BOUNDARY VALUE PROBLEMS ON A SMOOTH SURFACE WITH SMOOTH BOUNDARY

ROLAND DUDUCHAVA

Boundary value problems (BVPs) for tangent differential operators (TDOs) on smooth surfaces with smooth boundary encounter in applications rather often. The purpose of the present paper is to give a simplest possible approach for the investigation of such problems, which will be available not only for mathematicians with solid background in differential geometry and topology, but also for engineers and applied mathematicians. For this we use the Günter's and Stokes's tangent derivatives, defined with the help of the outer unit normal vector to the surface. We find explicit representation of the dual operator on the surface and explicit Green formula for TDOs and corresponding BVPs.

As an example we consider the Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ on the smooth closed surface in details: it is proved that $-\boldsymbol{\Delta}_{\mathscr{S}}$ is self adjoint and positive definite on non-constant functions. As a consequence it is proved that $\boldsymbol{\Delta}_{\mathscr{S}} - \nu I$ is invertible (has the fundamental solution) for arbitrary $\nu > 0$.

For the Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ on an open smooth surface $\mathscr{C} \subset \mathscr{S}$, with the smooth boundary $\Gamma := \partial\mathscr{C}$ an explicit Green formula is derived and proved that the Dirichlet boundary value problem has a unique solution in the Sobolev space $\mathbb{W}^1_2(\mathscr{C})$. For the Neumann boundary value problem the solvability is proved under the usual orthogonality condition on the data.

1 CALCULUS OF TANGENT DIFFERENTIAL OPERATORS

Boundary value problems (BVPs) for tangent differential operators (TDOs) on smooth surfaces with smooth boundary encounter in applications rather often (see [Ha1, § 72] for the heat conduction by surfaces and. [Zo1] for shell problems in elasticity). Theory of such BVPs for TDOs (and even theory of BVPs for pseudodifferential operators) is developed long ago and is rather complete. In the recent book [MMT1] one can find, together with a competent survey and references, a rather complete theory of BVPs (including the theory of potential operators) on non smooth Lipschitz surfaces. But all these results require a rather solid background in differential geometry and topology.

The purpose of the present paper is to give a simplest possible approach to the investigation of aforementioned problems, which will be available for engineers and applied mathematicians. For this we use the Günter's and Stokes's tangent derivatives, defined with the help of the outer unit normal vector to the surface (see [Du2, Gu1, KGBB1]). We find explicit representation of the dual operator on the surface and explicit Green formula for TDOs and corresponding BVPs.

Let $\mathscr{S}$ be a smooth surface of co–dimension 1 in $\mathbb{R}^n$, which divides $\mathbb{R}^n$ into a bounded $\Omega^+ \subset \mathbb{R}^n$ and the complementary unbounded $\Omega^- := \mathbb{R}^n \setminus \overline{\Omega}^+$ domains with the common boundary $\mathscr{S} := \partial\Omega^+$; let $\vec{\nu}(t) = (\nu_1(t), \dots, \nu_n(t))$, $t \in \mathscr{S}$ be the outward unit normal vector to Ω^+ at $t \in \mathscr{S}$ (see Fig.1).

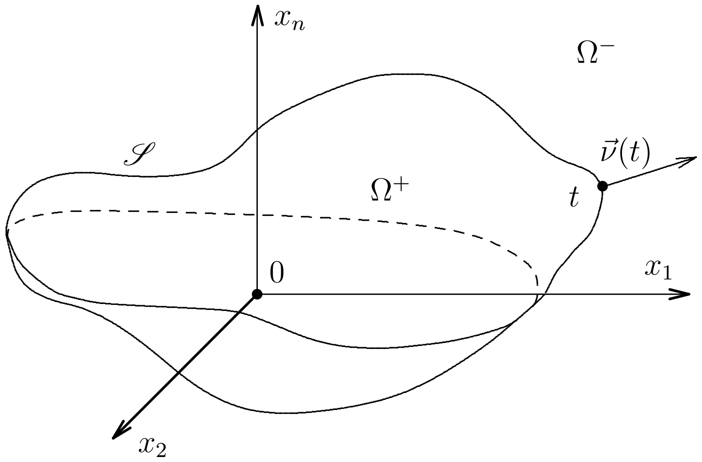

Fig. 1

We extend the outer unit normal continuously in some small neighborhood $U_{\mathscr{S}} \subset \mathbb{R}^n$ of the boundary$\mathscr{S}$ and define the normal derivative

$$\partial_{\vec{\nu}(x)} := \vec{\nu}(x) \cdot \nabla = \sum_{k=1}^{n} \nu_k(x)\partial_k, , \quad \nabla := (\partial_1, \dots, \partial_n) , \quad x \in \mathbb{R}^n , \quad j = 0, 1, \dots . \tag{1.1}$$

$\partial_{\vec{\nu}(x)}$ applies only to those functions which are defined in some neighborhood $U \subset U_{\mathscr{S}}$, but not to functions defined only on the surface $\mathscr{S}$.

In contrast to $\partial_{\vec{\nu}(t)}$ any first order linear differential operator

$$\vec{h}(t) \cdot \nabla := \sum_{k=1}^{n} h_k(t)\partial_k , \quad \vec{h}(t) = (h_1(t), \dots, h_n(t)) , \tag{1.2}$$

with the directing vector $\vec{h}(t)$ tangent to the surface $\mathscr{S}$

$$\forall t \in \mathscr{S} , \quad \vec{\nu}(t) \cdot \vec{h}(t) \equiv 0 ,$$

can be applied to a function $\varphi \in C^1(\mathscr{S})$ defined only on the surface $\mathscr{S}$. In fact, we define

$$\vec{h}(t) \cdot \nabla\varphi(t) := \lim_{\lambda \to 0} \frac{\varphi(t + \lambda\vec{h}_{\mathscr{S}}(t))}{\lambda} , \quad \varphi \in C^1(\mathscr{S}) ,$$

where $\lambda\vec{h}_{\mathscr{S}}(t)$ is the projection of the tangent vector $\lambda\vec{h}(t)$ onto the surface $\mathscr{S}$ (the projections are defined for small $|\lambda| < \varepsilon$).

The following two classes of differential operators are of special interest for us: Günter's derivatives

$$\mathscr{D}_x := (\mathscr{D}_1, \dots, \mathscr{D}_n) , \quad \mathscr{D}_j := \partial_j - \nu_j(x)\partial_{\vec{\nu}(x)} = \vec{d}_j \cdot \nabla \tag{1.3}$$

and Stokes's derivatives

$$\mathscr{M}_x := [\mathscr{M}_{j,k}]_{n \times n} , \quad \mathscr{M}_{j,k} := \nu_j(x)\partial_k - \nu_k(x)\partial_j = \vec{m}_{j,k} \cdot \nabla \tag{1.4}$$

(see [Du2, KGBB1]). The corresponding directing vectors are tangent to $\mathscr{S}$

$$\vec{\nu}(t) \cdot \vec{d}_j(t) \equiv \vec{\nu}(t) \cdot \vec{m}_{j,k}(t) \equiv 0, \quad t \in \mathscr{S} .$$

and, therefore, the Günter and the Stokes derivatives are tangent.

Only $n-1$ out of n derivatives $\mathscr{D}_1, \dots, \mathscr{D}_n$ and out of n^2 derivatives $\mathscr{M}_{1,1}, \dots, \mathscr{M}_{n,n}$ are linearly independent and the following relations are valid:

$$\begin{aligned} &\mathscr{D}_j := \sum_{k=1}^{n} \nu_k \mathscr{M}_{k,j} , \quad \sum_{k=1}^{n} \nu_k \mathscr{D}_k = 0 , \\ &\mathscr{M}_{j,k} = \nu_j \mathscr{D}_k - \nu_k \mathscr{D}_j , \quad \mathscr{M}_{j,j} = 0 , \quad \mathscr{M}_{j,k} = -\mathscr{M}_{k,j} . \\ &\sum_{i,j,k=m-1}^{m+1} \varepsilon_{i,j,k} \nu_i \mathscr{M}_{j,k} = 2 \sum_{\substack{i,j,k \in \{m-1,m,m+1\} \\ j<k}} \varepsilon_{i,j,k} \nu_i \mathscr{M}_{j,k} = 0 , \qquad m = 2, \dots, n-1 , \end{aligned} \tag{1.5}$$

where $\varepsilon_{i,j,k}$is the Levi-Chivita symbol: $\varepsilon_{i,j,k} = 0$, $\varepsilon_{i,j,k} = 1$ or $\varepsilon_{i,j,k} = -1$ depending whether two of three indices i, j, k have the same value, the triple (i, j, k) is obtained from the ordered triple $(m-1, m, m+1)$ by an even or an odd permutation, respectively.

The derivatives $\mathscr{D}_j$ were introduced in [Gu1, §1.3]). The derivatives $\mathscr{M}_{j,k}$ for $n = 3$ were intensively explored in [KGBB1, Ch. V] and for $n > 3$ see [Du2]. The derivatives $\mathscr{M}_{j,k}$ are natural entries of the Stokes formula.

In the sequel we use the following standard notation

$$\begin{aligned} \mathscr{D}_x^\alpha &:= \mathscr{D}_1^{\alpha_1} \dots \mathscr{D}_n^{\alpha_n} , \qquad \alpha \in \mathbb{N}_0^n , \\ \mathscr{M}_x^\beta &:= \mathscr{M}_1^{\beta_1} \dots \mathscr{M}_m^{\beta_m} , \quad \beta \in \mathbb{N}_0^m , \quad m = \frac{n(n-1)}{2} , \end{aligned} \tag{1.6}$$

where $\mathscr{M}_1 := \mathscr{M}_{1,2}, \dots, \mathscr{M}_m := \mathscr{M}_{n-1,n}$ are non–vanishing Stokes's derivatives $\mathscr{M}_{j,k}$, $j < k$, $j, k = 1, \dots, n$ (the other non–vanishing Stokes's derivatives differ from the selected ones only by the sign). In contrast to the case of the usual derivatives ∂^α it does really matters in which sequence we apply the derivatives $\mathscr{D}_j^{\alpha_j}$ and $\mathscr{M}_k^{\beta_k}$ in (1.6), because they have variable coefficients. In this connection let us write precisely what is meant under the dual operators:

$$\begin{aligned} (\mathscr{D}_x^*)^\alpha_{\mathscr{S}} &:= (\mathscr{D}_n^*)^{\alpha_n}_{\mathscr{S}} \dots (\mathscr{D}_1^*)^{\alpha_1}_{\mathscr{S}} , \quad \alpha \in \mathbb{N}_0^n , \\ (\mathscr{M}_x^*)^\beta_{\mathscr{S}} &:= (-1)^{|\beta|} (\mathscr{M}_m)^{\beta_m} \dots (\mathscr{M}_1)^{\beta_1}_{\mathscr{S}} , \quad \beta \in \mathbb{N}_0^m , \end{aligned} \tag{1.7}$$

where $\mathbf{A}^*_{\mathscr{S}}$ denotes the "surface" dual operator:

$$\oint_{\mathscr{S}} [\mathbf{A} u(\tau)]^\top \overline{v(\tau)} \, d_\tau \mathscr{S} = \oint_{\mathscr{S}} u^\top(\tau) \overline{\mathbf{A}^*_{\mathscr{S}} v(\tau)} \, d_\tau \mathscr{S} . \tag{1.8}$$

which differs from the formally adjoint operator (see the next lemma).

Note, that we use the same operators $(\mathscr{M}_1)^*_{\mathscr{S}} = -\mathscr{M}_1 = -\mathscr{M}_{1,2}, \dots, (\mathscr{M}_m)^*_{\mathscr{S}} = -\mathscr{M}_m := -\mathscr{M}_{n-1,n}$ for the "surface" dual operators to the Stokes derivatives, because these operators are skew-symmetric $(\mathscr{M}_{j,k})^* = -\mathscr{M}_{j,k}$ on the surface (see (1.13)). Therefore The skew-symmetric $\mathscr{M}_x^\top = -\mathscr{M}_x$ matrix operator in (1.4) is self adjoint

$$\mathscr{M}_x^* = [\mathscr{M}_{j,k}^*]_{n\times n}^\top = -[-\mathscr{M}_{j,k}]_{n\times n} = [\mathscr{M}_{j,k}]_{n\times n} = \mathscr{M}_x . \tag{1.9}$$

Lemma 1.1 (see [Du2, Lemma 1.8]). *Let* $\mathbf{G}(\mathscr{D})$ *be a "tangent" differential operator*

$$\mathbf{G}(\mathscr{D}) = \sum_{|\alpha|\leq k} g_\alpha(t)\mathscr{D}_t^\alpha = \sum_{|\beta|\leq k} f_\beta(t)\mathscr{M}_t^\beta, \qquad t\in\mathscr{S}\,. \tag{1.10}$$

Then

$$\oint\limits_{\mathscr{S}} [\mathbf{G}(\mathscr{D})u(\tau)]^\top\, \overline{v(\tau)}\; d_\tau\mathscr{S} = \oint\limits_{\mathscr{S}} u^\top(\tau)\overline{\mathbf{G}^*_{\mathscr{S}}(\mathscr{D})v(\tau)}\; d_\tau\mathscr{S}\,, \tag{1.11}$$

where

$$\mathbf{G}^*_{\mathscr{S}}(\mathscr{D}) = \sum_{|\alpha|\leq k} (\mathscr{D}_t^*)^\alpha_{\mathscr{S}} \left[\overline{g_\alpha(t)}\right]^\top I = \sum_{|\beta|\leq k} (-1)^{|\beta|}\mathscr{M}_t^\alpha \left[\overline{f_\beta(t)}\right]^\top I\,,$$

$$(\mathscr{D}_j)^*_{\mathscr{S}} u = -\sum_{k=1}^n \mathscr{M}_{k,j}\nu_k u\,, \tag{1.12}$$

$$(\mathscr{M}_{j,k})^*_{\mathscr{S}} u = -\mathscr{M}_{j,k} u\,. \tag{1.13}$$

Next lemma provides some additional properties of the derivatives.

Lemma 1.2 *The "surface" dual* $(\mathscr{D}_j)^*_{\mathscr{S}}$ *(cf.* (1.8)*) to the Günter derivative is written as follows:*

$$(\mathscr{D}_j)^*_{\mathscr{S}}\varphi = -\partial_j + \nu_j\partial^*_{\vec{\nu}} = -\mathscr{D}_j\varphi + \nu_j\mathscr{G}_{\mathscr{S}}\varphi\,, \tag{1.14}$$

where $\mathscr{G}_{\mathscr{S}}(t)$ *denotes the mean (Gaussian) curvature of the surface* $\mathscr{S}$*:*

$$\mathscr{G}_{\mathscr{S}}(t) := \operatorname{div}\vec{\nu}(t) := \sum_{k=1}^n \partial_k\nu_k(t) = \sum_{k=1}^n \mathscr{D}_k\nu_k(t)\,, \qquad t\in\mathscr{S}\,. \tag{1.15}$$

Proof. Due to (1.12) we get

$$\begin{aligned}
(\mathscr{D}_j)^*_{\mathscr{S}}\varphi &= -i\sum_{k=1}^n \mathscr{M}_{j,k}\nu_k\varphi = \sum_{k=1}^n (\nu_j\partial_k - \nu_k\partial_j)\nu_k\varphi \\
&= \sum_{k=1}^n \nu_j\partial_k\nu_k\varphi - \sum_{k=1}^n \nu_k^2\partial_j\varphi - \sum_{k=1}^n \nu_k\nu_k^{(j)}\varphi \\
&= \nu_j\partial^*_{\vec{\nu}}\varphi - \partial_j\varphi - \frac{1}{2}\partial_j\left(\sum_{k=1}^n \nu_k^2\right)\varphi \\
&= -\partial_j + \nu_j\partial^*_{\vec{\nu}} \\
&= -\partial_j + \nu_j\sum_{k=1}^n \nu_k\partial_k\varphi + \nu_j\sum_{k=1}^n \nu_k^{(j)}\varphi \\
&= -\partial_j + \nu_j\partial_{\vec{\nu}}\varphi + \nu_j\mathscr{G}_{\mathscr{S}}\varphi \\
&= -\mathscr{D}_j + \nu_j\mathscr{G}_{\mathscr{S}}\,, \qquad \nu_k^{(m)}(t) := \partial_m\nu_k(t)\,, \quad j,k=1,\dots,n\,, \quad t\in\mathscr{S}\,,
\end{aligned}$$

because

$$\sum_{k=1}^{n}\nu_k^2(t)\equiv 1 \quad\Longrightarrow\quad \partial_j\left(\sum_{k=1}^{n}\nu_k^2(t)\right)\equiv\partial_j 1\equiv 0 \tag{1.16}$$

and both equalities in (1.14) are proved.

To prove (1.15) we proceed as follows

$$\sum_{k=1}^{n}\mathscr{D}_k\nu_k=\sum_{k=1}^{n}\left(\nu_k^{(k)}-\nu_k\sum_{j=1}^{n}\nu_j\nu_k^{(j)}\right)=\mathscr{G}_{\mathscr{S}}-\sum_{j=1}^{n}\frac{\nu_j}{2}\partial_j\left(\sum_{k=1}^{n}\nu_k^2\right)=\mathscr{G}_{\mathscr{S}}$$

(see (1.16)) and the lemma is proved. ■

The formally adjoint $\mathbf{A}^*$ to a differential operator $\mathbf{A}$ and the "surface" adjoint $\mathbf{A}^*_{\mathscr{S}}$ are different, but the difference is a lower order operator.

We can define the Sobolev space on a smooth surface $\mathscr{S}$ as follows

$$\mathbb{W}^\ell_p(\mathscr{S}):=\{\varphi\in D'(\mathscr{S})\ :\ \mathscr{D}^\alpha\varphi\in\mathbb{L}_p(\mathscr{S})\,,\ \forall\,\alpha\in\mathbb{N}_0^n\,,\ |\alpha|\leq\ell\} \tag{1.17}$$

end endow it with the norm

$$\|\varphi\,\big|\,\mathbb{W}^\ell_p(\mathscr{S})\,\|:=\sum_{|\alpha|\leq\ell}\|\mathscr{D}^\alpha\varphi\,\big|\,\mathbb{L}_p(\mathscr{S})\|\,.$$

Lemma 1.3 *For Günter's gradient $\nabla_{\mathscr{S}}:=(\mathscr{D}_1,\dots,\mathscr{D}_n)$ we get the following:*

$$\|\nabla\,\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2=\|\nabla_{\mathscr{S}}\,\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2+\|\partial_{\vec{\nu}}\,\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2\,,\quad \varphi\in W^1_2(U_{\mathscr{S}})\,, \tag{1.18}$$

$$\varphi\in C^1(\mathscr{S})\,,\ \nabla_{\mathscr{S}}\,\varphi\equiv 0\quad\text{if and only if}\quad\varphi(t)\equiv\text{const}\,. \tag{1.19}$$

Proof. In fact,

$$\begin{aligned}
\|\nabla_{\mathscr{S}}\,\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2&=\sum_{j=1}^{n}\oint_{\mathscr{S}}\mathscr{D}_j\varphi(t)\overline{\mathscr{D}_j\varphi(t)}d_t\mathscr{S}\\
&=\sum_{j=1}^{n}\oint_{\mathscr{S}}(\partial_j\varphi(t)-\nu_j(t)\partial_{\vec{\nu}(t)}\varphi(t))\overline{(\partial_j\varphi(t)-\nu_j(t)\partial_{\vec{\nu}(t)}\varphi(t))}d_t\mathscr{S}\\
&=\sum_{j=1}^{n}\Bigg[\oint_{\mathscr{S}}(\partial_j\varphi(t)\overline{\partial_j\varphi(t)}d_t\mathscr{S}-\oint_{\mathscr{S}}\nu_j(t)\partial_j\varphi(t)\overline{\partial_{\vec{\nu}(t)}\varphi(t)}d_t\mathscr{S}\\
&-\oint_{\mathscr{S}}\partial_{\vec{\nu}(t)}\varphi(t)\overline{\nu_j(t)\partial_j\varphi(t)}d_t\mathscr{S}-\nu_j^2(t)\oint_{\mathscr{S}}\partial_{\vec{\nu}(t)}\varphi(t)\overline{\partial_{\vec{\nu}(t)}\varphi(t)}d_t\mathscr{S}\Bigg]\\
&=\|\nabla\,\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2-\|\partial_{\vec{\nu}}\varphi\,\big|\,\mathbb{L}_2(\mathscr{S})\|^2
\end{aligned}$$

and (1.18) follows.

To prove (1.19) we have to show only that $\nabla_{\mathscr{S}}\varphi\equiv 0$ implies $\varphi(t)\equiv\mathrm{const}$ (the inverse implication is trivial). If we suppose the contrary $\varphi(t)\neq\mathrm{const}$, there exists a smooth curve $\mathscr{L}\subset\mathscr{S}$ restricted to which $\varphi(t)$ is not constant: $\gamma_{\mathscr{L}}\varphi(t)\neq\mathrm{const}$. Since the tangent derivative $\vec{\ell}_{\mathscr{L}}\cdot\nabla$ along $\mathscr{L}$ is a linear combination of the Günter derivatives, we get $\partial_s\gamma_{\mathscr{L}}\varphi=0$ where ∂_s is the derivative with respect to the arc length on $\mathscr{L}$. Then $\gamma_{\mathscr{L}}\varphi=\mathrm{const}$ and we get a contradiction, which indicates that $\varphi\equiv\mathrm{const}$. ■

$\mathbb{W}^\ell_2(\mathscr{S})$ is a Hilbert space with the scalar product

$$(\varphi,\psi)^2_{\mathbb{W}^\ell_2(\mathscr{S})}:=\sum_{|\alpha|\leq\ell}\oint_{\mathscr{S}}\mathscr{D}^\alpha_x\varphi)(t)\overline{\mathscr{D}^\alpha_x\psi(t)}d_t\mathscr{S}\,. \tag{1.20}$$

Under the space $\mathbb{W}^{-\ell}_2(\mathscr{S})$ with a negative order $-\ell$, $\ell\in\mathbb{N}$, is understood, as usual, the dual space of distributions to the Sobolev space $\mathbb{W}^\ell_2(\mathscr{S})$.

We can indicate large classes of self adjoint operators on the surface. For example, special polynomials of the operators $\mathscr{M}_j,\ j=1,\ldots,m$ with variable coefficients

$$\mathbf{A}(x,\mathscr{M}_x)u=\sum_{j=1}^M b_j(x)\mathscr{M}_j^{m_j}\overline{b_j^\top(x)}u\,,\qquad b_j\in[C^\infty(\mathscr{S})]^{N\times N}\,, \tag{1.21}$$

or all polynomials with constant self adjoint $N\times N$ matrix coefficients

$$\mathbf{B}(\mathscr{M}_x)u=\sum_{j=1}^M a_j\mathscr{M}_j^{m_j}u\,,\qquad \overline{a_j^\top}=a_j=\mathrm{const}\quad\forall j=1,\ldots,M\,,\quad\forall m_j\in\mathbb{N}_0\,, \tag{1.22}$$

are self adjoint on the surface $\mathbf{A}^*_{\mathscr{S}}(\mathscr{M}_x)=\mathbf{A}(\mathscr{M}_x)$. An analogue of the Laplace operator

$$\widetilde{\mathbf{\Delta}}_{\mathscr{S}}:=\sum_{k=1}^m\mathscr{M}_k^2=\sum_{j<k,\ j,k=1}^n\mathscr{M}_{j,k}^2\,,\qquad m:=\frac{n(n-1)}{2}$$

is, obviously, self adjoint $(\widetilde{\mathbf{\Delta}}_{\mathscr{S}})^*_{\mathscr{S}}=\widetilde{\mathbf{\Delta}}_{\mathscr{S}}$ (see (1.12)) and even non–negative

$$(\widetilde{\mathbf{\Delta}}_{\mathscr{S}}\varphi,\varphi)_{\mathscr{S}}:=\sum_{k=1}^m\oint_{\mathscr{S}}(\mathscr{M}_k\varphi,\mathscr{M}_k\varphi)_{\mathscr{S}}\geq 0\,,\qquad\varphi\in\mathbb{W}^1_p(\mathscr{S})\,. \tag{1.23}$$

$\widetilde{\mathbf{\Delta}}_{\mathscr{S}}$ is an elliptic operator on the surface $\mathscr{S}$ as well. In fact, since $|\vec{\nu}(t)|=1$ and the symbol $\Delta_{\mathscr{S}}(t,\xi)$ is defined on the cotangent manifold

$$\mathscr{T}^*(\mathscr{S})=\{(t,\xi)\ :\ t\in\mathscr{S}\,,\ \xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n,\ (\vec{\nu}(t),\xi)=0\}\,, \tag{1.24}$$

we find easily that

$$\begin{aligned}\widetilde{\Delta}_{\mathscr{S}}(t,\xi)&=\sum_{k=1}^m\mathscr{M}_k^2(t,\xi)=\frac12\sum_{j,k=1}^m\mathscr{M}_{j,k}^2(t,\xi)=-\frac12\sum_{j,k=1}^n[\nu_j(t)\xi_k-\xi_j\nu_k(t)]^2\\&=|\xi|^2-(\vec{\nu}(t),\xi)^2=|\xi|^2\quad\text{for}\quad\xi\in\mathscr{T}^*(\mathscr{S})\,.\end{aligned} \tag{1.25}$$

As it is clear from (1.25) the operator $\widetilde{\mathbf{\Delta}}_{\mathscr{S}}$ is not elliptic on $\mathbb{R}^n$ because its symbol vanishes in the normal direction to the surface $\mathscr{S}$:

$$\widetilde{\Delta}_{\mathscr{S}}(t,\xi)=|\xi|^2-(\vec{\nu}(t),\xi)^2=0\quad\text{for all}\quad\xi=|\xi|\vec{\nu}(t)\,,\quad t\in\mathscr{S}\,. \tag{1.26}$$

2 THE GREEN FORMULA

Let $\mathscr{C} \subset \mathscr{S} \subset \mathbb{R}^n$ be a smooth surface of co–dimension 1 with the smooth boundary $\Gamma = \partial\mathscr{C}$ (see Fig. 2) and consider the following boundary value problem on the surface

$$\begin{cases} \mathbf{A}(t, \mathscr{D}_t)\varphi(t) = f(t), & t \in \mathscr{C}, \\ (\mathbf{B}_\ell(t, \mathscr{D}_t)\varphi)^+(s) = g_\ell(s), & \ell = 0, \dots, \mu - 1, \quad s \in \Gamma = \partial\mathscr{C}, \end{cases} \tag{2.1}$$

where $\mathbf{A}(t, \mathscr{D}_t)$ is the "basic" tangent differential operator and $\mathbf{B}_j(t, \mathscr{D}_t)$ are the "boundary" tangent partial differential operators:

$$\mathscr{A}(t, \mathscr{D}_t) = [\mathbf{A}_{jk}(t, \mathscr{D}_t)]_{N \times N} := \left[\sum_{|\alpha| \le m} a_{j,k,\alpha}(t)\mathscr{D}_t^\alpha \right]_{N \times N}, \qquad a_{j,k,\alpha} \in C^\infty(\overline{\mathscr{C}}),$$

$$\mathbf{B}_\ell(t, \mathscr{D}_t) = [\mathbf{B}_{\ell,j,k}(t, \mathscr{D}_t)]_{1 \times N} = \left[\sum_{|\alpha| \le m_\ell} b_{\ell,j,k,\alpha}(t)\mathscr{D}_t^\alpha \right]_{1 \times N}, \qquad b_{\ell,j,k,\alpha} \in C^\infty(U_\Gamma); \tag{2.2}$$

U_Γ denotes some neighborhood of $\Gamma \subset \overline{\mathscr{C}}$.

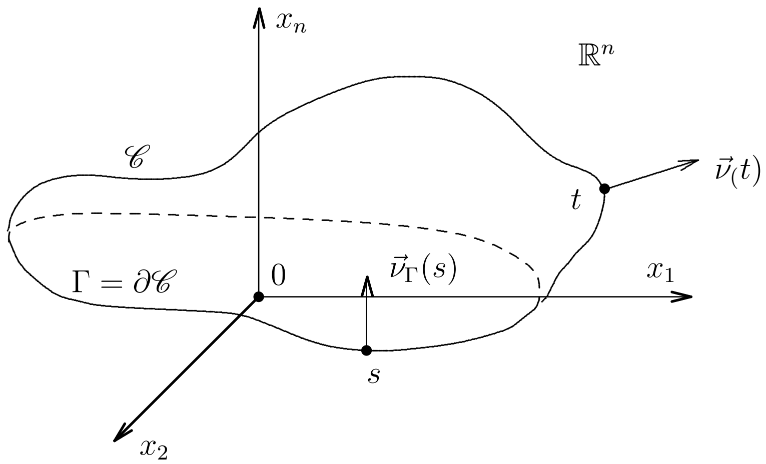

Fig. 2

Note, that in a "tangent" differential operator of order one

$$\mathbf{L}(t, D_t) := \sum_{k=1}^n \ell_k(t)\partial_k, \quad t \in U_{\mathscr{C}}, \qquad \vec{\nu}(t) \cdot \vec{\ell}(t) \equiv 0, \quad \forall t \in \mathscr{C}, \tag{2.3}$$

we can replace the usual derivatives ∂_j by Günter's tangent derivatives

$$\mathbf{L}(t, D_t) := \sum_{k=1}^n \ell_k(t)(\mathscr{D}_k + \nu_k(t)\partial_{\vec{\nu}(t)}) = \mathbf{L}(t, \mathscr{D}_t) + \vec{\nu}(t) \cdot \vec{\ell}(t) = \mathbf{L}(t, \mathscr{D}_t) \tag{2.4}$$

and write the surface dual as a formal adjoint operator

$$
\begin{aligned}
\mathbf{L}^*_{\mathscr{S}}(t,\mathscr{D}_t) &:= \sum_{k=1}^{n} (\mathscr{D}_k)^*_{\mathscr{S}} \ell_k(t) = \sum_{k=1}^{n} (-\mathscr{D}_k + \nu_k(t)\mathscr{G}_{\mathscr{S}}(t)\ell_k(t) \\
&= -\mathbf{L}^*(t,\mathscr{D}_t) + \vec{\nu}(t)\cdot\vec{\ell}(t)\mathscr{G}_{\mathscr{S}}(t) = -\mathbf{L}^*(t,\mathscr{D}_t)\,, \qquad (2.5)\\
\mathbf{L}^*(t,\mathscr{D}_t) &= \mathbf{L}^*(t,\partial_t) := \sum_{k=1}^{n} \mathscr{D}_k \ell_k(t) I = \sum_{k=1}^{n} \partial_k \ell_k(t) I
\end{aligned}
$$

(cf. (1.14)).

Lemma 2.1 *For a “tangent” differential operator of order one* (2.3) *we have the following rule for “integration by parts”:*

$$
\begin{aligned}
\oint_{\mathscr{C}} [\mathbf{L}(t,\mathscr{D}_t)\varphi(t)]^\top \overline{\psi(t)} d_t\mathscr{C} = &- \oint_{\Gamma} \vec{\ell}(s)\cdot\vec{\nu}_\Gamma(s)[\varphi(s)]^\top \overline{\psi(s)}\, d_s\Gamma \\
&+ \oint_{\mathscr{C}} [\varphi(t)]^\top\, \overline{\mathbf{L}^*(t,\mathscr{D}_t)\psi(t)} d_t\mathscr{C} \qquad (2.6)
\end{aligned}
$$

(cf. (2.4) *and* (2.5)*). Here $\vec{\nu}_\Gamma(s)$ is the unit inward normal vector to Γ at the point $s\in\Gamma$, which is tangent to the surface $\vec{\nu}(s)\cdot\vec{\nu}_\Gamma(s)\equiv 0$ for all $s\in\Gamma$. In particular,*

$$
\oint_{\mathscr{C}} [\mathbf{L}(t,\boldsymbol{D}_t)\varphi(t)]^\top \overline{\psi(t)} d_t\mathscr{C} = \oint_{\mathscr{C}} [\varphi(t)]^\top\, \overline{\mathbf{L}^*(t,\boldsymbol{D}_t)\psi(t)} d_t\mathscr{C} \qquad (2.7)
$$

if either $\mathbf{L}^*_{\mathscr{C}}(t,D_t)$ is tangent to the boundary $(\vec{\nu}_\Gamma(s),\vec{\ell}(s))\equiv 0$, $\forall s\in\Gamma$, or $\mathscr{C}$ is a closed surface $\Gamma=\partial\mathscr{C}=\emptyset$ (see Fig. 1).

Proof. We apply the Gauss formulae in the following form

$$
\mathscr{D}_k\chi_{\mathscr{C}}(t) = -\nu_{\Gamma,k}(t)\delta_\Gamma(t)\,, \qquad (2.8)
$$

where $\chi_{\mathscr{C}}(t)$, $t\in\mathscr{S}$ is the characteristic function of the surface $\mathscr{C}\subset\mathscr{S}$ and δ_Γ is a surface delta-function:

$$
\langle\delta_\Gamma, v\rangle := \oint_{\Gamma} v(s)\, d_s\Gamma\,, \qquad v\in C(\mathscr{S})\,. \qquad (2.9)
$$

In combination with formulae (2.4), (2.5) this leads to the following result:

$$
\begin{aligned}
\int_{\mathscr{C}} [\mathbf{L}(t,\mathscr{D}_t)\varphi(t)]^\top \overline{\psi(t)} d_t\mathscr{C} &= \int_{\mathscr{S}} [\mathbf{L}(t,\mathscr{D}_t)\varphi(t)]^\top \chi_{\mathscr{C}}\overline{\psi(t)} d_t\mathscr{S} \\
&= \int_{\mathscr{S}} \varphi^\top(t)\overline{\mathbf{L}^*(t,\mathscr{D}_t)\chi_{\mathscr{C}}\psi(t)} d_t\mathscr{S} = \sum_{k=1}^{n}\int_{\mathscr{S}} \varphi^\top(t)\overline{\mathscr{D}_k[\chi_{\mathscr{C}}\ell_k(t)\psi(t)]} d_t\mathscr{S}
\end{aligned}
$$

$$= -\sum_{k=1}^{n} \int\limits_{\mathscr{S}} \varphi^\top(t)[\nu_k(t)\delta_\Gamma(t)\ell_k(t)\psi(t) + \chi_{\mathscr{C}}\overline{\mathscr{D}_k[\ell_k(t)\psi(t)]}d_t\mathscr{S}$$

$$= -\oint\limits_{\Gamma} \vec{\ell}(s)\cdot\vec{\nu}_\Gamma(s)[\varphi(s)]^\top\overline{\psi(s)}\; d_s\Gamma + \oint\limits_{\mathscr{C}} [\varphi(t)]^\top\,\overline{\mathbf{L}^*(t,\mathscr{D}_t)\psi(t)}d_t\mathscr{C}$$

and (2.6) is proved. ■

Definition 2.2 *The operator* $\mathbf{A}(t, D_t)$ *in* (2.1) *is called* **normal** *on* $\Gamma = \mathscr{S}$ *if*

$$\inf|\det\mathscr{A}_0(s,\vec{\nu}_\Gamma(s))| \neq 0, \quad s\in\Gamma\,, \quad |\xi| = 1\,, \tag{2.10}$$

where $\mathscr{A}_0(t,\xi)$ *denotes the* **homogeneous principal** *symbol of* $\mathbf{A}$

$$\mathscr{A}_0(t,\xi) := \sum_{|\alpha|=m} a_\alpha(t)(-i\xi)^\alpha, \qquad (t,\xi)\in\mathscr{T}^*(\mathscr{C}). \tag{2.11}$$

Definition 2.3 *A system* $\{\mathbf{B}_j(t,\boldsymbol{D}_t)\}_{j=0}^{k-1}$ *of differential operators with matrix* $N\times N$ *coefficients is called a Dirichlet system of order* k *if all participating operators are normal on* $\mathscr{S}$ *(see Definition 2.2) and, after renumbering,* $\operatorname{ord}\mathbf{B}_j = j$*,* $j = 0,1,\dots,k-1$.

Definition 2.4 *A boundary value problem*

$$\begin{cases} \mathbf{A}^*(t,\mathscr{D}_t)v(t) = d(t), & t\in\mathscr{C}, \\ (\mathbf{C}_{m-\ell-1}(t,\mathscr{D}_t)v)^+(s) = h_\ell(s), \quad \ell = 0,\dots,\mu^*-1, & s\in\Gamma \end{cases} \tag{2.12}$$

(see (2.1)*), where* $\mu^*\le m$ *and* $\mathbf{C}_\ell(t,\mathscr{D}_t)$ *are some "boundary" differential operators*

$$\mathbf{C}_\ell(t,\mathscr{D}_t) = \sum_{|\alpha|\le\mu_j} c_{j,\alpha}(t)\mathscr{D}_t^\alpha\,, \qquad c_{\ell,\alpha}\in C^\infty(\Gamma,\mathbb{C}^{N\times N})$$

with $\operatorname{ord}\mathbf{C}_\ell = \ell \le m-1$*, is called a dual BVP to* (2.1)*, if there exist two systems of "boundary" differential operators*

$$\mathbf{B}_j(t,\mathscr{D}_t) = \sum_{|\alpha|\le m_j} b_{j\alpha}(t)\mathscr{D}_t^\alpha, \qquad \mathbf{C}_k(t,\mathscr{D}_t) = \sum_{|\alpha|\le\mu_k} c_{k\alpha}(t)\mathscr{D}_t^\alpha\,,$$

$$b_{j\alpha}, c_{k\alpha}\in C^\infty(\Gamma,\mathbb{C}^N), \quad j,k = 0,\dots,m-1\,,$$

which are extensions of systems $\{\mathbf{B}_j(t,\mathscr{D}_t)\}_{j=0}^{\mu-1}$ *and* $\{\mathbf{c}_j(t,\mathscr{D}_t)\}_{j=0}^{\mu^*-1}$*, respectively, such that the Green formula*

$$\int\limits_{\mathscr{C}} ((\mathbf{A}u)^\top\overline{v} - u^\top\overline{\mathbf{A}^*_{\mathscr{S}}v})d_t\mathscr{C} = -\sum_{j=0}^{m-1}\oint\limits_{\Gamma}(\mathbf{B}_j u)^\top\overline{\mathbf{C}_j v}d_\tau\Gamma \tag{2.13}$$

holds with $u,v\in C^\infty(\overline{\Omega^\pm},\mathbb{C}^N)$.

If (2.12) is formally adjoint to BVP (2.1), then

$$m_j + \mu_j = m - 1, \qquad j = 0, \dots .\omega - 1. \tag{2.14}$$

Theorem 2.5 *If either* $\{\mathbf{B}_j(t, \mathscr{D}_t)\}_{j=0}^{m-1}$ *or* $\{\mathbf{C}_j(t, \mathscr{D}_t)\}_{j=0}^{m-1}$ *is a fixed Dirichlet system of "boundary" operators, then the Green formula* (2.11) *holds. The related system (respectively,* $\{\mathbf{C}_j(t, \mathscr{D}_t)\}_{j=0}^{m-1}$ *or* $\{\mathbf{B}_j(t, \mathscr{D}_t)\}_{j=0}^{m-1}$*) is then unique and BVP* (2.12) *is formally adjoint to* (2.1).

The related system is a Dirichlet system if and only if the "basic" operator $\mathbf{A}(x, D_x)$ *is normal.*

Proof. The proof is verbatim to the proof of a similar [Du2, Theorem 1.6], if we apply the rule for "integration by parts" (2.6). ■

In the next theorem we consider a special case, which encounter in applications most frequent. For this we need the Sobolev–Slobodetski spaces $\mathbb{W}^s_2(\Gamma)$ and Bessel potential $\mathbb{H}^s_2(\Gamma)$ spaces for arbitrary $s \in \mathbb{R}$ (see. e.g., [Tr1] for the definition).

Note, that a function $\varphi \in \mathbb{W}^s_p(\mathscr{S})$ (and $\varphi \in \mathbb{H}^s_p(\mathscr{S})$) has the trace $\varphi^+ \in \mathbb{W}^{s-\frac{1}{p}}_p(\Gamma)$ on the boundary, provided $1 < p < \infty$ and $s > \dfrac{1}{p}$ (see [Tr1] for details).

For $p = 2$ the Sobolev–Slobodetski $\mathbb{W}^s_2(\Gamma)$ for $p = 2$, the Bessel potential $\mathbb{H}^s_2(\Gamma)$ and even the Besov $\mathbb{B}^s_{2,2}(\Gamma)$ spaces coincide (i.e., equivalent norms).

Theorem 2.6 *Let the "basic" tangent differential operator* $\mathbf{A}(t, \mathscr{D}_t)$ *in* (2.1) *be self adjoint on the surface*

$$\mathbf{A}^*_{\mathscr{S}}(t, \mathscr{D}_t) = \mathbf{A}(t, \mathscr{D}_t) \tag{2.15}$$

and elliptic of even order [(1)] $m = 2\mu$.

Let the "boundary" operators $\{\mathbf{B}_j(t, \mathscr{D}_t)\}_{j=0}^{\nu-1}$ *in* (2.1) *be normal and have different orders* $m_j \neq m_k < 2\mu$ *for* $j \neq k$*, Then there exist operators* $\{\mathbf{C}_j(t, \mathscr{D}_t)\}_{j=0}^{\nu-1}$ *such that:*

i. the system $\mathbf{B}_0(t, \mathscr{D}_t), \dots, \mathbf{B}_{\nu-1}(t, \mathscr{D}_t), \mathbf{C}_0(t, \mathscr{D}_t) \dots \mathbf{C}_{\nu-1}(t, \mathscr{D}_t)$*, is a Dirichlet system;*

ii. The following special Green formula holds

$$(\mathbf{A}u, v)_{\mathscr{S}} - \sum_{j=0}^{\nu-1}((\mathbf{B}_j u)^+, (\mathbf{C}_j v)^+)_\Gamma = (u, \mathbf{A}v)_{\mathscr{S}} - \sum_{j=0}^{\nu-1}((\mathbf{C}_j u)^+, (\mathbf{B}_j v)^+)_\Gamma\,, \tag{2.16}$$

with $u, v \in C^\infty(\overline{\Omega^\pm}, \mathbb{C}^N)$*, where*

$$(u, v)_{\mathscr{S}} := \oint_{\mathscr{S}} u^\top(t)\overline{v(t)} d_t\mathscr{S}\,, \qquad (\varphi, \psi)_\Gamma := \oint_\Gamma \varphi^\top(s)\overline{\psi(s)} d_s\Gamma\,. \tag{2.17}$$

iii. The BVP (2.1) *is self dual relative to the Green formula* (2.16) *(see Definition* 2.4*).*

---

(1) The order of an elliptic operator is automatically even if the dimension of the underlying surface is more than two: $n > 2$.

iv. The BVP (2.1) *is Fredholm in the space*

$$\mathfrak{A} := (\mathbf{A}, \mathbf{B}_0, \ldots, \mathbf{B}_{\mu-1}) \ : \ \mathbb{W}^{s+2\mu}_p(\mathscr{S}) \longrightarrow \mathbb{W}^s_p(\mathscr{S}) \times \prod_{j=0}^{\mu-1} \mathbb{W}_p^{s+2\mu-\frac{1}{p}-m_j}(\Gamma) \quad (2.18)$$

for arbitrary $s > 0$, $1 < p < \infty$ *and has a solution* $u \in \mathbb{W}^{s+2\mu}_p(\mathscr{S})$ *only for those data* $(f, g_0, \ldots g_{\mu-1}) \in \mathbb{W}^{s+2\mu}_p(\mathscr{S}) \times \prod_{j=0}^{\mu-1} \mathbb{W}_p^{s+2\mu-\frac{1}{p}-m_j}(\Gamma)$ *which satisfy the orthogonality condition*

$$(\mathbf{A}u, v)_{\mathscr{S}} - \sum_{j=0}^{\mu-1} ((\mathbf{B}_j u)^+, (\mathbf{C}_j v)^+)_\Gamma = 0 \quad (2.19)$$

for all solution $v \in C^\infty(\mathscr{S})$ *(see the Corollary* 2.7 *below) of the homogeneous BVP* (2.1) *with* $f = g_0 = \cdots = g_{\mu-1} = 0$.

Proof. The proof is verbatim to the proof of a similar Theorems in [LM1, Ro1]. To prove that BVP (2.1) is Fredholm, we apply the quasi-localization (see [CDS1, Du1, Ra1, Si1], "freezing" coefficients and rectifying the surface. The proof is then reduced to the case of BVPs with constant coefficients on $\mathbb{R}^n$. ■

Corollary 2.7 *Let conditions of Theorem* 2.6 *hold. Then all solutions of homogeneous BVP* (2.1) *with* $f = g_0 = \cdots = g_{\mu-1} = 0$ *are infinitely smooth*

$$\operatorname{Ker} \mathfrak{A} \subset C^\infty(\mathscr{S}). \quad (2.20)$$

Moreover, the Fredholm property of BVP (2.18)*, its kernel, cokernel (understood as the kernel of the dual BVP; cf.* (2.19)*) and indices are independent of* $s > 0$.

Proof. Proof is standard and based on the existence of the parametrix for the BVP (2.18). ■

Remark 2.8 *For conciseness we have avoided to formulate Theorem* 2.6 *and Corollary* 2.7 *for a general elliptic BVP* (2.1) *and its dual BVP* (2.12)*, which are similar (see [LM1, Ro1] for the case* $\mathscr{S} = \Omega \subset \mathbb{R}^n$*).*

3 BOUNDARY VALUE PROBLEMS FOR THE LAPLACE-BELTRAMI OPERATOR

Let $\mathscr{C} \subset \mathscr{S}$ be smooth surfaces as in § 2: $\mathscr{S}$ is cosed and $\mathscr{C}$ has a smooth boundary $\partial\mathscr{C} = \Gamma \neq \emptyset$:

The Laplace-Beltrami operator

$$\boldsymbol{\Delta}_{\mathscr{S}} \varphi(t) := \sum_{k=1}^n \mathscr{D}_k^2 \varphi(t), \qquad t \in \mathscr{S}, \quad (3.1)$$

where $\mathscr{D}_j := \partial_j - \nu_j(x)\partial_{\vec{\nu}(x)}$ are Günter's tangent derivatives, represents a projection of the Laplace operator

$$\boldsymbol{\Delta} u(x) := \sum_{k=1}^n \partial_k^2 u(x), \qquad x \in \mathbb{R}^n$$

onto the surface $\mathscr{S}$ (see [MP1] for the Laplace-Beltrami operator on the unit sphere).

Lemma 3.1 *Let $\mathscr{S}$ be μ–smooth and $\ell \in \mathbb{N}_0$, $\ell \leq \mu$. The Laplace-Beltrami operator $\mathbf{\Delta}_{\mathscr{S}}$ is elliptic on the surface $\mathscr{S}$ and self adjoint*

$$\mathbf{\Delta}_{\mathscr{S}}(t,\xi) \equiv |\xi|^2\,, \quad \forall (t,\xi) \in \mathscr{T}^*(\mathscr{S})\,, \qquad (\mathbf{\Delta}_{\mathscr{S}})^*_{\mathscr{S}} = \mathbf{\Delta}_{\mathscr{S}}\,. \tag{3.2}$$

For arbitrary $\ell = 0, \pm 1, \dots$ the operator

$$-\mathbf{\Delta}_{\mathscr{S}} \ : \ \mathbb{W}^{\ell+2}_2(\mathscr{S}) \to \mathbb{W}^{\ell}_2(\mathscr{S}) \tag{3.3}$$

is positive definite (coercive) on non–constant functions

$$(-\mathbf{\Delta}_{\mathscr{S}}\varphi,\varphi)_{\mathbb{W}^{\ell}_2(\mathscr{S})} = \sum_{k=1}^{n} (\mathscr{D}_k\varphi, \mathscr{D}_k\varphi)_{\mathbb{W}^{\ell}_2(\mathscr{S})} = \|\nabla_{\mathscr{S}}\varphi\big|\mathbb{W}^{\ell}_2(\mathscr{S})\| > 0 \tag{3.4}$$
$$\text{for} \quad \forall \varphi \in \mathbb{W}^{\ell+2}_2(\mathscr{S})\,,\ \varphi \neq \text{const}\,.$$

Proof. Let us prove that $\mathbf{\Delta}_{\mathscr{S}}$ is elliptic. We proceed straightforwardly [(2)] (cf. (1.25)):

$$\begin{aligned}\mathbf{\Delta}_{\mathscr{S}}(t,\xi) &= \sum_{k=1}^{n} \mathscr{D}^2_k(t,\xi) = \sum_{k=1}^{n} [\xi_k - \nu_k(t)(\vec{\nu}(t),\xi)]^2 \\ &= |\xi|^2 - 2(\vec{\nu}(t),\xi)^2 + |\vec{\nu}(t)|^2(\vec{\nu}(t),\xi)^2 \\ &= |\xi|^2 - (\vec{\nu}(t),\xi)^2 = |\xi|^2 \quad \text{for} \quad (t,\xi) \in \mathscr{T}^*(\mathscr{S})\,.\end{aligned} \tag{3.5}$$

Now let us prove that $\mathbf{\Delta}_{\mathscr{S}}$ is self adjoint on the surface (see the second equality in (3.2)). For this we apply (1.14) and proceed as follows:

$$\begin{aligned}(\mathbf{\Delta}_{\mathscr{S}})^*_{\mathscr{S}}\varphi &= \sum_{k=1}^{n} (\mathscr{D}^*_k)^2_{\mathscr{S}}\varphi = \sum_{k=1}^{n} (-\mathscr{D}_k + \nu_k\mathscr{G}_{\mathscr{S}})^2\varphi \\ &= \sum_{k=1}^{n} \mathscr{D}^2_k\varphi + \sum_{k=1}^{n} \nu^2_k\mathscr{G}^2_{\mathscr{S}}\varphi - \sum_{k=1}^{n} \nu_k\mathscr{G}_{\mathscr{S}}\mathscr{D}_k\varphi - \sum_{k=1}^{n} \mathscr{D}_k(\nu_k\mathscr{G}_{\mathscr{S}})\varphi \\ &= \mathbf{\Delta}_{\mathscr{S}}\varphi + \mathscr{G}^2_{\mathscr{S}}\varphi - \mathscr{G}_{\mathscr{S}}\sum_{k=1}^{n} \nu_k\mathscr{D}_k\varphi - \sum_{k=1}^{n} \nu_k\mathscr{D}_k(\mathscr{G}_{\mathscr{S}}\varphi) - \left(\sum_{k=1}^{n} \mathscr{D}_k\nu_k\right)\mathscr{G}_{\mathscr{S}}\varphi \\ &= \mathbf{\Delta}_{\mathscr{S}}\varphi + \mathscr{G}^2_{\mathscr{S}}\varphi - \mathscr{G}^2_{\mathscr{S}}\varphi = \mathbf{\Delta}_{\mathscr{S}}\varphi\,,\end{aligned} \tag{3.6}$$

because

$$\sum_{k=1}^{n} \nu_k\mathscr{D}_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{n} \mathscr{D}_k\nu_k = \mathscr{G}_{\mathscr{S}}$$

(see the second equality in (1.5) and (1.15)).

---

(2)Note that due to (3.5) the Laplace-Beltrami operator $\mathbf{\Delta}_{\mathscr{S}}$ is not elliptic on $\mathbb{R}^n$ and $\mathbf{\Delta}_{\mathscr{S}}(t,\xi)|\xi|^2 - (\vec{\nu}(t),\xi)^2 = 0$ in the normal direction to the surface $\mathscr{S}$ (cf. (1.26)).

To prove (3.4) we apply (1.14), the second equation in (1.5), and proceed as follows:

$$\begin{aligned}(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi,\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})} &= -\sum_{k=1}^n(\mathscr{D}_k\varphi,(\mathscr{D}_k)^*_{\mathscr{S}}\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}\\ &=\sum_{k=1}^n(\mathscr{D}_k\varphi,\mathscr{D}_k\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}-\sum_{k=1}^n(\mathscr{D}_k\varphi,\nu_k\mathscr{G}_{\mathscr{S}}\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}\\ &=\|\nabla_{\mathscr{S}}\varphi\big|\mathbb{W}^\ell_2(\mathscr{S})\|-\left(\sum_{k=1}^n\nu_k\mathscr{D}_k\varphi,\mathscr{G}_{\mathscr{S}}\varphi\right)_{\mathbb{W}^\ell_2(\mathscr{S})}\\ &=\|\nabla_{\mathscr{S}}\varphi\big|\mathbb{W}^\ell_2(\mathscr{S})\|>0\quad\text{provided}\quad\varphi\in\mathbb{W}^{\ell+2}_2(\mathscr{S})\,,\quad\varphi\neq\text{const}\end{aligned}$$

(cf. the second equality in (1.5) and (1.19)). ∎

Theorem 3.2 *Let $\mathscr{S}$ be μ-smooth and $\ell\in\mathbb{N}_0$, $\ell\leq\mu$. The perturbed Laplace-Beltrami operator*

$$\boldsymbol{\Delta}_{\mathscr{S}}-\nu I\ :\ \mathbb{W}^{s+2}_2(\mathscr{S})\to\mathbb{W}^s_2(\mathscr{S})\tag{3.7}$$

is invertible for arbitrary $s\in\mathbb{R}$, provided $\nu>0$ (i.e. $\boldsymbol{\Delta}_{\mathscr{S}}-\nu I$ has the fundamental solution).

Proof. As an elliptic operator on the closed surface $\boldsymbol{\Delta}_{\mathscr{S}}-\nu I$ in (3.7) is Fredholm for $s=0,1,\ldots$. On the other hand,

$$(-(\boldsymbol{\Delta}_{\mathscr{S}}-\nu)\varphi,\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}=\|\nabla_{\mathscr{S}}\varphi\big|\mathbb{W}^\ell_2(\mathscr{S})\|+\nu\|\varphi\big|\mathbb{W}^\ell_2(\mathscr{S})\|\quad\forall\,\varphi\in\mathbb{W}^{\ell+2}_2(\mathscr{S})\,.\tag{3.8}$$

and, therefore, $\mathrm{Ker}\,(\boldsymbol{\Delta}_{\mathscr{S}}-\nu I)=\emptyset$.

The same is true for the dual operator, which is the same and, therefore, $\mathrm{Coker}\,(\boldsymbol{\Delta}_{\mathscr{S}}-\nu I)=\emptyset$, which yields the invertibility.

The dual operator, which is again $\boldsymbol{\Delta}_{\mathscr{S}}-\nu I$, but between spaces $\mathbb{W}^{-s}_2(\mathscr{S})\to\mathbb{W}^{-s-2}_2(\mathscr{S})$, is also invertible. Then for non-integer $s\in\mathbb{R}$ the invertibility of the operator (3.7) follows by the interpolation (see [Tr1]. ∎

Remark 3.3 *$\boldsymbol{\Delta}_{\mathscr{S}}-\nu I$ is invertible as an operator between more general Sobolev-Slobodetski spaces $\mathbb{W}^{s+2}_p(\mathscr{S})\to\mathbb{W}^s_p(\mathscr{S})$ and the Bessel potential spaces $\mathbb{H}^{s+2}_p(\mathscr{S})\to\mathbb{H}^s_p(\mathscr{S})$ for arbitrary $s\in\mathbb{R}$, $1<p<\infty$ and $\nu>0$.*

Let us consider the Dirichlet

$$\begin{cases}(\boldsymbol{\Delta}_{\mathscr{C}}(t,\mathscr{D}_t)\varphi)(t)=f(t), & t\in\mathscr{C},\\ \varphi^+(s)=g(s), & s\in\Gamma=\partial\mathscr{C}\end{cases}\tag{3.9}$$

and the Neumann

$$\begin{cases}(\boldsymbol{\Delta}_{\mathscr{C}}(t,\mathscr{D}_t)\varphi)(t)=f(t), & t\in\mathscr{C},\\ (\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi)^+(s)=h(s), & s\in\Gamma=\partial\mathscr{C}\end{cases}\tag{3.10}$$

boundary value problems for the Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ (see (3.1)) on the open surface $\mathscr{C}$ with the boundary Γ (see Fig. 2). The normal derivative $\mathscr{D}_{\vec{\nu}_\Gamma(s)}$ is defined as follows

$$\begin{aligned}\mathscr{D}_{\vec{\nu}_\Gamma(s)} &:= \sum_{k=1}^{n} \nu_{\Gamma,k}(s)\mathscr{D}_k = \sum_{k=1}^{n} \nu_{\Gamma,k}(s)\partial_k\,,\\ \vec{\nu}_\Gamma(s) &:= (\nu_{\Gamma,1}(s),\ldots,\nu_{\Gamma,n}(s))\,,\quad s\in\Gamma\,.\end{aligned}\tag{3.11}$$

$\mathscr{D}_{\vec{\nu}_\Gamma(s)}$ is a tangent derivative on the surface $\mathscr{C}$ and is normal to the boundary Γ.

Note, that BVPs (3.9) and (3.10) describe the stationary [(3)] heat transfer process in a thin conductor having the shape of the surface $\mathscr{S}$ (see [Ha1, § 72]).

For the participating functions in (3.9) and (3.10) we suppose

$$\begin{gathered} f\in\mathbb{W}^s_p(\mathscr{S})\,,\qquad \varphi\in\mathbb{W}^{s+1}_p(\mathscr{S})\,,\\ g\in\mathbb{W}^{s-\frac{1}{p}}_p(\Gamma)\,,\qquad h\in\mathbb{W}^{s-1-\frac{1}{p}}_p(\Gamma)\,,\qquad 1<p<\infty\,,\quad s>0\,.\end{gathered}\tag{3.12}$$

Corollary 3.4 *For the Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ on the open surface $\mathscr{C}$ with the boundary $\partial\mathscr{S}:=\Gamma$ the following Green formulae are valid (see* (2.17) *for notations):*

$$(\boldsymbol{\Delta}_{\mathscr{C}}(t,D_t)\varphi,\psi)_{\mathscr{C}} + (\nabla_{\mathscr{C}}\varphi,\nabla_{\mathscr{C}}\psi)_{\mathscr{C}} = -(\mathscr{D}_{\vec{\nu}_\Gamma}\varphi,\psi)_\Gamma\,,\tag{3.13}$$

$$(\boldsymbol{\Delta}_{\mathscr{C}}(t,D_t)\varphi,\psi)_{\mathscr{C}} - (\mathscr{D}_{\vec{\nu}_\Gamma}\varphi^+,\psi^+)_\Gamma = (\varphi,\boldsymbol{\Delta}_{\mathscr{C}}(t,D_t)\psi)_{\mathscr{C}} - (\varphi^+,\mathscr{D}_{\vec{\nu}_\Gamma}\psi^+)_\Gamma\tag{3.14}$$

for arbitrary $\varphi,\psi\in C^\infty(\mathscr{M})$.

Proof. We apply (2.6), (1.14), the second equality in (1.5) and proceed as follows:

$$\begin{aligned}\oint_{\mathscr{C}} \boldsymbol{\Delta}_{\mathscr{C}}(t,D_t)\varphi(t)\overline{\psi(t)}d_t\mathscr{C} &= -\sum_{j=1}^{n}\oint_{\mathscr{C}} \mathscr{D}_j^2\varphi(t)\overline{\psi(t)}d_t\mathscr{C}\\ &= -\sum_{j=1}^{n}\oint_{\Gamma}[\nu_{\Gamma,j}(s)-\nu_j(s)(\vec{\nu}_\Gamma(s),\vec{\nu}(s))]\mathscr{D}_j\varphi(s)\overline{\psi(s)}d_s\Gamma\\ &\quad -\sum_{j=1}^{n}\oint_{\mathscr{C}}\mathscr{D}_j\varphi(t)\overline{\mathscr{D}_j\psi(t)}d_t\mathscr{C} + \oint_{\mathscr{C}}\mathscr{G}_{\mathscr{S}}(t)\sum_{j=1}^{n}\nu_j(t)\mathscr{D}_j\varphi(t)\overline{\psi(t)}d_t\mathscr{C}\\ &= -\oint_{\Gamma}\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi(s)\overline{\psi(s)}\,d_s\Gamma - \oint_{\mathscr{C}}\nabla_{\mathscr{C}}\varphi(t)\overline{\nabla_{\mathscr{C}}\psi(t)}d_t\mathscr{C}\,,\end{aligned}$$

because $(\vec{\nu}_\Gamma(s),\vec{\nu}(s))\equiv 0$. This proves (3.13). (3.14) follows if we apply (3.13) twice and take the difference. ∎

[(3)] We consider the stationary heat conduction only for simplicity. For the time dependent process, which is represented by a Hypoelliptic operator, one can obtain similar results.

Theorem 3.5 *Let* $1 < p < \infty$, $s > 0$. *The Dirichlet problem* (3.9), (3.12) *has a unique solution* $\varphi \in \mathbb{W}^s_2(\mathscr{S})$ *for arbitrary right–hand side* $g \in \mathbb{W}^{s-\frac{1}{p}}_p(\Gamma)$.

The Neumann problem (3.10), (3.12) *has a solution* $\varphi \in \mathbb{W}^s_p(\mathscr{S})$ *only for those right–hand sides* $h \in \mathbb{W}^{s-1-\frac{1}{p}}_p(\Gamma)$ *which satisfy the condition*

$$\oint\limits_{\Gamma} h(s) d_s\Gamma = 0\,. \tag{3.15}$$

If the condition (3.15) *holds, the Neumann problem has a solution* $\varphi_0 \in \mathbb{W}^s_p(\mathscr{S})$ *and a general solution reads* $\varphi = \varphi_0 + \mathrm{const}$.

Proof. First we prove the uniqueness. Taking $g = 0$ ($h = 0$) we should prove that the corresponding homogeneous Dirichlet BVP has only a trivial solution $\varphi = 0$ (the homogeneous Neumann BVP has only a constant solution $\varphi = \mathrm{const}$, respectively). In fact, due to (2.20) these solutions are infinitely smooth and by taking $\psi = \varphi$ in the Green formula (3.13) we get

$$\oint\limits_{\Gamma} |\nabla_{\mathscr{C}}\varphi(s)| d_s\Gamma = 0 \quad\Longrightarrow\quad \nabla_{\mathscr{C}}\varphi(s) \equiv 0 \quad\Longrightarrow \varphi(s) \equiv \mathrm{const} \tag{3.16}$$

(cf. (1.19)). For the Neumann BVP the result is proved. For the Dirichlet BVP we have in addition $\varphi(s) = 0\ \forall s \in \Gamma$ and this yields $\varphi(t) = 0\ \forall t \in \mathscr{C}$.

For the proof of existence we recall that the BVPs (3.9) and (3.10), with conditions(3.12), are Fredholm (see Theorem 2.6.) and their kernels coincide with cokernels. Therefore, by the part proved already, the Dirichlet BVP (3.9) is solvable uniquely, while for solvability of the Neumann problem there must hold the orthogonality condition (3.15) for the data with the solution $v(t) \equiv \mathrm{const}$ of the homogeneous equation (cf. (2.19)). ■

To implement another approach to the investigation of BVPs (3.9) and (3.10) (the **potential method**) we recall that we have proved existence of the fundamental solution $\mathscr{K}_\nu(t, t-\tau)$ to the perturbed Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}} - \nu I$, which is minded as the Schwartz kernel of the inverse operator (see Theorem 3.2). Then any solution the Dirichlet

$$\begin{cases} (\boldsymbol{\Delta}_{\mathscr{C}}(t, \mathscr{D}_t)\varphi)(t) - \nu\varphi(t) = f(t), & t \in \mathscr{C}, \\ \varphi^+(s) = g(s), & s \in \Gamma = \partial\mathscr{C} \end{cases} \tag{3.17}$$

and the Neumann

$$\begin{cases} (\boldsymbol{\Delta}_{\mathscr{C}}(t, \mathscr{D}_t)\varphi)(t) - \nu\varphi(t) = f(t), & t \in \mathscr{C}, \\ (\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi)^+(s) = h(s), & s \in \Gamma = \partial\mathscr{C} \end{cases} \tag{3.18}$$

boundary value problems with the perturbed Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}} - \nu I$, $\nu > 0$, and under the conditions (3.12), is represented as follows

$$\varphi(t) = (N_{\mathscr{S}} f)(t) + (W_\Gamma \varphi^+)(t) - (V_\Gamma(\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi)^+)(t)\,, \qquad t \in \mathscr{S}\,, \tag{3.19}$$

where

$$\begin{aligned}(\mathbf{N}_{\mathscr{S}}f)(t)\varphi(t) &:= \oint\limits_{\mathscr{S}} \mathscr{K}_\nu(t, t-\tau)f(\tau)\, d_\tau\mathscr{S}\,,\\ (\mathbf{W}_\Gamma\psi)(t) &:= \oint\limits_{\Gamma} \left[(\mathscr{D}_{\vec{\nu}_\Gamma(s)}\mathscr{K}_\nu)(t, s-t)\right]^\top \psi^+(s)\, d_s\Gamma\,,\\ (\mathbf{V}_\Gamma\psi)(t) &:= \oint\limits_{\Gamma} \mathscr{K}_\nu(t, t-s)\psi^+(s)\, d_s\Gamma\,, \qquad t\in\mathscr{S}\end{aligned} \tag{3.20}$$

are the volume (Newton), the double and the single layer potentials, respectively.

The proof of (3.19) is standard: by inserting the solution φ of $(\mathbf{\Delta}_{\mathscr{S}} - \nu I)\varphi = f$ and the fundamental solution $\psi = \mathscr{K}_\nu(t, t-\tau)$, $(\mathbf{\Delta}_{\mathscr{S}} - \nu I)\mathscr{K}_\nu(t, t-\tau) = \delta(t-\tau)$ truncated properly around the diagonal $t = \tau$ on the distance $\varepsilon > 0$, into the Green formula (3.14), written also for $\mathbf{\Delta}_{\mathscr{S}} - \nu I$, we get the representation formula (3.19) by sending $\varepsilon \to 0$.

The potentials have the standard mapping properties and for them are valid the standard Plemelj formulae

Theorem 3.6 *Let $1 < p < \infty$, $r \in \mathbb{R}$. Then the direct values of the double and the single layer potential operators are bounded between the spaces:*

$$\begin{aligned}\mathbf{N}_{\mathscr{S}} &: \mathbb{H}^s_p(\mathscr{S}) \longrightarrow \mathbb{H}^{s+2}_p(\mathscr{S})\,,\\ &: \mathbb{W}^s_p(\mathscr{S}) \longrightarrow \mathbb{W}^{s+2}_p(\mathscr{S}) \cap \mathbb{H}^{s+2}_p(\mathscr{S})\,,\\ \mathbf{V}_\Gamma &: \mathbb{H}^s_p(\Gamma) \longrightarrow \mathbb{H}^{s+1+\frac{1}{p}}_p(\mathscr{S})\,,\\ &: \mathbb{W}^s_p(\Gamma) \longrightarrow \mathbb{W}^{s+1+\frac{1}{p}}_p(\mathscr{S}) \cap \mathbb{H}^{s+1+\frac{1}{p}}_p(\mathscr{S})\,,\\ \mathbf{W}_\Gamma &: \mathbb{H}^s_p(\Gamma) \longrightarrow \mathbb{H}^{s+\frac{1}{p}}_p(\mathscr{S})\,,\\ &: \mathbb{W}^s_p(\Gamma) \longrightarrow \mathbb{W}^{s+\frac{1}{p}}_p(\mathscr{S}) \cap \mathbb{H}^{s+\frac{1}{p}}_p(\mathscr{S})\,.\end{aligned} \tag{3.21}$$

The following Plemelj formulae hold for the layer potentials:

$$\begin{aligned}(\mathbf{W}_\Gamma\varphi)^\pm(s) &= \pm\frac{1}{2}\varphi(s) + \mathbf{W}_0(s, \mathscr{D}_s)\varphi(s)\,,\\ (\mathscr{D}_{\vec{\nu}_\Gamma}\mathbf{V}_\Gamma\varphi)^\pm(s) &= \mp\frac{1}{2}\varphi(s) + \mathbf{W}^*_0(s, \mathscr{D}_s)\varphi(s)\,,\\ (\mathbf{V}\varphi)^-(s) &= (\mathbf{V}\varphi)^+(s) = \mathbf{V}_{-1}(s, \mathscr{D}_s)\varphi(s)\,,\\ (\mathscr{D}_{\vec{\nu}_\Gamma}\mathbf{W}\varphi)^-(s) &= (\mathscr{D}_{\vec{\nu}_\Gamma}\mathbf{W}\varphi)^+(s) = \mathbf{W}_{+1}(s, \mathscr{D}_s)\varphi(s)\,,\end{aligned} \tag{3.22}$$

where $\Phi^-(s)$ denotes the trace of $\Phi(t)$ on Γ from the surface $\mathscr{S}^c$, complemented to $\mathscr{S}$ (outer with respect of Γ, which is the common boundary $\Gamma = \partial\mathscr{S} = \partial\mathscr{S}^c$). The operators

$\mathbf{W}_0(s, \mathscr{D}_s)$ *and* $\mathbf{V}_{-1}(s, \mathscr{D}_s)$ *are the direct values of the corresponding double and the single layer potentials on the boundary* Γ *and represent PsDOs of order* ± 1. $\mathbf{W}_0^*(s, \mathscr{D}_s)$ *is the dual (adjoint) PsDO to* $\mathbf{W}_0(s, \mathscr{D}_s)$.

Proof. The proof is verbatim to the case of domains in $\mathbb{R}^n$ and we quote for details [Du2, Fi1, KGBB1, Le, Ma1] etc. ■

Following the "indirect potential method" one looks for a solution of the Dirichlet BVP (3.17) as the double layer potential and for the solution to the Neumann BVP (3.18) as the single layer potential with unknown densities. From boundary conditions we derive appropriate boundary integral equations, which are PsDOs of order 0 (or singular integral operators). These equations are Fredholm and BVPs can be investigated by a standard procedure (see, e.g., [Fi1, KGBB1, Ma1]). We get existence of solution of BVPs (3.17) and (3.18).

Recall that the BVPs (3.9) and (3.10) are Fredholm (see Theorem 2.6.iv) and we know their kernels (see (3.16)). Since BVPs (3.9) and (3.10) are the limits of Fredholm BVPs (3.17) and (3.18), respectively, as $\nu \to 0$, we can derive the second part of Theorem 3.5 on existence of solutions from the existence of solutions to BVPs (3.17) and (3.18).

Following the "direct potential method" one applies the representation formulae (3.19) and reduces BVPs to the equivalent boundary pseudodifferential equations of order -1 for the Dirichlet BVP and of order $+1$ for the Neumann BVP (see [CS1, DNS1, DW1, Ma1] etc.), which are Fredholm in appropriate spaces. Conclusions for BVPs (3.17) and (3.18) and then for BVPs (3.9) and (3.10) are similar as in the foregoing "indirect potential method".

BIBLIOGRAPHY

[CDS1] L.P. Castro, R. Duduchava, F.-O. Speck:
Localization and minimal normalization of mixed boundary value problem. To appear.

[CS1] M. Costabel, E. Stephan, An improved boundary element Galerkin method for three-dimensional crack problems, *Integral Equations and Operator Theory* **10**, 1987, 467-504.

[Du1] R. Duduchava, On multidimensional singular integral operators. I-II: The case of compact manifolds, *J. Oper. Theory*, **11**, 41-76, 199-214, 1984.

[Du2] R. Duduchava, The Green formula and layer potentials, *Integral Equations and Operator Theory* **41**, 2, 2001, 127–178.

[DNS1] R. Duduchava, D. Natroshvili, E. Shargorodsky, Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts I-II, *Georgian Mathematical journal*, **2**, 123–140, 259–276, 1995.

[DW1] R. Duduchava, W. Wendland, The Wiener-Hopf method for systems of pseudodifferential equations and its application to crack problems, *Integral Equations and Operator Theory* **23**, 294-335, 1995.

[Fi1] G. Fichera, Existence theorems in linear and semi-linear elasticity, *Z. Agnew. Math. Mech* **54**, Sonderheft, T24-T36 (1974).

[Gu1] N. Günter, *Potential Theory and its Application to the Basic Problems of Mathematical Physics*, Fizmatgiz, Moscow 1953 (Russian. Translation in French: Gauthier–Villars, Paris 1994).

[Ha1] W. Haack, *Elementare Differentialgeometrie* (German), Lehrbcher und Monographien aus dem Gebiete der exakten Wissenschaften. 20. Basel-Stuttgart: Birkhuser Verlag. VIII, 1955.

[Hn1] H. Han, The boundary integro-differential equations of three-dimensional Neumann problem in linear elasticity, *Numer. Math.*, **68**, No.2, 269-281 (1994).

[KGBB1] V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam 1979 (Russian edition: Nauka, Moscow 1976).

[Le] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*, Teubner, Stuttgart 1986.

[LM1] J.L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, Heidelberg 1972.

[MP1] S. Mikhlin, S. Pröeßdorf, *Singular Integral Operators*, Springer-Verlag., Berlin etc. 1986.

[MMT1] D. Mitrea, M. Mitrea, M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Memoirs Am. Math. Soc. 713, 2001.

[Ma1] V. Maz'ya, *Boundary Integral Equations*, vol. **27** of *Encyclopaedia of Mathematical Sciences, Analysis* 4, Editor R.Gamkrelidze, pp. 127–228, Springer–Verlag, Heidelberg 1992.

[Ra1] V. Rabinovich, Pseudodifferential equations in unbounded regions with conical structure at infinity, *Math. USSR, Sb.* **9**, 73-92, 1969; translation from Mat. Sb., **80(122)**, 77-96, 1969.

[Ro1] Ya.A. Roitberg, *Elliptic Boundary Value Problems in Generalized Functions*, Kluwer Academic Publishers, Dordrecht 1996.

[Si1] I. Simonenko, A new general method of investigating linear operator equations of the type of singular integral equations, *Sov. Math., Dokl.* **5**, 1323-1326, 1964; translation from *Dokl. Akad. Nauk SSSR* **158**, 790–793, 1964.

[Tr1] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig 1995.

[Zo1] V. Zozulja, Boundary integral equations for shells of arbitrary geometry. (Russian, English) *Int. Appl. Mech.*, **34**, No.5, 454-463, 1998; translation from Prikl. Mekh., Kiev 34, No.5, 57-67, 1998.

A.Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M.Alexidze str.
Tbilisi 93, Georgia

E–mail: duduch@rmi.acnet.ge

AMS classification: 47A68, 35J25, 35J55