times m)}\right)^*_{\mathscr{S}}(x,\mathscr{D}_x)\right]^{-1}\left[\left(\vec{\mathbf{D}}^{(m)}\right)^*(x,D_x)\right]^{\top}\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathscr{D}_x)\mathbb{S}_m\,, \tag{1.38}$$

where $\left(\mathbf{b}^{(m\times m)}\right)^*_{\mathscr{S}}(x,\mathscr{D}_x)$ *denotes the "surface" adjoint to* $\mathbf{b}^{(m\times m)}(x,\mathscr{D}_x)$ (*see* (1.24), (1.25)), *while* $\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathscr{D}_x)$ *is the formally adjoint* (*see* (1.26), (1.25)) *to the following lower–triangular matrix–operator*

$$\mathbf{A}^{(m\times m)}(x,\mathscr{D}_x) = \begin{bmatrix} \mathscr{A}_0(x,\vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{A}_1(x,\mathscr{D}_x) & \mathscr{A}_0(x,\vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{m-2}(x,\mathscr{D}_x) & \mathbf{A}_{m-3}(x,\mathscr{D}_x) & \cdots & 0 \\ \mathbf{A}_{m-1}(x,\mathscr{D}_x) & \mathbf{A}_{m-2}(x,\mathscr{D}_x) & \cdots & \mathscr{A}_0(x,\vec{\nu}(x)) \end{bmatrix} \tag{1.39}$$

(*cf.* [*Se1*, (*7a*)], [*Gr1*]) *compiled of "tangent" differential operators of the representation* (1.28) (*see also* (1.33)); $\mathbb{S}_m$ *in* (1.38) *is the skew–identity matrix of order m:*

$$\mathbb{S}_m = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{1.40}$$

Proof. The proof is a byproduct of the proof of Theorem 1.6 (see § 5.1). ■

Remark 1.12 *If a boundary operator* $\mathbf{B}_j(x,D_x)$ *has order* $\operatorname{ord}\mathbf{B}_{\rm j} > {\rm m}-1$*, by using representations* (1.28) *for the "basic" operator and representation* (1.35) *for a "boundary" operator* $\mathbf{B}_j(x,D_x)$ *then the boundary values* $\gamma_0^\pm\mathbf{B}_j(t,D_t)u(t)$ *of a solution to the "basic" equation* $\mathbf{A}(x,D_x)u=f$ *in* (1.10) *can be found provided the boundary values of the normal derivatives* $\left\{\gamma_0^\pm\partial_{\vec{\nu}(t)}u(t)\right\}_{j=0}^{m-1}$ *are known (or, due to Lemma* 4.7*, if the datae* $\left\{\gamma_0^\pm\mathbf{C}_j(x,D_x)u(t)\right\}_{j=0}^{m-1}$ *are known for some* Dirichlet *system* $\{\mathbf{C}_j(x,D_x)\}_{j=0}^{m-1}$*). Details can be found in* [*Hr2*, § 20.1]*. Therefore the orders of the "boundary" operators* $\mathbf{B}_j(x,D_x)$ *in* (1.10) *are restricted:* $\operatorname{ord}\mathbf{B}_{\rm j}\leq {\rm m}-1$ *for all* $j=0,ldots,\ell-1$*.*

2 Spaces

We proceed by recalling several definitions and properties of function spaces from [CD1, Tr1, Tr2] which are going to be needed in the sequel.

$\mathbb{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decaying functions and $\mathbb{S}'(\mathbb{R}^n)$ – the dual space of tempered distributions. Since the Fourier transform and its inverse, defined by

$$\mathscr{F}\varphi(\xi)=\int\limits_{\mathbb{R}^n}e^{i\xi x}\varphi(x)dx \quad \text{and} \quad \mathscr{F}^{-1}\psi(x)=(2\pi)^{-n}\int\limits_{\mathbb{R}^n}e^{-ix\xi}\psi(\xi)d\xi, \quad x,\xi\in\mathbb{R}^n \tag{2.1}$$

are continuous in both spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, the **convolution operator**

$$\mathbf{a}(D)\varphi=W_a^0\varphi:=\mathscr{F}^{-1}a\mathscr{F}\varphi \quad \text{with} \quad a\in\mathbb{S}'(\mathbb{R}^n), \quad \varphi\in\mathbb{S}(\mathbb{R}^n) \tag{2.2}$$

is a continuous transformation from $\mathbb{S}(\mathbb{R}^n)$ into $\mathbb{S}'(\mathbb{R}^n)$ (see [Du1, DS1]).

The Bessel potential space $\mathbb{H}_p^s(\mathbb{R}^n)$ is defined as a subset of $\mathbb{S}'(\mathbb{R}^n)$ and is endowed with the following norm (see [Tr1, Tr2]):

$$||u|\mathbb{H}_p^s(\mathbb{R}^n)||:=||\langle D\rangle^s u|L_p(\mathbb{R}^n)||, \text{ where } \langle\xi\rangle^s:=(1+|\xi|^2)^{\frac{s}{2}}\,. \tag{2.3}$$

For the definition of the Besov space $\mathbb{B}_{p,q}^s(\mathbb{R}^n)$ ($1\leq p\leq\infty$, $1\leq q\leq\infty$, $s\in\mathbb{R}$) see [Tr1]: the space $\mathbb{B}_{p,p}^s(\mathbb{R}^n)$ ($1<p<\infty$, $s>0$) coincides with the trace space $\gamma_{\mathbb{R}^n}^+\mathbb{H}_p^{s+\frac{1}{p}}(\mathbb{R}_+^{n+1})$ ($\mathbb{R}_+^{n+1}:=\mathbb{R}^n\otimes\mathbb{R}^+$) and is known also as the Sobolev–Slobodečkii space $W_p^s(\mathbb{R}^n)$. It is known also that $\mathbb{B}_{2,2}^s(\mathbb{R}^n)=\mathbb{H}_2^s(\mathbb{R}^n)$ for $s\geq 0$ (see [Tr1]).

The space $\mathbb{B}_{\infty,\infty}^s(\mathbb{R}^n)$ for $s>0$ coincides with the well known Zygmund space $\mathbb{Z}^s(\mathbb{R}^n)$, while for $s\in\mathbb{R}^+\setminus\mathbb{N}$ both $\mathbb{B}_{\infty,\infty}^s(\mathbb{R}^n)$ and $\mathbb{Z}^s(\mathbb{R}^n)$ coincide with the Hölder space $C^s(\mathbb{R}^n)$.

The space $\widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n)$ is defined as the subspace of $\mathbb{H}_p^s(\mathbb{R}^n)$ of those functions $\varphi\in\mathbb{H}_p^s(\mathbb{R}^n)$, which are supported in the half space, $\operatorname{supp}\varphi\subset\overline{\mathbb{R}_+^n}$, whereas $\mathbb{H}_p^s(\mathbb{R}_+^n)$ denotes the quotient space $\mathbb{H}_p^s(\mathbb{R}_+^n)=\mathbb{H}_p^s(\mathbb{R}^n)\big/\widetilde{\mathbb{H}}_p^s(\mathbb{R}_-^n)$, $\mathbb{R}_-^n:=\mathbb{R}^n\setminus\overline{\mathbb{R}_+^n}$ and can be identified with the space of distributions φ on $\mathbb{R}_+^n$ which admit extensions $\ell\varphi\in\mathbb{H}_p^s(\mathbb{R}^n)$. Therefore $r_+\mathbb{H}_p^s(\mathbb{R}^n)=\mathbb{H}_p^s(\mathbb{R}_+^n)$, where $r_+=r_{\mathbb{R}_+^n}$ denotes the restriction from $\mathbb{R}^n$ to the half–space $\mathbb{R}_+^n$.

The spaces $\widetilde{\mathbb{B}}_{p,q}^s(\mathbb{R}_+^n)$ and $\mathbb{B}_{p,q}^s(\mathbb{R}_+^n)$ are defined similarly [Tr1, Tr2].

Next we define the Bessel potential space with weight, (see [CD1, §1.3], [Es1, §§23 and 26]).

Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $1 < p < \infty$; by $\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)$ we denote the space of functions (of distributions for $s < 0$) endowed with the norm

$$\|u|\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)\| := \sum_{k=0}^{m} \|x^k_n u|\mathbb{H}^{s+k}_p(\mathbb{R}^n_+)\|. \tag{2.4}$$

Obviously, $\mathbb{H}^{s,0}_p(\mathbb{R}^n) = \mathbb{H}^s_p(\mathbb{R}^n)$. The space $\mathbb{B}^{s,m}_{p,q}(\mathbb{R}^n_+)$ is defined in a similar way:

$$||u|\mathbb{B}^{s,m}_{p,q}(\mathbb{R}^n_+)|| := \sum_{k=0}^{m} \|x^k_n u|\mathbb{B}^{s+k}_{p,q}(\mathbb{R}^n_+)||$$

Let

$$\mathbb{H}^{s,\infty}_p(\mathbb{R}^n_+) := \bigcap_{m\in\mathbb{N}_0} \mathbb{H}^{s,m}_p(\mathbb{R}^n_+), \quad \mathbb{B}^{s,\infty}_{p,q}(\mathbb{R}^n_+) := \bigcap_{m\in\mathbb{N}_0} \mathbb{B}^{s,m}_{p,q}(\mathbb{R}^n_+) \tag{2.5}$$

with an appropriate topology which turns them into Freshet spaces.

Let $\mathscr{M}$ be a compact, C^∞–smooth n–dimensional manifold with a smooth boundary $\Gamma := \partial\mathscr{M} \neq \emptyset$. The spaces $\mathbb{H}^s_p(\mathscr{M})$, $\widetilde{\mathbb{H}}^s_p(\mathscr{M})$, $\mathbb{B}^s_{p,q}(\mathscr{M})$, $\widetilde{\mathbb{B}}^s_{p,q}(\mathscr{M})$, $\mathbb{H}^{s,m}_p(\mathscr{M})$, $\widetilde{\mathbb{H}}^{s,m}_p(\mathscr{M})$, $\mathbb{B}^{s,m}_{p,q}(\mathscr{M})$ and $\widetilde{\mathbb{B}}^{s,m}_{p,q}(\mathscr{M})$ can be defined by a partition of unity $\{\psi_j\}^\ell_{j=1}$ subordinated to some covering $\{Y_j\}^\ell_{j=1}$ of $\mathscr{M}$ and local coordinate diffeomorphisms

$$æ_j : X_j \to Y_j, \qquad X_j \subset \mathbb{R}^n_+, \quad j = 1, \dots, \ell.$$

In particular, for a compact domain $\Omega^+ \subset \mathbb{R}^n$ and non–compact $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}^+$ the spaces $\mathbb{H}^{s,m}_p(\Omega^\pm)$, $\widetilde{\mathbb{H}}^{s,m}_p(\Omega^\pm)$, $\mathbb{H}^{s,m}_{p,com}(\overline{\Omega^\pm})$, $\widetilde{\mathbb{B}}^{s,m}_{p,q,loc}(\overline{\Omega^\pm})$ etc. are defined as described above. For a compact domain Ω^+ the subscripts *com* and *loc* can be omitted.

From the embedding theorems of Sobolev we get that

$$\mathbb{H}^{s,\infty}_{p,loc}(\overline{\Omega^\pm}), \mathbb{B}^{s,\infty}_{p,q,loc}(\overline{\Omega^\pm}) \subset C^\infty(\Omega^\pm) \;\; (\text{but} \not\subset C^\infty(\overline{\Omega^\pm}),$$

$$\varphi(x) = \mathscr{O}(1) \quad \text{as} \quad x \in \Omega^-, \quad |x| \to \infty \tag{2.6}$$

whatever the parameters $s \in \mathbb{R}$ and $1 < p < \infty$ are.

Let $\mathbb{L}(\mathbb{X}_1, \mathbb{X}_2)$ denote the space of all linear bounded operators between the Banach spaces, $\mathbf{A} : \mathbb{X}_1 \to \mathbb{X}_2$.

The next two theorems summarise some results on interpolation (see [BL1, Tr2]), which will be used later on.

Theorem 2.1 *Let* $\mathbf{Int}[\mathbb{X}_1, \mathbb{X}_2]$ *denote one of the interpolation methods either the real* $[\mathbb{X}_1, \mathbb{X}_2]_{\vartheta,q}$ *or the complex* $(\mathbb{X}_1, \mathbb{X}_2)_\vartheta$ *(see [BL1, Tr2]). Then*

$$\mathbb{X}' = \mathbf{Int}[\mathbb{X}'_1, \mathbb{X}'_2], \qquad \mathbb{X}'' = \mathbf{Int}[\mathbb{X}''_1, \mathbb{X}''_2] \tag{2.7}$$

imply

$$\mathbb{L}(\mathbb{X}', \mathbb{X}'') \subset \mathbb{L}(\mathbb{X}'_1, \mathbb{X}''_1) \bigcap \mathbb{L}(\mathbb{X}'_2, \mathbb{X}''_2),$$

i.e., the boundedness $A : \mathbb{X}'_1 \to \mathbb{X}''_1$ *and* $A : \mathbb{X}'_2 \to \mathbb{X}''_2$ *implies the boundedness* $A : \mathbb{X}' \to \mathbb{X}''$ *provided the spaces* $\mathbb{X}'$ *and* $\mathbb{X}''$ *are properly interpolated (see* (2.7)*).*

Theorem 2.2 (see [BL1, §§ 6.2,6.4]). *Let*

$$s=\vartheta s_1+(1-\vartheta)s_0\,,\quad s,s_0,s_1\in\mathbb{R}\,,\quad 0<\vartheta<1\,,$$
$$\frac{1}{p}=\frac{\vartheta}{p_1}+\frac{1-\vartheta}{p_0}\,,\quad 1\le p,p_0,p_1\le\infty\,,\quad \frac{1}{q}=\frac{\vartheta}{q_1}+\frac{1-\vartheta}{q_0}\,,\quad 1\le q,q_0,q_1\le\infty \tag{2.8}$$

and $\dfrac{\vartheta}{r}:=0$ *if* $r=\infty$. *Then*

$$\big(\mathbb{H}^{s_0}_{p_0}(\mathbb{M}),\mathbb{H}^{s_1}_{p_1}(\mathbb{M})\big)_\vartheta=\mathbb{H}^{s}_{p}(\mathbb{M})\,,$$
$$\big[\mathbb{H}^{s_0}_{p_0}(\mathbb{M}),\mathbb{H}^{s_1}_{p_1}(\mathbb{M})\big]_{\vartheta,q}=\mathbb{B}^{s}_{p,q}(\mathbb{M})\,,\quad \big(\mathbb{B}^{s_0}_{p_0,q_0}(\mathbb{M}),\mathbb{B}^{s_1}_{p_1,q_1}(\mathbb{M})\big)_\vartheta=\mathbb{B}^{s}_{p,q}(\mathbb{M})\,, \tag{2.9}$$

where $\mathbb{M}=\Omega^\pm\subset\mathbb{R}^n$ *or* $\mathbb{M}=\mathscr{M}$ *is a smooth manifold.*

The same interpolation results (2.9) *hold for the spaces* $\widetilde{\mathbb{H}}^s_p(\mathbb{M})$ *and* $\widetilde{\mathbb{B}}^s_{p,q}(\mathbb{M})$ *if* $\mathbb{M}$ *has the boundary* $\partial\mathbb{M}\neq\emptyset$.

Let us point out that a slight modification of the proof allows one to establish results, similar in spirit to those discussed in the theorem above, for weighted spaces $\mathbb{H}^{s,k}_p(\mathbb{M})$, $\widetilde{\mathbb{H}}^{s,k}_p(\mathbb{M})$, $\mathbb{B}^{s,k}_{p,q}(\mathbb{M})$ and $\widetilde{\mathbb{B}}^{s,k}_{p,q}(\mathbb{M})$, where k is arbitrary integer.

Let us agree to denote by $\mathbb{X}^{s,m}_p(\mathbb{M})$ (by $\widetilde{\mathbb{X}}^{s,m}_p(\mathbb{M})$) the following spaces

$$\text{either}\quad \mathbb{H}^{s,m}_p(\mathbb{M})\quad\text{or}\quad \mathbb{B}^{s,m}_{p,q}(\mathbb{M})\qquad \Big(\text{either}\quad \widetilde{\mathbb{H}}^{s,m}_p(\mathbb{M})\quad\text{or}\quad \widetilde{\mathbb{B}}^{s,m}_{p,q}(\mathbb{M})\Big)\,, \tag{2.10}$$

where $1\le q\le\infty$ is arbitrary.

3 Representation of solutions and layer potentials

Throughout the present section we assume that the differential operator $\mathbf{A}(x,D_x)$ in (1.1) is invertible on $\mathbb{R}^n$ or, in other words, has a fundamental solution (see [Hr2, § 4.4]), which is understood either as the inverse

$$\mathbf{F_A}=\mathbf{A}^{-1}(x,D_x): C^\infty_{\rm com}(\mathbb{R}^n)\longrightarrow\mathscr{D}'(\mathbb{R}^n)\,,$$
$$\mathbf{A}(x,D_x)\mathbf{F_A}\varphi=\mathbf{F_A}\mathbf{A}(x,D_x)\varphi=\varphi\,,\qquad \varphi\in C^\infty_0(\Omega^\pm)\,,$$

or as the distributional Schwartz kernel $\mathscr{K}_\mathbf{A}(x,y): C^\infty_0(\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^n)$ of the operator $\mathbf{F_A}$ (see [Hr2, Theorem 5.2.1])

$$\mathbf{A}(x,D_x)\mathscr{K}_\mathbf{A}(x,y)=\delta(x-y)I_N \tag{3.1}$$

with $\delta(x)$, I_N, standing for the Dirac function and the identity $N\times N$ matrix, respectively. The distributional kernel $\mathscr{K}_\mathbf{A}(x,y)$ is also called a fundamental matrix (for $A(x,D_x)$).

We suppose that $\mathbf{A}(x,D_x)$ is elliptic with even order, $\operatorname{ord}\mathrm{A}=\mathrm{m}=2\ell$ (see § 6.6). Then the inverse $\mathbf{F_A}=\mathbf{F_A}(x,D_x)$ is a pseudodifferential operator[5)] with a symbol from the Hörmander class $\mathbb{S}^{-m}(\Omega^\pm,\mathbb{R}^n)$ (see, e.g., [EgS1, Hr2, Sb1, Tv1]). This yields the inclusion $\operatorname{sing\,supp}\mathscr{K}_\mathbf{A}=\Delta_{\mathbb{R}^n}$ or, in other notation, $\mathscr{K}_\mathbf{A}\in C^\infty((\mathbb{R}^n\otimes\mathbb{R}^n)\setminus\Delta_{\mathbb{R}^n})$.

[5)]See § 4.1 for some elementary information about PsDOs.

Moreover, if $\mathbf{A}(x, D_x)$ is hypoelliptic (see § 4.1) a fundamental solution $\mathbf{F_A}(x, D_x)$ is a PsDO as well and[6)] sing supp $\mathscr{K}_{\mathbf{A}} = \Delta_{\mathbb{R}^n}$.

Since $\mathbf{A}(x, D_x)$ has a fundamental solution $\mathbf{F_A}$, the adjoint operator $\mathbf{A}^*(x, D_x)$ in (1.2) has it, too, and

$$\mathbf{F}_{\mathbf{A}^*} = \mathbf{F}^*_{\mathbf{A}}\,, \qquad \mathscr{K}_{\mathbf{A}^*}(x,y) = [\overline{\mathscr{K}_{\mathbf{A}}(y,x)}]^\top\,, \tag{3.2}$$

where $\mathscr{K}_{\mathbf{A}^*}(x,y)$ is the Schwartz kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}$ of the adjoint operator.

As a first application of the Green formula (1.13) we can get the representation of a solution of BVP (1.10). For this purpose let us consider $v_{\varepsilon,x}(y) = \chi_\varepsilon(x-y)\mathscr{K}_{\mathbf{A}^*}(y,x)$, where $\mathscr{K}_{\mathbf{A}^*}(x,y)$ is the kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}$ (see (3.2)) and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\chi_\varepsilon(x) = 1$, $\chi_\varepsilon(x) = 0$ for $|x| > \varepsilon$ and $|x| < \varepsilon/2$, respectively. Inserting each vector–column of the $N \times N$ matrix $v_{\varepsilon,x}(y)$ into the Green formula (1.13), sending $\varepsilon \to 0$ and recollecting the result as a vector, we find the following

$$\langle \delta(x-\cdot)I_N, u\rangle = \chi_{\Omega^\pm}(x)u(x) = \mathbf{N}_{\Omega^\pm}f(x) \pm \sum_{j=0}^{2\ell-1} \mathbf{V}_j \gamma^\pm_{\mathscr{S}} \mathbf{B}_j u(x)\,, \quad x \in \Omega^\pm\,, \tag{3.3}$$

$$\mathbf{N}_{\Omega^\pm}\varphi(x) := \int\limits_{\Omega^\pm} \left[\overline{\mathscr{K}_{\mathbf{A}^*}(y,x)}\right]^\top \varphi(y)dy = \int\limits_{\Omega^\pm} \mathscr{K}_{\mathbf{A}}(x,y)\varphi(y)dy\,, \tag{3.4}$$

where $\chi_{\Omega^\pm}$ is the characteristic function of $\Omega^\pm \subset \mathbb{R}^n$ and

$$\begin{aligned}\mathbf{V}_j\psi(x) &:= \oint\limits_{\mathscr{S}} \left[\overline{\mathbf{C}_j(\tau, D_\tau)\mathscr{K}_{\mathbf{A}^*}(\tau,x)}\right]^\top \varphi(\tau)d_\tau\mathscr{S} = \oint\limits_{\mathscr{S}} \left[\overline{\mathbf{C}_j(\tau, D_\tau)}\mathscr{K}^\top_{\mathbf{A}}(x,\tau)\right]^\top \varphi(\tau)d_\tau\mathscr{S} \\ &= \sum_{|\alpha|\le\mu_j} \oint\limits_{\mathscr{S}} \partial^\alpha_\tau \mathscr{K}_{\mathbf{A}}(x,\tau)\overline{c^\top_{j\alpha}(\tau)}\varphi(\tau)d_\tau\mathscr{S}\,, \quad j = 0, \cdots, 2\ell-1 \end{aligned}\tag{3.5}$$

(cf. (1.9), (1.11)) are the layer potentials.

The integrals in (3.3)–(3.5), as well as the similar ones considered later (see (3.11)) are understood as the functionals $\mathscr{K}_{\mathbf{A}}(x,\cdot)$, $(\partial^\alpha_t \mathscr{K}^{(\beta)}_{\mathbf{A}}(x,\cdot)$ etc.) with a parameter $x \in \mathbb{R}^n$ applied to the test function $\varphi(\tau)$ (to $\overline{c^\top_\alpha(\tau)}\varphi(\tau)$).

Summing up (3.3) for the domains $\Omega^\pm$ we get

$$\begin{aligned} u(x) &= \mathbf{F_A}f(x) + \sum_{j=0}^{2\ell-1} \mathbf{V}_j[\mathbf{B}_j u](x)\,, \\ [v](t) &:= \gamma^+_{\mathscr{S}}v(t) - \gamma^-_{\mathscr{S}}v(t)\,, \quad t \in \mathscr{S}\,, \quad x \in \mathbb{R}^n \setminus \mathscr{S} = \Omega^+ \cup \Omega^-\,, \end{aligned}\tag{3.6}$$

where $f = \mathbf{A}u\Big|_{\Omega^+\cap\Omega^-} = \mathbf{A}u\Big|_{\mathbb{R}^n\setminus\mathscr{S}}$ and

$$\mathbf{F_A}\varphi(x) := \mathbf{N}_{\Omega^-}v(x) + \mathbf{N}_{\Omega^+}v(x) = \int\limits_{\mathbb{R}^n} \mathscr{K}_{\mathbf{A}}(x,y)\varphi(y)dy \tag{3.7}$$

[6)]Almost all results of the present and forthcoming sections are valid for hypoelliptic operators, but operators might have odd order $m = 2\ell + 1$. Operators with odd order can be found also among properly elliptic systems (terminology from [Ag1, LM1, Ro1]; see § 6.6 and [Hr1, § 4.1], [Tv1, Ch.1, Theorem 2.2].

is the fundamental solution of $\mathbf{A}(x, D_x)$.

The pseudodifferential operators $\mathbf{a}(x, D)$ and $\mathbf{b}(x, D)$ are called locally equivalent at $x_0 \in \mathbb{R}^n$ if

$$\inf_{\chi} \|\chi\, [\mathbf{a}(\cdot, D) - \mathbf{b}(\cdot, D)]\, |\mathbb{H}^s_p(\mathbb{R}^n)\| = 0\,, \tag{3.8}$$

where the infimum is taken over the set of all smooth functions $\chi \in C_0^\infty(\mathbb{R}^n)$ which are equal to the identity, $\chi(x) \equiv 1$, in some neighbourhood of x_0. The local equivalence at x_0 is usually denoted as follows

$$\mathbf{a}(x, D) \overset{x_0}{\sim} \mathbf{b}(x, D)$$

(see [Sm1]) and we refer to [Du1] for the elementary properties of this local equivalence.

Lemma 3.1 *If the operator* $\mathbf{A}(x, D_x)$, *defined in* (1.1), *has constant matrix–coefficients* $a_\alpha =$ const, *then the fundamental solution* $\mathbf{F_A} = \mathbf{F_A}(D)$ *exists provided* [7] $\mathbf{A}(D) \not\equiv 0$. *If, in addition, the symbol*

$$\mathscr{A}(\xi) := \sum_{|\alpha|\le m} a_\alpha(-i\xi)^\alpha, \quad \xi \in \mathbb{R}^n$$

is elliptic, $\det \mathscr{A}(\xi) \neq 0$ *for all* $|\xi| \ge R$, *then the fundamental solution* $\mathbf{F_A} = \mathbf{F_A}(D)$ *is a convolution*

$$\mathbf{F_A}(D) = \mathscr{F}^{-1}_{\xi\to x}\left[\mathscr{A}^{-1}(\xi)\right] \tag{3.9}$$

and the Schwartz *kernel of* $\mathbf{F_A}(D)$ *depends on the difference of the arguments,* $\mathscr{K}_\mathbf{A}(x, y) = \mathscr{K}_\mathbf{A}(x - y)$.

In the general case of non–constant coefficients, a fundamental solution $\mathbf{F_A} : C_0^\infty(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n))$ *and the convolution operator* $\mathbf{F_{A_0}}(x_0, D)$, *which is the fundamental solution of the principal part* $\mathbf{A}_0(x_0, D)$ *(see* (1.4)*) with coefficients frozen at* x_0 *(cf.* (3.9)*), are locally equivalent*

$$\mathbf{F_A} \overset{x_0}{\sim} \mathbf{F_A}(x_0, D_x) \tag{3.10}$$

at an arbitrary point $x_0 \in \mathbb{R}^n$.

Proof. All claims, except (3.10), can be found in [Hr1, §§ 3,4], [Hr2, § 11].

Local equivalence (3.10) follows from the obvious equivalence $\mathbf{A}(x, D_x) \overset{x_0}{\sim} \mathbf{A}(x_0, D_x)$ (see [Du1]) and from the elementary property: if operators are locally equivalent and invertible, the inverses are locally equivalent as well. ∎

If $\mathbf{A}(x, D_x)$ is hypoelliptic and has a fundamental solution, we can only indicate the symbol of the fundamental solution, which is the symbol of a parametrix (see § 4.1). In particular, the principal symbol of the fundamental solution coincides with the inverse $\mathscr{A}_0^{-1}(x, \xi)$ of the principal symbol of $\mathbf{A}(x, D_x)$.

If $\mathbf{A}(x, D_x)$ has constant matrix–coefficients but is not an elliptic operator, the condition $\operatorname{sing\,supp} \mathscr{K}_\mathrm{A} = \Delta_{\mathbb{R}^n}$ might fail (see [Hr2, § 10.2]); if this is the case, the fundamental solution $\mathbf{F_A}$ is not a pseudodifferential operator.

Let $\beta \in \mathbb{N}_0^n$ and consider the following generalised layer potentials

$$\mathbf{V}^{(\beta)}_\mathbf{G}\varphi(x) := \oint_{\mathscr{S}} \left[\overline{\mathbf{G}(\tau, D_\tau)\mathscr{K}^{(\beta)}_{\mathbf{A}^*}(\tau, x)}\right]^\top \varphi(\tau)d_\tau\mathscr{S} = \sum_{|\alpha|\le\mu}\oint_{\mathscr{S}} \partial^\alpha_\tau \mathscr{K}^{(\beta)}_\mathbf{A}(x, \tau)\overline{c^\top_\alpha(\tau)}\varphi(\tau)d_\tau\mathscr{S}\,,$$

$$\mathbf{G}(t, D_t) = \sum_{|\alpha|\le\mu} c_\alpha(t)\partial^\alpha_t\,, c_\alpha \in C^\infty(\mathscr{S})\,, \quad \mathscr{K}^{(\beta)}_\mathbf{A}(x, y) := (x - y)^\beta\mathscr{K}_\mathbf{A}(x, y). \tag{3.11}$$

---

[7] Fundamental solutions exist also for operators with analytic coefficients $a_\alpha(x)$ (see [Jo1].

If $\mathbf{G}_0(t,D_t)=I$ and $\beta=0$, then we get the single layer potential

$$\mathbf{V}^{(0)}_{\mathbf{I}}\varphi(x)=\mathbf{V}\psi(x):=\mathbf{V}_0\psi(x)=\oint_{\mathscr{S}}\mathscr{K}_{\mathbf{A}}(x,\tau)\psi(\tau)d_\tau\mathscr{S}\,. \tag{3.12}$$

Theorem 3.2 *Let $\beta\in\mathbb{N}^n_0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, $m=2l$, $\mathbf{A}(x,D_x)$ be elliptic with a fundamental matrix $\mathscr{K}_A(x,y)$. The generalised layer potential $\mathbf{V}^{(\beta)}_{\mathbf{G}}$ with $\mu=\operatorname{ord}\mathbf{G}<2\ell$, $k=0,1,\dots,\infty$, has the following continuity properties:*

$$\mathbf{V}^{(\beta)}_{\mathbf{G}}\ :\ \mathbb{H}^s_p(\mathscr{S})\longrightarrow\mathbb{H}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm})\,, \tag{3.13}$$

$$:\ \mathbb{B}^s_{p,p}(\mathscr{S})\longrightarrow\mathbb{H}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm})\bigcap\mathbb{B}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,p,loc}(\overline{\Omega^\pm})\,, \tag{3.14}$$

$$:\ \mathbb{B}^s_{p,q}(\mathscr{S})\longrightarrow\mathbb{B}^{s+2\ell-1-\mu+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm})\,. \tag{3.15}$$

The result also holds for the Bessel potential spaces $\mathbb{B}^s_{p,q}(\mathscr{S})$ with $p=1,\infty$, $1\le q\le\infty$ provided $s>0$. In particular, it holds for the Zygmund *spaces (the case $p=q=\infty$):*

$$\mathbf{V}^{(\beta)}_{\mathbf{G}}:\mathbb{Z}^s(\mathscr{S})\longrightarrow\mathbb{Z}^{s+2\ell-1-\mu+|\beta|,k}(\overline{\Omega^\pm})\,. \tag{3.16}$$

The proof is deferred to § 5.3.

Remark 3.3 *If the operator* (1.1) *has constant matrix coefficients $a_\alpha(x)=\text{const}$, the restriction $\mu=\operatorname{ord}\mathbf{G}<2\ell$ in Theorem 3.2 turns out to be superfluous.*

In fact, the potential operators $\mathbf{V}^{(\beta)}_{\partial^k_{y_j}}=\partial^k_{x_j}\mathbf{V}^{(\beta)}$ are well defined even for $k\ge 2\ell+|\beta|$. Moreover, a potential–type operator $\mathbf{G}(x,D_x)\mathbf{V}^{(\beta)}$ (see (3.11) *for $\mathbf{G}(x,D_x)$) is well defined for arbitrary $\mu=\operatorname{ord}\mathbf{G}\in\mathbb{N}$ and restricted to the surface, $\gamma^\pm_{\mathscr{S}}\mathbf{G}(x,D_x)\mathbf{V}^{(\beta)}$ can be interpreted as a pseudodifferential operator of order $-2\ell+1+\mu-|\beta|$ on $\mathscr{S}$, although it has a hypersingular kernel when $-2\ell+1+\mu-|\beta|>0$ (see § 6.1).*

Corollary 3.4 *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, $m=2\ell$, $k=0,1,\dots,\infty$, $\mathbf{A}(x,D_x)$ be elliptic and have a fundamental solution. Then any solution $u(x)$ of the system*

$$\mathbf{A}(x,D_x)u=f\,,\quad f\in\mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^\pm})\,,$$

$f\in\mathbb{B}^{s-2\ell,k}_{p,q}(\overline{\Omega^\pm})$ (or $f\in\mathbb{Z}^{s-2\ell,k}(\overline{\Omega^\pm})$ with $s-2\ell>0$) satisfies a priori estimates

$$\|u|\mathbb{H}^{s,k}_p(\overline{\Omega^+})\|\le M\left[\|f|\mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^+})\|+\sum_{j=0}^{2\ell-1}\|\gamma^\pm_{\mathscr{S}}\partial^j_{\bar\nu}u|\mathbb{B}^{s-\frac{1}{p}-j,k}_{p,p}(\mathscr{S})\|\right],$$

$$\|u|\mathbb{B}^{s,k}_{p,q}(\overline{\Omega^+})\|\le M\left[\|f|\mathbb{B}^{s-2\ell,k}_{p,q}(\overline{\Omega^+}\|+\sum_{j=0}^{2\ell-1}\|\gamma^\pm_{\mathscr{S}}\partial^j_{\bar\nu}u|\mathbb{B}^{s-\frac{1}{p}-j,k}_{p,q}(\mathscr{S})\|\right] \tag{3.17}$$

$$\left(\|u|\mathbb{Z}^{s,k}(\overline{\Omega^\pm})\|\le M\left[\|f|\mathbb{Z}^{s-2\ell,k}(\overline{\Omega^\pm})\|+\sum_{j=0}^{2\ell-1}\|\gamma^\pm_{\mathscr{S}}\partial^j_{\bar\nu}u|\mathbb{Z}^{s-j,k}(\mathscr{S})\|\right]\right).$$

Similar estimates hold for the domain Ω^-, although we should replace u with χu where $\chi\in C^\infty_0(\Omega^-)$ is an arbitrary smooth function with a compact support.

Proof. The proof follows from Theorem 3.2 and the representation formula (3.3).

Remark 3.5 *When the operator* $\mathbf{A}(x, D_x)$ *is hypoelliptic and has no fundamental solution, then a parametrix* $\mathbf{R_A}(x, D_x)$ *can be used instead (see* § 4.1 *below). Specifically, inserting the truncated* Schwartz *kernel of the parametrix into the* Green *formulae similarly to* (3.3) *we get the following representation for the solution of the BVP* (1.14)*:*

$$\chi_{\Omega^\pm}(x)u(x) = \mathbf{N}_{\Omega^\pm}f(x) \pm \sum_{j=0}^{2\ell-1} \mathbf{V}_j\gamma^\pm_{\mathscr{S}}\mathbf{B}_j u(x) + \mathbf{T}u(x)\,, \quad x \in \Omega^\pm\,, \tag{3.18}$$

where the operator $\mathbf{T}$ *has order* $-\infty$*. From Theorem* 3.2 *and the representation formula* (3.18) *we get the following a priori estimate*

$$\|u|\mathbb{H}^{s,k}_p(\overline{\Omega^+})\| \le M\left[\|f|\mathbb{H}^{s-2\ell,k}_p(\overline{\Omega^+})\| + \sum_{j=0}^{2\ell-1}\|\gamma^\pm_{\mathscr{S}}\partial^j_{\overline{\nu}}u|\mathbb{H}^{s-\frac{1}{p}-j,k}_p(\mathscr{S})\| + \|u|\mathbb{H}^{s-m,k}_p(\overline{\Omega^+})\|\right] \tag{3.19}$$

for arbitrary $m = 1, 2, \dots$ *(cf.* (3.17)*). Similar inequalities hold for the spaces* $\mathbb{B}^{s-2\ell,k}_{p,q}(\overline{\Omega^+})$ *and* $\mathbb{Z}^{s-2\ell,k}(\overline{\Omega^\pm})$ *(with* $s - 2\ell > 0$*) as well. For the domain* Ω^- *we should replace* u *by* χu, $\chi \in C^\infty_0(\Omega^-)$ *(see Corollary* 3.4*).*

Remark 3.6 *Different a priory estimates have been proved, e.g., in* $[LM1, Ch.2, \S\, 4]$*. In contrast to* (3.19) *they contain half as many traces* $\gamma^\pm_{\mathscr{S}}\partial^{m_j}_{\overline{\nu}}u$, $j = 0, \dots, m-1$ *in the right–hand side. They are used in* $[LM1, Ch.2, \S\, 5]$ *in order to establish the* Fredholm *property of BVP* (1.14)*, provided the* Shapiro–Lopatinskii *conditions hold.*

4 Auxiliary propositions

4.1. On pseudodifferential operators. If the convolution operator in (2.2) admits a continuous extension

$$W^0_a : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n),$$

we write $a \in M_p(\mathbb{R}^n)$ and call $a(\xi)$ a (Fourier) L_p–multiplier. Let

$$M^{(\nu)}_p(\mathbb{R}^n) = \{\langle\xi\rangle^\nu a(\xi) : a \in M_p(\mathbb{R}^n)\}\ , \quad \nu \in \mathbb{R}\,,$$

where $\langle\xi\rangle$ is defined in (2.3). It is easy to observe, that the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}^n) \to \mathbb{H}^{s-\nu}_p(\mathbb{R}^n)$$

is continuous if and only if $a \in M^{(\nu)}_p(\mathbb{R}^n)$ (cf., e.g., [DS1, CD1]).

If $\partial^k_{\xi_n}a \in M^{(\nu-k)}_p(\mathbb{R}^n)$ for all $k = 0, \dots, m$, then W^0_a is continuous between weighted spaces

$$W^0_a : \mathbb{H}^{s,m}_p(\mathbb{R}^n) \to \mathbb{H}^{s-\nu,m}_p(\mathbb{R}^n)$$

(see [CD1, Theorem 1.6]).

As an example we consider the BESSEL potential operators

$$\begin{aligned} W^0_{\langle\xi\rangle^r} = \langle D\rangle^r \quad &: \quad \mathbb{X}^{s,m}_p(\mathbb{R}^n) \longrightarrow \mathbb{X}^{s-r,m}_p(\mathbb{R}^n)\,, \\ W^0_{(\xi_n - i\langle\xi'\rangle)^r} = r_+(D_n - i\langle D'\rangle)^r \ell \quad &: \quad \mathbb{X}^{s,m}_p(\mathbb{R}^n_+) \longrightarrow \mathbb{X}^{s-r,m}_p(\mathbb{R}^n_+)\,, \\ W^0_{(\xi_n + i\langle\xi'\rangle)^r} = (D_n + i\langle D'\rangle)^r \quad &: \quad \widetilde{\mathbb{X}}^{s,m}_p(\mathbb{R}^n_+) \longrightarrow \widetilde{\mathbb{X}}^{s-r,m}_p(\mathbb{R}^n_+)\,, \quad r \in \mathbb{R} \end{aligned} \tag{4.1}$$

(cf. (2.10)), where r_+ is the restriction operator (from $\mathbb{R}^n$ to $\mathbb{R}^n_+$), while ℓ is an arbitrary extension of a function $\varphi \in \mathbb{X}^{s,m}_p(\mathbb{R}^n_+)$ to $\ell\varphi \in \mathbb{X}^{s,m}_p(\mathbb{R})$ (a right inverse to r_+). The above considerations lead to results which are independent of the particular choice of the extension and restriction operators. In fact, $r_+(D_n - i\langle D'\rangle)^r\varphi_- = 0$ for $\varphi_- \in \widetilde{\mathbb{X}}^{s,m}_p(\mathbb{R}^n_-)$ due to the PALEY–WIENER theorem on the FOURIER transforms of functions supported on half spaces.

The operators in (4.1) are isomorphisms for arbitrary $r \in \mathbb{R}$ and the inverse isomorphisms are $\langle D\rangle^{-r}$ and $(D_n \pm i\langle D'\rangle)^{-r}$ (see, e.g., [CD1, §1.3]).

The next theorem is a slight modification of the MIKHLIN–HÖRMANDER–LIZORKIN multiplier theorem. The proof can be found in [Hr2, Theorem 7.9.5] and [Sr1].

Theorem 4.1 *If the inequality*

$$|\xi^\beta \partial^\beta a(\xi)| \le M\langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n\,, \quad |\beta| \le \left[\frac{n}{2}\right] + 1, \quad \beta \le 1\,,$$

holds for some $M > 0$, then $a \in \bigcap\limits_{1<p<\infty} M^{(\nu)}_p(\mathbb{R}^n)$. ■

Let $a \in M^{(\nu)}_p(\mathbb{R}^n)$. Then the operator

$$W_a := r_+\mathbf{a}(D) : \widetilde{\mathbb{X}}^s_p(\mathbb{R}^n_+) \to \mathbb{X}^{s-\nu}_p(\mathbb{R}^n_+)$$

is continuous. If a symbol $a(x,\xi)$ depends on the variable x and $a \in C(\mathbb{R}^n, \mathbb{S}'(\mathbb{R}^n))$, the corresponding operator (see (2.2))

$$\mathbf{a}(x,D)\varphi(x) = W^0_{a(x,\cdot)}\varphi(x) := \left(\mathscr{F}^{-1}_{\xi\to x} a(x,\xi)\mathscr{F}_{y\to\xi}\varphi(y)\right)(x)\,, \quad \varphi \in \mathbb{S}(\mathbb{R}^n) \tag{4.2}$$

is called a **pseudodifferential operator** (in brief PsDO). Here $C(\Omega, \mathscr{B})$ denotes the set of all continuous functions $a : \Omega \to \mathscr{B}$. Let $M^{(s,s-\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$ denote the class of symbols $a(x,\xi)$ for which the operator in (4.2) extends to a continuous mapping

$$\mathbf{a}(x,D) : \mathbb{H}^s_p(\mathbb{R}^n) \to \mathbb{H}^{s-\nu}_p(\mathbb{R}^n)$$

and $M^{(\nu)}_p(\mathbb{R}^n, \mathbb{R}^n) := \bigcup\limits_{s\in\mathbb{R}} M^{(s,s-\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$.

Theorem 4.2 *Let $\mathbb{N}_0 := \{0, 1, \ldots\}$. If the estimates*

$$\int\limits_{\mathbb{R}^n} |\xi^\beta \partial^\alpha_x \partial^\beta_\xi a(x,\xi)|dx \le M_\alpha\langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n \tag{4.3}$$

hold for some $M_\alpha > 0$ and all $\alpha, \beta \in \mathbb{N}^n_0$, $|\beta| \le \left[\frac{n}{2}\right] + 1$, $\beta \le 1$, then $a \in \bigcap\limits_{1<p<\infty} M^{(\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$.

Moreover, if (4.3) *holds for all* $\beta_n = 0, 1, \dots$ *and* $|\beta'| \leq \left[\frac{n}{2}\right] + 1$, *the PsDO*

$$\mathbf{a}(x, D) : \mathbb{H}_p^{s,m}(\mathbb{R}^n) \to \mathbb{H}_p^{s-\nu,m}(\mathbb{R}^n)$$

is continuous for arbitrary $m \in \mathbb{N}_0$.

Proof. The first part is proved in [Sh2, Theorems 4.1 and 5.1] and the second part in [CD1, Theorems 1.6]. ■

If the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta,K} \langle \xi \rangle^{\nu - |\beta|}, \quad \nu \in \mathbb{R}, \quad x \in K, \quad \xi \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{N}_0^n$$

hold for all compact $K \subset \overline{\Omega^\pm}$, we write $a \in \mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$ and call $\mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$ the Hörmander class. If r_Ω is the restriction to $\Omega \subset \mathbb{R}^n$ and $a \in \mathbb{S}^\nu(\Omega^\pm, \mathbb{R}^n)$, the operator

$$r_{\Omega^\pm} \mathbf{a}(x, D_x) : \widetilde{\mathbb{X}}_{p,com}^{s,m}(\Omega^\pm) \longrightarrow \mathbb{X}_{p,loc}^{s-\nu,m}(\overline{\Omega^\pm}), \quad s \in \mathbb{R}, \ 1 < p < \infty \tag{4.4}$$

(see (2.10)) is continuous.

The matrix–symbol $\mathscr{A}(x, \xi)$ (and the corresponding operator $\mathbf{A}(x, D_x)$) is called hypoelliptic $\mathscr{A} \in \mathbb{HS}^{\nu,\nu_0}(\Omega^\pm, \mathbb{R}^n) = \mathbb{HS}_{1,0}^{\nu,\nu_0}(\Omega^\pm, \mathbb{R}^n)$ if the following hold:

$$\begin{aligned} &a) \ \ C_{1,K} \, |\xi|^{\nu_0} \leq |\sigma(x, \xi)| \leq C_{2,K} \, |\xi|^\nu , \quad x \in K , \\ &b) \ \ \left| \left[\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi) \right] \sigma^{-1}(x, \xi) \right| \leq C_{\alpha,\beta,K} |\xi|^{-|\alpha|} , \ \ \xi \in \mathbb{R}^n \end{aligned}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and all compact sets $K \subset \overline{\Omega^\pm}$ (see [Hr1, § 4.1], [Sb1, § 5]). If hypoelliptic, $\mathbf{A}(x, D_x)$ has a parametrix

$$\mathbf{R}_\mathbf{A}(x, D_x) \mathbf{A}(x, D_x) = I - \mathbf{T}_1(x, D_x) , \quad \mathbf{A}(x, D_x) \mathbf{R}_\mathbf{A}(x, D_x) = I - \mathbf{T}_2(x, D_x) ,$$

where the PsDOs $\mathbf{T}_1(x, D_x)$ and $\mathbf{T}_2(x, D_x)$ have order $-\infty$, i.e. are continuous from $\mathbb{X}_{p,com}^s(\overline{\Omega^\pm})$ into $C^\infty(\Omega^\pm)$.

In [Hr2, § 7], [Sb1, § 5] the symbols of parametrices are written explicitly, expecially for classical PsDOs (see [Sb1, § 5.5]). We remind only the fact that the principal homogeneous symbol of a parametrix coincides with the inverse to the principal homogeneous symbol of the operator $(\mathscr{R}_\mathbf{A})_{pr}(x, \xi) = \mathscr{A}_{pr}^{-1}(x, \xi) = \mathscr{A}_0^{-1}(x, \xi)$.

Corollary 4.3 *Let* $\mathbf{A}(x, D_x)$ *be hypoelliptic with the symbol* $\mathscr{A} \in \mathbb{HS}^{\nu,\nu_0}(\mathbb{R}^n, \mathbb{R}^n)$ *and* $\mathbf{A}(x, D_x)$ *have a fundamental solution. Then the generalised fundamental solution*

$$\mathbf{F}_\mathbf{A}^{(\beta)} u(x) := \int_{\mathbb{R}^n} \mathscr{K}_\mathbf{A}^{(\beta)}(x, y) u(y) dy$$

(*cf.* (3.7)) *is continuous*

$$\mathbf{F}_\mathbf{A}^{(\beta)} : \mathbb{X}_{p,com}^s(\mathbb{R}^n) \longrightarrow \mathbb{X}_{p,loc}^{s+2\ell+|\beta|}(\mathbb{R}^n) \tag{4.5}$$

(*see* (2.10)) *provided* $\beta \in \mathbb{N}_0^n$, $\mu, s \in \mathbb{R}$, $m \in \mathbb{N}_0$, $1 < p < \infty$.

Proof. The symbol of PsDO $\mathbf{F}_{\mathbf{A}}^{(\beta)}(D)$ reads as

$$\mathscr{F}_{\mathbf{A}}^{(\beta)}(x,\xi)=(-i\partial_\xi)^\beta \mathscr{R}_{\mathbf{A}}(x,\xi)$$

where $\mathscr{R}_{\mathbf{A}}(x,\xi)$ is the symbol of a parametrix $\mathbf{R}_{\mathbf{A}}(x,D_x)$ of the hypoelliptic operator $\mathbf{A}(x,D_x)$ and $\mathscr{R}_{\mathbf{A}}\in\mathbb{S}^{-\nu}(\mathbb{R}^n,\mathbb{R}^n)$ (see [Sb1, § 5.5]). Therefore $\mathscr{F}_{\mathbf{A}}^{(\beta)}\in\mathbb{S}^{-\nu-|\beta|}(\mathbb{R}^n,\mathbb{R}^n)$ and continuity (4.6) follows from Theorem 4.2. ■

Remark 4.4 *The generalised volume potentials*

$$\mathbf{N}_{\Omega^\pm}^{(\beta)}u(x):=\int_{\Omega^\pm}\mathscr{K}_{\mathbf{A}}^{(\beta)}(x,y)u(y)dy \tag{4.6}$$

(*cf.* (3.6)) *are continuous, as usual PsDOs of order* $-2\ell-|\beta|$, *between the spaces*

$$\mathbf{N}_{\Omega^\pm}^{(\beta)}:\widetilde{\mathbb{X}}_{p,com}^{s,m}(\Omega^\pm)\longrightarrow\mathbb{X}_{p,loc}^{s+2\ell+|\beta|,m}(\overline{\Omega^\pm}).$$

Since the symbol of these operators are rational functions, they possess the transmission property and are also continuous in the following sense

$$\mathbf{N}_{\Omega^\pm}^{(\beta)}:\mathbb{X}_{p,com}^{s,m}(\Omega^\pm)\longrightarrow\mathbb{X}_{p,loc}^{s+2\ell+|\beta|,m}(\overline{\Omega^\pm})$$

(see [BS1, Bo1, GH1, RS1] for details).

Lemma 4.5 *Let* $\mathscr{S}=\partial\Omega^+$ *be* C^∞*–smooth and*

$$a(x,\xi)=a_\nu(x,\xi)+a_{\nu-1}(x,\xi)+\cdots+a_{\nu-k}(x,\xi)+\cdots,$$
$$a_{\nu-k}(x,\lambda\xi)=\lambda^{\nu-k}a_{\nu-k}(x,\xi),\quad x\in\Omega^\pm,\quad \xi\in\mathbb{R}^n,\quad \lambda>0$$

be a classical $N\times N$ *matrix–symbol* $a\in\mathscr{S}^\nu(\Omega^\pm,\mathbb{R}^n)$ *with* $\nu\le-1$. *Let* $\mathscr{K}_{\mathbf{a}}(x,y)$ *be the* SCHWARTZ *kernel of the corresponding PsDO* $\mathbf{a}(x,D)$ *and*

$$\mathbf{V}_{\mathbf{a}}\varphi(x):=\oint_{\mathscr{S}}\mathscr{K}_{\mathbf{a}}(x,\tau)\varphi(\tau)d_\tau\mathscr{S},\quad x\in\Omega^\pm \tag{4.7}$$

be the corresponding potential–type operator, i.e. the restriction of the domain of definition of the PsDO $\mathbf{a}(x,D)$ *to the boundary* $\mathscr{S}=\partial\Omega^\pm$.

If $\nu<-1$ *the trace*

$$\gamma_{\mathscr{S}}^\pm\mathbf{V}_{\mathbf{a}}\varphi(t)=\mathbf{a}_{\mathscr{S}}(t,D)\varphi(t)=\int_{\mathscr{S}}\mathscr{K}_{\mathbf{a}}(t,\tau)\varphi(\tau)d_\tau\mathscr{S},\quad t\in\mathscr{S}$$

from the domains Ω^+ *and* Ω^-, *respectively, coincides with the direct value of the potential–type operator* (4.7) *(i.e. with the full restriction of PsDO* $\mathbf{a}(x,D)$ *to* $\mathscr{S}$*) and represents a pseudodifferential operator*

$$\mathbf{a}_{\mathscr{S}}(t,D):\widetilde{\mathbb{H}}_p^{s,m}(\mathscr{S})\to\mathbb{H}_p^{s-\nu-1,m}(\mathscr{S}). \tag{4.8}$$

with the full classical symbol

$$a_{\mathscr{S}}(t,\xi') = \sum_{k=0}^{\infty} a_{\mathscr{S},\nu+1-k}(t,\xi')\,, \quad a_{\mathscr{S},\nu+1-k} \in \mathbb{S}^{\nu+1-k}(\mathscr{S},\mathbb{R}^n)$$

and the principal symbol

$$\begin{aligned} a_{\mathscr{S},pr}(\varkappa_j(x),\xi') &:= a_{\mathscr{S},\nu+1}(\varkappa_j(x),\xi') \\ &= \frac{\mathscr{G}_{\varkappa_j}(x)}{2\pi \det \mathscr{J}_{\varkappa_j}(0,x)} \int_{-\infty}^{\infty} a_\nu\left(\varkappa_j(x),\, \mathscr{J}^{-1}_{\varkappa_j}(0,x)^\top(\xi',\lambda)\right) d\lambda\,, \quad x \in U_j\,. \end{aligned}$$

Here $\mathscr{J}_{\varkappa_j}(t)$ *denotes the* JACOBIAN *and*

$$\mathscr{G}_{\varkappa_j} := \left(\det \|(\partial_k\varkappa_j, \partial_l\varkappa_j)\|_{(n-1)\times(n-1)}\right)^{\frac{1}{2}} \text{ with } \partial_k\varkappa_j := (\partial_k\varkappa_{j1},\ldots,\partial_k\varkappa_{jn})^\top$$

denotes the square root of the GRAM *determinant of the local (coordinate) diffeomorphisms* $\varkappa_j : U_j \to V_j$, $j = 1,2,\ldots,N$ *of* $U_j \subset \mathbb{R}^{n-1}$ *to* $V_j \subset \mathscr{S}$*.*

If $\nu = -1$*, the restriction of the potential–type operator* (4.7)

$$\mathbf{a}_{\mathscr{S}}(t,D_t)\varphi(t) := \oint_{\mathscr{S}} \mathscr{K}_{\mathbf{a}}(t,\tau)\varphi(\tau)d_\tau\mathscr{S}\,, \quad t \in \mathscr{S}$$

to the surface is understood in the CAUCHY *principal value sense (cf.* (6.30) *below) and* $\mathbf{a}_{\mathscr{S}}(t,D_t)\varphi(t)$ *represents a* CALDERÓN–ZYGMUND *singular integral operator (i.e.* $\mathbf{a}_{\mathscr{S}}(t,D_t)\varphi(t)$ *is a PsDO of order 0); The traces* $\gamma^{\pm}_{\mathscr{S}}\mathbf{V}_{\mathbf{a}}$ *and the restriction* $\mathbf{a}_{\mathscr{S}}(t,D_t)$ *are related as follows*

$$\gamma^{\pm}_{\mathscr{S}}\mathbf{V}_{\mathbf{a}}\varphi(t) := \pm\frac{1}{2} i a_{\mathrm{pr}}(t,\vec{\nu}(t))\varphi(t) + \mathbf{a}_{\mathscr{S}}(t,D_t)\varphi(t)\,, \quad t \in \mathscr{S}\,, \tag{4.9}$$

where $\vec{\nu}(t)$ *is the outer unit normal vector at* $t \in \mathscr{S}$ *and* $a_{\mathrm{pr}}(t,\xi)$*,* $\xi \in \mathbb{R}^n$*, denotes the homogeneous principal symbol of* $\mathbf{a}(t,D)$*.*

Proof. The proof, including a detailed description of the lower order terms of the asymptotic expansion of the symbol of a PsDO on the manifold $\mathscr{S}$, can be found in [CD1, § 1.4, Example 2] with two differences. First, the proof in [CD1] is carried out for pure convolution operators with symbols $a(\xi)$ but it can be extended to the case of PsDOs with classical symbols $a(x,\xi)$ by minor modifications. Second, for the coefficient in (4.9) there has to be quoted a (different) formula from [Es1, (3.26)].

A different proof of (4.9), including the formula for the coefficient, can be found in [MT1, Appendix C]. ■

4.2. On traces of functions. Let us recall the following theorem on traces, which will be generalised later in Theorem 6.4 for weighted spaces.

Theorem 4.6 *The trace operator*

$$\mathscr{R}^{\pm}_k u := \{\gamma^{0,\pm}_{\mathscr{S}}u, \gamma^{1,\pm}_{\mathscr{S}}u,\ldots,\gamma^{k,\pm}_{\mathscr{S}}u\}\,, \quad \gamma^{0,\pm}_{\mathscr{S}} := \gamma^{\pm}_{\mathscr{S}}\,, \quad \gamma^{j,\pm}_{\mathscr{S}} := \gamma^{\pm}_{\mathscr{S}}\partial^j_{\vec{\nu}}, \;\; u \in C^\infty_0(\overline{\Omega^\pm})\,, \tag{4.10}$$

$$\mathscr{R}_k^\pm \quad : \quad \mathbb{H}^s_{p,loc}(\overline{\Omega^\pm}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,p}(\mathscr{S}), \quad 1 < p < \infty\,,$$
$$\quad : \quad \mathbb{B}^s_{p,q,loc}(\overline{\Omega^\pm}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(\mathscr{S}), \quad 1 \le p,q \le \infty$$

is a retraction, provided $m \in \mathbb{N}_0$, $k < s - 1/p$, i.e. is continuous and has a continuous inverse from the right, called a coretraction:

$$\begin{aligned}(\mathscr{R}_k^\pm)^{-1} \quad &: \quad \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^s_{p,loc}(\overline{\Omega^\pm}),\\ &: \quad \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^s_{p,q,loc}(\overline{\Omega^\pm})\,.\end{aligned} \tag{4.11}$$

Proof. The proof can be found in [Tr1, § 2.7.2] ■

The next lemma generalises [LM1, Ch2, Lemma 2.1], proved there for the scalar case (see also [RS2, (11)].

Lemma 4.7 *Let*

$$\vec{\mathbf{Q}}^{(m)}(x, D_x) := \{\mathbf{Q}_0(x, D_x), \dots, \mathbf{Q}_{m-1}(x, D_x)\}^\top\,,$$
$$\vec{\mathbf{G}}^{(m)}(x, D_x) := \{\mathbf{G}_0(x, D_x), \dots, \mathbf{G}_{m-1}(x, D_x)\}^\top \tag{4.12}$$

be two Dirichlet *systems on $\mathscr{S}$. Then*

$$\vec{\mathbf{Q}}^{(m)}(x, D_x) = \mathbf{Q}^{(m\times m)}_{\mathbf{G}}(x, \mathscr{D}_x)\vec{\mathbf{G}}^{(m)}(x, D_x)\,, \tag{4.13}$$

where $\mathbf{Q}^{(m\times m)}_{\mathbf{G}}(x, \mathscr{D}_x)$ is the admissible matrix and thus invertible (see (4.15), (4.16), (4.18) *below).*

Proof. The following representations are similar to (1.28), (1.33)

$$\mathbf{Q}_j(x, D_x) = \vec{\mathbf{Q}}^{(j+1)}_j(x, \mathscr{D}_x) \cdot \vec{\mathbf{D}}^{(j+1)}(x, D_x)\,,$$
$$\mathbf{G}_j(x, D_x) = \vec{\mathbf{G}}^{(j+1)}_j(x, \mathscr{D}_x) \cdot \vec{\mathbf{D}}^{(j+1)}(x, D_x)\,, \quad j = 0, \dots, m-1\,, \tag{4.14}$$

where

$$\vec{\mathbf{Q}}^{(j+1)}_j(x, \mathscr{D}_x) := \{\mathbf{Q}_{j,j}(x, \mathscr{D}_x), \dots, \mathbf{Q}_{j,1}(x, \mathscr{D}_x), \mathscr{Q}_j(x, \vec{\nu}(x))\}^\top\,,$$
$$\vec{\mathbf{G}}^{(j+1)}_j(x, \mathscr{D}_x) := \{\mathbf{G}_{j,j}(x, \mathscr{D}_x), \dots, \mathbf{G}_{j,1}(x, \mathscr{D}_x), \mathscr{G}_j(x, \vec{\nu}(x))\}^\top\,.$$

Therefore the lower–triangular matrix–operators

$$\mathbf{q}^{(m\times m)}(x, \mathscr{D}_x) = \begin{bmatrix} \mathscr{Q}_{0,0}(x, \vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{Q}_{1,0}(x, \mathscr{D}_x) & \mathscr{Q}_{1,0}(x, \vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_{m-1,0}(x, \mathscr{D}_x) & \mathbf{Q}_{m-1,1}(x, \mathscr{D}_x) & \cdots & \mathscr{Q}_{m-1,0}(x.\vec{\nu}(x)) \end{bmatrix}, \tag{4.15}$$

$$\mathbf{g}^{(m\times m)}(x, \mathscr{D}_x) = \begin{bmatrix} \mathscr{G}_{0,0}(x, \vec{\nu}(x)) & 0 & \cdots & 0 \\ \mathbf{G}_{1,0}(x, \mathscr{D}_x) & \mathscr{G}_{1,0}(x, \vec{\nu}(x)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{G}_{m-1,0}(x, \mathscr{D}_x) & \mathbf{G}_{m-1,1}(x, \mathscr{D}_x) & \cdots & \mathscr{G}_{m-1,0}(x.\vec{\nu}(x)) \end{bmatrix}, \tag{4.16}$$

$$\det \mathscr{Q}_{\rm j}({\rm x}) \neq 0\,, \quad \det \mathscr{G}_{\rm j}({\rm x}) \neq 0\,, \quad {\rm t} \in \mathscr{S}\,, \quad {\rm j} = 0, \dots, {\rm m}-1$$

are admissible (see (1.34), (1.36)) and

$$\vec{\mathbf{Q}}^{(m)}(x, D_x) = \mathbf{q}^{(m\times m)}(x, \mathscr{D}_x)\vec{\mathbf{D}}^{(m)}(x, D_x)\,, \quad \vec{\mathbf{G}}^{(m)}(x, D_x) = \mathbf{g}^{(m\times m)}(x, \mathscr{D}_x)\vec{\mathbf{D}}^{(m)}(x, D_x)\,. \tag{4.17}$$

From (4.15)–(4.17) we get (4.13) with the following admissible matrix–operator

$$\mathbf{Q}_{\mathbf{G}}^{(m\times m)}(x, \mathscr{D}_x) := \mathbf{q}^{(m\times m)}(x, \mathscr{D}_x)\left[\mathbf{g}^{(m\times m)}(x, \mathscr{D}_x)\right]^{-1} \tag{4.18}$$

(cf. (1.35), (1.37)). ∎

Lemma 4.8 *Let $s > 0$, $s \notin \mathbb{N}$, $1 < p < \infty$, $1 \le p, q \le \infty$ and $\mathbf{A}(x, D_x)$ in (1.1) be a normal (not necessarily elliptic) operator; let further $\{\mathbf{B}_j(x, D_x)\}_{j=0}^{m-1}$ be a* Dirichlet *system of order $m-1$. Then there exists a continuous linear operator*

$$\mathscr{P} : \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,p}^{s+m-1-j}(\mathscr{S}) \longrightarrow \mathbb{H}_{p,loc}^{s+m-1+\frac{1}{p}}(\overline{\Omega^\pm}) \tag{4.19}$$

$$\left(\mathscr{P} : \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,q}^{s+m-1-j}(\mathscr{S}) \longrightarrow \mathbb{B}_{p,q,loc}^{s+m-1+\frac{1}{p}}(\overline{\Omega^\pm})\right)$$

such that

$$\gamma_{\mathscr{S}}^{\pm}\mathbf{B}_j\mathscr{P}\Phi = \varphi_j, \quad j = 0, 1, \ldots, m-1, \tag{4.20}$$

$$\mathbf{A}\mathscr{P}\Phi \in \widetilde{\mathbb{H}}_{p,loc}^{s-1+\frac{1}{p}}(\overline{\Omega^\pm}) \qquad (\mathbf{A}\mathscr{P}\Phi \in \widetilde{\mathbb{B}}_{p,q,loc}^{s-1+\frac{1}{p}}(\overline{\Omega^\pm})) \tag{4.21}$$

for arbitrary

$$\Phi = (\varphi_0, \ldots, \varphi_{m-1}) \in \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,p}^{s+m-1-j}(\mathscr{S}) \quad \left(\Phi \in \bigotimes_{j=0}^{m-1} \mathbb{B}_{p,q}^{s+m-1-j}(\mathscr{S})\right).$$

Proof. Let us recall the following property of the space $\widetilde{\mathbb{X}}_p^s(\Omega^\pm)$:

$$\widetilde{\mathbb{X}}_p^\mu(\Omega^\pm) = \{u \in \mathbb{X}_p^\mu(\Omega^\pm) : \mathscr{R}_\ell^\pm u = 0\} \tag{4.22}$$

(cf. (2.10)) which holds under the constraints $\frac{1}{p} + \ell < \mu < \frac{1}{p} + \ell + 1$ (see [Tr1] and [Sh1, Lemma 1.15]). Due to (4.22), the condition (4.21) can be reformulated as follows

$$\mathscr{R}_k^\pm A\mathscr{P}\Phi = \{\gamma_{\mathscr{S}}^{0,\pm}A\mathscr{P}\Phi, \ldots, \gamma_{\mathscr{S}}^{k,\pm}A\mathscr{P}\Phi\} = 0, \quad 0 < s-k-\frac{1}{p} < 1, \quad k \in \mathbb{N}_0 \tag{4.23}$$

(cf. (4.10)). For $0 < s \le \frac{1}{p}$, the condition (4.23) may be omitted. The operators

$$\mathbf{B}_{m+j}(x, D_x) := \partial_{\vec{\nu}(x)}^j \mathbf{A}(x, D_x)\,, \quad \mathrm{ord}\,\mathbf{B}_{\mathrm{m+j}} = \mathrm{m+j}\,, \quad \mathrm{j} = 0, \ldots, \mathrm{k}$$

are normal

$$\mathscr{B}_{m+j,0}(t,\vec{\nu}(t)) = \left(-i\sum_{s=1}^{n}\nu_s^2(t)\right)^j \mathscr{A}_0(t,\vec{\nu}(t)) = (-i)^j\mathscr{A}_0(t,\vec{\nu}(t)),$$

$$\det\mathscr{B}_{\mathrm{m+j,0}}(\mathrm{t},\vec{\nu}(\mathrm{t})) \neq 0, \quad \mathrm{t}\in\mathscr{S}, \quad \mathrm{j}=0,\dots,\mathrm{k}$$

and combining them with the above DIRICHLET system $\vec{\mathbf{B}}^{(m)}(x,D_x)$ we get a new DIRICHLET system $\vec{\mathbf{B}}^{(m+k+1)}(x,D_x)$. Therefore

$$\vec{\mathbf{B}}^{(m+k+1)}(x,D_x) = \mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)\vec{\mathbf{D}}^{(m+k+1)}(x,D_x), \tag{4.24}$$

and $\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)$ is admissible (see (1.34), (1.36)). By defining

$$\Phi_0 := (\varphi_0,\dots,\varphi_{m-1}, \underbrace{0,\dots,0}_{(k+1)-times}) \in \bigotimes_{j=0}^{m+k} \mathbb{B}_{p,q}^{s+m-1-j}(\mathscr{S}), \tag{4.25}$$

we can match conditions (4.20) and (4.23) (which replaces (4.21)) and reformulate the problem as follows: let us look for a continuous linear operator

$$\mathscr{P}_0 : \bigotimes_{j=0}^{m+k} \mathbb{B}_{p,p}^{s+m-1-j}(\mathscr{S}) \longrightarrow \mathbb{H}_{p,loc}^{s+m+k+\frac{1}{p}}(\overline{\Omega^\pm}) \tag{4.26}$$

$$\left(\mathscr{P}_0 : \bigotimes_{j=0}^{m+k} \mathbb{B}_{p,q}^{s+m-1-j}(\mathscr{S}) \longrightarrow \mathbb{B}_{p,q,loc}^{s+m+k+\frac{1}{p}}(\overline{\Omega^\pm})\right)$$

such that

$$\begin{aligned}\gamma_{\mathscr{S}}^{\pm}\vec{\mathbf{B}}^{(m+k+1)}\mathscr{P}_0\Phi_0 &= \mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)\gamma_{\mathscr{S}}^{\pm}\vec{\mathbf{D}}^{(m+k+1)}\mathscr{P}_0\Phi_0 \\ &= \mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)\mathscr{R}_{m+k+1}^{\pm}\mathscr{P}\Phi_0 = \Phi_0 .\end{aligned} \tag{4.27}$$

Here we have used the fact that $\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)$ is a "tangent" differential operator and $\mathscr{R}_{m+k+1}^{\pm} = \gamma_{\mathscr{S}}^{\pm}\vec{\mathbf{D}}^{(m+k+1)}$ (cf. (4.10)). Thus,

$$\mathscr{R}_{m+k+1}^{\pm}\mathscr{P}_0\Phi_0 = \left[\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)\right]^{-1}\Phi_0$$

and it remains to apply a coretraction (4.3): the function

$$\mathscr{P}_0\Phi_0 = \mathscr{R}_{m+k+1}^{-1}\left[\mathbf{b}^{((m+k+1)\times(m+k+1))}(x,\mathscr{D}_x)\right]^{-1}\Phi_0 \in \mathbb{H}_{p,loc}^{s+m+k+\frac{1}{p}}(\overline{\Omega^\pm}) \tag{4.28}$$

(in $\mathbb{B}_{p,q,loc}^{s+m+k+\frac{1}{p}}(\overline{\Omega^\pm})$) solves equation (4.27). ∎

Let us consider the following surface δ-function

$$(g\otimes\delta_{\mathscr{S}},v)_{\mathscr{S}} := \int_{\mathscr{S}} g(\tau)\gamma_{\mathscr{S}}^{\pm}v(\tau)d_\tau\mathscr{S}, \quad g\in C_0^\infty(\mathscr{S}), \quad v\in C_0^\infty(\Omega^\pm) \tag{4.29}$$

and its normal derivatives $\delta^{(k)}_{\mathscr{S}} := \partial^k_{\vec{\nu}}\delta_{\mathscr{S}}$:

$$(g\otimes\delta^{(k)}_{\mathscr{S}}, v)_{\mathscr{S}} := \int\limits_{\mathscr{S}} g(\tau)\gamma^{\pm}_{\mathscr{S}}((\partial^*_{\vec{\nu}})^k v)(\tau)d_\tau\mathscr{S}\,, \quad k=1,2,\dots\,, \tag{4.30}$$
$$\partial^*_{\vec{\nu}(x)}\varphi(x) = -\sum_{j=1}^{n}\partial_{x_j}[\nu_j(x)\varphi(x)] = -\partial_{\vec{\nu}(x)}\varphi(x) - \operatorname{div}\vec{\nu}(x)\,.$$

Obviously, $\operatorname{supp}(\mathrm{g}\otimes\delta^{(\mathrm{k})}_{\mathscr{S}}) = \operatorname{supp}\mathrm{g}\subset\mathscr{S}$ for arbitrary $k\in\mathbb{N}_0$.

The definition (4.29)–(4.30) can be extended to less regular functions (i.e. not necessarily C^∞). More precisely, the following lemma holds.

Lemma 4.9 *Let $1<p<\infty$ $(1\le q\le\infty)$, $s<0$, $g\in\mathbb{B}^s_{p,p}(\mathscr{S})$ (or $g\in\mathbb{B}^s_{p,q}(\mathscr{S})$). Then*

$$g\otimes\delta^{(k)}_{\mathscr{S}}\in\widetilde{\mathbb{H}}^{s-k-\frac{1}{p'},m}_{p,com}(\overline{\Omega^\pm})\bigcap\widetilde{\mathbb{B}}^{s-k-\frac{1}{p'},m}_{p,p,com}(\overline{\Omega^\pm})\,,\quad \left(g\otimes\delta^{(k)}_{\mathscr{S}}\in\widetilde{\mathbb{B}}^{s-k-\frac{1}{p'},m}_{p,q,com}(\overline{\Omega^\pm})\right)\,,$$

where $p'=p/(p-1)$ and $k,m\in\mathbb{N}_0$ are arbitrary.

Proof. We concentrate on the case $g\in\mathbb{B}^s_{p,p}(\mathscr{S})$ since the case $g\in\mathbb{B}^s_{p,q}(\mathscr{S})$ is very similar.

The distribution $g\otimes\delta^{(k)}_{\mathscr{S}}$ in (4.29) and (4.30) is a properly defined functional on the space $\mathbb{X}^{-s+k}_p(\Omega^\pm)$, where, for conciseness, $\mathbb{X}^\mu_p(\Omega^\pm)$ denotes either $\mathbb{H}^\mu_p(\Omega^\pm)$ or $\mathbb{B}^\mu_{p,p}(\Omega^\pm)$ (see Theorem 4.6). Moreover, relying again on Theorem 4.6 on traces we get the inequalities

$$|(g\otimes\delta^{(k)}_{\mathscr{S}}, v)| \le C_k(g)\|g|\mathbb{B}^s_{p,p}(\mathscr{S})\|\|v|\mathbb{X}^{-s+k+\frac{1}{p'}}_{p'}(\Omega^+)\|\,,$$
$$|(g\otimes\delta^{(k)}_{\mathscr{S}}, v)| \le C_k(g)\|g|\mathbb{B}^s_{p,p}(\mathscr{S})\|\|\chi v|\mathbb{X}^{-s+k+\frac{1}{p'}}_{p'}(\Omega^-)\|\,,$$

where $\chi\in C^\infty_0(\Omega^-)$ is a cut–off function, which equals 1 in the neighbourhood of $\mathscr{S}\subset\overline{\Omega^-}$. Therefore, by duality, $g\otimes\delta^{(k)}_{\mathscr{S}}\in\widetilde{\mathbb{X}}^{s-k-\frac{1}{p'}}_{p,com}(\overline{\Omega^\pm})$.

To prove the result for the weighted spaces let us note that

$$\partial^k_\nu\rho^m(x) = \frac{m!}{k!}\rho^{m-k}(x)\,, \tag{4.31}$$
$$\partial^k_\nu\rho^m(x) = \sum_{j=1}^{k}\frac{(-1)^k m!}{j!(k-j)!}(\operatorname{div}\vec{\nu}(x))^{k-j}\rho^{m-j}(x)\,,\quad m,k\in\mathbb{N}_0\,,$$

where $\rho=\rho(x):=\operatorname{dist}(x,\mathscr{S})$, $x\in\Omega^\pm$. In fact, if $t_x\in\mathscr{S}$ is a point for which the distance $\rho(x):=\operatorname{dist}(x,\mathscr{S})=\operatorname{dist}(x,t_x)$ from $x\in\Omega^\pm$ to the boundary $\mathscr{S}$ is attained then

$$\partial_{\vec{\nu}(t_x)}\rho(x) = \lim_{h\to0}\frac{\rho(x+h\vec{\nu}(t_x))-\rho(x)}{h} = 1\,,$$

because $\rho(x+h\vec{\nu}(t_x))-\rho(x)=h$ due to the obvious equality $x-t_x=\rho(x)\vec{\nu}(t_x)$. For arbitrary $m,k\in\mathbb{N}_0$ the first formula in (4.31) follows by a standard approach and is used to prove the second one.

Now we apply definition (4.30):

$$(\rho^\ell(g\otimes\delta^{(k)}_{\mathscr S}),v)_{\mathscr S}:=\int\limits_{\mathscr S} g(\tau)\gamma^\pm_{\mathscr S}((\partial^*_{\vec\nu})^k\rho^\ell v)(\tau)d_\tau\mathscr S$$

$$=\sum_{j=0}^k\frac{\ell!}{j!}\int\limits_{\mathscr S} g(\tau)\gamma^\pm_{\mathscr S}[(-\partial_{\vec\nu}-\operatorname{div}\vec\nu)^{\ell-j}\rho(\partial^*_{\vec\nu})^j v](\tau)d_\tau\mathscr S \tag{4.32}$$

$$=\begin{cases}(-1)^\ell\ell!\displaystyle\int\limits_{\mathscr S} g(\tau)\gamma^\pm_{\mathscr S}[(\partial^*_{\vec\nu})^{k-\ell}v](\tau)d_\tau\mathscr S,=(-1)^\ell\ell!(g\otimes\delta^{(k-\ell)}_{\mathscr S}),v)_{\mathscr S} & if \quad \ell\le k\,,\\ 0, & if\quad \ell>k\,.\end{cases}$$

According to the the portion of the lemma we have proved so far, $\rho^\ell(g\otimes\delta^{(k)}_{\mathscr S})\in\widetilde{\mathbb X}^{s-k+\ell-\frac{1}{p'},m}_{p,com}(\overline{\Omega^\pm})$ and the inclusion $g\otimes\delta^{(k)}_{\mathscr S}\in\widetilde{\mathbb X}^{s-k-\frac{1}{p'},m}_{p,com}(\overline{\Omega^\pm})$ for arbitrary $m\in\mathbb N$ follows from the definition of the weighted space. ∎

Particular cases of the foregoing lemma are well–known: e.g., see [Es1] for L_2–case, $m=k=0$ and [DW1, Gr3, Sh1, Tai1] for L_p–case and again $m=k=0$.

As a direct application of definition (4.29) we can write the generalised layer potential (3.11) as a volume potential

$$\mathbf V^{(\beta)}_{\mathbf G}\varphi(x)=\int\limits_{\mathbb R^n}\left[\overline{\mathbf G(y,D_y)}\mathscr K^{(\beta)}_{\mathbf A^*}(y,x)\right]^\top(\varphi\otimes\delta_{\mathscr S})(y)dy=:\mathbf F^{(\beta)}_{\mathbf A,\mathbf G}(\varphi\otimes\delta_{\mathscr S})(x),\qquad x\in\Omega^\pm. \tag{4.33}$$

The representation (4.33) has only one shortcoming: $\varphi\otimes\delta_{\mathscr S}\notin\mathbb X^s_{p,loc}(\overline{\Omega^\pm})$ for $s>-\dfrac{1}{p'}$ even for $\varphi\in C^\infty(\mathscr S)$ (i.e. Lemma 4.9 is precise). In fact, locally $\mathscr S$ can be interpreted as $\mathbb R^{n-1}$ and $\Omega^\pm$–as $\mathbb R^n_+$. Then $1\otimes\delta_{\mathbb R^{n-1}}=\delta(x_n)\notin\mathbb X^s_{p,loc}(\overline{\mathbb R^n_+})$ if $s>-1/p'$ (see [Es1] for $p=2$ and [Tr1, Tr2] for $1<p<\infty$).

5 Proofs

5.1. Proof of Theorem 1.6. It suffices to prove the theorem for the particular case of the BVP (1.10) and the corresponding Green formula (1.13) because of the following argument. First we extend the system $\{\mathbf b_j(t,D_t)\}_{j=0}^{\omega-1}$ of "boundary" differential operators up to a Dirichlet system

$$\{\mathbf B_j(t,D_t)\}_{j=0}^{m-1}=\mathscr H_0\{\mathbf b_j(t,D_t)\}_{j=0}^{mN-1}$$

of order m (see Lemma 1.4). If the Green formula (1.13) is proved we get that

$$\int\limits_{\Omega^\pm}((\mathbf Au)^\top\overline v-u^\top\overline{\mathbf A^*v})dy=\pm\sum_{j=0}^{m-1}\oint\limits_{\mathscr S}(\mathbf B_ju)^\top\overline{\mathbf C_jv}d_\tau\mathscr S=\pm\sum_{j=0}^{mN-1}\oint\limits_{\mathscr S}(\mathbf b_ju)^\top\overline{\mathbf c_jv}d_\tau\mathscr S\,, \tag{5.1}$$

where $\{\mathbf c_j(t,D_t)\}_{j=0}^{mN-1}=\mathscr H_0^\top\{\mathbf C_j(t,D_t)\}_{j=0}^{m-1}$ is a decomposition in rows.

Thus, we can concentrate on BVP (1.10) and the corresponding Green formula (1.13). Moreover, we suppose that this choice of extension is made and $\vec{\mathbf B}^{(m)}(x,D_x)$ (see

(1.31)) is the fixed DIRICHLET system of order $m-1$. Without loss of generality we can suppose that $\operatorname{ord}\mathbf{B}_{\rm j} = {\rm j}$, $j=0,\dots,m-1$; otherwise we have just to renumber these operators.

In Theorem 1.10 we have already proved the GREEN formula

$$\int\limits_{\Omega^\pm} \left[(\mathbf{A}u)^\top\overline{v} - u^\top\overline{\mathbf{A}^*v}\right] dy = \pm \oint\limits_{\mathscr{S}} (\vec{\mathbf{D}}^{(m)}u)^\top \cdot \overline{\vec{\mathbf{G}}^{(m)}v}\, d_\tau\mathscr{S} \tag{5.2}$$

(see (1.32)) with the special operators $\vec{\mathbf{D}}^{(m)}(x,D_x)$ defined in (1.31) and

$$\begin{aligned}\vec{\mathbf{G}}^{(m)}(x,D_x) &:= \{\mathbf{G}_0(x,D_x),\dots,\mathbf{G}_{m-1}(x,D_x)\}^\top\\ &= \left[\left(\vec{\mathbf{D}}^{(m)}\right)^*(x,D_x)\right]^\top \left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathscr{D}_x)\mathbb{S}_m\,,\end{aligned} \tag{5.3}$$

(see (1.29)) with skew identity matrix $\mathbb{S}^*_m = \mathbb{S}_m$ (see (1.40)) and the formally adjoint matrix–operator $\left(\mathbf{A}^{(m\times m)}\right)^*(x,\mathscr{D}_x)$ to (1.39). $\vec{\mathbf{B}}^{(m)} = \left\{\partial^j_{\vec{\nu}}\right\}_{j=0}^{m-1}$ is a DIRICHLET system. Due to Lemma 4.7

$$\vec{\mathbf{D}}^{(m)}(t,D_t) = \left[\left(\mathbf{b}^{(m\times m)}\right)(t,\mathscr{D}_t)\right]^{-1}\vec{\mathbf{B}}^{(m)}(t,D_t)\,,\quad t\in\mathscr{S} \tag{5.4}$$

(see (1.36)). Inserting (5.4) into (5.2), taking into account (5.3), and applying the partial integration formula (1.24) we get

$$\begin{aligned}\int\limits_{\Omega^\pm} \left[(\mathbf{A}u)^\top\overline{v} - u^\top\overline{\mathbf{A}^*v}\right] dy &= \pm \oint\limits_{\mathscr{S}} (\vec{\mathbf{D}}^{(m)}u)^\top \cdot \overline{\vec{\mathbf{G}}^{(m)}v}\, d_\tau\mathscr{S}\\ &= \pm \oint\limits_{\mathscr{S}} \left[\left(\mathbf{b}^{(m\times m)}\right)\right]^{-1}\vec{\mathbf{B}}^{(m)}u)^\top \cdot \overline{\left[\left(\vec{\mathbf{D}}^{(m)}\right)^*\right]^\top \left(\mathbf{A}^{(m\times m)}\right)^*\mathbb{S}_m v}\, d_\tau\mathscr{S}\\ &= \pm \oint\limits_{\mathscr{S}} (\vec{\mathbf{B}}^{(m)}u)^\top \cdot \overline{\vec{\mathbf{C}}^{(m)}v}\, d_\tau\mathscr{S}\,,\end{aligned} \tag{5.5}$$

where $\vec{\mathbf{C}}^{(m)}(x,D_x)$ is defined by (1.38) and is unique. Due to this formula the operators $\mathbf{C}_k(t,D_t)$, $k=0,1,\dots,m-1$ are normal iff the matrix $\overline{\mathscr{A}_0^\top(t,\vec{\nu}(t))}$ on the main diagonal of the block–matrix $\left(\mathbf{A}^{(m\times m)}(x,\mathscr{D}_x)\right)^*$ is invertible for all $t\in\mathscr{S}$, i.e. iff the "basic" operator $\mathbf{A}(x,D_x)$ is normal (see Definition 1.1).

If the DIRICHLET system $\{\mathbf{C}_j(t,D_t)\}_{j=0}^{m-1}$ is fixed (instead of $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$), the proof proceeds similarly with a single difference–instead of $\mathbf{A}(x,D_x)$ the proof starts with the formally adjoint operator $\mathbf{A}^*(x,D_x)$.

Now let us suppose that the "basic" operator is normal and that the systems $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{k-1}$ and $\{\mathbf{C}_{m-j-1}(t,D_t)\}_{j=0}^{m-k-1}$ are fixed.

If one of them is a DIRICHLET system (of order k or $m-k$, respectively,), we extend it up to a DIRICHLET system $\{\mathbf{B}_{j,0}(t,D_t)\}_{j=0}^{m-1}$ (or $\{\mathbf{C}_{j,0}(t,D_t)\}_{j=0}^{m-1}$) of order m and write the GREEN formula (1.13) (see (5.5)). Next we replace the system $\{\mathbf{C}_{m-j-1,0}(t,D_t)\}_{j=0}^{m-k-1}$, $\operatorname{ord}\mathbf{C}_{\rm m-j-1,0} = {\rm j}$ (or the system $\{\mathbf{B}_{j,0}(t,D_t)\}_{j=0}^{k-1}$, $\operatorname{ord}\mathbf{B}_{\rm j,0} = {\rm j}$; see (1.14), (1.30)) by the fixed system $\{\mathbf{C}_{m-j-1}(t,D_t)\}_{j=0}^{m-k-1}$ (by the system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$, respectively) with the help of a matrix $\left[c^{((m-k)\times((m-k))}(t,\mathscr{D}_t)\right]^\top$ transposed to an admissible[8)] (with an admissible matrix $b^{(k\times k)}(t,\mathscr{D}_t)$, respectively; see Lemma 4.7). Another part of the system remains unchanged. The relation between the entire systems has the form

$$\vec{\mathbf{C}}_0^{(m)}(t,D_t)=c^{(m\times m)}(t,\mathscr{D}_t)\vec{\mathbf{C}}^{(m)}(t,D_t)\quad(\vec{\mathbf{B}}_0^{(m)}(t,D_t)=b^{(m\times m)}(t,\mathscr{D}_t)\vec{\mathbf{B}}^{(m)}(t,D_t))\,,$$

where the participating block–matrices are defined as follows

$$c^{(m\times m)}(t,\mathscr{D}_t)=\begin{bmatrix} I_k & 0\\ 0 & \left[c^{(m-k)\times(m-k))}(t,\mathscr{D}_t)\right]^\top\end{bmatrix}\quad\left(b^{(m\times m)}(t,\mathscr{D}_t)=\begin{bmatrix} b^{(k\times k)}(t,\mathscr{D}_t) & 0\\ 0 & I_{m-k}\end{bmatrix}\right),$$

and by I_ℓ we denote the identity matrix of order ℓ.

Inserting the obtained representations into the Green formula we find

$$\int\limits_{\Omega^\pm}\left[(\mathbf{A}u)^\top\overline{v}-u^\top\overline{\mathbf{A}^*v}\right]dy=\pm\oint\limits_{\mathscr{S}}(\vec{\mathbf{B}}_0^{(m)}u)^\top\cdot\overline{\vec{\mathbf{C}}_0^{(m)}v}\,d_\tau\mathscr{S}$$

$$=\pm\oint\limits_{\mathscr{S}}(\vec{\mathbf{B}}_0^{(m)}u)^\top\cdot\overline{c^{(m\times m)}\vec{\mathbf{C}}^{(m)}}\,d_\tau\mathscr{S}=\pm\oint\limits_{\mathscr{S}}\left(\vec{\mathbf{B}}^{(m)}u\right)^\top\cdot\overline{\vec{\mathbf{C}}^{(m)}}\,d_\tau\mathscr{S}\,.$$

Due to the structure of the relation matrix $c^{(m\times m)}(t,\mathscr{D}_t)$, the first part of the transformed system $\vec{\mathbf{B}}^{(m)}:=\left[c^{(m\times m)}\right]^\top\vec{\mathbf{B}}_0^{(m)}$ remains unchanged and coincides with the system $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{k-1}$ fixed at the beginning.

Similarly, if the system $\{\mathbf{B}_{j,0}(t,D_t)\}_{j=0}^{k-1}$ is changed, the second part of the transformed system $\vec{\mathbf{C}}^{(m)}:=\left[b^{(m\times m)}\right]^\top\vec{\mathbf{C}}_0^{(m)}$ in the Green formula remains unchanged and coincides with the second part $\{\mathbf{C}_{m-j-1}(t,D_t)\}_{j=m-k}^{m}$ of the system fixed at the beginning.

The uniqueness of the full Dirichlet systems $\{\mathbf{B}_j(t,D_t)\}_{j=0}^{m-1}$ and $\{\mathbf{C}_j(t,D_t)\}_{j=0}^{m-1}$ follows from the proved part of the theorem. In fact, after one full Dirichlet systems is fixed and another one is chosen, we can replace the chosen full Dirichlet system by a new one with the help of an admissible matrix. If the admissible matrix is not identity, it will change the fixed full system.

Assume that $\{\mathbf{b}_j(t,D_t)\}_{j=0}^{kN-1}$ (or $\{\mathbf{c}_{mN-j}(t,D_t)\}_{j=0}^{(m-k)N-1}$) is not a Dirichlet system. Then (see Definition 1.2):

i. if the linear independence of rows is missing, then the Green formula (1.14) can not be valid because, by the first part of the Theorem, both systems of “boundary” operators must be Dirichlet systems;

ii. if one or several orders are missing, then the structure of the connection matrix ($c^{(m\times m)}(t,\mathscr{D}_t)$ (of $b^{(m\times m)}(t,\mathscr{D}_t)$, respectively) does not allow to maintain fixed parts of “boundary” systems in the Green formula. ■

5.2. Proof of Theorem 1.7. If we apply (1.26), we get

$$\mathscr{A}(u,v):=\int\limits_{\Omega^\pm}\sum_{|\alpha|,|\beta|\le\ell}\left[a_{\alpha,\beta}(y)\partial_y^\beta u(y)\right]^\top\overline{\partial_y^\alpha v(y)}dy$$

$$=\int\limits_{\Omega^\pm}(\mathbf{A}u)^\top\overline{v}dy\pm\sum_{j=0}^{\ell-1}\int\limits_{\mathscr{S}}(\widetilde{\mathbf{C}}_ju)^\top\overline{\widetilde{\mathbf{B}}_jv}d_\tau\mathscr{S}\,,\quad u,v\in C_0^\infty(\overline{\Omega^\pm},\mathbb{C}^N)\tag{5.6}$$

[8)] It is easy to ascertain that the relation between Dirichlet systems with diminishing orders is established by a transposed (and therefore upper triangular) admissible matrix; see Lemma 4.7.

with some systems $\{\widetilde{\mathbf{B}}_j\}_{j=0}^{\ell-1}$ and $\{\widetilde{\mathbf{C}}_j\}_{j=0}^{\ell-1}$, which we can not control and, therefore, can not change. Therefore we start again and proceed with the help of the representations

$$\partial_x^\beta = \mathbf{b}_{\beta,0}(x)\partial_{\vec{\nu}(x)}^{|\beta|} + \sum_{j=1}^{|\beta|} \mathbf{b}_{\beta,|\beta|-j}(x,\mathscr{D}_x)\partial_{\vec{\nu}(x)}^j\,, \quad \mathbf{b}_{\beta,0}(x) = \vec{\nu}^\beta(x) := \nu_1^{\beta_1}(x)\ldots\nu_n^{\beta_n}(x)$$

($\beta \in \mathbb{N}_0$; cf. (1.28)); by inserting them into (1.16) and applying (1.26) we get

$$\begin{aligned}
\mathscr{A}(u,v) &:= \int\limits_{\Omega^\pm} \sum_{|\alpha|,|\beta|\le\ell} \left[a_{\alpha,\beta}(x)\partial_y^\alpha u(y)\right]^\top \overline{\sum_{j=0}^{|\beta|} \mathbf{b}_{\beta,|\beta|-j}(y,\mathscr{D}_y)\partial_{\vec{\nu}(y)}^j v(y)}dy \\
&= \sum_{|\alpha|,|\beta|\le\ell} \sum_{j=0}^{|\beta|} \int\limits_{\Omega^\pm} \left[\mathbf{b}^*_{\beta,|\beta|-j}(y,\mathscr{D}_y)a_{\alpha,\beta}(x)\partial_y^\alpha u(y)\right]^\top \overline{\partial_{\vec{\nu}(y)}^j v(y)}dy \\
&= \int\limits_{\Omega^\pm} [\mathbf{A}(y,D_y)u(y)]^\top \overline{v(y)}dy \pm \sum_{j=0}^{\ell}\sum_{k=0}^{j-1} \int\limits_{\mathscr{S}} (-1)^k \left[\mathbf{C}_{1,j}(\tau,D_\tau)u(\tau)\right]^\top \overline{\partial_{\vec{\nu}(\tau)}^{j-k-1} v(\tau)}d_\tau\mathscr{S} \\
&= \int\limits_{\Omega^\pm} [\mathbf{A}(y,D_y)u(y)]^\top \overline{v(y)}dy \pm \sum_{j=0}^{\ell-1} \int\limits_{\mathscr{S}} \left[\mathbf{C}_{2,j}(\tau,D_\tau)u(\tau)\right]^\top \overline{\partial_{\vec{\nu}(\tau)}^j v(\tau)}d_\tau\mathscr{S}\,,
\end{aligned} \tag{5.7}$$

because we can not get anything different in the first group of summands than in (5.6). Thus, we get the GREEN formula (1.17) with special "boundary" operators $\mathbf{C}_j = \mathbf{C}_{2,j}$ and $\mathbf{B}_j = \partial_{\vec{\nu}(y)}^j$ ($j = 0,\ldots,\ell-1$). Now we can apply (5.4) with $m = \ell$ and replace $\{\partial_{\vec{\nu}(y)}^j\}_{j=0}^{\ell-1}$ in (5.7) by another DIRICHLET system $\{\mathbf{B}_j\}_{j=0}^{\ell-1}$ (see (5.5)), which gives us the claimed formula (1.17).

If $\mathbf{A}$ is formally self–adjoint, $\mathbf{A} = \mathbf{A}^*$, then $\mathscr{A}(u,v) = \overline{\mathscr{A}(v,u)}$ and from (1.17) written for pairs u,v and v,u we get the simplified GREEN formula (1.18). ■

5.3. Proof of Theorem 3.2. Due to Theorem 1.6 we can suppose that the GREEN formula (1.13) is valid and let $\{\mathbf{C}_j(x,D_x)\}_{j=0}^{2\ell-1}$ be the DIRICHLET system, participating in formula (1.13). Without loss of generality we can assume that $\operatorname{ord}\mathbf{C}_{\rm j} = 2\ell - \operatorname{ord}\mathbf{B}_{\rm j} - 1 = 2\ell - {\rm j} - 1$ (see (1.14)). Due to Lemma 4.7 we have that

$$\mathbf{G}(x,D_x) = \sum_{j=0}^{\mu} \mathbf{G}_{\mu-j}(x,\mathscr{D}_x)\mathbf{C}_{2\ell-j-1}(x,D_x)\,, \tag{5.8}$$

$$\mathbf{G}_k(x,\mathscr{D}_x) = \sum_{|\alpha|\le k} c^0_{k\alpha}(x)\mathscr{D}_x^\alpha, \quad x\in\Omega^\pm, \quad k = 1,2,\ldots,\mu\,.$$

Then (see Lemma 1.8)

$$\begin{aligned}
\mathbf{V}_{\mathbf{G}}^{(\beta)}\varphi(x) &= \sum_{j=0}^{\mu} \oint\limits_{\mathscr{S}} \left[\overline{\mathbf{G}_{\mu-j}(\tau,\mathscr{D}_\tau)\gamma_{\mathscr{S}}^\pm\mathbf{C}_{2\ell-j-1}(\tau,D_\tau)\mathscr{K}_{\mathbf{A}^*}^{(\beta)}(x,\tau)}\right]^\top \varphi(\tau)d_\tau\mathscr{S} \\
&= \sum_{j=0}^{\mu} \oint\limits_{\mathscr{S}} \left[\overline{\gamma_{\mathscr{S}}^\pm\mathbf{C}_{2\ell-j-1}(\tau,D_\tau)\mathscr{K}_{\mathbf{A}^*}^{(\beta)}(x,\tau)}\right]^\top (\mathbf{G}_{\mu-j})^*_{\mathscr{S}}(\tau,\mathscr{D}_\tau)\varphi(\tau)d_\tau\mathscr{S}
\end{aligned}$$

and it suffices to concentrate on the case of generalised layer potentials

$$\mathbf{V}_j^{(\beta)}\varphi(x) := \oint_{\mathscr{S}} \left[\overline{\gamma_{\mathscr{S}}^{\pm}\mathbf{C}_j(\tau, D_\tau)\mathscr{K}_{\mathbf{A}^*}^{(\beta)}(x,\tau)}\right]^\top \varphi(\tau)d_\tau\mathscr{S} = \mathbf{F}_{\mathbf{A},\mathbf{j}}^{(\beta)}(\varphi\otimes\delta_{\mathscr{S}})(x)\,,$$

$$\mathbf{F}_{\mathbf{A},\mathbf{j}}^{(\beta)}\psi(x) := \int_{\Omega^\pm} \left[\overline{\gamma_{\mathscr{S}}^{\pm}\mathbf{C}_j(\tau, D_\tau)\mathscr{K}_{\mathbf{A}^*}^{(\beta)}(x,\tau)}\right]^\top \psi(y)dy\,, \qquad x\in\Omega^\pm \tag{5.9}$$

(see (3.5), (4.33)). Let us consider the symbol

$$\mathscr{F}_{\mathbf{A},\mathbf{j}}^{(\beta)}(x,\xi) = (-i\partial_\xi)^\beta\mathscr{F}_{\mathbf{A}}(x,\xi)\left[\overline{\mathscr{C}_j(x,\xi)}\right]^\top\,, \quad \mathscr{F}_{\mathbf{A},\mathbf{j}}^{(\beta)} \in \mathbb{S}^{-2\ell+j-|\beta|}(\mathbb{R}^n,\mathbb{R}^n)$$

of the PsDO $\mathbf{F}_{\mathbf{A},\mathbf{j}}^{(\beta)}$ (see Corollary 4.3), where $\mathscr{F}_{\mathbf{A}}(x,\xi)$ is the symbol of the fundamental solution of $\mathbf{A}(x,D_x)$. If $\varphi\in\mathbb{B}_{p,p}^s(\mathscr{S})(\varphi\in\mathbb{B}_{p,q}^s(\mathscr{S}))$ and $s<0$, then

$$\varphi\otimes\delta_{\mathscr{S}} \in \widetilde{\mathbb{H}}_{p,com}^{s-\frac{1}{p'},\infty}(\overline{\Omega^\pm})\bigcap\widetilde{\mathbb{B}}_{p,q,com}^{s-\frac{1}{p'},\infty}(\overline{\Omega^\pm}),$$

where $p'=p/(p-1)$. From (5.9) and (4.4) we derive the continuity results (3.13)–(3.15) for $s<0$.

Next we take $s>0$, $s\notin\mathbb{N}$. We define the operators

$$\mathscr{P}_j\varphi := \mathscr{P}\Psi_j, \quad \Psi_j := (0,\dots,0,\varphi,0,\dots,0)\,,$$

where φ stands at j-th place and $\mathscr{P}$ is from Lemma 4.8 (see (4.12), (4.14)).Easy to ascertain, that these operators and their composition with A, considered between the following spaces

$$\begin{aligned}\mathscr{P}_j \;\; &: \;\; \mathbb{B}_{p,p}^{s+2\ell-m_j-1}(\mathscr{S}) \longrightarrow \mathbb{H}_{p,loc}^{s+2\ell-1+\frac{1}{p}}(\overline{\Omega^\pm}),\\ &: \;\; \mathbb{B}_{p,q}^{s+2\ell-m_j-1}(\mathscr{S}) \longrightarrow \mathbb{B}_{p,q,loc}^{s+2\ell-1+\frac{1}{p}}(\overline{\Omega^\pm}),\end{aligned} \tag{5.10}$$

$$\begin{aligned}\mathbf{A}\mathscr{P}_j \;\; &: \;\; \mathbb{B}_{p,p}^{s+2\ell-m_j-1}(\mathscr{S}) \longrightarrow \widetilde{\mathbb{H}}_{p,loc}^{s-1+\frac{1}{p}}(\overline{\Omega^\pm}),\\ &: \;\; \mathbb{B}_{p,q}^{s+2\ell-m_j-1}(\mathscr{S}) \longrightarrow \widetilde{\mathbb{B}}_{p,q,loc}^{s-1+\frac{1}{p}}(\overline{\Omega^\pm})\end{aligned} \tag{5.11}$$

are continuous. Moreover, $\gamma_{\mathscr{S}}^{\pm}\mathbf{B}_k\mathscr{P}_j=0$ for $k\neq j$ and $\gamma_{\mathscr{S}}^{\pm}\mathbf{B}_j\mathscr{P}_j=I$.

Let us consider $v_{\varepsilon,x}^{(\beta)}(y) := \chi_\varepsilon(x-y)\mathscr{K}_{\mathbf{A}^*}^{(\beta)}(y,x)$, where $\mathscr{K}_{\mathbf{A}^*}(x,y)$ is the kernel of the fundamental solution $\mathbf{F}_{\mathbf{A}^*}(x,D_x)$ and $\chi_\varepsilon\in C^\infty(\mathbb{R}^n)$, $\chi_\varepsilon(x)=1$, $\chi_\varepsilon(x)=0$ for $|x|>\varepsilon$ and $|x|<\varepsilon/2$, respectively. By inserting

$$v(y)=v_{\varepsilon,x}^{(\beta)}(y)\,, \quad u=\mathscr{P}_j\varphi\,, \quad \varphi\in\mathbb{B}_{p,p}^{s+\mu_j}(\mathscr{S}) \quad \big(\varphi\in\mathbb{B}_{p,q}^{s+\mu_j}(\mathscr{S})\big)$$

into the GREEN formula (1.13) and sending $\varepsilon\to 0$, similarly to (3.3) we find the following

$$\pm\mathbf{V}_j^{(\beta)}\varphi(x) = \chi_{\Omega^\pm}^{(\beta)}(x)\mathscr{P}_j\varphi(x) - \mathbf{N}_{\Omega^\pm}^{(\beta)}\mathbf{A}\mathscr{P}_j\varphi(x)$$

$$+\sum_{\substack{\alpha+\gamma\le 2\ell\\ 0\neq\alpha\le\beta}}\int_{\Omega^\pm} c_{\alpha\beta\gamma}^1(y)(x-y)^{\beta-\alpha}(\partial_y^\gamma\mathscr{K}_{\mathbf{A}})(x,y)c_{\alpha\beta\gamma}^2(y)\mathscr{P}_j\varphi(y)dy \tag{5.12}$$

(see (4.6) for $\mathbf{N}^{(\beta)}_{\Omega^\pm}$), where $c^1_{\alpha\beta\gamma}$, $c^2_{\alpha\beta\gamma} \in C^\infty(\mathbb{R}^n)$, and $\chi^{(\beta)}_\pm(x) = 0$ for $\beta \neq 0$, $\chi^{(0)}_\pm(x) = \chi_\pm(x)$, $x \in \Omega^\pm$ $(j = 0, \dots, 2\ell - 1)$.

Applying Remark 4.4 and Lemma 4.8 from (5.12) we derive the following continuity results:

$$\begin{aligned} \mathbf{V}^{(\beta)}_j \quad &: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \bigcap \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,p,loc}(\overline{\Omega^\pm}), \\ &: \quad \mathbb{B}^{s+2\ell-m_j-1}_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm}). \end{aligned} \tag{5.13}$$

Since $m_j + \mu_j = 2\ell - 1$ (see (1.14)), (5.13) implies the continuity

$$\begin{aligned} \mathbf{V}^{(\beta)}_j \quad &: \quad \mathbb{B}^{s}_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,loc}(\overline{\Omega^\pm}) \bigcap \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,p,loc}(\overline{\Omega^\pm}), \\ &: \quad \mathbb{B}^{s}_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p}}_{p,q,loc}(\overline{\Omega^\pm}) \quad j = 0, \dots, 2\ell - 1, \end{aligned} \tag{5.14}$$

provided

$$s > 2\ell - \mu^0 - 1, \quad s \neq 2\ell - m_j + k, \quad k \in \mathbb{N}, \quad \mu^0 := \min\{\mu_0, \dots, \mu_{2\ell-1}\}. \tag{5.15}$$

Continuity in (5.14), in its turn, yields the following continuity

$$\begin{aligned} \mathbf{V}^{(\beta)}_j \quad &: \quad \mathbb{B}^{s}_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm}) \bigcap \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p},k}_{p,p,loc}(\overline{\Omega^\pm}), \\ &: \quad \mathbb{B}^{s}_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^{s+2\ell-\mu_j-1+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm}), \end{aligned} \tag{5.16}$$

because $\beta \in \mathbb{N}_0$ is an arbitrary multi–index in (5.14) and $\rho^k \mathbf{V}^{(\beta)}_j = \mathbf{V}^{(\beta', \beta_n + k)}_j$ in local coordinates, in which $\rho^k(x) := [\,\mathrm{dist}\,(x, \mathscr{S})]^k = x^k_n$.

The continuity (3.14), (3.15) for $s < 0$ and the cases (5.16) is proved. The missing cases in (3.14), (3.15) are filled in with the interpolation (2.9). The continuity (3.16) follows almost automatically.

Let us consider the remaining case, i.e. the continuity (3.13) for $s \geq 0$. We recall that (3.13) has been proved for $s < 0$, while the continuity (3.13) follows from (3.14) for $p = 2$ and $s \geq 0$, because $\mathbb{B}^r_{2,2}(\mathscr{S}) = \mathbb{H}^r_2(\mathscr{S})$ and $\mathbb{B}^{r,k}_{2,2}(\overline{\Omega^\pm}) = \mathbb{H}^{r,k}_2(\overline{\Omega^\pm})$ (see § 2). At this stage, we only need to observe that the desired continuity claim follows from the first interpolation result in (2.9). ∎

6 Consequences and related results

6.1. Traces of generalised potentials on the boundary. Let $\mathbf{A}(x, D_x)$ in (1.1) be an elliptic differential operator with even order $m = 2\ell$ and $\mathbf{F_A} = \mathbf{F_A}(x, D)$ be its fundamental solution. $\mathscr{K}_\mathbf{A}(x, y)$ is the corresponding SCHWARTZ kernel (i.e, the fundamental matrix of $\mathbf{A}(x, D_x)$).

Let us consider a **Potential–type** operator

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x, D_x) := \mathbf{B}(x, D_x)\mathbf{V}^{(\beta)}\mathbf{C}(t, \mathscr{D}_t), \quad x \in \Omega^\pm, \quad t \in \mathscr{S} = \partial\Omega^+ \tag{6.1}$$

where $\mathbf{V}^{(\beta)}$, $\beta \in \mathbb{N}^n_0$, is a generalised single layer potential

$$\mathbf{V}^{(\beta)}\psi(x) := \oint_{\mathscr{S}} \mathscr{K}^{(\beta)}_\mathbf{A}(x, \tau)\varphi(\tau) d_\tau\mathscr{S} \tag{6.2}$$

(cf. (3.12), (3.11)) and

$$\begin{aligned}\mathbf{B}(x,D_x) &= \sum_{|\alpha|\le m} b_\alpha(x)\partial_x^\alpha, \quad b_\alpha\in C^\infty(\Omega^\pm,\mathbb{C}^{N\times N}), \quad x\in\Omega^\pm, \\ \mathbf{C}(t,\mathscr{D}_t) &= \sum_{|\alpha|\le \mu} c_\alpha(t)\mathscr{D}_t^\alpha, \quad c_\alpha\in C^\infty(\mathscr{S},\mathbb{C}^{N\times N}), \quad t\in\mathscr{S}\end{aligned} \tag{6.3}$$

are some differential operators of orders $m,\mu=0,1,\cdots$. $\mathbf{C}(t,\mathscr{D}_t)$ is a tangent differential operator and it can be restricted to the boundary $\mathscr{S}$ (see (1.21)–(1.22)).

Theorem 6.1 *Let $\beta\in\mathbb{N}_0^n$, $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, $m,\mu\in\mathbb{N}_0$. Then the potential–type operators*

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x) \;:\; \mathbb{B}^s_{p,p}(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2\ell-1-m-\mu+|\beta|+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm}), \tag{6.4}$$

$$\;:\; \mathbb{B}^s_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^{s+2\ell-1-m-\mu+|\beta|+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm}) \tag{6.5}$$

are continuous for all $k=0,1,\dots,\infty$.

Moreover, the traces $\gamma^\pm_{\mathscr{S}}\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x)$ exist and are classical pseudodifferential operators with symbols

$$\mathscr{V}^{(\beta)}_{\mathscr{B},\mathscr{C}}(t,\xi) \simeq \sum_{k=0}^{N}\mathscr{V}^{(\beta)}_{\mathscr{B},\mathscr{C},k}(t,\xi) + \widetilde{\mathscr{V}}^{(\beta)}_{\mathscr{B},\mathscr{C}.N+1}(t,\xi), \quad t\in\mathscr{S},\ \ \xi\in\mathbb{R}^n, \tag{6.6}$$

$$\widetilde{\mathscr{V}}_{\mathscr{B},\mathscr{C},N+1}\in\mathbb{S}^{-2\ell+1+m+\mu-|\beta|-N-1}(\mathscr{S}),$$

where $N\in\mathbb{N}_0$ is arbitrary and $\mathscr{V}^{(\beta)}_{\mathscr{B},\mathscr{C},k}(t,\xi)$ are homogeneous of order $-2\ell+1+m+\mu-|\beta|-k$ ($k=0,1,\cdots,N$).

The result is also valid for $s>0$ and $1\le p,q\le\infty$. In particular, it is valid for the Zygmund *spaces (the case $p=q=\infty$):*

$$\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x) : \mathbb{Z}^s(\mathscr{S}) \longrightarrow \mathbb{Z}^{s+2\ell-1-m-\mu+|\beta|,k}(\Omega^\pm). \tag{6.7}$$

Proof. Continuity in (6.4), (6.5) and (6.7) follows from Theorem 3.2 and we shall concentrate on the traces $\gamma^\pm_{\mathscr{S}}\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x,D_x)$. Without loss of generality we can suppose $\mathbf{C}(x,\mathscr{D}_x)=I$ because a composition of classical PsDOs is classical. Decomposing $\mathbf{B}(x,D_x)$ similarly to (1.28)

$$\mathbf{B}(x,D_x)=\sum_{k=0}^{m}\mathbf{B}^{(2\ell-k)}(x,\mathscr{D}_x)\partial^k_{\vec{\nu}(x)},$$

where $\mathbf{B}^{(k)}(x,\mathscr{D}_x)$ is a tangent differential operator of order k, we find

$$\begin{aligned}\mathbf{V}^{(\beta)}_{\mathbf{B}}(x,D_x) &= \sum_{k=0}^{m}\mathbf{B}^{(2\ell-k)}(x,\mathscr{D}_x)\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x), \\ \widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x) &:= \mathbf{V}^{(\beta)}_{\partial^k_{\vec{\nu}(x)}}(x,D_x) := \partial^k_{\vec{\nu}(x)}\mathbf{V}^{(\beta)}(x,D_x).\end{aligned} \tag{6.8}$$

If $k=0,1,\cdots,2\ell-1$ the generalised potentials $\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ are PsDOs due to Lemma 4.5 and the traces $\gamma^\pm_{\mathscr{S}}\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ are well defined classical PsDOs on $\mathscr{S}$.

Let us consider the representation

$$\mathbf{A}(x,D_x)=\mathscr{A}_0(x,\vec{\nu}(x))\partial^{2\ell}_{\vec{\nu}(x)}+\sum_{k=0}^{2\ell-1}\mathbf{A}_{2\ell-k}(x,\mathscr{D}_x)\partial^{k}_{\vec{\nu}(x)} \tag{6.9}$$

(cf. (1.28)), where $\mathscr{A}_0(x,\xi)$ is the principal symbol of $\mathbf{A}(x,D_x)$ (cf. (1.13)) and

$$\mathbf{A}_j(t,\mathscr{D}_t)=\sum_{|\alpha|\le j}a_{j,\alpha}(t)\mathscr{D}^{\alpha}_t\,,\quad t\in\mathscr{S}\quad j=0,1,\dots,2\ell-1$$

are tangent differential operators. Since $\mathscr{K}_{\mathbf{A}}(x,y)$ is the kernel of the fundamental solution, we get

$$\begin{aligned}\mathbf{A}(x,D_x))\mathscr{K}^{(\beta)}_{\mathbf{A}}(x,y)&=\mathbf{A}(x,D_x)(x-y)^{\beta}\mathscr{K}_{\mathbf{A}}(x,y)\\&=(x-y)^{\beta}\mathbf{A}(x,D_x)\mathscr{K}_{\mathbf{A}}(x,y)+\mathbf{E}(x,D_x)\mathscr{K}_{\mathbf{A}}(x,y)\\&=(x-y)^{\beta}\delta(x-y)+\mathbf{E}(x,D_x)\mathscr{K}_{\mathbf{A}}(x,y)=\delta_{|\beta|,0}\delta(x-y)+\mathbf{E}(x,D_x)\mathscr{K}_{\mathbf{A}}(x,y)\end{aligned} \tag{6.10}$$

(cf. (3.1) and (6.9)), where

$$\mathbf{E}(x,D_x)=\begin{cases}0\,, & \text{if}\quad \beta=0\,,\\ \displaystyle\sum_{k=0}^{2\ell-1}\mathbf{E}_{2\ell-k-1}(x,\mathscr{D}_x)\partial^{k}_{\vec{\nu}(x)}\,, & \text{if}\quad \beta\neq 0\end{cases}$$

and $\operatorname{ord}\mathbf{E}_{\mathrm{j}}=\mathrm{j}$. On the other hand, by invoking (6.9), we find

$$\begin{aligned}\mathbf{A}(x,D_x)\mathscr{K}^{(\beta)}_{\mathbf{A}}(x,y)=\mathscr{A}_0(x,\vec{\nu}(x))\partial^{2\ell}_{\vec{\nu}(x)}\mathscr{K}^{(\beta)}_{\mathbf{A}}(x,y)+\sum_{k=0}^{2\ell-1}\sum_{\gamma\le\beta}\mathbf{G}_{2\ell-k,\gamma}(x,\mathscr{D}_x)\partial^{k}_{\vec{\nu}(x)}\mathscr{K}^{(\gamma)}_{\mathbf{A}}(x,y)\\=\delta_{|\beta|,0}\,\delta(x-y)+\mathbf{E}(x,D_x)\mathscr{K}_{\mathbf{A}}(x,y)\,.\end{aligned} \tag{6.11}$$

Now we recall that $\mathbf{A}(x,D_x)$ is elliptic, which implies $\det\mathscr{A}_0(x,\vec{\nu}(x))\neq 0$ in the neighbourhood of the boundary $\mathscr{S}$ (see (1.5)). This ensures solvability of the equation (6.11) and we find:

$$\begin{aligned}\partial^{2\ell}_{\vec{\nu}(x)}\mathscr{K}^{(\beta)}_{\mathbf{A}}(x,y)=\delta_{|\beta|,0}\,\delta(x-y)\,[\mathscr{A}_0(x,\vec{\nu}(x))]^{-1}\\+\sum_{k=0}^{2\ell-1}\sum_{\gamma\le\beta}\mathbf{H}^{(2\ell)}_{2\ell-k,\gamma}(x,\mathscr{D}_x)\partial^{k}_{\vec{\nu}(x)}\mathscr{K}^{(\gamma)}_{\mathbf{A}}(x,y)\,.\end{aligned} \tag{6.12}$$

Applying the mathematical induction and invoking (6.12) we obtain the representation

$$\partial^{m}_{\vec{\nu}(x)}\mathscr{K}^{(\beta)}_{\mathbf{A}}(x,y)=\sum_{k=0}^{2\ell-1}\sum_{\gamma\le\beta}\mathbf{H}^{(m)}_{m-k,\gamma}(x,\mathscr{D}_x)\partial^{k}_{\vec{\nu}(x)}\mathscr{K}^{(\gamma)}_{\mathbf{A}}(x,y) \tag{6.13}$$

for arbitrary $m=2\ell,2\ell+1,\cdots$.

The representation (6.13), inserted into (6.8), shows that all generalised potentials $\widetilde{\mathbf{V}}^{(\beta)}_k(x,D_x)$ have traces on $\mathscr{S}$ which are classical PsDOs (see [CD1, § 1, Example 2]). ∎

Remark 6.2 *The representation* (6.12) *for* $\beta = 0$ *is well known in the literature (see, e.g., [KGBB1, § 6.7] and [Na1]).*

Remark 6.3 *In the definition of the potential–type operators* $\mathbf{V}^{(\beta)}_{\mathbf{B},\mathbf{C}}(x, D_x)$ *in* (6.1), $\mathbf{C}(t, \mathscr{D}_t)$ *can be an arbitrary classical pseudodifferential operator on the boundary* $\mathscr{S}$*.*

6.2. The trace theorem for weighted spaces

The next Theorem generalises Theorem 4.6.

Theorem 6.4 *The trace operator*

$$\mathscr{R}^{\pm}_k u := \{\gamma^{0,\pm}_{\mathscr{S}} u, \gamma^{1,\pm}_{\mathscr{S}} u, \ldots, \gamma^{k,\pm}_{\mathscr{S}} u\}, \quad u \in C^\infty_0(\overline{\Omega^\pm})$$

(see (4.10)*) is a retraction*

$$\begin{aligned} \mathscr{R}^{\pm}_k \ &: \ \mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,p}(\mathscr{S}), \\ &: \ \mathbb{B}^{s,m}_{p,q,loc}(\overline{\Omega^\pm}) \longrightarrow \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(\mathscr{S}), \end{aligned} \tag{6.14}$$

provided $1 \leq p, q \leq \infty$, $m \in \mathbb{N}_0$, $k < s - \dfrac{1}{p}$ *and has a coretraction.*

We will present two different proofs of this assertion.

Proof 1. If $m = 1, 2, \ldots$ the continuity results in (6.14) follow directly from Theorem 4.6 since $\mathbb{H}^{s,m}_{p,loc}(\overline{\Omega^\pm})$ and $\mathbb{B}^{s,m}_{p,q,loc}(\overline{\Omega^\pm})$ are subspaces of $\mathbb{H}^{s}_{p,loc}(\overline{\Omega^\pm})$ and of $\mathbb{B}^{s}_{p,q,loc}(\overline{\Omega^\pm})$, respectively.

To find a continuous coretraction $\mathscr{R}^{-1}_k$ we use the representation formulae (3.6), setting there $\mathbf{A}u(x) = 0$:

$$\mathscr{R}^{-1}_{2\ell}(\gamma^{\pm}_{\mathscr{S}} u)(x) := u(x) = \sum_{j=0}^{\ell-1} \left\{ [\mathbf{V}_{\ell+j}\mathbf{B}_j]\, u(x) - [\mathbf{V}_j \mathbf{B}_{\ell+j}]\, u(x) \right\} \tag{6.15}$$

for $x \in \Omega^+ \cup \Omega^-$. Now the continuity of $\mathscr{R}^{-1}_{2\ell}$ follows from Theorem 3.2.

Proof 2. Let us dwell on the case of the half–spaces $\Omega^\pm = \mathbb{R}^n_\pm$ and $k = 0$, because the cases $k \neq 0$ and of arbitrary domains $\Omega^\pm$ can be treated as in [Tr1, Theorem 2.7.2, Steps 6–7] and [Tr1, Theorem 3.3.3].

Let us recall an alternative definition of (equivalent) norms in the spaces $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbb{H}^s_p(\mathbb{R}^n) = \mathbb{F}^s_{p,2}(\mathbb{R}^n)$:

$$\|\varphi|\mathbb{B}^s_{p,q}(\mathbb{R}^n)\| = \| \{2^{sj}\mathscr{F}^{-1}\chi_j\mathscr{F}\varphi\}_{j=0}^{\infty} \Big| \ell_q\left(L_p(\mathbb{R}^n)\right) \| \,,$$

$$\|\varphi|\mathbb{F}^s_{p,q}(\mathbb{R}^n)\| = \| \{2^{sj}\mathscr{F}^{-1}\chi_j\mathscr{F}\varphi\}_{j=0}^{\infty} \Big| L_p(\mathbb{R}^n, \ell_q)\| \tag{6.16}$$

(see [Tr1, §§ 2.3.1,2.5.6], where

$$\chi_j \in C^\infty_0(\mathbb{R}^n)\,, \quad \operatorname{supp}\chi_0 \subset \{\mathrm{x} \in \mathrm{R^n} \ : \ |\mathrm{x}| \leq 2\}\ ,$$

$$\operatorname{supp}\chi_j \subset \{x \in \mathbb{R}^n \ : \ 2^{j-1} < |x| < 2^{j+1}\}\ , \quad \sum_{j=0}^{\infty} \chi_j(x) \equiv 1\,.$$

In [Tr1, § 2.3.1, Step 5] the coretraction $\mathscr{R}_0^{-1}$ is defined as follows

$$\mathscr{R}_0^{-1}\varphi(x',x_n)=\sum_{j=0}^{\infty}2^{-j}\mathscr{F}^{-1}_{\lambda_n\to x_n}\psi_j(\lambda_n)\mathscr{F}^{-1}_{\lambda'\to x'}\chi_j(\lambda')\mathscr{F}_{y'\to\lambda'}[\varphi(y')]\,, \tag{6.17}$$

where

$$\psi_j(\lambda_n)=\psi(2^{-j}\lambda_n)\,,\quad j\in\mathbb{N}\,,\quad \psi_0,\psi\in C_0^\infty(\mathbb{R})\,,$$
$$\operatorname{supp}\psi_0\in(0,1)\,,\quad \operatorname{supp}\psi\in(1,2)\,,\quad \mathscr{F}^{-1}\psi_0(0)=\mathscr{F}^{-1}\psi(0)=1\,.$$

Then $\mathscr{F}^{-1}\psi_j(0)=2^j$ which yields $(\mathscr{R}_0^{-1}\varphi)(x',0)=\psi(x',0)$. We proceed as in [Tr1, § 2.7.2–(30)]

$$\begin{aligned}\|x_n^m\mathscr{R}_0^{-1}\varphi|\mathbb{B}^{s+m+\frac{1}{p}}_{p,q}(\mathbb{R}^n)\| &\le C_1\Big\|\Big\{2^{(s+m+\frac{1}{p})j}\mathscr{F}^{-1}_{\lambda_n\to x_n}\left[(-i\partial_{\lambda_n})^m\psi_j(\lambda_n)\right]\\ &\qquad\times\mathscr{F}^{-1}_{\lambda'\to x'}\chi_j(x)\mathscr{F}_{y'\to\lambda'}[\varphi(y')]\Big\}_{j=0}^{\infty}\Big|\ell_q\left(L_p(\mathbb{R}^n)\right)\Big\|\\ &= C_1\Big\|\Big\{2^{(s+\frac{1}{p})j}\mathscr{F}^{-1}_{\lambda_n\to x_n}\psi_j^{(m)}(\lambda_n)\mathscr{F}^{-1}_{\lambda'\to x'}\chi_j(x)\mathscr{F}_{y'\to\lambda'}[\varphi(y')]\Big\}_{j=0}^{\infty}\ell_q\left(L_p(\mathbb{R}^n)\right)\Big\|\\ &\le C_2\|\left\{\mathscr{F}^{-1}\chi_j\mathscr{F}\varphi\right\}_{j=0}^{\infty}\Big|\ell_q\left(L_p(\mathbb{R}^n)\right)\|=\left\|\varphi|\mathbb{B}^s_{p,q}(\mathbb{R}^n)\right\|\,,\end{aligned}$$

where $\psi^{(m)}(t):=\partial_t^m\psi(t)$. Similarly we find

$$\|x_n^m\mathscr{R}_0^{-1}\varphi|\mathbb{H}^{s+m+\frac{1}{p}}_{p}(\mathbb{R}^n)\|\le C_3\|\left\{2^{sj}\mathscr{F}^{-1}\chi_j\mathscr{F}\varphi\right\}_{j=0}^{\infty}|L_p(\mathbb{R}^n,\ell_2)\|\le C_3\|\varphi|\mathbb{H}^s_p(\mathbb{R}^n)\|\,. \quad\blacksquare$$

Corollary 6.5 *Let $s>0$, $s\notin\mathbb{N}$, $1<p<\infty$ ($1\le p,q\le\infty$) and $\mathbf{A}(x,D_x)$ in (1.1) be a normal (not necessarily elliptic) operator; let further $\{\mathbf{B}_j(x,D_x)\}_{j=0}^{m-1}$ be a* DIRICHLET *system of order m (see Definition 1.2).*

Then for arbitrary $k\in\mathbb{N}_0$ there exists a continuous linear operator

$$\mathscr{P}^{(k)}:\ \bigotimes_{j=0}^{m-1}\ \mathbb{B}^{s+m-j-1}_{p,p}(\mathscr{S})\longrightarrow\mathbb{H}^{s+m-1+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm})$$

$$\left(\mathscr{P}^{(k)}:\ \bigotimes_{j=0}^{m-1}\ \mathbb{B}^{s+m-j-1}_{p,q}(\mathscr{S})\longrightarrow\mathbb{B}^{s+m-1+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm})\right)$$

such that

$$\gamma^\pm_{\mathscr{S}}\mathbf{B}_j\mathscr{P}^{(k)}\Phi=\varphi_j,\quad j=0,1,\dots,m-1,$$

$$\mathbf{A}\mathscr{P}^{(k)}\Phi\in\widetilde{\mathbb{H}}^{s-1+\frac{1}{p},k}_{p,loc}(\overline{\Omega^\pm})\qquad\left(\mathbf{A}\mathscr{P}^{(k)}\Phi\in\widetilde{\mathbb{B}}^{s-1+\frac{1}{p},k}_{p,q,loc}(\overline{\Omega^\pm})\right)$$

for arbitrary

$$\Phi=(\varphi_0,\dots,\varphi_{m-1})\in\ \bigotimes_{j=0}^{m-1}\ \mathbb{B}^{s+m-j-1}_{p,p}(\mathscr{S})\quad\left(\Phi\in\mathbb{B}^{s+m-j-1}_{p,q}(\mathscr{S})\right).$$

Proof. The proof emploies Theorem 6.4 and proceeds as in Lemma 4.8. ■

6.3. The Calderón projections. Throughout this subsection it is assumed that the conditions of Theorem 1.6 hold and the GREEN formula (1.13) is valid. Let

$$\begin{aligned}\mathbb{H}^{s,\pm}_p(\mathbf{A},\mathbf{B_j},\mathscr{S}) &:= \left\{\gamma^\pm_{\mathscr{S}}\mathbf{B}_j\varphi \ : \ \varphi\in\mathbb{H}^{s+j+\frac{1}{p}}_p(\Omega^\pm)\,,\ \ \mathbf{A}(x,D_x)\varphi=0\right\}\,,\\ \mathbb{B}^{s,\pm}_{p,q}(\mathbf{A},\mathbf{B_j},\mathscr{S}) &:= \left\{\gamma^\pm_{\mathscr{S}}\mathbf{B}_j\varphi \ : \ \varphi\in\mathbb{B}^{s+j+\frac{1}{p}}_{p,q}(\Omega^\pm)\,,\ \ \mathbf{A}(x,D_x)\varphi=0\right\}\end{aligned} \tag{6.18}$$

for $j=0,\dots,2\ell-1\,,\quad s\in\mathbb{R}\,,\ \ 1<p<\infty\,,\ \ 1\le q\le\infty$, where $\gamma^\pm_{\mathscr{S}}u$ denote the traces (see Introduction).

Theorem 6.6 *The decompositions*

$$\begin{aligned}\mathbb{H}^s_p(\mathscr{S}) &= \mathbb{H}^{s,-}_p(\mathbf{A},\mathbf{B_j},\mathscr{S})\oplus\mathbb{H}^{s,+}_p(\mathbf{A},\mathbf{B_j},\mathscr{S})\,,\\ \mathbb{B}^s_{p,q}(\mathscr{S}) &= \mathbb{B}^{s,-}_{p,q}(\mathbf{A},\mathbf{B_j},\mathscr{S})\oplus\mathbb{B}^{s,+}_{p,q}(\mathbf{A},\mathscr{S})\,,\end{aligned} \tag{6.19}$$

$$\mathbb{H}^{s,-}_p(\mathbf{A},\mathbf{B_j},\mathscr{S})\cap\mathbb{H}^{s,+}_p(\mathbf{A},\mathbf{B_j},\mathscr{S})=\emptyset\,,\quad \mathbb{B}^{s,-}_{p,q}(\mathbf{A},\mathbf{B_j},\mathscr{S})\cap\mathbb{B}^{s,+}_{p,q}(\mathbf{A},\mathbf{B_j},\mathscr{S})=\emptyset \tag{6.20}$$

hold and the corresponding CALDERÓN *projections*

$$\begin{aligned}\mathbf{P}^\pm_{\mathbf{A,j}} \ &: \ \mathbb{H}^s_p(\mathscr{S})\longrightarrow\mathbb{H}^{s,\pm}_p(\mathbf{A},\mathbf{B_j},\mathscr{S})\,,\\ &: \ \mathbb{B}^s_{p,q}(\mathscr{S})\longrightarrow\mathbb{B}^{s,\pm}_{p,q}(\mathbf{A},\mathbf{B_j},\mathscr{S})\end{aligned} \tag{6.21}$$

are defined as follows

$$\mathbf{P}^\pm_{\mathbf{A,j}}=\pm\gamma^\pm_{\mathscr{S}}\mathbf{B}_j\mathbf{V}_j \quad \text{for} \quad j=0,\dots,2\ell-1\,. \tag{6.22}$$

Proof (see [Se1, Lemmata 5 and 6] for a simpler case). We will prove (6.19)–(6.20) for the BESOV spaces. For the BESSEL potential spaces we have to prove only the continuity property (6.21) while the others (including (6.22)) follow from the embedding $\mathbb{B}^s_{p,q}(\mathscr{S})\subset\mathbb{H}^s_r(\mathscr{S})$ for $1<r<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$.

First we note that $\mathbf{P}^\pm_{\mathbf{A,j}}$ are PsDOs of order 0 (see Lemma 4.5). The continuity (6.21) follow from the boundedness of PsDOs (see, e.g., Theorem 4.2) provided the inclusions

$$\operatorname{Im}\mathbf{P}^\pm_{\mathbf{A,j}}\subset\mathbb{H}^{\mathrm{s},\pm}_{\mathrm{p}}(\mathbf{A},\mathbf{B_j},\mathscr{S})\subset\mathbb{H}^{\mathrm{s}}_{\mathrm{p}}(\mathscr{S})\,,\quad \operatorname{Im}\mathbf{P}^\pm_{\mathbf{A,j}}\subset\mathbb{B}^{\mathrm{s},\pm}_{\mathrm{p,q}}(\mathbf{A},\mathbf{B_j},\mathscr{S})\subset\mathbb{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathscr{S}) \tag{6.23}$$

hold; here $\operatorname{Im}\mathbf{P}^\pm_{\mathbf{A,j}}$ denotes the image in appropriate spaces. The inclusions (6.23) follow because $\mathbf{A}\mathbf{V}_j\varphi(x)=0$ for $x\in\Omega^-\cup\Omega^+$ and $j=0,\dots,2\ell-1$.

Inserting $u=\mathscr{P}_j\varphi$, $f=\mathbf{A}u=\mathbf{A}\mathscr{P}_j\varphi$ (cf. (5.10), (5.11)) into (3.3) we get

$$\chi_{\Omega^+}\mathscr{P}_j\varphi(x)=\mathbf{N}_{\Omega^+}\mathbf{A}\mathscr{P}_j\varphi(x)+\sum_{k=0}^{2\ell-1}\mathbf{V}_k\mathbf{B}_k\mathscr{P}_j\varphi(x)=\mathbf{N}_{\Omega^+}\mathbf{A}\mathscr{P}_j\varphi(x)+\mathbf{V}_j\varphi(x) \tag{6.24}$$

for all $j=0,\dots,2\ell-1$ and all $x\in\Omega^-\cup\Omega^+$. Since the first summand in (6.24) and its derivatives are continuous across the surface $\mathscr{S}$

$$(\gamma^-_{\mathscr{S}}\partial^\alpha_x\mathbf{N}_{\Omega^+}\mathbf{A}\mathscr{P}_j\varphi)(t)=(\gamma^+_{\mathscr{S}}\partial^\alpha_x\mathbf{N}_{\Omega^+}\mathbf{A}\mathscr{P}_j\varphi)(t)\quad t\in\mathscr{S}\,,\quad \alpha\in\mathbb{N}^n_0\,,$$

by invoking (5.10), (5.11) we get

$$(\gamma^+_{\mathscr{S}}\mathbf{B}_k\mathbf{V}_j\varphi)(t) - (\gamma^-_{\mathscr{S}}\mathbf{B}_k\mathbf{V}_j\varphi)(t) = \mathbf{B}_k\mathscr{P}_j\varphi(t) = \delta_{kj}\varphi(t)\,, \tag{6.25}$$

where $j, k = 0, \cdots, 2\ell - 1$. Formula (6.25) yield

$$\mathbf{P}^-_{\mathbf{A},\mathbf{j}}\varphi + \mathbf{P}^+_{\mathbf{A},\mathbf{j}}\varphi = \gamma^+_{\mathscr{S}}\mathbf{B}_j\mathbf{V}_j\varphi - \gamma^-_{\mathscr{S}}\mathbf{B}_j\mathbf{V}_j\varphi = \varphi\,, \quad \varphi \in \mathbb{B}^s_{p,q}(\mathscr{S}) \tag{6.26}$$

and with (6.21) they imply (6.19).

To prove (6.20) (for the Besov spaces) let us apply formula (6.24), written for the homogeneous equation $f = \mathbf{A}u = \mathbf{A}\mathscr{P}_j\varphi = 0$ and a similar one for the outer domain Ω^-:

$$\chi_{\Omega^\pm}\mathscr{P}_j\varphi(x) = \pm\mathbf{V}_j\varphi(x)\,, \quad j = 0, \ldots, 2\ell - 1\,, \quad x \in \Omega^- \cup \Omega^+\,.$$

Taking the sum, applying the operator $\mathbf{B}_j$ and invoking (5.10), (5.11) we find the representation of a function $\varphi \in \mathbb{B}^{s,-}_{p,q}(\mathbf{A}, \mathbf{B}_\mathbf{j}, \mathscr{S}) \cap \mathbb{B}^{s,+}_{p,q}(\mathbf{A}, \mathbf{B}_\mathbf{j}, \mathscr{S})$

$$\varphi(x) = \mathbf{B}_j\mathbf{V}_j[\varphi](x)\,, \quad j = 0, \ldots, 2\ell - 1\,, \quad x \in \Omega^- \cup \Omega^+\,, \tag{6.27}$$

where $[\varphi](t) := \gamma^+_{\mathscr{S}}\varphi(t) - \gamma^-_{\mathscr{S}}\varphi(t)$. Thus $[\varphi](t) = 0$ on $\mathscr{S}$ implies $\varphi(x) = 0$ for all $x \in \mathbb{R}^n$.

From (6.20), (6.23) and (6.26) we get that $\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}$ are projections:

$$\left(\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}\right)^2 = \mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}\left(\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}} + \mathbf{P}^\mp_{\mathbf{A},\mathbf{j}}\right) = \mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}\,. \qquad \blacksquare$$

Example 6.7 *If in Example* 1.9 *we take the Laplacian* $\mathbf{A}(x, D_x)u(x) = \mathbf{\Delta}u(x) = 0$ *in the plane domains* $\Omega^\pm \subset \mathbb{R}^2$ *(see* (1.1)*), the spaces* $\mathbb{H}^s_p(\mathscr{S})$ *and* $\mathbb{B}^s_{p,q}(\mathscr{S})$ *are decomposed into the spaces of harmonic functions in* Ω^+ *and in* Ω^-.

6.4. The Plemelji formulae for layer potentials.
Let

$$\mathbf{V}_{j,k}(t, D_t)\varphi(t) := \oint_{\mathscr{S}} \mathbf{B}_j(t, D_t)\left[\overline{\mathbf{C}_k(\tau, D_\tau)}\mathscr{K}^\top_{\mathbf{A}}(t, \tau)\right]^\top \varphi(\tau)d_\tau\mathscr{S} \tag{6.28}$$

for $j = 0, \ldots, 2\ell-1$ denote the restriction of the potential–type operator $\mathbf{B}_j\mathbf{V}_k$ on the surface $t \in \mathscr{S}$ (see (3.5)). According to Theorems 3.2 and 6.1, $\mathbf{V}_{j,k}$ is a pseudodifferential operator and

$$\begin{aligned} \mathbf{V}_{j,k} \quad &: \quad \mathbb{H}^s_p(\mathscr{S}) \longrightarrow \mathbb{H}^{s-m_j-\mu_k+2\ell-1}_p(\mathscr{S}), \\ &: \quad \mathbb{B}^s_{p,q}(\mathscr{S}) \longrightarrow \mathbb{B}^{s-m_j-\mu_k+2\ell-1}_{p,q}(\mathscr{S}) \end{aligned} \tag{6.29}$$

are continuous provided $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ ($1 \le p \le \infty$, $1 \le q \le \infty$ if $s > 0$).

We have already explained in § 6.1 in what sense the operator $V_{j,k}$ should be understood when its order is strictly positive, i.e. $\operatorname{ord}\mathbf{V}_{\mathrm{j,k}} = \mathrm{m_j} + \mu_\mathrm{k} - 2\ell + 1 > 0$. Since $\operatorname{ord}\mathbf{V}_{\mathrm{j,j}} = 0$ (see (1.14)), $\mathbf{V}_{j,j}$ becomes a Calderón–Zygmund singular integral operator and the integral in (6.28) is understood in the Cauchy principal value sense:

$$\mathbf{V}_{j,j}(t, D_t)\varphi(t) := \lim_{\varepsilon\to 0} \oint_{\mathscr{S}\setminus\mathscr{S}(t,\varepsilon)} \mathbf{B}_j(t, D_t)\left[\overline{\mathbf{C}_j(\tau, D_\tau)}\mathscr{K}^\top_{\mathbf{A}}(t, \tau)\right]^\top \varphi(\tau)d_\tau\mathscr{S}\,. \tag{6.30}$$

Here $\mathscr{S}(t, \varepsilon) := S^{n-1}(t, \varepsilon) \cap \mathscr{S}$ is the part of the surface $\mathscr{S}$ inside the sphere $S^{n-1}(t, \varepsilon)$ with radius ε centred at $t \in \mathscr{S}$. Then $\mathbf{V}_{j,j}$ is continuous in the spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\mathbb{B}^s_{p,q}(\mathscr{S})$ (see (6.29)).

Theorem 6.8 *Let the BVP* (1.11) *be formally adjoint to* (1.10) *and suppose that the* GREEN *formula* (1.13) *holds . Then, for the traces* $\gamma^{\pm}_{\mathscr{S}}\mathbf{B}_j\mathbf{V}_k$ *we have the following* PLEMELJI *formulae:*

$$(\gamma^{-}_{\mathscr{S}}\mathbf{B}_j(x,D_x)\mathbf{V}_k\varphi)(t) = (\gamma^{+}_{\mathscr{S}}\mathbf{B}_j(x,D_x)\mathbf{V}_k\varphi)(t) \quad \text{for} \quad k \neq j\,, \tag{6.31}$$

$$(\gamma^{\pm}_{\mathscr{S}}\mathbf{B}_j(x,D_x)\mathbf{V}_j\varphi)(t) = \pm\frac{1}{2}\varphi(t) + \mathbf{V}_{j,j}(t,D_t)\varphi(t)\,, \quad t \in \mathscr{S}\,, \tag{6.32}$$

$$k,j = 0,\dots 2\ell-1\,, \quad \varphi \in \mathbb{H}^s_p(\mathscr{S})\,.$$

Proof. (6.31) follows from (6.25).

Let $t \in \mathscr{S}$ be the projection of $x \in \Omega^{\pm}$, i.e. $x \in \mp\vec{\nu}(t)$ (recall that $\vec{\nu}(t)$ is the unit normal vector, directed outwards, into Ω^-). The potential–type operator

$$\begin{aligned} \mathbf{V}_{j,j}\varphi(x) &:= \oint_{\mathscr{S}} \mathscr{K}_{j,\mathbf{A}}(x,x-\tau)\varphi(\tau)d_\tau\mathscr{S}\,, \\ \mathscr{K}_{j,\mathbf{A}}(x,x-y) &:= \mathbf{B}_j(x,D_x)\left[\overline{\mathbf{C}_j(\tau,D_\tau)}\mathscr{K}^{\top}_{\mathbf{A}}(x,y)\right]^{\top}\,, \quad x,y \in \Omega^{\pm}\,, \end{aligned} \tag{6.33}$$

restricted to $\mathscr{S}$) has order 0 and has the following CALDERÓN–ZYGMUND kernel

$$\mathscr{K}_{j,\mathbf{A}} \in C^{\infty}(\mathbb{R}^n \otimes \mathbb{R}^n \setminus \Delta_{\mathbb{R}^n})\,, \tag{6.34}$$

$$|\mathscr{K}_{j,\mathbf{A}}(x,x-y)| \leq M_0|x-y|^{1-n}\,, \qquad x,y \in \mathbb{R}^n\,, \quad x \neq y\,. \tag{6.35}$$

Then the truncated operator

$$\mathbf{V}^0_{j,j,\varepsilon}\varphi(x) := \oint_{\mathscr{S}\setminus\mathscr{S}(t,\varepsilon)} \mathscr{K}_{j,\mathbf{A}}(x,x-\tau)\varphi(\tau)d_\tau\mathscr{S}\,, \quad \varepsilon > 0 \tag{6.36}$$

(see (6.30)) has C^{∞}-smooth kernel (see (6.34)) and

$$\lim_{\varepsilon\to 0}\left(\gamma^{-}_{\mathscr{S}}\mathbf{V}^0_{j,j,\varepsilon}\varphi\right)(t) = \lim_{\varepsilon\to 0}(\gamma^{+}_{\mathscr{S}}\mathbf{V}^0_{j,j,\varepsilon}\varphi)(t)\,. \tag{6.37}$$

Due to the definition (6.30) and the continuity property (6.37),

$$\left(\gamma^{\pm}_{\mathscr{S}}\mathbf{B}_j(x,D_x)\mathbf{V}_j\varphi\right)(t) = \left(\mathbf{V}_{j,j}(t,D_t)\varphi\right)(t) + \lim_{\varepsilon\to 0}\left(\gamma^{\pm}_{\mathscr{S}}\mathbf{V}_{j,j,\varepsilon}\varphi\right)(t)\,, \tag{6.38}$$

$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) = \oint_{\mathscr{S}(t,\varepsilon)} \mathscr{K}_{j,\mathbf{A}}(x,x-\tau)\varphi(\tau)d_\tau\mathscr{S}\,, \quad x \in \Omega^{\pm}\,, \quad \varphi \in C^{\infty}(\mathscr{S})\,.$$

Since $\varepsilon > 0$ is sufficiently small there exists a diffeomorphism

$$\begin{aligned} \varkappa\ :\ \mathscr{S}_0(t,\varepsilon) \longrightarrow \mathscr{S}(t,\varepsilon)\,, \quad \varkappa(x') = (x',g(x')) \in \mathscr{S}(t,\varepsilon) \subset \mathscr{S}\,, \\ x' = (x_1,\dots,x_{n-1}) \in \mathscr{S}_0(t,\varepsilon) \subset \mathbb{R}^{n-1}_t\,, \\ g(t) = t \in \mathscr{S}\,, \quad (\partial_k g)(t) = 0\,, \quad k = 1,\dots,n-1 \end{aligned} \tag{6.39}$$

and $\mathscr{S}_0(t,\varepsilon)$ is the projection of the part $\mathscr{S}(t,\varepsilon)$ into the tangent plane $\mathbb{R}^{n-1}_t$ to $\mathscr{S}$ at $t\in\mathscr{S}$. By changing the variable $\tau=\varkappa(y')$, $y'\in\mathscr{S}_0(t,\varepsilon)$ in the integral (6.38) we find the following

$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) := \oint\limits_{\mathbb{R}^{n-1}} \mathscr{K}_{j,\mathbf{A}}(x,x-\varkappa(y'))\mathscr{G}_{\varkappa}(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy'\,, \quad |x-t|<2\varepsilon\,, \quad x\notin\mathscr{S}_0(t,\varepsilon)\,,$$

where χ_ε is the characteristic function of the part $\mathscr{S}_0(t,\varepsilon)\subset\mathbb{R}^{n-1}$ and

$$\mathscr{G}_{\varkappa}(y') := \sqrt{|\operatorname{grad}\,\mathrm{g(y')}|^2+1} = 1+\mathscr{O}\left(|y'-t|\right) \tag{6.40}$$

is the Gram determinant (see [Sc1, §IV.10.38], [Si1, §3.6]).

Next we note that

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \oint\limits_{\mathbb{R}^{n-1}} \mathscr{K}_{j,\mathbf{A}}(x,x-y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy' + o\,(1) \tag{6.41}$$

as $\varepsilon\to 0$ uniformly for $x\in\mathbb{R}^n$ in the vicinity of $\mathscr{S}_0(t,\varepsilon)$.

In fact, the remainder kernel

$$\mathscr{K}^0_{j,\mathbf{A}}(x,y') := \mathscr{K}_{j,\mathbf{A}}(x,x-\varkappa(y'))\mathscr{G}_{\varkappa}(y') - \mathscr{K}_{j,\mathbf{A}}(x,x-y')$$

is weakly singular

$$\left|\mathscr{K}^0_{j,\mathbf{A}}(x,y')\right| \leq M_1|x-y|^{2-n}\,, \qquad x,y\in\mathbb{R}^n\,, \quad x\neq y \tag{6.42}$$

(cf. (6.34); see (6.37) and [CD1, § 1.4]) and it is almost obvious that

$$\lim_{\varepsilon\to0}\gamma^{\pm}_{\mathscr{S}} \oint\limits_{\mathscr{S}_0(t,\varepsilon)} \mathscr{K}^0_{j,\mathbf{A}}(x,x-y')\mathscr{G}_{\varkappa}(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy' = 0$$

for arbitrary $\varphi\in C^\infty(\mathscr{S})$. By the same reasons

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(t) \oint\limits_{\mathscr{S}_0(t,\varepsilon)} \mathscr{K}_{j,\mathbf{A}}(x,x-y')dy' + o\,(1) \quad \text{as} \quad \varepsilon\to 0\,, \tag{6.43}$$

because $|\varphi(\varkappa(y'))-\varphi(t)|\leq M_2|y'-t|$.

If in the definition of the kernel $\mathscr{K}_{j,\mathbf{A}}(x,x-y')$ in (6.33) the differential operators $\mathbf{B}_j(x,D_x)$, $\mathbf{C}_j(x,D_x)$ and $\mathbf{A}(x,D_x)$ are replaced by their principal parts $\mathbf{B}_{j,0}(t,D_x)$, $\mathbf{C}_{j,0}(t,D_x)$ and $\mathbf{A_0}(t,D_x)$, respectively, the remainder kernel is weakly singular and admits an estimate similar to (6.42). Therefore, as in (6.43),

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(t) \oint\limits_{\mathscr{S}_0(t,\varepsilon)} \mathscr{K}_{j,0,\mathbf{A}}(x,x-y')dy' + o\,(1) \quad \text{as} \quad \varepsilon\to 0\,, \tag{6.44}$$

where the kernel is homogeneous of order $1-n$:

$$\mathscr{K}_{j,0,\mathbf{A}}(x,\lambda z) = \lambda^{1-n}\mathscr{K}_{j,0,\mathbf{A}}(x,z)\,, \quad x,z\in\mathbb{R}^n\,, \quad z\neq 0\,. \tag{6.45}$$

We can simplify the integral (6.44) furthermore:

1. First we replace the domain of integration $\mathscr{S}_0(t,\varepsilon)$ by the ball

$$\mathscr{B}(t,\varepsilon) := \left\{|y'-t| \le \varepsilon \ : \ y' \in \mathbb{R}^{n-1}\right\}.$$

   Observe that $\operatorname{mes}\mathscr{B}(\mathrm{t},\varepsilon) - \operatorname{mes}\mathscr{S}_0(\mathrm{t},\varepsilon) = \mathscr{O}(\varepsilon)$, while the corresponding integrals differ by $\mathscr{o}(1)$ as $\varepsilon \to 0$.

2. Next it is possible, by freezing coefficients at $t_0 \in \mathscr{S}$ as $\varepsilon \to 0$, to consider a pure convolution kernel $\mathscr{K}_{j,0,\mathbf{A}}(t_0, x-y')$ which is translation invariant; the remainder has a weak singularity and contributes a term $\mathscr{o}(1)$ in (6.44).

3. Due to the described simplifications, the domain of integration $|y'-t| \le \varepsilon$ can be translated (shifted) to the origin and stretched up to the unit ball $|y'| \le 1$; the integral is invariant with respect to translations and dilations (stretching).

Finally, taking the traces, we get the following

$$\left(\gamma^{\pm}_{\mathscr{S}}\mathbf{V}_{j,\varepsilon}\varphi\right)(t) := \pm c_0\, \varphi(t) + \mathscr{o}(1) \quad \text{as} \quad \varepsilon \to 0\,, \tag{6.46}$$

where $\gamma^{\pm}$ denote the traces on different faces of the surface; the integral

$$c_0 := \oint\limits_{|y'|\le 1} \mathscr{K}_{j,0,\mathbf{A}}(t_0, y')dy'$$

is independent of $\varepsilon > 0$ and $t_0 \in \mathscr{S}$. Invoking (6.26) we find $c_0 = \dfrac{1}{2}$. Now (6.38) and (6.46) yield (6.32). ■

Remark 6.9 *Applied to the operator* $\mathbf{B}_j(x,D_x)\mathbf{V}_j$, (4.9) *gives*

$$(\gamma^{\pm}_{\mathscr{S}}\mathbf{B}_j(x,D_x)\mathbf{V}_j\varphi)(t) = \pm\frac{c_0(t)}{2}\varphi(t) + \mathbf{V}_{j,j}(t,D_t)\varphi(t) \quad t \in \mathscr{S}\,, \tag{6.47}$$

where $c_0(t) = i\mathscr{B}_j(t,\vec{\nu}(t))\mathscr{N}_{j,j}(t,\vec{\nu}(t))$ *and* $\mathscr{N}_{j,j}(t,\vec{\nu}(t))$ *is the symbol of the pseudodifferential operator on* $\mathbb{R}^n$

$$\mathbf{N}_{j,j}(x,D_x)\varphi(t) := \int\limits_{\mathbb{R}^n} \mathbf{B}_j(x,D_x)\left[\overline{\mathbf{C}_j(y,D_y)}\mathscr{K}^{\top}_{\mathbf{A}}(x,y)\right]^{\top}\varphi(y)dy\,, \tag{6.48}$$

associated with the potential operator $\mathbf{V}_{j,j}$ *in* (6.30)*. From* (6.26) *we find* $c_0(t) \equiv 1$.

It is possible to find the symbol $\mathscr{B}_j(t,\vec{\nu}(t))\mathscr{N}_{j,j}(t,\vec{\nu}(t))$ *directly by invoking* (1.38).

6.5. On smoothness of solutions and coefficients. It is possible to diminish substantially the smoothness requirements, imposed in § 2 on the coefficients and on the boundary. We need only to ensure an invariant definition of the relevant spaces $\mathbb{H}_p^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathscr{S})$, $\mathbb{B}_{p,q}^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathscr{S})$ etc. and the continuity of operator (1.1) and of its formal adjoint (1.2) in appropriate spaces. For more refined results for the second order equations on domains with Lipschitz boundary we refer to [MMT1, MT1] and the literature cited therein.

Let the boundary $\partial\Omega = \mathscr{S}$ be C^{ω}-smooth.

If the integers $\omega, \ell_0, \dots, \ell_m$ and the coefficients $a_\alpha(x)$ of the operator $\mathbf{A}(x, D_x)$ in (1.1) satisfy the following conditions

$$\omega > \left|\vartheta + \frac{m}{2} - \frac{1}{p}\right| > 0, \quad a_\alpha \in C^{\ell_{|\alpha|}}(\mathbb{R}^n, \mathbb{C}^{N\times N}), \tag{6.49}$$

$$\ell_k \begin{cases} > |\vartheta + \frac{m}{2} - k| & \text{for} \quad \vartheta - \frac{m}{2} \geq 0\,, \\ = 0 & \text{for} \quad \vartheta + \frac{m}{2} - k \geq 0\,,\ \ \vartheta - \frac{m}{2} \leq 0\,, \\ > k - \vartheta - \frac{m}{2} & \text{for} \quad \vartheta + \frac{m}{2} - k < 0 \end{cases} \tag{6.50}$$

for all $k = 0, 1, \dots, m$, then the spaces $\mathbb{H}_p^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathscr{S})$, $\mathbb{B}_{p,q}^{\vartheta\pm\frac{m}{2}-\frac{1}{p}}(\mathscr{S})$ are well–defined, the traces $\mathbb{B}_{p,q}^{\vartheta+\frac{m}{2}-\frac{1}{p}}(\mathscr{S}) = \gamma_{\mathscr{S}}^{\pm}\mathbb{B}_{p,q,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm})$ exist and the operators

$$\begin{aligned} \mathbf{A}(x, D_x) \quad &: \quad \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm}) \longrightarrow \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})\,, \\ &: \quad \mathbb{B}_{p,q,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm}) \longrightarrow \mathbb{B}_{p,q,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm}) \end{aligned} \tag{6.51}$$

are continuous.

In fact, let $\vartheta - \dfrac{m}{2} \geq 0$. Since $\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm})$ we get $a_\alpha\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C^{\vartheta+\frac{m}{2}-|\alpha|}(\mathbb{R}^n, C^{N\times N})$, $|\alpha| \leq m$ (we remind that a multiplication operator aI is continuous in $\mathbb{H}_p^\nu(\mathscr{L})$, $\mathbb{B}_{p,q}^\nu(\mathscr{L})$ provided $a \in C^\mu(\mathscr{L})$ and $\mu > \nu$; see [Tr1, Corollary 2.8.2]).

Now let $\vartheta - \dfrac{m}{2} < 0$. If $\vartheta + \dfrac{m}{2} - |\alpha| \geq 0$ we have $\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset L_{p,loc}(\Omega^\pm)$ and $a_\alpha\partial^\alpha\varphi \in L_{p,loc}(\Omega^\pm) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C(\mathbb{R}^n, C^{N\times N})$. If $\vartheta + \dfrac{m}{2} - |\alpha| < 0$, then $a_\alpha\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$ for $a_\alpha \in C^{|\alpha|-\vartheta-\frac{m}{2}}(\mathbb{R}^n, C^{N\times N})$, $|\alpha| \leq m$. This yields the boundedness result (6.51).

The condition (6.49) can be slightly improved, provided the condition $\omega > \left|\vartheta + \dfrac{m}{2} - \dfrac{1}{p}\right| > 0$ holds: if $s + \dfrac{m}{2} + \dfrac{1}{p} \geq 0$ and $s - \dfrac{m}{2} < 0$ we can take

$$c_\alpha \in \mathbb{H}_{p,loc}^\mu(\overline{\Omega^\pm}) \quad \text{for} \ \ \vartheta + \frac{m}{2} - |\alpha| > \frac{n}{2}\,,\ \ \vartheta - \frac{m}{2} \leq 0\,, \tag{6.52}$$

$$\mu := \max\left\{-\vartheta - \frac{m}{2} + |\alpha| + \frac{n}{p}, \vartheta - \frac{m}{2}\right\}.$$

In fact, under the conditions (6.52) and $\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}}(\overline{\Omega^\pm})$ we get $\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^{\vartheta+\frac{m}{2}-|\alpha|}(\overline{\Omega^\pm}) \subset C^{\vartheta+\frac{m}{2}-|\alpha|-\frac{n}{p}+\varepsilon}(\overline{\Omega^\pm})$ for a small $\varepsilon > 0$. Therefore, $a_\alpha\partial^\alpha\varphi \in \mathbb{H}_{p,loc}^\mu(\overline{\Omega^\pm}) \subset \mathbb{H}_{p,loc}^{\vartheta-\frac{m}{2}}(\overline{\Omega^\pm})$.

Under the conditions (6.49) and (6.50), Lemma 4.8 with $s = \vartheta - \dfrac{m}{2} - \dfrac{1}{p} > 0$ (which implies $\vartheta > \dfrac{m}{2} + \dfrac{1}{p}$ and $\omega > m$) remains valid.

Theorem 3.2 can also be extended, based on Lemma 4.8 with weaker smoothness requirements. We leave these results for forthcoming publications.

6.6. Concluding Remarks. As we have already mentioned, if $\mathbf{A}(x, D_x)$ in (1.1) is scalar ($N = 1$), elliptic and has real valued matrix–coefficients (or complex valued coefficients and $n > 2$), then it is proper elliptic and has even order $\operatorname{ord}\mathbf{A}(\mathrm{x}, \mathrm{D_x}) = \mathrm{m} = 2\ell$ (see [LM1, Ch.2, §§ 1.1]).

For the non–scalar case $N = 2, 3, \dots$ matters are different. The operator

$$\mathbf{A}(D_x) = \begin{pmatrix} i\partial_3 & -i\partial_1 - \partial_2 \\ i\partial_1 - \partial_2 & i\partial_3 \end{pmatrix} \tag{6.53}$$

is elliptic

$$\mathscr{A}(\xi) = \begin{pmatrix} \xi_3 & -\xi_1 + i\xi_2 \\ \xi_1 + i\xi_2 & \xi_3 \end{pmatrix}, \quad \det \mathscr{A}(\xi) = |\xi|^2 \neq 0 \quad \text{for} \ \ \xi \neq 0$$

and has order 1.

Let us consider the BVP (1.10) with an elliptic "basic" operator $\mathbf{A}(x, D_x)$, $\operatorname{ord}\mathbf{A}= m$, with quasi–normal "boundary" operators $\mathbf{b}_0(x, D_x), \dots, \mathbf{b}_{\omega-1}(x, D_x)$ and the following conditions:

$$u \in \mathbb{H}^s_p(\Omega^\pm)\,, \quad f \in \mathbb{H}^{s-m}_p(\Omega^\pm)\,, \quad s \in \mathbb{R}\,,\ 1 < p, \infty\,, \quad s - \frac{1}{p} > m - 1\,. \tag{6.54}$$

The Fredholm properties and the solvability of the BVP (1.10) is completely determined by the factorisation of the "lifted" principal homogeneous symbol

$$\mathscr{A}_{(m)}(t, \xi', \lambda) := (\lambda - i|\xi'|)^{-m}\mathscr{A}_0(t, \xi' + \lambda\vec{\nu}(t)), \quad t \in \mathscr{S}\,, \ \ \xi' \in \mathscr{T}(t, \mathscr{S})\,, \ \ \lambda \in \mathbb{R}\,, \tag{6.55}$$

where $\mathscr{T}(t, \mathscr{S}) := \{\xi' \in \mathbb{R}^n \ : \ \xi' \cdot \vec{\nu}(t) = 0\}$ is the tangent space to $\mathscr{S}$ at $t \in \mathscr{S}$, and $\mathscr{A}_0(x, \xi)$ is the principal homogeneous symbol of $\mathbf{A}(x, D_x)$ (see (1.4)).

The symbol $\mathscr{A}_{(m)}(t, \xi', \lambda)$ in (6.55) admits the following factorisation

$$\mathscr{A}_{(m)}(t, \xi', \lambda) = \mathscr{A}_-(t, \xi', \lambda) \left(\frac{\lambda + i|\xi'|}{\lambda - i|\xi'|}\right)^{\frac{m}{2}} \mathscr{A}_+(t, \xi', \lambda)\,, \tag{6.56}$$

where $\mathscr{A}^{\pm}_-(t, \xi', \lambda)$ and $\mathscr{A}^{\pm}_+(t, \xi', \lambda)$ are rational, uniformly bounded (with their derivatives) and have analytic continuation in the lower ($\operatorname{Im}\lambda < 0$) and the upper ($\operatorname{Im}\lambda > 0$) complex half–planes, respectively (see [Du1, Es1, Lo2, Sh1] and the most recent paper [CD1, § 1.7]). The factors $\mathscr{A}^{\pm 1}_{\pm}$ in (6.56) do not influence the Fredholm and solvability properties of the equation and we are left with the middle factor. This leads locally to the problem of invertibility of a PsDO (or of a convolution operator) with the symbol $\left(\frac{\lambda + i|\xi'|}{\lambda - i|\xi'|}\right)^{\frac{m}{2}}$ in the space $\mathbb{H}^s_p(\mathbb{R}^n_+)$ (details see, e.g., in [CD1, § 1.7], [Es1, Sh1]). If $m = 2$ this PsDO has a kernel which is eliminated by the Schapiro–Lopatinskii condition; this condition in the scalar case $N = 1$ can be written as follows

$$\det\left[\mathrm{b_j}(\mathrm{t}, \xi', \lambda^+_{\mathrm{k}})\right]_{\frac{m}{2}\times\frac{m}{2}} \neq 0\,, \quad \mathrm{t} \in \mathscr{S}\,, \quad \xi' \in \mathscr{T}(\mathrm{t}, \mathscr{S})\,, \tag{6.57}$$

where $\lambda^+_0, \dots, \lambda^+_{\frac{m}{2}-1}$ are all roots of the polynomial equation $\mathscr{A}_0(t, \xi', \lambda) = 0$, $\operatorname{Im}\lambda > 0$ (see, e.g., [LM1, Ro1] and [EgS1, Ch.2, § 2]). As we see, the number of boundary conditions in the BVP (1.10) in the scalar case equals $\frac{m}{2}$ and is independent of the space where BVP is considered.

For the matrix case, conditions are formulated in terms of unique solvability of the initial boundary value problem for ordinary differential equations (see [Ag1, Es1, Hr2, Ro1]). If the "basic" operator in the BVP (1.10) has even order (see (6.53)), a problem arises: the values of the parameters

$$s - \frac{1}{p} = \text{integer} + \frac{1}{2} \tag{6.58}$$

are critical and the BVP (1.10) under the conditions (6.54) is not FREDHOLM (moreover, $\mathbf{A}(x, D_x)$ has a non–closed range; see [CD1, § 1.5]). In the case when (6.58) does not hold, the number of boundary conditions ω in (1.10) also depends on the space parameters $s - \frac{1}{p}$. The details will be discussed in further publications.

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AMS classification: 47A68, 35J25, 35J55