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COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Vol. 28, Nos. 5 & 6, pp. 869–926, 2003

Asymptotics Without Logarithmic Terms for Crack Problems†

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ABSTRACT

We consider boundary value problems for elliptic systems in a domain complementary to a smooth surface with boundary, which models a crack with its edge. The same boundary conditions are prescribed on both sides of the surface. We prove that the singular functions appearing in the expansion of the solution along the crack edge all have the form $r^{k+1/2}\psi(\theta)$ in local polar coordinates: The logarithmic shadow terms predicted by the general theory do not appear. Moreover, we obtain that, for a smooth right hand side, the jump of the displacement across the crack surface is the product of $r^{1/2}$ with a smooth vector function. We present two different, but complementing, approaches leading to these results, and providing distinct generalizations. The first one is based on a Wiener–Hopf factorization of the pseudodifferential symbol on the surface obtained after reduction of the boundary value problem. The second approach concerns directly the boundary value problem and is based on a closer look at the Mellin symbol at each point of the crack edge.

†Dedicated to Prof. W. L. Wendland for his 65th anniversary.

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DOI: 10.1081/PDE-120021180 0360-5302 (Print); 1532-4133 (Online)

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Key Words: Edge asymptotics; Crack surface; Boundary value problems; Pseudodifferential operators; Wiener–Hopf factorization; Mellin symbol.

INTRODUCTION

Solutions of elliptic partial differential or pseudo-differential equations have many opportunities of being non-smooth: In certain points they can have a singular asymptotics instead of a regular Taylor expansion. This happens for elliptic boundary value problems if the domain has conical or edge singularities, see works by Costabel and Dange (1993a), Dauge (1998), Kondrat'ev (1967), Maz'ya and Plamenevskii (1984), Maz'ya and Rossmann (1988), Nazarov and Plamenevskii (1994), or even if the domain is smooth but the operator has some degeneracy, see Andersson and Chrusciel (1993) and Mazzeo (1992). This also happens for elliptic pseudo-differential equations posed on a domain with boundary, as soon as they do not satisfy the *transmission condition*, see works by Bennish (1993), Chkadna and Duduchava (2001), Costabel and Stephan (1987), Dnduchava and Natroshvili (1998), Duduchava and Wendland (1995), and Eskin (1981).

In the case of degenerate Fuchsian equations or pseudo-differential equations on a smooth n-dimensional manifold $\mathscr{M}$ with boundary, the terms of the solution asymptotics in general have the form

$$r^{\lambda+k} \log^q r \, d(x')$$

where $r = |x_n|$ and x', x_n are the tangential and normal variables to the boundary, respectively. Here the exponent λ belongs to a finite set of complex numbers and k, q are non-negative integers. In the case of elliptic boundary value problems on domains with edges, the solution asymptotics have the more general form

$$r^{\lambda+k} \log^q r \, \psi(\theta, x')$$

where (r, θ, x') are cylindrical coordinates around the edge and λ belongs to a discrete, but in general infinite, set of complex numbers.

In this article, we study a quite general class of elliptic pseudo-differential equations on the manifold $\mathscr{M}$, together with general elliptic boundary value problems posed on $\mathbb{R}^{n+1}\backslash\overline{\mathscr{M}}$. In the latter situation the set of generating exponents λ is reduced to $\{1/2\}$. Our special concern is the absence of logarithmic terms $\log^q r$ in the corresponding asymptotics (this issue is also the topic of Andersson and Chrusciel (1993), for a nonlinear degenerate problem). In fact, these logarithmic terms seem to have good reasons to appear because of resonances between asymptotics due to the principal part of the operator and its Taylor expansion near the boundary $\mathscr{E}$ of $\mathscr{M}$ (or the edge of $\mathbb{R}^{n+1}\backslash\overline{\mathscr{M}}$).

Up to now, the general results on the absence of logarithms concerned the *first term* in the asymptotics, those generated by the principal part of the operator only. For scalar ΨDO see Eskin (1981), for systems of ΨDO see Chkadua and

Duduchava (2001) and for Agmon–Douglis–Nirenberg systems see Costabel and Dauge (2002): Logarithms are absent from the first asymptotic term

- For systems of classical ΨDO with principal symbol $\mathbf{a}_0 = \mathbf{a}_0(\mathscr{x}', \mathscr{x}_n; \xi', \xi_n)$ if $\mathbf{a}_0(\mathscr{x}', 0\,; 0, +1)^{-1}\mathbf{a}_0(\mathscr{x}', 0\,; 0, -1)$ is diagonal for all $\mathscr{x}' \in \mathscr{E}$,
- For Agmon–Douglis–Nirenberg systems if the same boundary conditions are applied on both sides of the crack surface $\mathscr{M}$.

In the present work, we exhibit quite general conditions for the *total* absence of these logarithmic terms in the *whole asymptotics* for both families of problems: In Part B for pseudo-differential operators and in Part C for boundary value problems. It turns out that both results can be applied to a sub-class of these boundary value problems which is of high practical interest: This is explained in Part A. The most important model in this class is the crack problem in three-dimensional linear elasticity, either isotropic or anisotropic: There the boundary conditions are Neumann, i.e., tractions are prescribed on both faces of the crack surface.

Part A: Elasticity-like operators. We consider in this part homogeneous second order coercive systems with constant coefficients in the domain $\Omega := \mathbb{R}^{n+1}\backslash\overline{\mathscr{M}}$, associated with Dirichlet or Neumann boundary conditions on both sides of the crack surface $\mathscr{M}$. The prototype of such operators is the system of linear elasticity. We motivate and illustrate the more general results obtained in the rest of the article by their application to this case. For such operators we can indeed apply both approaches (PDE or ΨDO) to obtain that the asymptotics of solutions around the crack edge $\mathscr{E}$ is logarithm free:

- Either we reduce the problem to a pseudo-differential equation on $\mathscr{M}$ and we prove that the symbol of this equation satisfies our "*continuity property*" which ensures the absence of logarithm for its solution; the asymptotics of the solution in the full space is then deduced by a representation formula from the asymptotics of the solution on $\mathscr{M}$.
- Or we apply directly our result on general Agmon–Douglis–Nirenberg systems, based on an investigation of properties of the Mellin symbol at each point of the crack edge $\mathscr{E}$.

Part B: ΨDO. We consider classical $N \times N$ matrix symbols $\mathbf{a}(\mathscr{x}; \xi) = \mathbf{a}_0(\mathscr{x}; \xi) + \mathbf{a}_1(\mathscr{x}; \xi) + \cdots$ of order $\nu \in \mathbb{R}$, defined on the cotangent manifold $\mathscr{T}^*\mathscr{M}$, and with elliptic principal symbols $\mathbf{a}_0$. The asymptotics of solutions to the equation

$$\mathbf{a}(\mathscr{x}, D_{\mathscr{x}})\, \boldsymbol{\phi}(\mathscr{x}) = \boldsymbol{g}(\mathscr{x}), \quad \mathscr{x} \in \mathscr{M}$$

with a smoth $\boldsymbol{g}$, contains logarithms in general. We introduce what we call "*generalized continuity property*" which states that there exists a non-zero complex number Λ such that for all $\mathscr{x}' \in \mathscr{E}$, $j \in \mathbb{N}_0$, $\alpha' \in \mathbb{N}_0^{n-1}$, $m \in \mathbb{N}_0$:

$$\partial^m_{\mathscr{x}_n}\partial^{\alpha'}_{\xi'}\mathbf{a}_j(\mathscr{x}', 0\,; 0, -1) = (-1)^{j+|\alpha'|}\Lambda\, \partial^m_{\mathscr{x}_n}\partial^{\alpha'}_{\xi'}\mathbf{a}_j(\mathscr{x}', 0\,; 0, +1).$$

We prove that under this condition, the asymptotics of $\boldsymbol{\phi}$ only contains terms of the form (where the variable r coincides with $|\mathscr{x}_n|$)

$$r^{(\nu/2)+\delta+k} \quad \text{with} \quad k \in \mathbb{N}_0 \quad \text{and} \quad \delta \in \mathbb{C} \ \text{ such that } \ e^{2i\pi\delta} = \Lambda.$$

For integer ν and $\Lambda = (-1)^{\nu}$, our condition coincides with the usual *transmission condition*, see Boutet de Monvel (1971)—note that, in this case, the exponents $\nu/2 + \delta$ are integer. For $\Lambda = 1$, this condition is our *continuity property* which we prove to be satisfied with $\nu = 1$ or -1 by the symbols obtained after reduction to the boundary of a second order elliptic system with constant coefficients—the exponents $\nu/2 + \delta$ are then integer translates of $1/2$.

Part C: Boundary value problems. We study directly elliptic boundary value problems on Ω by means of a two-level representation formula for the edge asymptotics of their solutions. The first level is the classical Cauchy integral involving the inverse Mellin symbol at each point of the crack edge $\mathscr{E}$. The second level, that we call Caley representation formula, concerns the angular variable θ: The Mellin symbol is proved to act between special subspaces of angular functions. We prove that this fact precludes the appearance of logarithmic terms. This approach yields logarithmic free asymptotics for any *Agmon–Douglis–Nirenberg system* with smooth coefficients (provided a classical ellipticity condition holds along the edge, cf Maz'ya and Plamenevskii (1980) and Schulze (1998), which ensures the existence of a general asymptotic expansion).

PART A. SCOPE AND COMMON PRINCIPAL RESULTS

A.I. The Crack Domain and the Elasticity-like Boundary Value Problem

Let $\mathscr{M}$ be a bounded $\mathscr{C}^\infty$ orientable surface of codimension 1 in $\mathbb{R}^{n+1}$. We assume that the boundary $\mathscr{E}$ of $\mathscr{M}$ is $\mathscr{C}^\infty$. Let

$$\Omega := \mathbb{R}^{n+1} \backslash \overline{\mathscr{M}}.$$

be the domain where the boundary value problems are set. For the equations of *linear elasticity* (with isotropic or anisotropic material law), the solutions of such boundary value problems yield the stresses in the domain Ω around $\mathscr{M}$ which represents a *crack* with front $\mathscr{E}$. For the equations of *electromagnetism* (Helmholtz or Maxwell), the solutions represent the diffracted field around the *screen* $\mathscr{M}$.

We are going to set our problem and describe our results in a framework including such problems, which is also covered by the hypotheses of our two methods.

We denote by $\boldsymbol{x} = (x_1, \ldots, x_{n+1})$ cartesian coordinates in $\mathbb{R}^{n+1}$ and by $\partial_{\boldsymbol{x}}^{\alpha}$ the partial derivative $\partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}$. Let b be a homogeneous integrodifferential form of degree 1 with constant coefficients acting on N component vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{H}^1(\Omega)^N$

$$b(\boldsymbol{u}, \boldsymbol{v}) = \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{|\alpha|, |\beta|=1} \int_{\Omega} a_{jk}^{\alpha\beta} \, \partial_{\boldsymbol{x}}^{\alpha} u_j \, \partial_{\boldsymbol{x}}^{\beta} \overline{v}_{\mathbf{k}} \, d\boldsymbol{x}.$$

Here $\boldsymbol{u} = (u_1, \ldots, u_N)$, $\boldsymbol{v} = (v_1, \ldots, v_N)$. The coefficients $a_{j,k}^{\alpha,\beta}$ are supposed constant. We assume that the form b is coercive on $\mathrm{H}^1(\Omega)^N$, i.e., that for some constants c, $C > 0$ there holds

$$\forall \boldsymbol{u} \in \mathrm{H}^1(\Omega)^N, \quad \mathrm{Re}\, b(\boldsymbol{u}, \boldsymbol{u}) + C\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}^2 \geq c\|\boldsymbol{u}\|_{\mathrm{H}^1(\Omega)}^2. \qquad (\mathfrak{H}_{\mathrm{A1}})$$

Moreover, we suppose that b is symmetric on $\mathrm{H}^1(\Omega)^N$:

$$\forall \boldsymbol{u}, \ \boldsymbol{v} \in \mathrm{H}^1(\Omega)^N, \quad b(\boldsymbol{u}, \boldsymbol{v}) = \overline{b(\boldsymbol{v}, \boldsymbol{u})}. \qquad (\mathfrak{H}_{\mathrm{A2}})$$

The partial differential operator associated with the form b is

$$L = (L_{jk})_{j,k} \quad \text{with} \quad L_{jk} = - \sum_{|\alpha|, |\beta|=1} \partial_{\boldsymbol{x}}^{\beta} \, a_{jk}^{\alpha\beta} \, \partial_{\boldsymbol{x}}^{\alpha}.$$

Hypotheses $(\mathfrak{H}_{\mathrm{A1}})$ and $(\mathfrak{H}_{\mathrm{A2}})$ are satisfied for the Laplace equation ($N = 1$), for the equations of general elasticity, including the anisotropic case, (N is equal to the dimension of the space) and for equations of thermoelasticity and electroelasticity (N is the dimension of the space plus 1).

Since $\mathscr{M}$ is orientable, we can define a smooth unit normal vector field $\boldsymbol{n}$ on $\mathscr{M}$, which is unique if we choose the direction of the normal at some fixed point. After fixing the field $\boldsymbol{n}$ we can fix the traces $\gamma_{\pm}$, taking γ_{+} opposite to the direction of $\boldsymbol{n}$ (i.e., from "above" if we consider $\boldsymbol{n}$ as pointing upward) and taking γ_{-} in the direction of $\boldsymbol{n}$ (i.e., from "below").

The Neumann operator T associated with b and the normal field $\boldsymbol{n}$ is defined as $T = (T_{jk})_{j,k}$ with

$$T_{jk} = \sum_{|\alpha|, |\beta|=1} \boldsymbol{n}^{\beta} \, a_{jk}^{\alpha\beta} \, \partial_{\boldsymbol{x}}^{\alpha}, \quad \boldsymbol{n}^{\beta} = n_1^{\beta_1} \cdots n_{n+1}^{\beta_{n+1}}.$$

Let B denote either the identity (which will be associated with the Dirichlet operator) or the Neumann operator T on $\mathscr{M}$. We consider solutions $\boldsymbol{u} \in \mathrm{H}^1(\Omega)^N$ of the problem

$$\begin{cases} L\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \gamma_{\pm} B\boldsymbol{u} = 0 & \text{on } \mathscr{M}, \end{cases} \qquad (\mathrm{A.1})$$

with, possibly, conditions at infinity (note that we may relax the condition $\boldsymbol{u} \in \mathrm{H}^1(\Omega)^N$ into $\boldsymbol{u} \in \mathrm{H}^1(\Omega \cap \mathscr{B}_R)^N$ for any $R > 0$, with $\mathscr{B}_R$ the ball of center 0 and radius R). We assume that $\boldsymbol{f}$ is a $\mathscr{C}^{\infty}$ vector function on $\mathbb{R}^{n+1}$, with compact support.

A.II. State of the Art and Motivations

Due to the presence of the edge $\mathscr{E}$, the domain is highly non-smooth and this yields strong singularities for the solutions of problem (A.1) along this edge. The general structure of these singularities is known, and addressed by many works, see Chkadua and Duduchava (2000), Costabel and Dauge (1993a), Dauge (1988), Duduchava and Wendland (1995), Grisvard (1985), Kondrat'ev (1970), Maz'ya amd Rossmann (1988), Nazarov and Plamenevskii (1994), Nazarov and Plamenevskii (1995a), and Nikishkin (1979). The generic form of these singularities is

$$c(\mathscr{x}')\sum_{q=0}^{Q} r^{\lambda(\mathscr{x}')}\log^q r\,\boldsymbol{\psi}_q(\mathscr{x}',\theta)$$

where $\mathscr{x}'$ represents coordinates in $\mathscr{E}$, and (r,θ) polar coordinates in the planes normal to $\mathscr{E}$, centered on $\mathscr{E}$. The structure of $\boldsymbol{u}$ in a neighborhood of the boundary $\mathscr{E}$ of $\mathscr{M}$ is very important in applications.

For example, in **elasticity**, the asymptotics of $\boldsymbol{u}$ provides essential data for the investigation of crack propagation in the quasi-static case. The propagation criterion is based on the stress intensity factors (the coefficients $c(\mathscr{x}')$ of the leading terms in asymptotics) and on the "polarization operator" (which involves the second terms in asymptotics), see Nazarov and Plamenevskii (1995b). But the application of these tools requires that the asymptotic expansion of $\boldsymbol{u}$ contains neither oscillatory terms (i.e., *non real exponents* λ) nor logarithmic terms (i.e., $\log^q r$ with $q \geq 1$).

- Concerning oscillations, it is known that the solution of the crack problems never oscillates provided the crack is inside an homogeneous material, even if the material is anisotropic, see Duduchava and Wendland (1995) and Nazarov and Plamenevskii (1994).
- Concerning logarithms, although absence of logarithms in the leading terms was known long ago for isotropic materials (Grisvard, 1989; Nazarov and Plamenevskii, 1994), the same was not proved for further terms, where logarithms could appear as shadow singularities.

The main scope of the present investigation is to establish that the structure of the solution $\boldsymbol{u}$ of the general problem (A.1) is *simpler* than the general theory would predict. The main result can be summarized in one sentence:

"*The edge asymptotics of* $\boldsymbol{u}$ *does not contain any logarithmic term* $\log r$."

Still in the framework of elasticity, this was observed in the case of a curved crack in the isotropic elastic plane $\mathbb{R}^2$ for the second term in the asymptotics in Wendland and Stephan (1990, Theorem 2.4) and in the case of a half plane crack $\mathbb{R}^2_+$ in the anisotropic elastic space $\mathbb{R}^3$ in Duduchava and Wendland (1995, Theorem 4.3); For curved cracks the conjecture was first formulated by S. A. Nazarov.

Moreover, it has been shown Costabel and Dauge (2002) and Duduchava and Natroshvili (1998), that even in the very general framework of Agmon–Douglis–

Nirenberg systems with the same boundary conditions on both sides of the crack $\mathscr{M}$, the *principal part of the asymptotics* contains only powers of r with half-integer exponents (i.e., $\lambda = 1/2 + k$, $k \in \mathbb{N}_0$), and without any $\log r$ term, see also Kozlov (1990), for scalar operators of order $2m$ with Dirichlet condition. In this work, we prove that, in fact, this simple structure extends to the *complete asymptotics*.

The result that the whole asymptotics does not contain $\log r$ terms is by no way obvious, and is not an easy consequence of the simple structure of its principal part. Indeed, because the exponents $1/2 + k$ of the whole asymptotics are all translated from each other by integers, we should expect $\log r$ terms, due to the interaction between the non-principal terms in the operator and the principal singularities (see, for example, Kozlov et al. (2001, Remark 10.5.1), where this interaction is explained).

A.III. Reduction to the Crack Surface and Representation Formulas

One of the essential features of our crack-type boundary value problem (A.1) is that *all information* on the singular behavior of $\boldsymbol{u}$ is contained in an N-component vector function $\boldsymbol{\phi}$, defined on the crack surface by the jump of $\boldsymbol{u}$ across $\mathscr{M}$

$$\boldsymbol{\phi} = [C\boldsymbol{u}] := \gamma_+ C\boldsymbol{u} - \gamma_- C\boldsymbol{u}$$

where C denotes the complementing trace of B, i.e., the Dirichlet trace if B is Neumann and $C = T$ if B is Dirichlet. Of course, the asymptotics of $\boldsymbol{u}$ will yield the asymptotics of $\boldsymbol{\phi}$. But even more important is that $\boldsymbol{\phi}$ can be directly obtained as the solution of a pseudodifferential equation on $\mathscr{M}$ of the form

$$\mathbf{a}(\mathscr{x}, D_{\mathscr{x}})\,\boldsymbol{\phi}(\mathscr{x}) = \boldsymbol{g}(\mathscr{x}), \quad \mathscr{x} \in \mathscr{M}, \tag{A.2}$$

and analyzed in this respect. The relation between the boundary value problem (A.1) and the pseudodifferential Eq. (A.2) will be fully explained in § B.II. Let us only mention that in the case of the Dirichlet problem $\mathbf{a} = V$, where $V := \gamma_+ \mathscr{V} = \gamma_- \mathscr{V}$ is the trace of the single layer potential $\mathscr{V}$ associated with the operator L, and in the case of the Neumann problem, $\mathbf{a} = W$, where $W := \gamma_- T\mathscr{D} = \gamma_+ T\mathscr{D}$ is the Neumann trace of the double layer potential $\mathscr{D}$. Then $\boldsymbol{u}$ can be reconstructed by the representation formula

$$\forall \boldsymbol{x} \in \Omega, \quad \boldsymbol{u}(\boldsymbol{x}) = N\boldsymbol{f}(\boldsymbol{x}) + \mathscr{D}[\boldsymbol{u}](\boldsymbol{x}) - \mathscr{V}[T\boldsymbol{u}](\boldsymbol{x}), \tag{A.3}$$

where $[\boldsymbol{u}] := \gamma_+ \boldsymbol{u} - \gamma_- \boldsymbol{u}$ and $[T\boldsymbol{u}] := \gamma_+ T\boldsymbol{u} - \gamma_- T\boldsymbol{u}$ denote the jumps of the functions $\boldsymbol{u}(\boldsymbol{x})$ and $T\boldsymbol{u}(\boldsymbol{x})$ across the surface $\mathscr{M}$ and N denotes the Newton (volume) potential. Thus, the asymptotics of $\boldsymbol{u}$ depends only on $\boldsymbol{\phi}$ because the volume potential part $N\boldsymbol{f}$ is smooth and $[B\boldsymbol{u}] = 0$ on $\mathscr{M}$.

Note that the coerciveness hypothesis ($\mathfrak{H}_{\mathrm{A1}}$) ensures the Fredholm property of both problems (A.1) and (A.2) in appropriate spaces.

Thus, two different approaches are available to us: either first study the solution $\boldsymbol{\phi}$ of equation (A.2), then derive the asymptotics of $\boldsymbol{u}$, or first study the solution $\boldsymbol{u}$ of problem (A.1), then derive the asymptotics of $\boldsymbol{\phi} = [C\boldsymbol{u}]$.

First approach. The results are a consequence of Part B where we develop the potential operator technique based on the Wiener–Hopf factorization, according to the three main following steps:

Step 1 The boundary value problem (A.1) is reduced to a pseudodifferential equation of type (A.2) on the crack surface $\mathscr{M}$ by invoking the representation of solutions (A.3), see § B.II.

Step 2 Asymptotics of solutions $\boldsymbol{\phi}$ of the pseudodifferential equation on the crack surface are found using the Wiener–Hopf factorization, see § B.III–§ B.VII.

Step 3 By inserting the surface asymptotics into the representation formula (A.3), the full spatial asymptotic expansion of $\boldsymbol{u}$ is derived, see § B.IX.

G. Eskin (1981) was the first who applied the Wiener–Hopf factorization to investigations of asymptotics. The method received then many contributions (Bennish, 1993; Chkadua and Duduchava, 1998; 2000; 2001; Costabel and Stephan, 1987; Duduchava and Natroshvili, 1998; Duduchava and Wedland, 1995).

Second approach. It is developed in Part C: it relies on the classical Mellin transform, cf Kondrat'ev (1967), and more recent representation formulas for the angular part of singular functions, cf Costabel and Dauge (1994). The main steps are:

Step 1 By separation of variables and Mellin transform in r, the problem is transformed into systems of ordinary differential equations in the angular variable θ with the parameters $\mathscr{x}'$ and λ, the dual variable of r, see § C.I.

Step 2 The solutions of these systems are represented by contour integrals around the unit circle with the *Cayley symbols* of the principal part of the operator, see § C.IV.

Step 3 By the Cayley representation formulae, the condition of absence of logarithm is reduced to compatibility conditions between traces of a series of right hand sides in the Mellin calculus, see § C.V.

A.IV. Results

In order to state our results, let us introduce *local coordinates* in a neighborhood of the edge $\mathscr{E}$ which is the crack front.

Definition A.1.

(i) Let $\mathscr{x}' = (\mathscr{x}_1, \ldots, \mathscr{x}_{n-1})$ denote local coordinates in $\mathscr{E}$.

(ii) For $\mathscr{x}' \in \mathscr{E}$, let $\Pi_{\mathscr{x}'}$ denote the normal plane to $\mathscr{E}$ containing $\mathscr{x}'$. We take polar coordinates (r, θ) in $\Pi_{\mathscr{x}'}$ such that $r = 0$ is the intersection $\Pi_{\mathscr{x}'} \cap \mathscr{E}$, $\theta = -\pi$ is $\Pi_{\mathscr{x}'} \cap \mathscr{M}$ from below and $\theta = \pi$ is $\Pi_{\mathscr{x}'} \cap \mathscr{M}$ from above.
(iii) We set $\mathscr{x}_n = r\cos\theta$ and $\mathscr{x}_{n+1} = r\sin\theta$. The n coordinates $(\mathscr{x}', \mathscr{x}_n)$ are local coordinates in $\mathscr{M}$ and the $n+1$ coordinates $\underline{\mathscr{x}} := (\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1})$ are local coordinates in Ω in a neighborhood of $\mathscr{E}$.
(iv) The local cylindrical coordinates are $(\mathscr{x}', r, \theta)$ and we shall use $(\mathscr{x}', r, 0) = \mathscr{x} \in \mathscr{M}$ and $(\mathscr{x}', 0, 0) = \mathscr{x}' \in \mathscr{E}$.
(v) The dual variables of $\underline{\mathscr{x}} = (\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1})$ are denoted by $\underline{\xi} = (\xi', \xi_n, \xi_{n+1})$.
(vi) We denote by κ: $\underline{\mathscr{x}} \mapsto \boldsymbol{x}$ the generic map of an atlas on $\mathscr{M}$, and by $\mathscr{J}_\kappa(\underline{\mathscr{x}}) := [D\kappa(\underline{\mathscr{x}})^\top]^{-1}$, the inverse of its Jacobian matrix. ■

From the combination of general edge asymptotics (Costabel and Dauge, 1993a; Dauge, 1988; Maz'ya and Plamenevskii, 1980; Maz'ya and Rossmann, 1988; Nazarov and Plamenevskii, 1994), and of the particular structure of the principal part for crack problems (Costabel and Dauge, 2002; Daduchava and Wendland, 1995), we may derive that there holds the following general statement, see § B.2 and C.2.

Proposition A.2.

(i) *Any solution* $\boldsymbol{u}$ *of the boundary value problem* (A.1) *with a smooth right hand side* $\boldsymbol{f}$ *has the following asymptotic expansion as* $r \to 0$: *For any integer* $K \geq 0$

$$\begin{aligned}\boldsymbol{u} = &\sum_{j=1}^{N} c_j^0(\mathscr{x}')\, r^{1/2} \boldsymbol{\psi}_j^0(\mathscr{x}', \theta)\\ &+ \sum_{k=1}^{K} \sum_{q=0}^{q(k)} \sum_{j=1}^{j(k)} c_j^{k,q}(\mathscr{x}')\, r^{(1/2)+k} \log^q r\, \boldsymbol{\psi}_j^{k,q}(\mathscr{x}', \theta)\\ &+ \boldsymbol{u}_{\mathrm{reg},K} + \boldsymbol{u}_{\mathrm{rem},K}. \end{aligned} \tag{A.4}$$

The coefficients c_j^0, $c_j^{k,q}$ *are* $\mathscr{C}^\infty(\mathscr{E})$ *functions depending on* $\boldsymbol{f}$. *The regular part* $\boldsymbol{u}_{\mathrm{reg},K}$ *is a linear combination of terms of the form* $c(\mathscr{x}')\, p(\mathscr{x}_n, \mathscr{x}_{n+1})$, *with polynomial* p *and* $\mathscr{C}^\infty(\mathscr{E})$ *coefficient* c. *The remainder* $\boldsymbol{u}_{\mathrm{rem},K}$ *satisfies* $\partial^\beta \boldsymbol{u}_{\mathrm{rem},K} = \mathscr{o}\,(r^{K-|\beta|+1/2})$ *as* $r \to 0$ *for any multi-index* $\beta \in \mathbb{N}_0^{n+1}$. *The* $\boldsymbol{\psi}_j^0$ *and* $\boldsymbol{\psi}_j^{k,q}$ *are* N*-component vector functions in* $\mathscr{C}^\infty([-\pi, \pi] \times \mathscr{E})$ *and depend only on the domain* Ω *and the operators* (L, B).

(ii) *Any solution* $\boldsymbol{\phi} = [C\boldsymbol{u}]$ *of the pseudodifferential Eq.* (A.2) *with a smooth right hand side* $\mathbf{g}$ *has the following asymptotic expansion as* $r \to 0$: *For any integer* K

$$\begin{aligned}\boldsymbol{\phi} = &\, r^{\nu/2}\, \boldsymbol{d}^0(\mathscr{x}') + \sum_{k=1}^{K} \sum_{q=0}^{q(k)} r^{(\nu/2)+k} \log^q r\, \boldsymbol{d}^{k,q}(\mathscr{x}')\\ &+ \boldsymbol{\phi}_{\mathrm{rem},K}. \end{aligned} \tag{A.5}$$

Here ν *is the order of the pseudodifferential operator* $\mathbf{a}$. *The* $\boldsymbol{d}^0$ *and* $\boldsymbol{d}^{k,q}$ *are* N*-component vector functions in* $\mathscr{C}^\infty(\mathscr{E})$. *The remainder* $\boldsymbol{\phi}_{\mathrm{rem},K}$ *satisfies* $\partial^\beta \boldsymbol{\phi}_{\mathrm{rem},K} = \mathscr{o}\,(r^{K-|\beta|+\nu/2})$ *as* $r \to 0$ *for any multi-index* $\beta \in \mathbb{N}_0^{n+1}$.

Our main result in this article is that there are no logarithmic terms at all in expansions (A.4) and (A.5):

Theorem A.3.

(i) *Any solution* $\boldsymbol{u}$ *of the boundary value problem* (A.1) *with smooth right hand side* $\boldsymbol{f}$ *has the following asymptotic expansion as* $r \to 0$: *For any integer* $K \geq 0$

$$\boldsymbol{u} = \sum_{j=1}^{N} c_j^0(x')\, r^{1/2} \boldsymbol{\psi}_j^0(x', \theta) + \sum_{k=1}^{K} \sum_{j=1}^{j(k)} c_j^k(x')\, r^{(1/2)+k}\, \boldsymbol{\psi}_j^k(x', \theta) + \boldsymbol{u}_{\mathrm{reg},K} + \boldsymbol{u}_{\mathrm{rem},K}. \tag{A.6}$$

The scalar coefficients c_j^k *belong to* $\mathscr{C}^\infty(\mathscr{E})$ *and depend on* $\boldsymbol{f}$, *while the N-component vector functions* $\boldsymbol{\psi}_j^k$ *depend only on the domain* Ω *and the operators* (L, B).

(ii) *Any solution* $\boldsymbol{\phi} = [C\boldsymbol{u}]$ *of the pseudodifferential equation* (A.2) *with smooth right hand side* $\mathbf{g}$ *has the following asymptotic expansion as* $r \to 0$: *For any integer* $K \geq 0$

$$\boldsymbol{\phi} = r^{\nu/2}\, \boldsymbol{d}^0(x') + \sum_{k=1}^{K} r^{(\nu/2)+k}\, \boldsymbol{d}^k(x') + \boldsymbol{\phi}_{\mathrm{rem},K}. \tag{A.7}$$

The coefficients $\boldsymbol{d}^0$ *and* $\boldsymbol{d}^{k,q}$ *are N-component vector functions in* $\mathscr{C}^\infty(\mathscr{E})$.

This result is a consequence of the more general results that we prove in Parts B and C. Moreover both approaches allow precise representation formulas for the "angular" vector functions $\boldsymbol{\psi}_j^k(x', \theta)$ as linear combinations of simple trigonometric functions, see §B.IX and C.VII.

Because of the relation $\boldsymbol{\phi} = [C\boldsymbol{u}]$ between $\boldsymbol{u}$ and $\boldsymbol{\phi}$, it is quite simple to link the first terms in expansions (A.6) and (A.7).

- For Neumann: $C = \mathrm{Id}$ and:

$$\boldsymbol{d}^0(x') = \sum_{j=1}^{N} c_j^0(x') \left[\boldsymbol{\psi}_j^0(x', \theta)\right]_\pi,$$

  where $[\boldsymbol{\psi}(\theta)]_\pi$ denotes the jump $\boldsymbol{\psi}(\pi) - \boldsymbol{\psi}(-\pi)$.
- For Dirichlet: $C = T$, and let $r^{-1} T_0(x', \theta; r\partial_r, \partial_\theta) + T_1(x')\partial_{x'}$ be the expression of T in cylindrical coordinates. Then there holds

$$\boldsymbol{d}^0(x') = \sum_{j=1}^{N} c_j^0(x') \left[T_0\left(x', \theta; \frac{1}{2}, \partial_\theta\right) \boldsymbol{\psi}_j^0(x', \theta)\right]_\pi.$$

- Defining $\boldsymbol{s}_j^0(x') \in \mathscr{C}^\infty(\mathscr{E}) \otimes \mathbb{C}^N$ by

$$\begin{cases} \boldsymbol{s}_j^0(x') = \left[\boldsymbol{\psi}_j^0(x', \theta)\right]_\pi & \text{if } C = \mathrm{Id}, \\ \boldsymbol{s}_j^0(x') = \left[T_0\left(\theta, x'; \frac{1}{2}, \partial_\theta\right) \boldsymbol{\psi}_j^0(x', \theta)\right]_\pi & \text{if } C = T, \end{cases}$$

we get the common relation

$$\boldsymbol{d}^0(\mathscr{x}') = \sum_{j=1}^{N} c_j^0(\mathscr{x}')\, \boldsymbol{s}_j^0(\mathscr{x}'). \tag{A.8}$$

The vectors $\boldsymbol{s}_j^0(\mathscr{x}')$, $j = 1, \dots, N$ are independent of the right hand side and *form a basis of* $\mathbb{C}^N$ *for each fixed* $\mathscr{x}'$. We will address in a forthcoming article formulae and numerical methods for computing the scalar coefficients $c_j^0(\mathscr{x}')$.

Conversely, as a consequence of the representation formula (A.3), we obtain the inverse relation between the coefficients involved in Eq. (A.8): all coefficients $c_j^0(\mathscr{x}')$ are defined as a composition of some matrices with $\boldsymbol{d}^0(\mathscr{x}')$, see Chkadua and Duduchava (2000).

A.V. Modular Representation

The asymptotics (A.6) and (A.7) give the possibility of representing $\boldsymbol{u}$ and $\boldsymbol{\phi}$ as finite linear combination of non-smooth functions with smooth coefficients: As a straightforward consequence of (A.7), we obtain the following factorization of the density $\boldsymbol{\phi}$:

Corollary A.4. *Any solution* $\boldsymbol{\phi}$ *of the boundary pseudodifferential Eq.* (A.2) *with a smooth right hand side* $\boldsymbol{g}$ *satisfies*

$$r^{-\nu/2}\, \boldsymbol{\phi} \in \mathscr{C}^\infty(\overline{\mathscr{M}})^N. \tag{A.9}$$

As a further consequence of the expansion (A.6), we can prove that a simple splitting of $\boldsymbol{u}$ holds in local cylindrical coordinates. For this, we first introduce $\mathscr{U}$, a closed tubular neighborhood of the edge $\mathscr{E}$ where the local cartesian coordinates are well defined. We may take $\mathscr{U}$ as a set of the form

$$\mathscr{U} = \left\{(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1});\, r \le r_0,\ \mathscr{x}' \in \mathscr{E}\right\}.$$

Then we denote by $\breve{\mathscr{U}}$ its expression in local cylindrical coordinates

$$\breve{\mathscr{U}} = \left\{(\mathscr{x}', r, \theta);\, 0 \le r \le r_0,\ \theta \in [-\pi, \pi],\ \mathscr{x}' \in \mathscr{E}\right\}.$$

Note that we distinguish the two faces $\theta = -\pi$ and $\theta = \pi$ of $\breve{\mathscr{U}}$.

Corollary A.5. *Let* $\boldsymbol{u}$ *be any solution of the problem* (A.1) *with a smooth right hand side* $\boldsymbol{f}$ *and denote by* $\breve{\boldsymbol{u}}$ *its expression in local cylindrical coordinates*: $\boldsymbol{u}(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1}) = \breve{\boldsymbol{u}}(\mathscr{x}', r, \theta)$. *Then* $\breve{\boldsymbol{u}}$ *admits a splitting in two parts*

$$\breve{\boldsymbol{u}}(\mathscr{x}', r, \theta) = \breve{\boldsymbol{u}}_0(\mathscr{x}', r, \theta) + r^{1/2}\, \breve{\boldsymbol{u}}_1(\mathscr{x}', r, \theta), \tag{A.10}$$

where $\breve{\boldsymbol{u}}_0$ *and* $\breve{\boldsymbol{u}}_1$ *are* $\mathscr{C}^\infty(\breve{\mathscr{U}})$ *in the variables* r, θ *and* $\mathscr{x}'$.

Now, we write Eq. (A.10) in local cartesian coordinates and obtain

$$\boldsymbol{u}(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1}) = \boldsymbol{u}_0(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1}) + r^{1/2}\, \boldsymbol{u}_1(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1}). \tag{A.11}$$

The part $\boldsymbol{u}_0$ is in fact $\mathscr{C}^\infty(\overline{\mathscr{U}})$ in the coordinates $(\mathscr{x}', \mathscr{x}_n, \mathscr{x}_{n+1})$. Now we may wonder if $\boldsymbol{u}_1$ is also a $\mathscr{C}^\infty(\overline{\mathscr{U}})$ function. This is not true. For example, for the Laplace operator with Dirichlet boundary conditions we have $u_1 = c_1 \sin(1/2)\theta + c_2 r \sin(3/2)\theta + \cdots$ Replacing the factor $r^{1/2}$ by another function does not help. We need to split $r^{1/2}u_1$ into new parts. Again, when $L = \Delta$ and $n = 2$, we simply have

$$r^{1/2}\, u_1 = c_1(\zeta^{1/2} - \bar{\zeta}^{1/2}) + c_2(\zeta^{3/2} - \bar{\zeta}^{3/2}) + c_3(\zeta^{5/2} - \bar{\zeta}^{5/2}) \cdots$$

with $\zeta = re^{i\theta}$. Therefore

$$r^{1/2}\, u_1 = \zeta^{1/2}(c_1 + c_2\zeta + c_3\zeta^2 + \cdots) + \bar{\zeta}^{1/2}(c_1 + c_2\bar{\zeta} + c_3\bar{\zeta}^2 + \cdots)$$

which means that $r^{1/2}u_1$ can be written as $\zeta^{1/2}u_1' + \bar{\zeta}^{1/2}u_2'$ with $\mathscr{C}^\infty(\overline{\mathscr{U}})$ functions u_1' and u_2'. This result extends to the wider class of problems satisfying hypotheses $(\mathfrak{H}_{A1})$ and $(\mathfrak{H}_{A2})$, provided a condition on the symbol of the interior operator L: the symbol $\xi \mapsto L(\xi)$ of L is defined so that $L = L(D_{\boldsymbol{x}})$, where $D_x = i\partial_x$. We require that this symbol satisfies,

$$\begin{cases} \forall \mathscr{x}' \in \mathscr{E}, \quad \text{the roots } \tau \in \mathbb{C} \text{ of} \\ \det L\big(\mathscr{J}_\kappa(\mathscr{x}')(0,1,\tau)\big) = 0 \quad \text{are simple,} \end{cases} \qquad (\mathfrak{H}_{A3})$$

where we recall, cf Definition A.1, that $\mathscr{x}'$ stands for $\underline{\mathscr{x}} = (\mathscr{x}', 0, 0)$ and $(0, 1, \tau)$ is the value of the dual variable $\underline{\xi} = (\xi', \xi_n, \xi_{n+1})$. Note that $L\big(\mathscr{J}_\kappa(\mathscr{x}')\,\underline{\xi}\big)$ is the principal part of the symbol of the operator L written in local variables $(\underline{\mathscr{x}}\,;\underline{\xi})$.

Theorem A.6. *If hypotheses* $(\mathfrak{H}_{A1})$–$(\mathfrak{H}_{A3})$ *are satisfied, there exist* $2N$ *scalar singular functions* $\sigma_\ell = r^{1/2}\varphi_\ell(\mathscr{x}', \theta)$ *for* $\ell = 1, \ldots, 2N$, with $\varphi_\ell \in \mathscr{C}^\infty(\mathscr{E} \times [-\pi, \pi])$ *such that any solution* $\boldsymbol{u}$ *of the problem* (A.1) *with smooth right hand side* $\boldsymbol{f}$ *can be split as follows*

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sigma_1 \boldsymbol{u}_1' + \cdots + \sigma_{2N}\boldsymbol{u}_{2N}', \qquad (A.12)$$

where $\boldsymbol{u}_0, \boldsymbol{u}_1', \ldots, \boldsymbol{u}_{2N}'$ *are* $\mathscr{C}^\infty(\overline{\mathscr{U}})$*-smooth vector functions in local cartesian variables.*

PART B. THE WIENER–HOPF APPROACH

In this part we investigate the asymptotics of solutions of a class of Pseudo-Differential Equations (ΨDE) on the manifold $\mathscr{M}$; we also study how these asymptotics are transformed by representation formulas and how they give back asymptotics for our class of Boundary Value Problems (BVP).

After fixing in § B.I notations for more or less classical functional spaces, including anisotropic Bessel potential spaces, we recall in § B.II, how the boundary value problem (A.1) with Dirichlet or Neumann boundary conditions can be reduced to the ΨDE (A.2) on the manifold $\mathscr{M}$. The feedback is governed by the representation formulas which reconstruct the solution of the BVP in Ω from the solution of the ΨDE on $\mathscr{M}$.

In § B.III, independently from the previous section, we consider the class of classical ΨDE on $\mathscr{M}$ and recall from Chkadua and Duduchava (2001) and Eskin (1981) the general form of asymptotics of the solutions $\boldsymbol{\phi}$ of such equations near the boundary $\mathscr{E}$ of $\mathscr{M}$.

In § B.IV, we introduce the sub-class of classical ΨDE where the full symbol satisfies what we call the "continuity property" ($\mathfrak{H}_{B4}$) with respect to the conormal variable. We state in Theorem B.4 the main result of Part B: the asymptotics of the solutions do not contain any logarithmic term. We also prove that the ΨDE (A.2) obtained from the BVP (A.1) satisfies the continuity property.

In § B.V, before proving the main theorem in its full general framework, we investigate the simpler situation of *scalar* ΨDO in dimension 1. We find a necessary and sufficient condition, denoted ($\mathfrak{H}_{B5}$) and called "generalized continuity property" for the absence of logarithms from the whole asymptotics: the continuity property ($\mathfrak{H}_{B4}$) appears as a particular case of ($\mathfrak{H}_{B5}$).

In § B.VI, we give useful auxiliary propositions relating to ΨDO in one variable acting on functions of n variables and in § B.VII, we prove the main Theorem B.4. Then, in § B.VIII, we show how this latter statement extends to $N \times N$ matrix symbols satisfying the generalized continuity property on $\mathscr{M}$.

In § B.IX, relying on results from Chkadua and Duduchava (2000), we give, as a consequence of the simple structure of the solutions $\boldsymbol{\phi}$ of ΨDE, the form of vector functions $\boldsymbol{u}$ defined in Ω by a certain type of representation formula acting on $\boldsymbol{\phi}$. We prove that the representation formulae (A.3) belong to this type. As a result we have the statement of Theorem A.3.

B.I. Sobolev and Bessel Potential Spaces

1. Standard spaces. We first recall the definition of the Fourier transform and Sobolev spaces. Let $\mathscr{S}(\mathbb{R}^{n+1})$ denote the Schwartz space of all rapidly decreasing functions and $\mathscr{S}'(\mathbb{R}^{n+1})$ the dual space of tempered distributions. For $\varphi \in \mathscr{S}'(\mathbb{R}^{n+1})$ let

$$\mathscr{F}\varphi(\boldsymbol{\xi}) = \mathscr{F}_{y\to\xi}\varphi(\boldsymbol{\xi}) := \int_{\mathbb{R}^{n+1}} e^{i\boldsymbol{\xi}\cdot\boldsymbol{y}}\,\varphi(\boldsymbol{y})\,d\boldsymbol{y}, \quad \boldsymbol{\xi} \in \mathbb{R}^{n+1}$$

denote its Fourier transform in $\mathbb{R}^{n+1}$. The inverse Fourier transform $\mathscr{F}^{-1}_{\xi\to y}$ in $\mathbb{R}^{n+1}$ is defined as

$$\mathscr{F}^{-1}_{\xi\to y}\psi(x) := \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-iy\cdot\xi}\,\psi(\xi)\,d\boldsymbol{\xi}.$$

We denote by $\mathscr{F}_{x\to\xi}$ and $\mathscr{F}^{-1}_{\xi\to x}$ the Fourier and inverse Fourier transforms in $\mathbb{R}^n$. The Sobolev space $\mathrm{H}^s(\mathbb{R}^{n+1})$ is defined as the subspace of $\mathscr{S}'(\mathbb{R}^{n+1})$ endowed with the norm

$$\|\varphi\|^2_{\mathrm{H}^s(\mathbb{R}^{n+1})} := \int_{\mathbb{R}^{n+1}} (1+|\boldsymbol{\xi}|^2)^s \left|\mathscr{F}_{y\to\xi}\varphi(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi}$$

For an integer $s = m$ an equivalent norm on the space $\mathrm{H}^m(\mathbb{R}^{n+1})$ is

$$\left(\sum_{|\boldsymbol{\alpha}|\le m} \int_{\mathbb{R}^{n+1}} |\partial_y^{\boldsymbol{\alpha}}\varphi(\boldsymbol{y})|^2\,d\boldsymbol{y}\right)^{1/2}.$$

For a domain $\Omega \subset \mathbb{R}^{n+1}$ with a smooth boundary (Ω can be, for example, one of the half-spaces $\mathbb{R}^{n+1}_{\pm} := \mathbb{R}^n \times \mathbb{R}^{\pm}$), two families of spaces can be defined:

(i) The subspace $\widetilde{\mathrm{H}}^s(\Omega) \subset \mathrm{H}^s(\mathbb{R}^{n+1})$ of the distributions φ which are supported inside $\overline{\Omega}$. The extension by 0 outside $\overline{\Omega}$ of such a distribution yields an element in $\mathrm{H}^s(\mathbb{R}^{n+1})$.

(ii) The quotient space $\mathrm{H}^s(\Omega) := \mathrm{H}^s(\mathbb{R}^{n+1})/\widetilde{\mathrm{H}}^s(\Omega^c)$, where $\Omega^c := \mathbb{R}^{n+1}\backslash\overline{\Omega}$ is the complementary domain. The space $\mathrm{H}^s(\Omega)$ can also be interpreted as the space of restrictions $p_\Omega\varphi$ of functions $\varphi \in \mathrm{H}^s(\mathbb{R}^{n+1})$. The space is endowed with the factor-norm, i.e., the minimal norm of all possible extensions to $\mathbb{R}^{n+1}$.

By $\widetilde{\mathrm{H}}^s(\Omega)^N$, $\mathrm{H}^s(\Omega)^N$, we will denote the spaces of N-vector functions.

For a surface $\mathscr{M} \subset \mathbb{R}^{n+1}$ of codimension 1, with a smooth boundary $\partial\mathscr{M}$, the spaces $\mathrm{H}^s(\mathscr{M})$ and $\widetilde{\mathrm{H}}^s(\mathscr{M})$ are defined in a standard way, involving some fixed finite covering $\{U_j\}_{j=1}^J$ of $\mathscr{M}$, appropriate diffeomorphisms $\varkappa_j : U_j \to V_j \subset \mathbb{R}^n_+$ and partition of a unity subordinate to the fixed covering, see, e.g., Eskin (1981) and Hörmander (1983).

2. Anisotropic weighted spaces. Besides the above classical spaces, we need a 3-parameter class of anisotropic Sobolev spaces with weight. The weight appears as integer powers of one particular coordinate. We first define these spaces on $\mathbb{R}^n$, then on $\mathbb{R}^n_+$, finally on $\mathscr{M}$.

Let μ, $s \in \mathbb{R}$ and $\kappa \in \mathbb{N}_0$. We denote by $\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)$ the Hilbert space of distributions u with finite norm

$$\|u\|^2_{\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)} := \sum_{k=0}^{\kappa} \|\langle D'\rangle^{\mu}\langle D\rangle^{s+k} x_n^k u\|^2_{L_2(\mathbb{R}^n)}$$
$$\simeq \sum_{k=0}^{\kappa} \|\langle \xi'\rangle^{\mu}\langle \xi\rangle^{s+k} \mathscr{F}[D_n^k u]\|^2_{L_2(\mathbb{R}^n)}$$

where $x = (x', x_n)$ are cartesian coordinates in $\mathbb{R}^n$, $D_n := i\partial_n$, $\xi = (\xi', \xi_n)$ are the corresponding dual variables,

$$\langle \xi \rangle := (1 + |\xi|^2)^{1/2},$$

and where

$$\langle D'\rangle^{\mu} := \mathscr{F}^{-1}_{\xi'\to x'}\langle \xi'\rangle^{\mu}\mathscr{F}_{y'\to\xi'}, \quad \langle D\rangle^{\nu} := \mathscr{F}^{-1}_{\xi\to x}\langle \xi\rangle^{\nu}\mathscr{F}_{y\to\xi}.$$

are the Bessel potential operators. For integer μ, $s \in \mathbb{N}_0$ we have the equivalent norm

$$\|u\|_{\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)} \simeq \sum_{k=0}^{\kappa} \sum_{\substack{\alpha' \in \mathbb{N}^{n-1} \\ |\alpha'| \le \mu}} \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \le s+k}} \|\partial_{x'}^{\alpha'}\partial_x^{\beta}[x_n^k u]\|_{L_2(\mathbb{R}^n)}.$$

We define the Frechet spaces

$$\mathrm{H}^{(\infty,s),\kappa}(\mathbb{R}^n) := \bigcap_{\mu \in \mathbb{N}} \mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)$$

and

$$\mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n) := \bigcap_{\kappa\in\mathbb{N}} \mathrm{H}^{(\infty,s),\kappa}(\mathbb{R}^n).$$

The functions in these spaces are H^s globally on $\mathbb{R}^n$ and $\mathscr{C}^\infty$ in $\mathbb{R}^n\backslash(\mathbb{R}^{n-1}\times\{0\})$.

On the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1}\times\mathbb{R}_+$, we define $\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n_\pm)$ as the space of restrictions to $\mathbb{R}^n_+$ of distributions in $\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)$. The space $\widetilde{\mathrm{H}}^{(\mu,s),\kappa}(\mathbb{R}^n_+)$ denotes the subspace of $\mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n)$ of distributions with support in $\overline{\mathbb{R}}^n_+$.

The spaces $\mathrm{H}^{(\mu,s),\kappa}(\mathscr{M})$ and $\widetilde{\mathrm{H}}^{(\mu,s),\kappa}(\mathscr{M})$ for a smooth compact manifold $\mathscr{M}$ with a smooth boundary $\partial\mathscr{M}$ are defined in a standard way, involving some fixed finite covering of $\mathscr{M}$, appropriate diffeomorphisms and partition of a unity subordinate to the covering, so that the particular coordinate x_n corresponds to the distance to $\partial\mathscr{M}$ in $\mathscr{M}$, see Chkadua and Duduchava (2001, §1.1).

B.II. Reduction to the Boundary

In this section, we explain in more detail the way from BVP (A.1) to ΨDE (A.2) and back.

We start from the first Green formula for all $\boldsymbol{u}\in\mathrm{H}^2(\Omega)^N$ and $\boldsymbol{v}\in\mathrm{H}^1(\Omega)^N$:

$$b(\boldsymbol{u},\boldsymbol{v}) = -\int_\Omega L\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,d\boldsymbol{y} + \int_{\mathscr{M}} \gamma_+(T\boldsymbol{u})\cdot\gamma_+\overline{\boldsymbol{v}}\,d\sigma - \int_{\mathscr{M}} \gamma_-(T\boldsymbol{u})\cdot\gamma_-\overline{\boldsymbol{v}}\,d\sigma. \tag{B.1}$$

Under the symmetry hypothesis ($\mathfrak{H}_{\mathrm{A2}}$) we have the simplified second Green formula for all $\boldsymbol{u},\boldsymbol{v}\in\mathrm{H}^2(\Omega)^N$

$$\int_\Omega \left(\boldsymbol{u}\cdot\overline{L\boldsymbol{v}} - L\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\right)d\boldsymbol{y} = \int_{\mathscr{M}} \Big(\gamma_+\boldsymbol{u}\cdot\gamma_+(\overline{T\boldsymbol{v}}) - \gamma_-\boldsymbol{u}\cdot\gamma_-(\overline{T\boldsymbol{v}}) - \gamma_+(T\boldsymbol{u})\cdot\gamma_+\overline{\boldsymbol{v}} + \gamma_-(T\boldsymbol{u})\cdot\gamma_-\overline{\boldsymbol{v}}\Big)\,d\sigma. \tag{B.2}$$

Let us recall a construction for the fundamental matrix of the operator $L(D_{\mathscr{X}})$, i.e., the distribution $F_L\in\mathscr{S}'(\mathbb{R}^{n+1})$ such that

$$\forall\boldsymbol{x}\in\mathbb{R}^{n+1},\quad L(D_{\mathscr{X}})F_L(\boldsymbol{x}) = \delta(\boldsymbol{x})\mathrm{Id}, \tag{B.3}$$

where Id is the identity matrix and δ is the Dirac distribution at 0

$$\forall\varphi\in\mathscr{C}^\infty(\mathbb{R}^{n+1}),\quad (\delta,\varphi) = \varphi(0).$$

After choosing in $\mathbb{R}^{n+1}$ a system of coordinates $\boldsymbol{x} = (x, x_{n+1})\in\mathbb{R}^n\times\mathbb{R}$ which particularizes one coordinate, the fundamental matrix of Eq. (B.3) can be written in the following form, see Hörmander (1983):

$$F_L(\boldsymbol{x}) := \mathscr{F}^{-1}_{\xi\to x}\left[\frac{1}{2\pi}\int_{\mathscr{L}_\pm} L^{-1}(\xi,\tau)e^{-i\tau x_{n+1}}\,d\tau\right]\quad\text{if}\quad \mp x_{n+1}>0 \tag{B.4}$$

where $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$ represents the dual variables of (x, x_{n+1}). The contour $\mathscr{L}_+$ ($\mathscr{L}_-$) is situated in the upper (in the lower) complex half-plane $\mathbb{C}_+ := \mathbb{R} \oplus i\mathbb{R}_+$ (in $\mathbb{C}_- := \mathbb{R} \oplus i\mathbb{R}_-$) and is oriented counterclockwise (clockwise, respectively) encircling all roots of the polynomial $\det L(\xi, \tau)$ with respect to the variable τ in the corresponding half-planes $\tau \in \mathbb{C}_\pm$.

Taking the columns of the matrix $F_L(\boldsymbol{x} - \boldsymbol{y})$ as test functions $\boldsymbol{v}(\boldsymbol{x})$ and inserting the equation $L(D_{\boldsymbol{x}})\boldsymbol{u} = \boldsymbol{f}$ into the second Green formula (B.2), we easily obtain a representation formula for any $\boldsymbol{u}$ satisfying the equation $L(D_{\boldsymbol{x}})\boldsymbol{u} = \boldsymbol{f}$:

$$\forall \boldsymbol{x} \in \Omega, \quad \boldsymbol{u}(\boldsymbol{x}) = N\boldsymbol{f}(\boldsymbol{x}) + \mathscr{D}[\boldsymbol{u}](\boldsymbol{x}) - \mathscr{V}[T\boldsymbol{u}](\boldsymbol{x}), \tag{B.5}$$

where

$$\forall \mathscr{x} \in \mathscr{M}, \quad [\boldsymbol{u}](\mathscr{x}) := \gamma_+\boldsymbol{u}(\mathscr{x}) - \gamma_-\boldsymbol{u}(\mathscr{x}), \quad [T\boldsymbol{u}](\mathscr{x}) := \gamma_+T\boldsymbol{u}(\mathscr{x}) - \gamma_-T\boldsymbol{u}(\mathscr{x})$$

denote the jumps of the functions $\boldsymbol{u}(\mathscr{x})$ and $T\boldsymbol{u}(\mathscr{x})$ across the surface $\mathscr{M}$; the operators $\mathscr{V}$, $\mathscr{D}$ and N are the well-known single layer, double layer and volume (Newton) potentials:

$$\mathscr{V}\boldsymbol{\phi}(\boldsymbol{x}) = \int_{\mathscr{M}} F_L(\boldsymbol{x} - \sigma)\boldsymbol{\phi}(\sigma)\,d\sigma, \tag{B.6}$$

$$\mathscr{D}\boldsymbol{\phi}(\boldsymbol{x}) = \int_{\mathscr{M}} (TF_L)^*(\sigma - \boldsymbol{x})\boldsymbol{\phi}(\sigma)\,d\sigma, \tag{B.7}$$

$$N\boldsymbol{f}(\boldsymbol{x}) = \int_{\mathbb{R}^{n+1}} (F_L)^*(\boldsymbol{x} - \boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\,d\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega. \tag{B.8}$$

Here $\mathscr{A}^* := \overline{\mathscr{A}^\top}$ denotes the hermitian conjugate of the matrix $\mathscr{A}$.

Solving the boundary value problem (A.1) with the help of the representation formula (B.5) we have to find only one density, either $\boldsymbol{\varphi} = [\boldsymbol{u}] \in \widetilde{\mathrm{H}}^{1/2}(\mathscr{M})$ for the Neumann problem or $\boldsymbol{\psi} = [T\boldsymbol{u}] \in \widetilde{\mathrm{H}}^{-1/2}(\mathscr{M})$ for the Dirichlet problem (due to the boundary conditions in Eq. (B.5) the other density vanishes on $\mathscr{M}$). Invoking the well–known jump relations ("Plemelj formulae") (see, e.g., (Kupradze et al., 1979, Chazarain and Piriou, 1982)) we get the following pseudodifferential equations on the crack surface (compare with Eq. (A.2))

$$W(\mathscr{x}, D_{\mathscr{x}})\boldsymbol{\varphi}(\mathscr{x}) = -\gamma_+TN\boldsymbol{f}(\mathscr{x}), \quad \mathscr{x} \in \mathscr{M}, \quad \text{for Neumann}, \tag{B.9}$$

$$V(\mathscr{x}, D_{\mathscr{x}})\boldsymbol{\psi}(\mathscr{x}) = \gamma_+N\boldsymbol{f}(\mathscr{x}), \quad \mathscr{x} \in \mathscr{M}, \quad \text{for Dirichlet}. \tag{B.10}$$

Here $W(\mathscr{x}, D_{\mathscr{x}}) = \gamma_+T\mathscr{D} = \gamma_-T\mathscr{D}$ is the trace of the composition of the Neumann operator with the double layer potential and is a hypersingular operator, understood as a pseudodifferential operator of order 1. $V(\mathscr{x}, D_{\mathscr{x}}) = \gamma_+\mathscr{V} = \gamma_-\mathscr{V}$ is the trace of the single layer potential on the surface $\mathscr{M}$ and is a weakly singular integral operator (pseudodifferential operator of order -1).

Thus, by solving the Eq. (B.9) or (B.10), and inserting the solution into the representation formula

$$\boldsymbol{u}(\boldsymbol{x}) = N\boldsymbol{f}(\boldsymbol{x}) + \mathscr{D}\boldsymbol{\varphi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \text{for Neumann}, \tag{B.11}$$

$$\boldsymbol{u}(\boldsymbol{x}) = N\boldsymbol{f}(\boldsymbol{x}) - \mathscr{V}\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \text{for Dirichlet}, \tag{B.12}$$

we obtain a solution of the boundary value problem (A.1).

B.III. Asymptotics of Solutions of ΨDE with Classical Symbols

In this section we recall general results on asymptotics of solutions to ΨDE on a manifold with smooth boundary from Chkadua and Duduchava (2001) and Eskin (1981) obtained by the Wiener–Hopf approach.

Let us consider a classical $N \times N$ matrix symbol $\mathbf{a}(\mathscr{x};\xi)$ of order $\nu \in \mathbb{R}$, defined on the cotangent manifold $\mathscr{T}^*\mathscr{M}$ to $\mathscr{M} \subset \mathbb{R}^n$:

$$\mathbf{a} \in S^{\nu}_{\mathrm{cl}}(\mathscr{T}^*\mathscr{M})^{N\times N} \;\Leftrightarrow\; \mathbf{a}(\mathscr{x};\xi) = \mathbf{a}_0(\mathscr{x};\xi) + \mathbf{a}_1(\mathscr{x};\xi) + \cdots,$$

where $\mathbf{a}_j(\mathscr{x},\theta)$ are $\mathscr{C}^\infty$ on the bundle of cotangent unit spheres $\overline{\mathscr{M}} \times \mathbb{S}^{n-1} \subset \mathscr{T}^*\mathscr{M}$ (see Chkadua and Duduchava (2001, § 1.2) and Hörmander (1983)) and satisfy

$$\forall\lambda > 0,\ \forall\mathscr{x} \in \mathscr{M},\ \forall\xi \in \mathbb{R}^n,\quad \mathbf{a}_j(\mathscr{x};\lambda\xi) = \lambda^{\nu-j}\mathbf{a}_j(\mathscr{x};\xi). \tag{B.13}$$

For any Sobolev exponent $s \in \mathbb{R}$, the corresponding $N \times N$ system of ΨDE on $\mathscr{M}$ with symbol $\mathbf{a}(\mathscr{x};\xi)$ is continuous from $\widetilde{\mathrm{H}}^s(\mathscr{M})^N$ into $\mathrm{H}^{s-\nu}(\mathscr{M})^N$. We are interested in the structure of any $\boldsymbol{\phi}$ satisfying for some $s \in \mathbb{R}$ and an integer $K > 0$:

$$\boldsymbol{\phi} \in \widetilde{\mathrm{H}}^s(\mathscr{M})^N \quad \text{such that} \quad \mathbf{a}(\mathscr{x};D_{\mathscr{x}})\boldsymbol{\phi} = \boldsymbol{g}, \quad \text{with} \quad \boldsymbol{g} \in \mathrm{H}^{s-\nu+K}(\mathscr{M})^N. \tag{B.14}$$

Further we suppose that the principal homogeneous part $\mathbf{a}_0(\mathscr{x};\xi)$, which we will also denote by $\mathbf{a}_{\mathrm{pr}}(\mathscr{x};\xi)$ is **elliptic**, which reads

$$\det \mathbf{a}_{\mathrm{pr}}(\mathscr{x};\xi) \neq 0, \quad \mathscr{x} \in \overline{\mathscr{M}}, \quad \xi \in \mathbb{R}^n\backslash\{0\}. \tag{$\mathfrak{H}_{\mathrm{B1}}$}$$

The following $N \times N$ matrix plays a fundamental role in the structure of the solutions $\boldsymbol{\phi}$ satisfying (B.14): for $\mathscr{x}' \in \mathscr{E}$

$$\mathbf{b}(\mathscr{x}') := [\mathbf{a}_{\mathrm{pr}}(\mathscr{x}',0\,;0,\,+1)]^{-1}\,\mathbf{a}_{\mathrm{pr}}(\mathscr{x}',0\,;0,\,-1), \tag{B.15}$$

where we recall that $\mathscr{x} := (\mathscr{x}',\mathscr{x}_n) \in \mathscr{M}$ are the local and $\xi = (\xi',\xi_n)$ are the dual coordinates, with $\mathscr{x}' \in \mathscr{E} = \partial\mathscr{M}$ the edge variable. Note that for all $\xi' \in \mathbb{R}^{n-1}$:

$$\mathbf{a}_{\mathrm{pr}}(\mathscr{x}',0\,;0,\,\pm 1) = \lim_{t\to\pm\infty} |t|^{-\nu}\mathbf{a}_{\mathrm{pr}}(\mathscr{x}',0\,;\xi',t).$$

For any $\mathscr{x}' \in \mathscr{E}$, let us denote by

$$\lambda_1(\mathscr{x}'),\ldots,\lambda_N(\mathscr{x}') \quad \text{the eigenvalues of} \quad \mathbf{b}(\mathscr{x}'),$$

where each eigenvalue is repeated according to its *algebraic* multiplicity.

The assumption which will ensure the absence of logarithms in the principal term of the asymptotics of the $\boldsymbol{\phi}$ satisfying Eq. (B.14) is that $\mathbf{b}$ is *diagonalizable* in each point $\mathscr{x}'$ in $\mathscr{E}$, and that the eigenvalues are $\mathscr{C}^\infty(\mathscr{E})$, which is written as:

$$\begin{cases} \forall\mathscr{x}' \in \mathscr{E}, \quad \exists \text{ an invertible matrix } \mathscr{K}(\mathscr{x}') \text{ so that:} \\ \mathbf{b}(\mathscr{x}') = \mathscr{K}(\mathscr{x}')\big(\mathrm{diag}\{\lambda_1(\mathscr{x}'),\ldots,\ \lambda_N(\mathscr{x}')\}\big)\mathscr{K}^{-1}(\mathscr{x}') \\ \mathscr{x}' \mapsto \mathscr{K}(\mathscr{x}'), \quad \mathscr{x}' \mapsto \lambda_j(\mathscr{x}') \quad \text{are} \quad \mathscr{C}^\infty(\mathscr{E}). \end{cases} \tag{$\mathfrak{H}_{\mathrm{B2}}$}$$

We need one more assumption on the eigenvalues of $\mathbf{b}(\mathscr{x}')$: let us set

$$\delta_j(\mathscr{x}') = (2\pi i)^{-1} \log \lambda_j(\mathscr{x}'), \quad j = 1, \ldots, N.$$

We assume that

$$\begin{cases} \exists \eta \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \exists \text{ a } \mathscr{C}^\infty(\mathscr{E}) \text{ determination of the } \delta_j(\mathscr{x}') \\ \text{such that} \quad \forall \mathscr{x}' \in \mathscr{E}, \quad \eta - \frac{1}{2} < \mathrm{Re}\delta_j(\mathscr{x}') < \eta + \frac{1}{2}. \end{cases} \qquad (\mathfrak{H}_{\mathrm{B3}})$$

While *locally* a consequence of $(\mathfrak{H}_{\mathrm{B2}})$, this assumption has to be required to hold globally on $\mathscr{E}$.

The following result, see Duduchava et al. (1999, Lemma A.6), provides a general framework where assumptions $(\mathfrak{H}_{\mathrm{B2}})$ and $(\mathfrak{H}_{\mathrm{B3}})$ are satisfied.

Lemma B.1. *If for any $\mathscr{x}' \in \mathscr{E}$ the two matrices $\mathbf{a}_{\mathrm{pr}}(\mathscr{x}', 0; 0, \pm 1)$ in Eq.* (B.15) *are positive definite, then the matrix $\mathbf{b}(\mathscr{x}')$ is diagonalizable with unitary $\mathscr{K}(\mathscr{x}')$, its eigenvalues are real, which means that the numbers $\delta_j(\mathscr{x}')$ can be chosen purely imaginary*:

$$\delta_j \in \mathscr{C}^\infty(\mathscr{E}), \quad \mathrm{Re}\, \delta_j(\mathscr{x}') = 0 \quad \textit{for all} \quad j = 1, \ldots, N. \qquad \text{(B.16)}$$

The main result in this section is the asymptotic structure of solutions $\boldsymbol{\phi}$ of Eq. (B.14), whose first term *does not contain logarithms*. We recall that $(\mathscr{x}', r) = (\mathscr{x}', r, \pm\pi)$ denotes the local cylindrical coordinate system on $\mathscr{M}$ in a closed tubular neighborhood of the edge $\mathscr{E} = \partial\mathscr{M}$ (see Definition A.I).

Theorem B.2. (*See Chkadua and Duduchava* (2001) *and Eskin* (1981, *Chap*. 26).
We assume hypotheses $(\mathfrak{H}_{\mathrm{B1}})$, $(\mathfrak{H}_{\mathrm{B2}})$ *and* $(\mathfrak{H}_{\mathrm{B3}})$. *We choose*

- *A determination of the δ_j, $j = 1, \ldots, N$,*
- *A real Sobolev exponent s,*

such that there holds for all $\mathscr{x}' \in \mathscr{E}$ and all $j = 1, \ldots, N$:

$$-1 < \frac{\nu}{2} + \mathrm{Re}\, \delta_j(\mathscr{x}'), \qquad \text{(B.17)}$$

$$-\frac{\nu}{2} + s - \frac{1}{2} < \mathrm{Re}\, \delta_j(\mathscr{x}') < -\frac{\nu}{2} + s + \frac{1}{2}. \qquad \text{(B.18)}$$

Let $\boldsymbol{\phi} \in \widetilde{\mathrm{H}}^s(\mathscr{M})^N$ be a solution of the equation $\mathbf{a}(\mathscr{x}; D_{\mathscr{x}})\boldsymbol{\phi} = \boldsymbol{g}$ where the right hand side $\boldsymbol{g}$ is $\mathscr{C}^\infty(\mathscr{M})^N$. Then, for any integer $K > 0$ the solution $\boldsymbol{\phi}(\mathscr{x}', r)$ has the following *asymptotic expansion*

$$\mathscr{K}(x')\, r^{(\nu/2)+\Delta(x')}\chi(r)\Big[\boldsymbol{d}^0(x') + \sum_{k=1}^{K-1} r^k \sum_{q=0}^{\sigma(k)} \boldsymbol{d}^{k,q}(x')\,\log^q r\Big] + \boldsymbol{\phi}_{\mathrm{rem},K}(x', r), \qquad \boldsymbol{\phi}_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{s+K}(\mathscr{M})^N \tag{B.19}$$

with N-vector coefficients $\boldsymbol{d}^0$, $\boldsymbol{d}^{k,q}$ *in* $\mathscr{C}^\infty(\mathscr{E})^N$. *Here, the vector* Δ *is defined as* $(\delta_1, \ldots, \delta_N)^\top$ *and for any* $\mu \in \mathbb{R}$, $r^{\mu+\Delta}$ *is understood as the diagonal* $N \times N$ *matrix*

$$r^{\mu+\Delta} := \mathrm{diag}\{\, r^{\mu+\delta_1}, \,\ldots,\, r^{\mu+\delta_N}\}. \tag{B.20}$$

Remark B.3.

(i) In Eskin (1981, Chap. 26), it is proved that the asymptotics of $\boldsymbol{\phi}$ has no logarithmic term in its leading summand, and in Chkadua and Duduchava (2001) the more explicit formula (B.19) is proved.

(ii) It is possible to extend hypothesis $(\mathfrak{H}_{\mathrm{B2}})$ to certain cases where $\mathbf{b}(x')$ is not diagonalizable: then we assume that we have a canonical Jordan decomposition with a $\mathscr{C}^\infty(\mathscr{E})$ dependence. This implies in particular that the geometrical multiplicities are constant along $\mathscr{E}$. Then it is proved in Chkadua and Duduchava (2001) that there holds a decomposition like Eq. (B.19), with explicit logarithmic terms in the leading summand of the asymptotics. This means that the condition $(\mathfrak{H}_{\mathrm{B2}})$ is necessary and sufficient so that logarithms are absent in the leading summand of the asymptotic of a solution (B.19).

(iii) It is possible to get the first term of the asymptotic expansion without the smoothness properties on $\mathscr{K}$ and δ_j, but the further terms are not available so far, see Chkadua and Duduchava (2001). ■

B.IV. Asymptotics of ΨDE with Symbols Satisfying the Continuity Property

Here is a condition which ensures that logarithms disappear from the entire asymptotics (B.19). This condition, called continuity property, applies to the *full* symbol $\sum_{j\geq 0} \mathbf{a}_j(x', x_n\,;\, \xi', \xi_n)$:

$$\begin{cases} \forall x' \in \mathscr{E}, \quad \forall j \in \mathbb{N}_0, \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \quad m \in \mathbb{N}_0, \\ \partial_{x_n}^m \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, -1) = (-1)^{j+|\alpha'|} \partial_{x_n}^m \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, +1). \end{cases} \tag{$\mathfrak{H}_{\mathrm{B4}}$}$$

We note that the above condition implies that for all $\beta \in \mathbb{N}_0^n$,

$$\partial_x^\beta \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, -1) = (-1)^{j+|\alpha'|} \partial_x^\beta \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, +1). \tag{B.21}$$

On the other hand, concerning the principal symbol, the above condition implies that

$$\mathbf{a}_{\mathrm{pr}}(x', 0\,; 0, -1) = \mathbf{a}_{\mathrm{pr}}(x', 0\,; 0, +1),$$

whence for all $\mathscr{x}' \in \mathscr{E}$, $\mathbf{b}(\mathscr{x}') = \mathrm{Id}$. Thus condition $(\mathfrak{H}_{B4})$ implies conditions $(\mathfrak{H}_{B2})$ and $(\mathfrak{H}_{B3})$. Here follows the main result about asymptotics of ΨDE with symbols satisfying the continuity property.

Theorem B.4. *Let* $\mathbf{a}(\mathscr{x};\xi)$ *be an elliptic classical symbol* (B.13) *of order* $\nu > -2$ *and let its homogeneous components* $\mathbf{a}_j$ *satisfy the continuity property* $(\mathfrak{H}_{B4})$ *on the boundary* $\mathscr{E}$. *Let s be a Sobolev exponent such that*

$$\frac{\nu}{2} - \frac{1}{2} < s < \frac{\nu}{2} + \frac{1}{2}. \tag{B.22}$$

Any solution $\boldsymbol{\phi} \in \widetilde{\mathrm{H}}^s(\mathscr{M})^N$ *of the equation* $\mathbf{a}(\mathscr{x}; D_{\mathscr{x}})\boldsymbol{\phi} = \boldsymbol{g}$ *where the right hand side* $\boldsymbol{g}$ *is* $\mathscr{C}^\infty(\mathscr{M})^N$,[a] *has the following asymptotic expansion for any integer* $K > 0$

$$\boldsymbol{\phi} = \sum_{k=0}^{K-1} r^{\nu/2+k}\chi(r)\boldsymbol{d}^k(\mathscr{x}') + \boldsymbol{\phi}_{\mathrm{rem},K}, \quad \boldsymbol{\phi}_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{s+K}(\mathscr{M})^N, \tag{B.23}$$

where the N-vectors $\boldsymbol{d}^k$, $k = 0, 1, \ldots$ *belong to* $\mathscr{C}^\infty(\mathscr{E})$.

We postpone the proof of the theorem until § B.VII.

The assumptions of Theorem B.4 hold for the boundary ΨDE (B.9) and (B.10) corresponding to the BVP (A.1). This follows from the following theorem.

Theorem B.5. *The symbols of the boundary* ΨDE (B.9) *of order* $\nu = 1$ *and* (B.10) *of order* $\nu = -1$ *are positive definite and satisfy the continuity property* $(\mathfrak{H}_{B4})$. *Moreover, for any volume data* $\boldsymbol{f} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$, *the right hand sides of Eqs.* (B.9) *and* (B.10) *are in* $\mathscr{C}^\infty(\mathscr{M})^N$, *and Eqs.* (B.9) *and* (B.10) *have unique solutions* $\boldsymbol{\varphi} \in \widetilde{\mathrm{H}}^s(\mathscr{M})^N$ *and* $\boldsymbol{\psi} \in \widetilde{\mathrm{H}}^{s-1}(\mathscr{M})^N$, *respectively, for any* $s \in (0,1)$. *Thus asymptotics* (B.23) *hold for these solutions.*

Proof. We quote Chkadua and Duduchava (2001), Costabel and Stephan (1987), Duduchava and Wendland (1995) and Duduchava et al. (1995), for the proofs of positive definiteness of the symbols and unique solvability (also in more general spaces) of ΨDE (B.9) and (B.10) and concentrate on the proof of the continuity property (B.21).

In Chkadua and Duduchava (2001, Example 1.17) it is proved that the symbols of both Eqs. (B.9) and (B.10) are classical[b] and the components of the asymptotic

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[a]If the requirement $\boldsymbol{g} \in \mathscr{C}^\infty(\mathscr{M})^N$ is relaxed into $\boldsymbol{g} \in \mathrm{H}^{(\infty, s-\nu+K),\kappa}(\mathscr{M})^N$ for an integer $K > 0$ and $\kappa \geq K$, we still obtain the asymptotics (B. 23) for the same value K.

[b]In Chkadua and Duduchava (2001, Example 1.17) is considered the restriction of a ΨDO on $\mathbb{R}^{n+1}$ with a classical symbol onto the smooth surface $\mathscr{M}$ of codimension 1 and proved that the restricted operator is again a classical ΨDO; explicit formulae for the components of the asymptotic expansion of the symbol are indicated.

representation of the symbols have the following form

$$W(\mathscr{x};\xi) = W_0(\mathscr{x};\xi) + W_1(\mathscr{x};\xi) + \cdots + W_j(\mathscr{x};\xi) + \cdots,$$
$$V(\mathscr{x};\xi) = V_0(\mathscr{x};\xi) + V_1(\mathscr{x};\xi) + \cdots + V_j(\mathscr{x};\xi) + \cdots,$$

where the homogeneous components $W_j(\mathscr{x};\xi)$ and $V_j(\mathscr{x};\xi)$ (of orders $1-j$ and $-1-j$ respectively) are generated by an explicit symbol $\boldsymbol{W}$ and $\boldsymbol{V}$ respectively

$$W_j(\mathscr{x};\xi) = \sum_{\substack{|\alpha|-|\beta|=j\geq 0\\ 2\beta\leq\alpha}} \mathbf{a}_{\alpha,\beta}(\mathscr{x})\,\xi^\beta\partial_\xi^\alpha \boldsymbol{W}(\mathscr{x};\xi), \tag{B.24}$$

$$V_j(\mathscr{x};\xi) = \sum_{\substack{|\alpha|-|\beta|=j\geq 0\\ 2\beta\leq\alpha}} \mathbf{a}_{\alpha,\beta}(\mathscr{x})\,\xi^\beta\partial_\xi^\alpha \boldsymbol{V}(\mathscr{x};\xi), \tag{B.25}$$

where the sums are finite since $|\alpha|-|\beta|=j$ and $2\beta\leq\alpha$ imply $2|\beta|\leq|\alpha|\leq 2nj$, and where the matrices $\mathbf{a}_{\alpha,\beta}(\mathscr{x})$ have $\mathscr{C}^\infty(\mathscr{M})$ coefficients. The generating symbols $\boldsymbol{W}$, $\boldsymbol{V}$ are defined for $\mathscr{x}\in\mathscr{M}$ and $\xi\in\mathbb{R}^n$ as follows—the contour $\mathscr{L}_+$ is the same as in Eq. (B.4) and the Jacobian $\mathscr{J}_\kappa(\mathscr{x}')$ as in Definition A.1 (*vi*):

$$\boldsymbol{W}(\mathscr{x};\xi) := \int_{\mathscr{L}_+} T\big(\mathscr{x};\mathscr{J}_\kappa(\mathscr{x})(\xi,\tau)\big)L^{-1}\big(\mathscr{J}_\kappa(\mathscr{x})(\xi,\tau)\big)^\top \times T\big(\mathscr{x};\mathscr{J}_\kappa(\mathscr{x})(\xi,\tau)\big)^\top\,d\tau \tag{B.26}$$

$$\boldsymbol{V}(\mathscr{x};\xi) := \int_{-\infty}^{\infty} L^{-1}\big(\mathscr{J}_\kappa(\mathscr{x})(\xi,\tau)\big)\,d\tau. \tag{B.27}$$

In particular, the principal symbols $W_0(\mathscr{x};\xi)$ and $V_0(\mathscr{x};\xi)$ both have the following coefficient

$$\mathbf{a}_{0,0}(\mathscr{x}) := \frac{\Gamma_\varkappa(\mathscr{x})}{2\pi\det D\varkappa(\mathscr{x})},$$

where $\Gamma_\varkappa(\mathscr{x})$ is the Gram determinant of the local coordinate diffeomorphisms $\varkappa$.

Since the elliptic differential operator $L(D_\mathbf{x})$ in Eq. (A.1) is supposed to be *homogeneous of degree* 2, its symbol $L(\xi,\xi_{n+1})$ is even

$$\forall\boldsymbol{\xi}=(\xi,\xi_{n+1})\in\mathbb{R}^{n+1},\quad L(-\xi,-\xi_{n+1}) = L(\xi,\xi_{n+1}).$$

As a consequence, with the change of variable $\tau\mapsto-\tau$ in integrals (B.26) and (B.27), we find that the generating symbols $\boldsymbol{V}$ and $\boldsymbol{W}$ are even,3,[c] : For all $\mathscr{x}\in\mathscr{M}$ and $\xi\in\mathbb{R}^n$

$$\boldsymbol{V}(\mathscr{x},-\xi) = \boldsymbol{V}(\mathscr{x},\xi)\quad\text{and}\quad \boldsymbol{W}(\mathscr{x},-\xi) = \boldsymbol{W}(\mathscr{x},\xi).$$

[c]For this, we use in particular that any contour integral of the integrand in Eq. (B.26) surrounding all the roots τ of $\det L(\mathscr{J}_\kappa(\mathscr{x}'),(\xi,\tau))=0$, is zero, which allows to replace in Eq. (B.26) $\mathscr{L}_+$ by $\mathscr{L}_-$.

Therefore, as a consequence of formulas (B.26) and (B.27), for all $\mathscr{x} \in \mathscr{M}$, for all $\lambda \in \mathbb{R}$, for all integers $j, m = 0, 1, \ldots$ and all multiindices $\alpha' \in \mathbb{N}_0^{n-1}$, there holds

$$(\partial_{\mathscr{x}_n}^m \partial_{\xi'}^{\alpha'} W_j)(\mathscr{x}', 0\,;0, -\lambda) = (-1)^{j+|\alpha'|}(\partial_{\mathscr{x}_n}^m \partial_{\xi'}^{\alpha'} W_j)(\mathscr{x}', 0\,;0, \lambda),$$
$$(\partial_{\mathscr{x}_n}^m \partial_{\xi'}^{\alpha'} V_j)(\mathscr{x}', 0\,;0, -\lambda) = (-1)^{j+|\alpha'|}(\partial_{\mathscr{x}_n}^m \partial_{\xi'}^{\alpha'} V_j)(\mathscr{x}', 0\,;0, \lambda). \qquad \blacksquare$$

B.V. ΨDE in Dimension 1

Before we start the proof of the main theorem B.4, we want to explain the principal mechanism responsible for the absence of logarithmic terms by presenting the result in a very simple situation, namely the case of a scalar elliptic pseudodifferential equation with constant coefficients on the half-line $\mathbb{R}_+$. This simple one-dimensional situation allows us to stay free of many of the technical difficulties of the higher-dimensional case and to concentrate on the essential feature, namely the role of the continuity condition for the asymptotic expansion of the symbol. We can show in this case that a natural generalization of this condition is not only sufficient, but also necessary for the absence of logarithmic terms in the asymptotics of the solution. The class of operators considered here can be larger than the one obtained from the 2D crack problem.

We need the following well-known Fourier transform of distributions supported in the positive half-line, see for instance Eskin (1981). By χ_+ and χ_- we denote the characteristic functions of $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Lemma B.6.

(i) $\mathscr{F}_{t\to\lambda}(\chi_+(t)\, t^{\mu-1}e^{-\tau t}) = \Gamma(\mu)e^{i(\pi/2)\mu}(\lambda + i\tau)^{-\mu}, \quad \tau > 0$

(ii) $\mathscr{F}_{t\to\lambda}(\chi_+(t)\, \log t\, t^{\mu-1}e^{-\tau t}) = (\lambda + i\tau)^{-\mu}(c\log(\lambda + i\tau) + d)$

with $c = -\Gamma(\mu)\, e^{i(\pi/2)\mu}$ *and* $d = (d/d\mu)\big(\Gamma(\mu)\, e^{i(\pi/2)\mu}\big)$.

Another crucial result concerns the additive decomposition of homogeneous distributions into "plus" and "minus" terms.

Lemma B.7. *Let* a^+, a^-, $\gamma \in \mathbb{C}$. *Then*

(i) *If* $\gamma \notin \mathbb{Z}$, *we have the representation*

$$\big(a^+\chi_+(t) + a^-\chi_-(t)\big)\,|t|^\gamma = \frac{a^- - e^{-i\pi\gamma}a^+}{e^{i\pi\gamma} - e^{-i\pi\gamma}}\,(t + i0)^\gamma - \frac{a^- - e^{i\pi\gamma}a^+}{e^{i\pi\gamma} - e^{-i\pi\gamma}}\,(t - i0)^\gamma.$$

(ii) $\gamma \in \mathbf{Z}$, *we have the representation*

$$\big(a^+\chi_+(t) + a^-\chi_-(t)\big)\,|t|^\gamma = a^+\,(t + i0)^\gamma + \frac{(-1)^\gamma a^- - a^+}{2i\pi} \times ((t + i0)^\gamma \log(t + i0) - (t - i0)^\gamma \log(t - i0)).$$

Proof. It suffices to use the identities

$$(t \pm i0)^{\gamma} = \chi_{+}(t)\, t^{\gamma} + \chi_{-}(t)\, e^{\pm i\pi\gamma} |t|^{\gamma}$$
$$\log(t \pm i0) = \chi_{+}(t) \log t + \chi_{-}(t)\big(\log |t| \pm i\pi\big). \qquad \blacksquare$$

Let $a \in \mathscr{C}^{\infty}(\mathbb{R}; \mathbb{C})$ be a classical elliptic symbol of order $\nu \in \mathbb{R}$ with constant coefficients, i.e., $a(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, and a has an asymptotic expansion in homogeneous terms

$$a(\xi) \sim \sum_{j=0}^{\infty} a_j(\xi) \quad \text{with} \quad \forall j \in \mathbb{N}, \quad a_j(t\xi) = t^{\nu-j} a_j(\xi), \tag{B.28}$$

for all $t > 0$ and $\xi \in \mathbb{R}$.

In one dimension, homogeneous functions are determined by two values:

$$a_j(\xi) = \big(a_j^{+} \chi_{+}(\xi) + a_j^{-} \chi_{-}(\xi)\big)\, |\xi|^{\nu-j}. \tag{B.29}$$

From the ellipticity follows that $a_0^{+} a_0^{-} \neq 0$, and we can define

$$\Lambda = \frac{a_0^{-}}{a_0^{+}} \in \mathbb{C}.$$

By $p_{+}a(D)u$ we denote the restriction of

$$a(D)u(x) := \mathscr{F}^{-1}_{\xi \mapsto x}\, a(\xi)\, (\mathscr{F} u)(\xi)$$

to the half-line $\mathbb{R}_{+}$.

Theorem B.8. *Let a be a classical elliptic symbol of order ν as above, and let $\delta \in \mathbb{C}$, $s \in \mathbb{R}$ be chosen such that*

$$e^{2i\pi\delta} = \Lambda \quad \textit{and} \quad \frac{\nu}{2} + \operatorname{Re}\delta - \frac{1}{2} < s < \frac{\nu}{2} + \operatorname{Re}\delta + \frac{1}{2} \tag{B.30}$$

and $\nu/2 + \delta \notin \{-1, -2, \ldots\}$. Let $u \in \widetilde{\mathrm{H}}^{s}(\mathbb{R}_{+})$ be solution of

$$p_{+}a(D)u = g \quad \textit{on} \quad \mathbb{R}_{+} \tag{B.31}$$

with $g \in \mathscr{C}^{\infty}(\overline{\mathbb{R}}_{+}) \cap \mathrm{H}^{s-\nu}(\mathbb{R}_{+})$. Then u has an asymptotic expansion as $x \to 0$:

$$u(x) \sim \sum_{k \geq 0} \sum_{q=0}^{q_k} c_{kq}\, x^{(\nu/2)+\delta+k} \log^{q} x.$$

This asymptotic expansion for any such u is free of logarithms, i.e., $q_k = 0$ for all $k \geq 0$, if and only if the following condition $(\mathfrak{H}_{\mathrm{B5}})$ is satisfied

$$\forall j \geq 0, \quad a_j^{-} = (-1)^{j} \Lambda\, a_j^{+}. \tag{$\mathfrak{H}_{\mathrm{B5}}$}$$

Note that, reduced to the case of dimension 1 with scalar operators, condition ($\mathfrak{H}_{B5}$) is a generalization of condition ($\mathfrak{H}_{B4}$) which corresponds to taking $\Lambda = 1$.

Proof.

(i) We first show the sufficiency of condition ($\mathfrak{H}_{B5}$).
If ($\mathfrak{H}_{B5}$) is satisfied, then we can write

$$a(\xi) = a_0(\xi)\, q(\xi), \tag{B.32}$$

where $q(\xi)$ has an asymptotic expansion of the form

$$q(\xi) \sim 1 + \sum_{j\geq 1} q_j\, \xi^{-j} \quad \text{with} \quad q_j = \frac{a_j^+}{a_0^+}. \tag{B.33}$$

Thus q is a symbol of *rational type*.

For a_0 we find the factorization

$$a_0(\xi) = a_0^+\, (\xi + i0)^{\nu/2+\delta}\, (\xi - i0)^{\nu/2-\delta}. \tag{B.34}$$

Let us introduce the corresponding $\mathscr{C}^\infty(\mathbb{R})$ symbol

$$a^\infty(\xi) = a_0^+\, (\xi + i)^{\nu/2+\delta}\, (\xi - i)^{\nu/2-\delta}. \tag{B.35}$$

Then we have a global representation of the symbol a as the product

$$a(\xi) = a^\infty(\xi)\, q^\infty(\xi) \tag{B.36}$$

with a symbol of rational type

$$q^\infty(\xi) \sim 1 + \sum_{j\geq 1} q_j^\infty\, (\xi + i)^{-j}. \tag{B.37}$$

Formula (B.36) is deduced from identities (B.32)–(B.34) by Taylor expansion at $\xi + i = \infty$, which allows to expand the functions

$$\xi^{-j}, \quad \left(\frac{\xi + i0}{\xi + i}\right)^{\nu/2+\delta} \quad \text{and} \quad \left(\frac{\xi - i0}{\xi - i}\right)^{\nu/2-\delta}$$

in negative powers of $(\xi + i)$.

There is also an expansion for

$$q^{-\infty} \sim \frac{1}{q^\infty(\xi)} \sim 1 + \sum_{j\geq 1} q_j^{-\infty}\, (\xi + i)^{-j} \tag{B.38}$$

so that $q^\infty\, q^{-\infty}$ is a symbol of order $-\infty$.

The following result is well known from Eskin's version (1981) of the Wiener–Hopf method:

Proposition B.9. *For $h \in \mathrm{H}^{s-\nu}(\mathbb{R}_+)$, the equation*

$$p_+ a^\infty(D) v = h \tag{B.39}$$

has a unique solution $v \in \widetilde{\mathrm{H}}^s(\mathbb{R}_+)$. *This solution is given by*

$$v = (D+i)^{-(\nu/2)-\delta} p_+ (D-i)^{-(\nu/2)+\delta} [a_0^+]^{-1} \tilde{h}, \tag{B.40}$$

where $\tilde{h} \in \mathrm{H}^{s-\nu}(\mathbb{R})$ *is an extension of h to the whole line.*
For $K \in \mathbb{N}$ *and* $h \in \mathrm{H}^{s-\nu+K}(\mathbb{R}_+)$, *this solution v has the asymptotic expansion*

$$v(x) = \sum_{k=0}^{K-1} \chi_+(x)\, x^{\nu/2+\delta+k}\, e^{-x} d_k + v_{\mathrm{rem},K}(x) \tag{B.41}$$

with the remainder $v_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{s+K}(\mathbb{R}_+)$ *given by*

$$v_{\mathrm{rem},K} = (D+i)^{-(\nu/2)-\delta-k} p_+ (D-i)^{-(\nu/2)+\delta+k} [a_0^+]^{-1} \tilde{h}, \tag{B.42}$$

and the coefficients d_k *by*

$$d_k = \frac{e^{-i(\pi/2)(\nu/2+\delta+k+1)}}{\Gamma(\nu/2+\delta+k+1)} \Big((D-i)^{-(\nu/2)+\delta+k} \big([a_0^+]^{-1} \tilde{h} \big) \Big)(0). \tag{B.43}$$

Let now $u \in \widetilde{\mathrm{H}}^s(\mathbb{R}_+)$ be a solution of (B.31). For $K \in \mathbb{N}$, let v be defined by

$$v = q^K(D)u \quad \text{with} \quad q^K(\xi) = 1 + \sum_{j=1}^{K-1} q_j^{\infty}\, (\xi+i)^{-j}. \tag{B.44}$$

Note that $(D+i)^{-j}$ is a convolution operator with kernel

$$\frac{(-i)^j}{(j-1)!}\, x^{j-1}\, e^{-x}\, \chi_+(x),$$

cf Lemma B.6
(i) Thus $v \in \widetilde{\mathrm{H}}^s(\mathbb{R}_+)$, and v is solution of

$$p_+ a^{\infty}(D) v = g - p_+ a^{\infty}(D) \big(q^{\infty}(D) - q^K(D)\big) u =: h,$$

where h belongs to $\mathrm{H}^{s-\nu+K}(\mathbb{R}_+)$. Therefore v has the expansion (B.41), and we can recover the expansion of

$$\begin{aligned} u &\equiv q^{-\infty}(D) \left(v + (q^{\infty}(D) - q^K(D))u \right) \quad (\mathrm{mod}\ \mathscr{C}^{\infty}) \\ &\equiv q^{-K}(D) v \quad (\mathrm{mod}\ \widetilde{\mathrm{H}}^{s+K}(\mathbb{R}_+)) \\ &\quad \text{with} \quad q^{-K}(\xi) = 1 + \sum_{j=1}^{K-1} q_j^{-\infty}\, (\xi+i)^{-j} \end{aligned} \tag{B.45}$$

by simply integrating (B.41):

$$(D+i)^{-j}\big[\chi_+(x)\,x^{\nu/2+\delta+k}\,e^{-x}\big] = (-1)^j \frac{\Gamma\Big(\dfrac{\nu}{2}+\delta+k+1\Big)}{\Gamma\Big(\dfrac{\nu}{2}+\delta+k+j+1\Big)}\,\chi_+(x)\,x^{\nu/2+\delta+k+j}\,e^{-x}, \tag{B.46}$$

except if $\nu/2+\delta+k \in \{-1,\ldots,-j\}$, where logarithms will appear.

Thus we obtain the asymptotics of u up to regularity $\widetilde{\mathrm{H}}^{s+K}(\mathbb{R}_+)$, and since we assumed that $\nu/2+\delta$ is not a negative integer, no logarithm will appear. We have shown that condition $(\mathfrak{H}_{\mathrm{B5}})$ implies that the asymptotics of u is free of logarithms.

(ii) Let us show the converse. We assume that the equality in $(\mathfrak{H}_{\mathrm{B5}})$ is violated for some $j \geq 1$. Let M be the first such j, so that

$$a(\xi) = a_0(\xi)\,q^M(\xi) + a_{M+1}(\xi) \tag{B.47}$$

with

$$q^M(\xi) = 1 + \sum_{j=1}^{M-1} q_j \xi^{-j} + \big(q_M^+\chi_+(\xi) + q_M^-\chi_-(\xi)\big)\,|\xi|^{-M}$$

and $a_{M+1}(\xi) = \mathrm{O}\big(|\xi|^{-M-1}\big)$ as $|\xi| \to \infty$.

We will show that there exist $g \in \mathrm{H}^{s-\nu+M+1}(\mathbb{R}_+)$ and $u \in \widetilde{\mathrm{H}}^s(\mathbb{R}_+)$ solution of (B.31) such that

$$u(x) = c_0\,\chi_+(x)\,x^{\nu/2+\delta} + c_M\,\chi_+(x)\,x^{\nu/2+\delta+M}\log x \tag{B.48}$$

near $x=0$. The question of regularity of $g = p_+a(D)u$ is local at $x=0$. We can therefore stay within the framework of quasi-homogeneous distributions and homogeneous symbols, discard lower order terms such as $a_{M+1}(\xi)$, and replace ξ^{-j} by $(\xi+i0)^{-j}$.

Since the Fourier transform of $\chi_+(x)\,x^\gamma$ is $c(\xi+i0)^{-1-\gamma}$, and the Fourier transform of $\chi_+(x)\,x^\gamma\log x$ is $(\xi+i0)^{-1-\gamma}(c\log(\xi+i0)+d)$, see Lemma B.6, we shall construct the Fourier transform $\hat{u}$ of u in the form

$$\hat{u}(\xi) = (\xi+i0)^{-\nu/2-\delta-1} + \hat{c}_M(\xi+i0)^{-\nu/2-\delta-M-1}\log(\xi+i0) + \hat{d}_M(\xi+i0)^{-\nu/2-\delta-M-1}. \tag{B.49}$$

We shall show that there exists $\hat{c}_M \neq 0$ (hence $c_M \neq 0$) such that

$$p_+a_0(D)q^M(D)u \in \mathrm{H}^{s-\nu+M+1}_{\mathrm{loc}}(\overline{\mathbb{R}}_+). \tag{B.50}$$

Since there holds $p_+(D-i0)^{\nu/2-\delta}(1-p_+)=0$, we have the identities

$$p_+a_0(D)q^M(D)u = p_+a_0^+\,(D-i0)^{\nu/2-\delta}\,(D+i0)^{\nu/2+\delta}\,q^M(D)u = p_+a_0^+\,(D-i0)^{\nu/2-\delta}\,p_+\,(D+i0)^{\nu/2+\delta}\,q^M(D)u.$$

Therefore, if we prove that $p_+(D+i0)^{\nu/2+\delta}q^M(D)u$ belongs to the space $\mathrm{H}^{s-(\nu/2)-\delta+M+1}_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$, we have proved (B.50). Consider therefore the Fourier transform $w(\xi)$ of $(D+i0)^{\nu/2+\delta}q^M(D)u$ if $\hat{u}$ has the form (B.49):

$$\begin{aligned} w(\xi) &= (\xi+i0)^{\nu/2+\delta}q^M(\xi)\hat{u}(\xi) \\ &= \hat{c}_M(\xi+i0)^{-M-1}\log(\xi+i0) \\ &\quad + \left(q_M^+\chi_+(\xi)+q_M^-\chi_-(\xi)\right)|\xi|^{-M}(\xi+i0)^{-1}+w_M(\xi) \end{aligned}$$

where

$$\begin{aligned} w_M(\xi) &= \sum_{j=0}^{M-1} q_j\,(\xi+i0)^{-j-1}+\hat{d}_M\,(\xi+i0)^{-M-1} \\ &\quad + \hat{c}_M\log(\xi+i0)\,\mathrm{O}\left(|\xi|^{-M-2}\right)+\mathrm{O}\left(|\xi|^{-M-2}\right). \end{aligned}$$

Thus we can discard w_M, because $p_+\mathscr{F}^{-1}w_M$ is sufficiently regular. Now we use the additive decomposition, see Lemma B.7, for $\xi\neq 0$:

$$\begin{aligned} &\left(q_M^+\chi_+(\xi)+q_M^-\chi_-(\xi)\right)|\xi|^{-M}(\xi+i0)^{-1} \\ &\quad = q_M^+(\xi+i0)^{-M-1}+\frac{1}{2i\pi}\left((-1)^M q_M^- - q_M^+\right)\Big((\xi+i0)^{-M-1}\log(\xi+i0) \\ &\qquad -(\xi-i0)^{-M-1}\log(\xi-i0)\Big). \end{aligned}$$

The only non-regular contribution to $p_+\mathscr{F}^{-1}w$ comes from the term

$$(\xi+i0)^{-M-1}\log(\xi+i0)\left[\hat{c}_M+\frac{1}{2i\pi}\left((-1)^M q_M^- - q_M^+\right)\right].$$

This term is absent if

$$\hat{c}_M+\frac{1}{2i\pi}\left((-1)^M q_M^- - q_M^+\right)=0. \tag{B.51}$$

We see that the possibility of having $\hat{c}_M\neq 0$ together with condition (B.51) is a consequence of the violation of equality $(\mathfrak{H}_{\mathrm{B5}})$ for $j=M$. The proof is complete. ■

B.VI. Auxiliary Results on ΨDO

We need some results for pseudodifferential operators (ΨDO) of one variable acting on functions of n variables, and also the connection between ΨDO in n variables and reduced ΨDO in one variable. The suitable function spaces were introduced in Section B.I. Here, we only need the "model" domain for the boundary of $\mathscr{M}$, that is $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times\mathbb{R}_+$ with coordinates $x=(x',x_n)$ and dual coordinates $\xi=(\xi',\xi_n)$.

The following lemma is a particular case of Theorem 1.11 and Lemma 2.9 in Chkadua and Duduchava (2001).

Lemma B.10. *Let the symbol* $\mathrm{b} = \mathrm{b}(x;\xi_n)$ *satisfy* $\partial_x^\alpha \partial_{\xi_n}^k \mathrm{b}(x;\xi_n) = \mathrm{O}(|\xi_n|^{\nu-k})$ *as* $|\xi_n| \to \infty$ *for all* $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $\xi_n \in \mathbb{R}$. *Let* $\kappa \in \mathbb{N}_0$, $s \in \mathbb{R}$. *Then the pseudodifferential operator* $\mathrm{b}(x;D_n)$ *is continuous between anisotropic Bessel potential spaces*:

$$\mathrm{b}(x;D_n) : \mathrm{H}^{(\infty,s),\kappa}(\mathbb{R}^n) \longrightarrow \mathrm{H}^{(\infty,s-\nu),\kappa}(\mathbb{R}^n). \tag{B.52}$$

If, in addition, $\operatorname{supp}\mathrm{b}(x,\cdot)$ *is compact, for all* $x \in \mathbb{R}^n$, *then the operator* $\mathrm{b}(x;D_n)$ *is a smoothing operator*:

$$\mathrm{b}(x;D_n) : \mathrm{H}^{(\infty,s),\kappa}(\mathbb{R}^n) \longrightarrow \mathscr{C}^\infty(\mathbb{R}^n) \tag{B.53}$$

Proof. It is easy to check that the operator

$$\mathrm{b}(x;D_n) : \mathrm{H}^{(\mu,s),\kappa}(\mathbb{R}^n) \longrightarrow \mathrm{H}^{(\mu-\sigma(\nu),s-\nu),\kappa}(\mathbb{R}^n). \tag{B.54}$$

is bounded, where

$$\sigma(\nu) := \begin{cases} 0 & \text{for} \quad \nu > 0, \\ |\nu| & \text{for} \quad \nu < 0. \end{cases}$$

In fact, the boundedness (B.54) follows from the Mikhlin–Hörmander theorem on multipliers since for all $|\alpha| \le n$

$$\forall \xi = (\xi', \xi_n) \in \mathbb{R}^n, \quad \xi^\alpha \partial_\xi^\alpha \left[\frac{\langle \xi' \rangle^{\mu-\sigma(\nu)} \langle \xi \rangle^{s-\nu} \langle \xi \rangle^\nu}{\langle \xi' \rangle^\mu \langle \xi \rangle^s} \right] \le 1.$$

The boundedness (B.52) is a consequence of Eq. (B.54).

As for Eq. (B.53), it follows from Eq. (B.52) because the symbol b satisfies $\partial_x^\alpha \partial_{\xi_n}^k \mathrm{b}(x;\xi_n) = \mathrm{O}(|\xi_n|^{\nu-k})$ for arbitrary $\nu < 0$. ■

The following lemma generalizes Eskin's Wiener–Hopf technique from the scalar one-dimensional case, see Proposition B.9, to systems of multidimensional pseudodifferential equations.

Lemma B.11. *Let us consider the principal part* $\mathbf{a}_{\mathrm{pr}}$ *of the symbol* $\mathbf{a}$ *in Eq.* (B.13) *with the ellipticity condition* ($\mathfrak{H}_{\mathrm{B1}}$). *We introduce*

$$\mathbf{a}^\infty(x';\xi_n) := \langle \xi_n \rangle^\nu \mathbf{a}_{\mathrm{pr}}(x', 0; 0, +1). \tag{B.55}$$

Let $s, \nu \in \mathbb{R}$ *such that* $(\nu/2) - (1/2) < s < (\nu/2) + (1/2)$, $\kappa \in \mathbb{N}_0$. *Then the system of equations*

$$\boldsymbol{p}_+ \mathbf{a}^\infty(x';D_n)\boldsymbol{u} = \boldsymbol{g}, \quad \boldsymbol{g} \in \mathrm{H}^{(\infty,s-\nu),\kappa}(\mathbb{R}^n_+)^N, \tag{B.56}$$

where $\boldsymbol{p}_+$ *is the restriction from* $\mathbb{R}^n$ *to* $\mathbb{R}^n_+$, *has a unique solution* $\boldsymbol{u} \in \widetilde{\mathrm{H}}^{(\infty,s),\kappa}(\mathbb{R}^n_+)^N$, *represented by the formula*

$$\boldsymbol{u} = (D_n + i)^{-(\nu/2)} \chi_+ (D_n - i)^{-(\nu/2)} [\mathbf{a}_{\mathrm{pr}}(x', 0; 0, 1)]^{-1} \boldsymbol{g}, \tag{B.57}$$

where $\chi_+(x_n)$ *is the characteristic function of the half space* $\mathbb{R}^n_+$.

For arbitrary $K \in \mathbb{N}$, $K \le \kappa$, *and* $\boldsymbol{g} \in \mathrm{H}^{(\infty, s-\nu+K), \kappa}(\mathbb{R}^n_+)$ *this solution has the following asymptotic expansions*

$$\begin{aligned} \boldsymbol{u}(x', x_n) &= \sum_{k=0}^{K-1} x_n^{\nu/2+k} e^{-x_n} \boldsymbol{d}^k(x') + \boldsymbol{u}_{\mathrm{rem},K}(x', x_n), \quad \boldsymbol{u}_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{(\infty, s+K), \kappa}(\mathbb{R}^n_+)^N \\ &= \sum_{k=0}^{K-1} x_n^{\nu/2+k} \boldsymbol{d}_0^k(x') + \boldsymbol{u}^0_{\mathrm{rem},K}(x', x_n), \quad \boldsymbol{u}^0_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{(\infty, s+K), \kappa}(\mathbb{R}^n_+)^N. \end{aligned}$$

with the $\mathscr{C}^\infty(\mathbb{R}^{n-1})$ *coefficients* $\boldsymbol{d}^k(x')$ *defined as*

$$\frac{e^{-(\pi/2)i(\nu/2+k+1)}}{\Gamma(\nu/2+k+1)} \Big((D_n - i)^{-(\nu/2)+k} [\mathbf{a}_{\mathrm{pr}}((x', 0; 0, 1)]^{-1} \boldsymbol{g} \Big)(x', 0),$$

and

$$\boldsymbol{d}_0^k(x') := \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{(k-\ell)!} \boldsymbol{d}^\ell(x'), \quad k = 1, \ldots, M.$$

For the proof see Chkadua and Duduchava (2001, Lemma 2.6). Note that, by its mere definition, $\mathbf{a}^\infty$ satisfies itself condition $(\mathfrak{H}_{\mathrm{B4}})$.

The following Lemma B.12 will serve for the evaluation of the terms and the remainders in the Taylor expansions which will provide the next Lemma B.13.

Lemma B.12. *Let* $\mathrm{b}(x; D)$ *be a* ΨDO *such that for a* $\overline{m} \in \mathbb{N}$ *its symbol satisfies* $\mathrm{b}(x; \xi) = x_n^{\overline{m}} \breve{\mathrm{b}}(x; \xi)$ *with* $\breve{\mathrm{b}}$ *in* $S^\nu(\overline{\mathbb{R}^n_+} \times \mathbb{R}^n)$. *We suppose that, moreover, there exists an integer* $\overline{k} \ge 0$ *such that* $\partial_x^\alpha \partial_\xi^\gamma \breve{\mathrm{b}}(x; \xi) = \mathrm{O}(|\xi'|^{\overline{k}-|\gamma'|} |\xi_n|^{\nu-\overline{k}-\gamma_n})$ *for all* α *and* $\gamma = (\gamma', \gamma_n) \in \mathbb{N}^n$. *Then for all* μ, $s \in \mathbb{R}$ *and* $\kappa \ge m$, $\mathrm{b}(x; D)$ *is bounded between the spaces*:

$$\mathrm{b}(x; D) : \mathrm{H}^{(\mu, s), \kappa}(\mathbb{R}^n) \longrightarrow \mathrm{H}^{(\mu-\overline{k}, s+\overline{k}+\overline{m}-\nu), \kappa-\overline{m}}(\mathbb{R}^n).$$

Lemma B.13. *Let* $j \in \mathbb{N}_0$ *and let us consider the homogeneous part* $\mathbf{a}_j$ *of degree* $\nu - j$ *of the symbol* $\mathbf{a}$ *in Eq.* (B.13). *For any* $K \in \mathbb{N}$, *there holds the expansion of the symbol* $\mathbf{a}_j$

$$\mathbf{a}_j(x; \xi) = \sum_{m+|\gamma'| \le K-1} x_n^m (\xi')^{\gamma'} \breve{\mathbf{a}}_{j;m,\gamma'}(x'; \mathrm{sgn}\ \xi_n)\, \xi_n^{-j-|\gamma'|} |\xi_n|^\nu + \mathbf{a}_{j;\mathrm{rem},K} \tag{B.58}$$

with $\breve{\mathbf{a}}_{j;m,\gamma'}(x'; \omega) = (1/m!)\,(1/\gamma'!)\, \omega^{j+|\gamma'|} \partial^m_{x_n} \partial^{\gamma'}_{\xi'} \mathbf{a}_j(x', 0; 0, \omega)$, $x' \in \mathbb{R}^{n-1}$, $\omega = \pm 1$, *and* $\mathbf{a}_{j;\mathrm{rem},K}$ *bounded between the spaces*

$$\mathbf{a}_{j;\mathrm{rem},K}(x; D) : \mathrm{H}^{(\infty, s), \infty}(\mathbb{R}^n)^N \longrightarrow \mathrm{H}^{(\infty, s+K-\nu), \infty}(\mathbb{R}^n)^N.$$

If condition $(\mathfrak{H}_{\mathrm{B4}})$ *holds, then* $\breve{\mathbf{a}}_{j;m,\gamma'}(x'; \omega) = \breve{\mathbf{a}}_{j;m,\gamma'}(x')$ *does not depend on* ω.

Proof. The Taylor formula, applied at $x_n = 0$, and at $|\xi_n|^{-1}\xi' = 0$, gives that $\mathbf{a}_j(x', r; \xi', \xi_n)$ is equal to

$$\sum_{m=0}^{K-1} \frac{x_n^m}{m!} (\partial_{x_n}^m \mathbf{a}_j)(x', 0; |\xi_n|^{-1}\xi', \operatorname{sgn} \xi_n)|\xi_n|^{\nu-j} + x_n^K \mathbf{a}_{j;\,\mathrm{rem},\,K}^{(1)}(x;\xi)$$
$$= \sum_{m=0}^{K-1} \frac{x_n^m}{m!} \sum_{|\gamma'|=0}^{K-1-m} |\xi_n|^{\nu-j-|\gamma'|} \frac{(\xi')^{\gamma'}}{(\gamma')!} (\partial_{x_n}^m \partial_{\xi'}^{\gamma'} \mathbf{a}_j)(x', 0; 0, \operatorname{sgn} \xi_n) + \mathbf{a}_{j;\,\mathrm{rem},\,K}(x;\xi),$$

where the remainder can be written as

$$\mathbf{a}_{j;\mathrm{rem},\,K}(x;\xi) = x^K \mathbf{a}_{j;\mathrm{rem},\,K}^{(1)}(x;\xi) + \sum_{m=0}^{K-1} x_n^m \mathbf{a}_{j;\mathrm{rem},\,m,\,K-m}^{(2)}(x';\xi),$$

where $x_n^K \mathbf{a}_{j;\mathrm{rem},\,K}^{(1)}$ satisfies the assumptions of Lemma B.12 with $\overline{m} = K$ and $\overline{k} = 0$, and $x_n^m \mathbf{a}_{j;\mathrm{rem},\,m,\,K-m}^{(2)}$ with $\overline{m} = m$ and $\overline{k} = K - m$. Taking $\mu = \infty$ and $\kappa = \infty$, we obtain the lemma. ■

A standard Taylor expansion of the function $\langle \xi_n \rangle^\nu$ at $\xi_n = \pm\infty$ yields the following expansion of the symbol $\mathbf{a}^\infty$, Eq. (B.55):

Lemma B.14. *Let us consider the symbol* $\mathbf{a}^\infty$ *defined in* Eq. (B.55). *For any integer* $K \in \mathbb{N}$, *there holds the expansion*

$$\mathbf{a}^\infty(x';\xi_n) = \sum_{j \le K-1} \breve{\mathbf{a}}_j^\infty(x')\, \xi_n^{-j} |\xi_n|^\nu + \mathbf{a}_{\mathrm{rem},K}^\infty(x';\xi_n) \tag{B.59}$$

with $\breve{\mathbf{a}}_0^\infty(x') = \mathbf{a}_0(x', 0; 0, +1)$, $\breve{\mathbf{a}}_j^\infty(x') = c_j \mathbf{a}_0(x', 0; 0, +1)$ *with* $c_j \in \mathbb{R}$, *and* $\mathbf{a}_{\mathrm{rem},K}^\infty$ *is a bounded operator between the spaces*

$$\mathbf{a}_{\mathrm{rem},K}^\infty(x'; D_n) : \mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n)^N \longrightarrow \mathrm{H}^{(\infty,s+K-\nu),\infty}(\mathbb{R}^n)^N.$$

B.VII. Proof of the Main Theorem of Part B

We are going to prove Theorem B.4. Let us start by reformulation of the conditions of Eq. (B.14): we consider

$$\boldsymbol{\phi} \in \widetilde{\mathrm{H}}^{(\mu,s),\kappa}(\mathscr{M})^N \quad \text{such that} \quad \mathbf{a}(\mathscr{x}; D_{\mathscr{x}})\boldsymbol{\phi} = \boldsymbol{g},$$
$$\text{with} \quad \boldsymbol{g} \in \mathrm{H}^{(\mu,s-\nu),\kappa}(\mathscr{M})^N \tag{B.60}$$

for arbitrary $-\infty < \mu \le \infty$. In Chkadua and Duduchava (2001, Theorem 1.12) it is proved that the system (B.14) is Fredholm (or is uniquely solvable) if and only if the system (B.60) is Fredholm (is uniquely solvable) and these equations have equal dimensions of kernels and cokernels.

Since the assertion is local, we can suppose that our domain is the half-space $\mathbb{R}^n_+$, and all functions and symbols are compactly supported in the variable $x \in \mathbb{R}^n_+$. We recall that $x = (x', x_n)$ and its dual variable is $\xi = (\xi', \xi_n)$.

Homogeneous symbols and the kernels of the corresponding ΨDO with negative order have singularities at 0. Multiplying them by a function $\chi^0 \in \mathscr{C}^\infty(\mathbb{R})$, where $\chi^0(\xi_n) = 0$ for $|\xi_n| < 1$ and $\chi^0(\xi_n) = 1$ for $|\xi_n| > 2$ we cut the singularity off. The perturbation operator is smoothing: $[I - \chi^0(D_n)]\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ for arbitrary $\varphi \in \mathrm{H}^{(\infty,\mu),\kappa}(\mathbb{R}^n)$ (see Lemma B.10), and will be ignored. Although we do not write the cutoff function, we suppose it is present and forget about singularities of symbols at $\xi_n = 0$.

Since $\mathscr{C}_0^\infty(\overline{\mathbb{R}^n_+})^N \subset \mathrm{H}^{(\infty, s-\nu+M+1),\infty}(\mathbb{R}^n_+)^N$, for any $M \in \mathbb{N}_0$, it is sufficient to derive the asymptotics for a solution of Eq. (B.60). Relying on the expansion of the classical symbol $\mathbf{a}(x;\xi)$:

$$\mathbf{a} = \sum_{j=0}^{M} \mathbf{a}_j + \mathbf{a}_{\mathrm{rem},\,M+1}, \quad \mathbf{a}_j \in \mathrm{S}^{\nu-j}_{\mathrm{hom}}(\mathbb{R}^n_+ \times \mathbb{R}^n)^{N\times N},$$

$$\mathbf{a}_{\mathrm{rem},\,M+1} \in \mathrm{S}^{\nu-M-1}_{\mathrm{cl}}(\mathbb{R}^n_+ \times \mathbb{R}^n)^{N\times N} \tag{B.61}$$

we will apply induction on M, starting with the case $M = 0$.

For $M = 0$, the Eq. (B.60) (with $\mathscr{M} = \mathbb{R}^n_+$, as agreed) is written in the following equivalent form

$$\boldsymbol{p}_+ \mathbf{a}^\infty(x'; D_n)\boldsymbol{\phi} = \boldsymbol{g}^\infty, \tag{B.62}$$

where $\mathbf{a}^\infty(x';\xi_n)$ is defined in Eq. (B.55) and

$$\boldsymbol{g}^\infty := \boldsymbol{g} - \mathbf{a}_{\mathrm{rem},1}(x; D_x)\boldsymbol{\phi} - [\mathbf{a}_0(x; D_x) - \mathbf{a}^\infty(x'; D_n)]\boldsymbol{\phi}.$$

We observe

(i) The remainder

$$\mathbf{a}_{\mathrm{rem},1}(x; D_x) \colon \mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n)^N \longrightarrow \mathrm{H}^{(\infty,s+1-\nu),\infty}(\mathbb{R}^n)^N$$

is bounded.

(ii) Lemma B.13 for $j = 0$, $K = 1$ gives that

$$\mathbf{a}_0(x;\xi) = \mathbf{a}_0(x', 0; 0, \operatorname{sgn}\xi_n)\,|\xi_n|^\nu + \mathbf{a}_{0;\mathrm{rem},1}(x,\xi)$$

with $\mathbf{a}_{0;\mathrm{rem},1}(x; D_x) \colon \mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n)^N \longrightarrow \mathrm{H}^{(\infty,s+1-\nu),\infty}(\mathbb{R}^n)^N$ bounded.

(iii) Lemma B.14 for $j = 0$, $K = 1$ gives that

$$\mathbf{a}^\infty(x;\xi_n) = \mathbf{a}_0(x', 0; 0, +1)\,|\xi_n|^\nu + \mathbf{a}^\infty_{\mathrm{rem},1}(x,\xi_n)$$

with $\mathbf{a}^\infty_{\mathrm{rem},1}(x; D_n) : \mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n)^N \longrightarrow \mathrm{H}^{(\infty,s+1-\nu),\infty}(\mathbb{R}^n)^N$ bounded.

Condition $(\mathfrak{H}_{\mathrm{B4}})$ yields that $\mathbf{a}_0(x', 0; 0, \operatorname{sgn}\xi_n) = \mathbf{a}_0(x', 0; 0, +1)$.

Therefore $\mathbf{a}_0 - \mathbf{a}^\infty = \mathbf{a}_{0;\mathrm{rem},1} - \mathbf{a}^\infty_{\mathrm{rem},1}$. Finally, we deduce from the regularity $\boldsymbol{\phi} \in \mathrm{H}^{(\infty,s),\infty}(\mathbb{R}^n)^N$, that

$$\boldsymbol{g}^\infty \in \mathrm{H}^{(\infty,s+1-\nu),\infty}(\mathbb{R}^n)^N. \tag{B.63}$$

Invoking Lemma B.11 we derive the expansion (B.23) for $K = 1$:

$$\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \boldsymbol{\phi}_{\text{rem},1},$$

with

$$\begin{aligned} &\boldsymbol{\phi}_0(x', x_n) = \boldsymbol{d}^0(x')x_n^{\nu/2}e^{-x_n}, \boldsymbol{d}^0 \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N, \\ &\quad \boldsymbol{\phi}_{\text{rem},1} \in \widetilde{\mathrm{H}}^{(\infty, s+1), \infty}(\mathbb{R}^n_+)^N. \end{aligned} \tag{B.64}$$

Now let $M \geq 1$ and suppose we have proved

$$\boldsymbol{\phi} = \sum_{k=0}^{M-1} \boldsymbol{\phi}_k + \boldsymbol{\phi}_{\text{rem},M},$$

with

$$\begin{aligned} &\boldsymbol{\phi}_k(x', x_n) = \boldsymbol{d}^k(x')x_n^{\nu/2+k}e^{-x_n}, \quad \boldsymbol{d}^k \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N, \\ &\quad \boldsymbol{\phi}_{\text{rem},\,M} \in \widetilde{\mathrm{H}}^{(\infty, s+M), \infty}(\mathbb{R}^n_+)^N. \end{aligned} \tag{B.65}$$

It can be prove that $\boldsymbol{\phi}_k \in \widetilde{\mathrm{H}}^{(\infty, s+k), \infty}(\mathbb{R}^n_+)^N$, because $\nu/2 - s > -(1/2)$, see Chkadua and Duduchava, (2001, Eq. (2.30)). Then the right hand side $\boldsymbol{g}^\infty$ of Eq. (B.62) can be represented as follows

$$\boldsymbol{g}^\infty = \boldsymbol{g}^1_{\text{rem},\,M+1} - \sum_{j=1}^{M}\sum_{k=0}^{M-j} \boldsymbol{p}_+\mathbf{a}_j(x; D_x)\boldsymbol{\phi}_k - \sum_{k=0}^{M-1} \boldsymbol{p}_+[\mathbf{a}_0(x; D_x) - \mathbf{a}^\infty(x'; D_n)]\boldsymbol{\phi}_k, \tag{B.66}$$

where

$$\begin{aligned} \boldsymbol{g}^1_{\text{rem},\,M+1} &= \boldsymbol{g} - \boldsymbol{p}_+\mathbf{a}_{\text{rem},\,M+1}(x; D_x)\boldsymbol{\phi} - \sum_{j=1}^{M} \boldsymbol{p}_+\mathbf{a}_j(x; D_x)\boldsymbol{\phi}_{\text{rem},\,M-j+1} \\ &\quad - \boldsymbol{p}_+[\mathbf{a}_0(x; D_x) - \mathbf{a}^\infty(x'; D_n)]\boldsymbol{\phi}_{\text{rem},\,M}. \end{aligned}$$

It is clear from the arguments used for the step $M = 0$ that $\boldsymbol{g}^1_{\text{rem},\,M+1}$ belongs to the space $\mathrm{H}^{(\infty, s-\nu+M+1), \infty}(\mathbb{R}^n_+)^N$.

We now use the expansion (B.58) with $K = M + 1 - j - k$ for the term $\mathbf{a}_j(x; D_x)\boldsymbol{\phi}_k$, and the expansion (B.59) with $K = M + 1 - k$ for the term $\mathbf{a}^\infty(x'; D_n)\boldsymbol{\phi}_k$. Taking into account that condition ($\mathfrak{H}_{\text{B4}}$) holds, we obtain

$$\boldsymbol{g}^\infty = \boldsymbol{g}^2_{\text{rem},\,M+1} - \sum_{k=0}^{M-1} \sum_{\substack{j,\,m,\,\gamma' \\ 0<j+m+|\gamma'|\leq M-k}} \mathbf{b}_{j;\,m,\,\gamma'}(x; D_x)\,\boldsymbol{\phi}_k, \tag{B.67}$$

where $\boldsymbol{g}^2_{\text{rem},\,M+1}$ belongs to $\mathrm{H}^{(\infty, s-\nu+M+1), \infty}(\mathbb{R}^n_+)^N$ and

$$\mathbf{b}_{j;\,m,\,\gamma'}(x; \xi) = x_n^m\,(\xi')^{\gamma'}\,\check{\mathbf{b}}_{j;\,m,\,\gamma'}(x')\,\xi_n^{-j-|\gamma'|}|\xi_n|^{\,\nu},$$

with $\breve{\mathbf{b}}_{j;\,m,\,\gamma'}(x')$ defined for $x' \in \mathbb{R}^{n-1}$ as follows

$$\breve{\mathbf{b}}_{j;\,m,\,\gamma'}(x') := \begin{cases} \breve{\mathbf{a}}_{j;\,m,\,\gamma'}(x') & \text{if} \quad m + |\gamma'| \neq 0, \\ \breve{\mathbf{a}}_{j;\,0,\,0}(x') - \breve{\mathbf{a}}_j^{\infty}(x') & \text{if} \quad m = 0, \ \gamma' = 0. \end{cases}$$

Now we use formula (B.57), Lemma B.11 to invert the operator $\mathbf{a}^{\infty}(x'; D_n)$, and from Eq. (B.62) $\boldsymbol{p}_+ \mathbf{a}^{\infty}(x'; D_n)\boldsymbol{\phi} = \boldsymbol{g}^{\infty}$ with the expansion (B.67) of $\boldsymbol{g}^{\infty}$, we find

$$\boldsymbol{\phi} = (D_n + i)^{-(\nu/2)} \boldsymbol{p}_+ (D_n - i)^{-(\nu/2)} \breve{\mathbf{a}}_0^{-1}(x') \left[\sum_{k=0}^{M-1} \sum_{\substack{j,\,m,\,\gamma' \\ 0<j+m+|\gamma'|\leq M-k}} \mathbf{b}_{j;\,m,\,\gamma'}(x; D_x)\, \boldsymbol{\phi}_k \right] + \left[\boldsymbol{p}_+(\mathbf{a}^{\infty}(x'; D_n)\right]^{-1} \boldsymbol{g}^2_{\text{rem},\,M+1}, \tag{B.68}$$

where $\breve{\mathbf{a}}_0(x') = \mathbf{a}_{\text{pr}}(x', 0; 0, 1)$. Recalling Lemma B.6 (i), and using a Taylor expansion at $\xi_n = \infty$, we find the following, cf (B.65), for $\boldsymbol{\phi}_k(x', x_n)$:

$$\begin{aligned} \mathscr{F}_{x_n \to \xi_n}[\boldsymbol{\phi}_k] &= \mathscr{F}_{x_n \to \xi_n}\left[x_n^{\nu/2+k} \boldsymbol{d}^k(x') e^{-x_n}\right] \\ &= (\xi_n + i)^{-\nu/2-k-1} e^{(\pi/2)(\nu/2+k+1)i}\, \Gamma\Big(\frac{\nu}{2} + k + 1) \boldsymbol{d}^k(x'\Big) \\ &= \sum_{q=0}^{M} \boldsymbol{d}^{kq}(x')(\xi_n + i0)^{-(\nu/2)-k-q-1} \\ &\quad + (\xi_n + i)^{-(\nu/2)-M-q-2}\, \boldsymbol{d}^k_{\text{rem},\,M}(x'). \end{aligned} \tag{B.69}$$

and the last summand is ignored in the sequel because it contributes into the smooth remainder term. From Eqs. (B.68) and (B.69), we see that modulo a remainder $\boldsymbol{\phi}^1_{\text{rem};\,M+1}$ in the space $\mathrm{H}^{(\infty,\,s+M+1),\,\infty}(\mathbb{R}^n_+)^N$, $\boldsymbol{\phi}$ is a finite sum of terms $\boldsymbol{\varphi}$ which have the generic form

$$\boldsymbol{\varphi} = (D_n + i)^{-(\nu/2)} \boldsymbol{\psi} \tag{B.70}$$

with

$$\boldsymbol{\psi} = \boldsymbol{p}_+ (D_n - i)^{-(\nu/2)} \breve{\mathbf{a}}_0^{-1}(x')\, \boldsymbol{h}(x) \tag{B.71}$$

with

$$\boldsymbol{h} = x_n^m \mathscr{F}_{\xi \to x}\left\{(\xi')^{\gamma'} \breve{\mathbf{b}}(x')\, \xi_n^{-\ell} |\xi_n|^{\,\nu} \times \mathscr{F}_{x' \to \xi'}\left[\boldsymbol{d}(x')\right](\xi_n + i0)^{-(\nu/2)-q-1}\right\}, \tag{B.72}$$

for $m,\ \ell,\ q \in \mathbb{N}_0,\ \gamma' \in \mathbb{N}^{n-1}$, and $\breve{\mathbf{b}} \in \mathscr{C}^{\infty}(\mathbb{R}^{n-1})^{N \times N}$, $\boldsymbol{d} \in \mathscr{C}^{\infty}(\mathbb{R}^{n-1})$. Let us study $\boldsymbol{h}(x)$ first:

$$\begin{aligned} \boldsymbol{h} &= x_n^m\, \breve{\mathbf{b}}(x') \left[(i\partial_{x'})^{\gamma'} \boldsymbol{d}\right](x') \times \mathscr{F}_{\xi_n \to x_n}\left\{(\xi_n + i0)^{-(\nu/2)-q-1}\, \xi_n^{-\ell} |\xi_n|^{\,\nu}\right\} \\ &= \boldsymbol{d}_1(x')\, \mathscr{F}_{\xi_n \to x_n}\left\{\partial^m_{\xi_n}\left[(\xi_n + i0)^{\nu/2-q-\ell-1} \theta_\nu(\xi_n)\right]\right\}, \end{aligned} \tag{B.73}$$

with $\boldsymbol{d}_1(x') := (-i)^m \breve{\mathbf{b}}(x')\,[(i\partial_{x'})^{\gamma'}\boldsymbol{d}](x') \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N$, and where we have used the formula, cf Lemma B.7,

$$|t|^\sigma = \theta_\sigma(t)\,(t+i0)^\sigma \ \text{ with } \ \theta_\sigma(t) = \chi_+(t) + e^{-i\pi\sigma}\chi_-(t). \tag{B.74}$$

We note that although we have taken derivatives $\partial^m_{\xi_n}|\xi_n|^\sigma$, the δ-functions do not appear due to cutoff functions (see the beginning of the proof).

Inserting expression (B.73) of $\boldsymbol{h}$ into Eq. (B.71) we find that

$$\boldsymbol{\psi}(x) = \boldsymbol{p}_+\mathscr{F}_{\xi_n\to x_n}\Big\{(\xi_n - i)^{-(\nu/2)}\,\breve{\mathbf{a}}_0^{-1}(x')\,\mathscr{F}_{x_n\to\xi_n}\big[\boldsymbol{h}(x',x_n)\big]\Big\}.$$

Using Eq. (B.73), we find that $\boldsymbol{\psi}(x)$ is equal to

$$\boldsymbol{p}_+\mathscr{F}_{\xi_n\to x_n}\Big\{(\xi_n - i)^{-(\nu/2)}(\xi_n+i0)^{\nu/2-q-\ell-m-1}\theta_\nu(\xi_n)\Big\}\boldsymbol{d}_2(x') \tag{B.75}$$

with $\boldsymbol{d}_2(x') = c\breve{\mathbf{a}}_0^{-1}(x')\boldsymbol{d}_1(x') \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N$. By expanding the function $(\xi_n - i)^{-(\nu/2)}$ as a Taylor series in $(\xi_n - i0)^{-(\nu/2)-p}$, cf (B.69), and applying the equality

$$(\xi_n - i0)^{-(\nu/2)-p}\,\theta_\nu(\xi_n) = (\xi_n + i0)^{-(\nu/2)-p}, \quad p = 0,1,\ldots$$

(see Eq. (B.74)), we get

$$\boldsymbol{\psi}(x) = \sum_{p=0}^{M}\mathscr{F}_{\xi_n\to x_n}\{(\xi_n+i0)^{-q-\ell-m-p-1}\}\boldsymbol{d}_{2,p}(x') + \boldsymbol{\psi}_{\text{rem},\,M+1}(x), \tag{B.76}$$

with $\boldsymbol{d}_{2,p} \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N$ and $\boldsymbol{\psi}_{\text{rem},\,M+1} \in \widetilde{\mathrm{H}}^{(\infty,\,s-(\nu/2)+M+1),\,\infty}(\mathbb{R}^n_+)^N$.

The restriction operator $\boldsymbol{p}_+$ in front in Eq. (B.75) was eliminated since the Fourier transform of the analytic function is supported on $\mathbb{R}_+$.

From Eqs. (B.68)–(B.76) we find

$$\begin{aligned}\boldsymbol{\phi}(x) &= \sum_{k=0}^{M-1}(D_n+i)^{-(\nu/2)}\mathscr{F}_{\xi_n\to x_n}\Big\{(\xi_n+i0)^{-k-1}\Big\}\boldsymbol{d}_{3,k}(x') + \big[\boldsymbol{p}_+\mathbf{a}^\infty(x';D_n)\big]^{-1}\boldsymbol{g}^3_{\text{rem},\,M+1}\\ &= \sum_{k=0}^{M-1}\mathscr{F}_{\xi_n\to x_n}\Big\{(\xi_n+i)^{-(\nu/2)}(\xi_n+i0)^{-k-1}\Big\}\boldsymbol{d}_{3,k}(x') + \big[\boldsymbol{p}_+\mathbf{a}^\infty(x';D_n)\big]^{-1}\boldsymbol{g}^3_{\text{rem},\,M+1}.\end{aligned}$$

By transforming $(\xi_n + i0)^{-k-1}$ into $(\xi_n + i)^{-k-1}$ as above, and using the asymptotics of $[\boldsymbol{p}_+\mathbf{a}^\infty(x';D_n)]^{-1}\boldsymbol{g}^3_{\text{rem},\,M+1}$ from Lemma B.11, we finally obtain the desired expansion

$$\boldsymbol{\phi}(x) = \sum_{k=0}^{M-1} x_n^{\nu/2+k}e^{-x_n}\,\boldsymbol{d}^k(x') + \boldsymbol{\phi}_{\text{rem},\,M+1}(x), \tag{B.77}$$

with $\boldsymbol{\phi}_{\text{rem},\,M+1} \in \mathrm{H}^{(\infty,\,s+M+1),\,\infty}(\mathbb{R}^n_+)^N$, and $\boldsymbol{d}^k \in \mathscr{C}^\infty(\mathbb{R}^{n-1})^N$. The theorem is proved. ∎

B.VIII. Symbols Satisfying the Generalized Continuity Property

From our investigation of scalar problems in dimension 1, we can conclude that ($\mathfrak{H}_{B4}$) is not the most general condition ensuring the absence of logarithms in the asymptotics (B.19). In this section, we prove that the result of Theorem B.4 together with its proof remains valid if we replace condition ($\mathfrak{H}_{B4}$) the following generalized continuity property, inspired from our 1D scalar condition ($\mathfrak{H}_{B5}$): $\exists \Lambda \in \mathbb{C}\backslash\{0\}$

$$\begin{cases} \forall x' \in \mathscr{E}, \quad \forall j \in \mathbb{N}_0, \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \quad m \in \mathbb{N}_0, \\ \partial_{x_n}^m \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, -1) = (-1)^{j+|\alpha'|} \Lambda \partial_{x_n}^m \partial_{\xi'}^{\alpha'} \mathbf{a}_j(x', 0; 0, +1). \end{cases} \tag{$\mathfrak{H}_{B6}$}$$

Condition ($\mathfrak{H}_{B6}$) also implies conditions ($\mathfrak{H}_{B2}$) and ($\mathfrak{H}_{B3}$). The corresponding result about asymptotics follows.

Theorem B.15. *Let* $\mathbf{a}(x; \xi)$ *be an elliptic classical symbol* (B.13) *of order* $\nu > -2$ *satisfying the generalized continuity property* ($\mathfrak{H}_{B6}$) *on the boundary* $\mathscr{E}$. *Let* $\delta \in \mathbb{R}$ *be such that* $e^{2i\pi\delta} = \Lambda$ *and let* s *be a Sobolev exponent such that* Eq. (B.30) *holds. Any solution* $\boldsymbol{\phi} \in \widetilde{\mathrm{H}}^s(\mathscr{M})^N$ *of the equation* $\mathbf{a}(x; D_x)\boldsymbol{\phi} = \boldsymbol{g}$ *where the right hand side* $\boldsymbol{g}$ *is* $\mathscr{C}^\infty(\mathscr{M})^N$, *has the following asymptotic expansion for any integer* $K > 0$

$$\boldsymbol{\phi} = \sum_{k=0}^{K-1} r^{\nu/2+\delta+k} \chi(r)\, \boldsymbol{d}^k(x') + \boldsymbol{\phi}_{\mathrm{rem},K}, \quad \boldsymbol{d}^k \in \mathscr{C}^\infty(\mathscr{E})^N, \quad \boldsymbol{\phi}_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{s+K}(\mathscr{M})^N. \tag{B.78}$$

Remark B.16. Since $\mathbf{a}_0$ is homogeneous elliptic of order ν, the symbol $\boldsymbol{q} := \mathbf{a}_0^{-1}\mathbf{a}$ is an elliptic symbol of order 0. It is easy to see that the condition ($\mathfrak{H}_{B6}$) on $\mathbf{a}$ is equivalent to the transmission condition on $\boldsymbol{q}$—compare with (B.32)–(B.33). ■

The proof of Theorem B.15 follows the same lines as the proof of Theorem B.4 with a few obvious modifications. For example the vector version of Proposition B.9 comes in replacement of Lemma B.11. Besides, everywhere the functions

$$(\xi_n + i0)^{\nu/2+\delta}(\xi_n - i0)^{\nu/2-\delta} \quad \text{and} \quad (\xi_n + i)^{\nu/2+\delta}(\xi_n - i)^{\nu/2-\delta}$$

replace $|\xi_n|^\nu$ and $\langle \xi_n \rangle^\nu$, respectively.

B.IX. Spatial Asymptotics of Solutions to BVP

We have already performed the first two steps of the analysis of asymptotics of the solution of the boundary value problem (A.1) by the Wiener–Hopf method: (i) the reduction to a ΨDE (A.2) on the boundary, (ii) the asymptotics of the solution of this ΨDE. There remains to derive the spatial asymptotics of the solution $\boldsymbol{u}$ to BVP (A.1), represented by the formula (B.5),

$$\boldsymbol{u} = N\boldsymbol{f} + \mathscr{D}[\boldsymbol{u}] - \mathscr{V}[T\boldsymbol{u}],$$

if we know the asymptotics of the densities $[\boldsymbol{u}]$ or $[T\boldsymbol{u}]$. Note, that since $\boldsymbol{f} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$, the summand $N\boldsymbol{f}$ only contributes to the regular part of $\boldsymbol{u}$.

Therefore, we only need to apply either the single layer potential $\mathscr{V}$, or the double layer potential $\mathscr{D}$ to a function $\boldsymbol{\phi}$ defined on $\mathscr{M}$, the asymptotic expansion of which being of the form (B.23).

Thus, let us by denote by $\mathscr{A}$ either the single layer potential $\mathscr{V}$, or the double layer potential $\mathscr{D}$, see (B.6), associated with an homogeneous elliptic second order $N \times N$ system $L(D_x)$ in $\mathbb{R}^{n+1}$ with constant real coefficients.[d] Let q be the order of $\mathscr{A}$ ($q = -1$ if $\mathscr{A} = \mathscr{V}$ and $q = 1$ if $\mathscr{A} = \mathscr{D}$). We consider $\boldsymbol{u}$ defined on Ω by

$$\boldsymbol{u}(\boldsymbol{x}) = \mathscr{A}\boldsymbol{\phi}(\boldsymbol{x}), \quad \operatorname{supp}\boldsymbol{\phi} \subset \overline{\mathscr{M}}, \quad \boldsymbol{x} \in \Omega. \tag{B.79}$$

For any $\mathscr{x}' \in \mathscr{E}$, let $\tau_1(\mathscr{x}'), \dots, \tau_\ell(\mathscr{x}')$ be all different roots of the polynomial equation

$$\det L\big(\mathscr{I}_\varkappa(\mathscr{x}')(0, 1, \tau)\big) = 0, \qquad \operatorname{Im}\tau < 0. \tag{B.80}$$

We recall that $(0, 1, \tau)$ represents the value of the dual variable $\underline{\xi} = (\xi', \xi_n, \xi_{n+1})$ and that $\mathscr{I}_\varkappa(\mathscr{x}')$ is the Jacobian of the local coordinate diffeomorphism κ, cf Definition A.1.

We assume that it is possible to enumerate $\tau_1(\mathscr{x}'), \dots, \tau_\ell(\mathscr{x}')$ so that

$$\begin{cases} \text{The multiplicities } n_1, \dots, n_\ell \text{ of } \tau_1(\mathscr{x}'), \dots, \tau_\ell(\mathscr{x}') \\ \text{are constant on } \mathscr{E}. \end{cases} \tag{$\mathfrak{H}_{B7}$}$$

Therefore the τ_m are $\mathscr{C}^\infty(\mathscr{E})$.

Since L is a $N \times N$ elliptic system of order 2, there holds

$$n_1 + \cdots + n_\ell = N$$

and since its coefficients are real, the roots of Eq. (B.80) with $\operatorname{Im}\tau > 0$ are the conjugate of the $\tau_m(\mathscr{x}')$. Let for $\mathscr{x}' \in \mathscr{E}$ and $m = 1, \dots, \ell$ the angular functions $\psi_{m,\pm}$ be defined as

$$\begin{aligned} \psi_{m,-1}(\mathscr{x}', \theta) &:= \cos\theta + \tau_m(\mathscr{x}')\sin\theta, \\ \psi_{m,+1}(\mathscr{x}', \theta) &:= \cos\theta + \overline{\tau}_m(\mathscr{x}')\sin\theta. \end{aligned} \tag{B.81}$$

Theorem B.17. *Let $\boldsymbol{\phi}$ be a N-vector function on $\mathscr{M}$ with the following infinite asymptotics without logarithms*: $\exists \mu \in \mathbb{R}$, $\forall K > 0$

$$\boldsymbol{\phi} = \sum_{k=0}^{K-1} r^{\mu+k}\chi(r)\boldsymbol{d}^k(\mathscr{x}') + \boldsymbol{\phi}_{\mathrm{rem},K},$$

[d] Here we restrict consideration to the potential operators related to a second order system. For more general results we quote Chkadua and Duduchava (2001).

with

$$\boldsymbol{d}^k \in \mathscr{C}^\infty(\mathscr{E})^N, \quad \boldsymbol{\phi}_{\mathrm{rem},K} \in \widetilde{\mathrm{H}}^{\mu+K}(\mathscr{M})^N.$$

We assume that μ is not an integer and that the $N \times N$ second order system L satisfies hypothesis $(\mathfrak{H}_{\mathrm{B7}})$. Let $\mathscr{A}$ denote either the single or the double layer potential associated with L and q its order, and let $\boldsymbol{u}$ be defined on Ω by $\mathscr{A}\boldsymbol{\phi}$.

Then for arbitrary $K \in \mathbb{N}$, the potential-type function u has in local cylindrical coordinates (x', r, θ) the following asymptotic expansion free of logarithms as well

$$\boldsymbol{u} = \sum_{m=1}^{\ell} \sum_{\omega=\pm 1} \chi(r) \Bigg[\sum_{j=0}^{n_m - 1} r^{\mu-q} \sin^j\theta\, \psi_{m,\omega}^{\mu-q-j}(x', \theta)\, \boldsymbol{d}_{m,\omega}^j(x') + \sum_{k=1}^{K+q-1} \sum_{j=0}^{p(m,k)} \sum_{|\alpha| \le N(m,k)} r^{\mu-q+k}\, \psi_{m,\omega}^{\mu-q-j+k} \sin^{\alpha_1}\theta \cos^{\alpha_2}\theta\, \boldsymbol{d}_{m,\omega}^{k,j,\alpha} \Bigg] + \boldsymbol{u}_{\mathrm{rem},K} \tag{B.82}$$

where $\boldsymbol{u}_{\mathrm{rem},K} \in \mathrm{H}_{\mathrm{loc}}^{\mu+K}(\mathbb{R}^{n+1})^N$ and the coefficients $\boldsymbol{d}_{m,\omega}^j$ and $\boldsymbol{d}_{m,\omega}^{j,k,\alpha}$ are $\mathscr{C}^\infty(\mathscr{E})$.

The proof is a direct adaptation of proofs in Chkadua and Duduchava (2000, 2001).
As a straightforward corollary of Theorems B.5 and B.17 combined with

- formulas (B.9) and (B.11) for Neumann conditions,
- formulas (B.10) and (B.12) for Dirichlet conditions,

we obtain:

Theorem B.18. *Let the $N \times N$ second order system L satisfy hypotheses $(\mathfrak{H}_{\mathrm{A1}})$, $(\mathfrak{H}_{\mathrm{A2}})$, and $(\mathfrak{H}_{\mathrm{B7}})$. Then any solution $\boldsymbol{u}$ of BVP (A.1) with $\boldsymbol{f} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$ has the following asymptotic expansion in local cylindrical coordinates (x', r, θ)*

$$\boldsymbol{u} = \sum_{m=1}^{\ell} \sum_{\omega=\pm 1} \chi(r) \Bigg[\sum_{j=0}^{n_m - 1} r^{1/2} \sin^j\theta\, \psi_{m,\omega}^{1/2-j}(x', \theta)\, \boldsymbol{d}_{m,\omega}^j(x') + \sum_{k=1}^{K-1} \sum_{j=0}^{p(m,k)} \sum_{|\alpha| \le N(m,k)} r^{1/2+k}\, \psi_{m,\omega}^{1/2+k-j}(x', \theta) \sin^{\alpha_1}\theta \cos^{\alpha_2}\theta\, \boldsymbol{d}_{m,\omega}^{k,j,\alpha} \Bigg] + \boldsymbol{u}_{\mathrm{rem},K} \tag{B.83}$$

where $\boldsymbol{u}_{\mathrm{rem},K} \in \mathrm{H}_{\mathrm{loc}}^{K}(\mathbb{R}^{n+1})^N$ and the coefficients $\boldsymbol{d}_{m,\omega}^j$ and $\boldsymbol{d}_{m,\omega}^{j,k,\alpha}$ are $\mathscr{C}^\infty(\mathscr{E})$.

For the particular case of isotropic elasticity we have to deal with the Lamé equation

$$L(D_{\boldsymbol{x}})\boldsymbol{u} = \mu \Delta \boldsymbol{u} + (\lambda + \mu)\operatorname{grad}\operatorname{div} \boldsymbol{u} = \boldsymbol{f}, \tag{B.84}$$

with a right hand side $\boldsymbol{f} \in \mathscr{C}_0^\infty(\mathbb{R}^3)$. Equation (B.80) has one triple root $\tau_1 = -i$ and for the singular functions (B.81) we get

$$\psi_{1,-1}(\theta) = e^{i\theta} \quad \text{and} \quad \psi_{1,1}(\theta) = e^{-i\theta}.$$

The asymptotics of the displacement $\boldsymbol{u}(\boldsymbol{x})$ has the form

$$\begin{aligned}\boldsymbol{u}(\mathscr{x}', r, \theta) = \sum_{\omega=\pm 1} \Bigg[\sum_{j=0}^{2} r^{1/2} \sin^j \theta e^{i\omega(1/2-j)\theta}\, \boldsymbol{d}_{\omega}^{j}(\mathscr{x}') \\ + \sum_{k=1}^{K-1} \sum_{j=0}^{p_k} \sum_{|\alpha|\le N_k} r^{1/2+k} e^{i\omega(1/2-j+k)\theta} \sin^{\alpha_1}\theta \cos^{\alpha_2}\theta\, \boldsymbol{d}_{\omega}^{k,j,\alpha}(\mathscr{x}') \Bigg] \\ + \boldsymbol{u}_{\mathrm{rem},K}(\mathscr{x}', r, \theta). \end{aligned} \tag{B.85}$$

The stress $T(\boldsymbol{x}, D_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x})$ has a similar asymptotics as the displacement, starting with the exponent $r^{-(1/2)}$ instead of $r^{1/2}$.

PART C. THE MELLIN APPROACH

C.I. General Edge Asymptotics

In our second approach, we consider the boundary value problem (A.1) as a special case of boundary value problems on domains with edges. For such problems, the method of Mellin transformation is a well-developed technique that allows precise descriptions of the solutions in the neighborhood of the edge.

The general description of solutions of problems like (A.1) on a wedge originates from Kontratiev's work (1967) and was developed in the subsequent works (Maz'ya and Plamenevskii, 1980; Maz'ya and Rossmann, 1988; Nazarov and Plamenevskii, 1994) and (Costabel and Dauge, 1993a; Dauge, 1988), among other contributions. As a preparation for our proof on the absence of logarithm, we are going to explain the general edge structure in the framework of the above papers.

We keep the local cylindrical coordinates $(\mathscr{x}', r, \theta)$ around the edge $\mathscr{E}$, see Definition A.1. As this will be of constant use, we introduce the notation y for the two normal cartesian coordinates $(\mathscr{x}_n, \mathscr{x}_{n+1})$, which will be also alternatively denoted by (y_1, y_2). Let us consider as domain for the boundary value problem the wedge $W_\omega = \mathscr{E} \times \Gamma_\omega$ where Γ_ω is the plane sector $\{y \sim (r,\theta) \mid \theta \in (-\omega, \omega)\}$ of opening 2ω. Let $\partial_\pm \Gamma_\omega$ be the two sides of Γ_ω. They correspond to the two sides $\partial_\pm W_\omega$ of W_ω. The situation which is the aim of our investigation corresponds to taking $\omega = \pi$.

But for a while, let us consider the more general case of an elliptic $N \times N$ system $L = (L_{k\ell})$ of order $2d$ complemented by two sets $B_\pm$ of $m := dN$ boundary conditions on $\partial_\pm W_\omega$. The general framework of edge asymptotics demands a supplementary condition of ellipticity along the edge, see Maz'ya and Plamenevskii (1980), Maz'ya and Rossmann (1988). A natural way to satisfy this condition is to suppose that (L, B_-, B_+) is associated with a coercive form b on H^d, see Dauge (1988), as stated in Part A (but now with order $2d$ and more general boundary conditions).

Thus, let us consider $\boldsymbol{u}$ solution in $\mathrm{H}^d(W_\omega)^N$ of the following boundary value problem with a right hand side $\boldsymbol{f} \in \mathscr{C}^\infty(\overline{W}_\omega)^N$

$$\begin{cases} L\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad W_\omega \\ \gamma_\pm B_\pm \boldsymbol{u} = 0 & \text{on} \quad \partial_\pm W_\omega. \end{cases} \tag{C.1}$$

The solution $\boldsymbol{u}$ has an infinite edge asymptotics, mainly determined by the expansion of the problem (L, B_-, B_+) in "homogeneous components" (L^j, B_-^j, B_+^j), $j \geq 0$, with respect to the variables y normal to the edge $\mathscr{E}$.

In the coordinates $(x', y) \in \mathscr{E} \times \Gamma_\omega$, the system L has variable coefficients, in general. We write it with the notation

$$L = L(x', y; \partial_{x'}, \partial_y).$$

For any $x' \in \mathscr{E}$, let $L^0[x']$ be the principal part of the operator $L(x', 0; 0, \partial_y)$. We denote similarly the boundary operators in local coordinates by $B_\pm(x', r; \partial_{x'}, \partial_y)$ and their principal parts in $y = 0$ by $B_\pm^0[x']$.

For each x' fixed in $\mathscr{E}$, the *singular exponents* associated with x' are the complex numbers λ such that there exists nonzero solutions $\boldsymbol{\psi} = \boldsymbol{\psi}(\theta)$ to the problem

$$\begin{cases} L^0[x'](r^\lambda \boldsymbol{\psi}) = 0 & \text{in} \quad \Gamma_\omega \\ \gamma_\pm B_\pm^0[x'](r^\lambda \boldsymbol{\psi}) = 0 & \text{on} \quad \partial_\pm \Gamma_\omega. \end{cases} \tag{C.2}$$

Due to the dependency on x' of the coefficients of $(L^0, B_\pm^0)$, the set $\Lambda[x']$ of such λ varies in general with $x' \in \mathscr{E}$, see Maz'ya and Rossmann (1988), Costabel and Dauge (1993a).

The Ansatz for solutions in the form $r^\lambda \boldsymbol{\psi}(\theta)$ has a close relation with the *Mellin transform* which allows a diagonalization of $(L^0, B_\pm^0)[x']$ for each x'. Let us recall the Mellin transform $\lambda \mapsto \mathfrak{M}(f)(\lambda)$ of a function f defined on $\mathbb{R}_+$:

$$\mathfrak{M}(f)(\lambda) = \int_0^\infty r^\lambda f(r) \,\frac{dr}{r}.$$

We have the formula $\mathfrak{M}(r\partial_r f)(\lambda) = \lambda \mathfrak{M}(f)(\lambda)$ which is the foundation of the Mellin symbolic calculus. Thus the Mellin symbol $\lambda \mapsto \mathfrak{A}^0[x'](\lambda)$ of problem (C.2) is defined after writing L^0 and $B_\pm^0$ in cylindrical coordinates as

$$r^{-2d} \mathscr{L}^0[x'](\theta; r\partial_r, \partial_\theta)$$

and

$$r^{-\rho_{\pm,h}} \mathscr{B}_{\pm,h}^0[x'](\theta; r\partial_r, \partial_\theta), \quad \rho_{\pm,h} = \deg B_{\pm,h}, \ (h = 1, \ldots, m),$$

by $\mathfrak{A}^0[x'](\lambda)$:

$$\begin{aligned} &\mathrm{H}^{2d}(-\omega, \omega)^N \longrightarrow \mathrm{L}^2(-\omega, \omega)^N \times \mathbb{C}^{2m} \\ &\varphi \longmapsto (\mathscr{L}^0[x'](\lambda, \partial_\theta)\varphi, \gamma_\pm \mathscr{B}_\pm^0[x'](\lambda, \partial_\theta)\varphi). \end{aligned}$$

For each $x' \in \mathscr{E}$, $\lambda \mapsto \mathfrak{A}^0[x'](\lambda)^{-1}$ is meromorphic in $\mathbb{C}$ and the set of its poles is $\Lambda[x']$.

It is possible to classify the singularities occurring in the asymptotics of a solution $\boldsymbol{u}$ of problem (C.1) in

(i) Leading singularities,
(ii) Shadow singularities.

(i) The leading singularities. They are denoted $\boldsymbol{s}^0$ and are directly obtained from the Mellin transform $\lambda \mapsto \mathfrak{M}(\boldsymbol{f})[\mathscr{x}'](\lambda)$ of $\boldsymbol{f}$,[e] via the Mellin symbol $\mathfrak{A}^0[\mathscr{x}']$ of problem (C.2) by the inverse Mellin formula

$$\boldsymbol{s}^0(\mathscr{x}', y) = \frac{1}{2i\pi}\int_{\gamma^0} r^{\mu}\left[\mathfrak{A}^0[\mathscr{x}'](\mu)\right]^{-1}\mathfrak{M}(r^{2d}\boldsymbol{f}, 0, 0)[\mathscr{x}'](\mu)\, d\mu, \tag{C.3}$$

where the 0 in $(\boldsymbol{f}, 0, 0)$ stand for the zero boundary conditions and γ^0 is a suitable contour surrounding the poles $\lambda \in \Lambda[\mathscr{x}']$ in the right half plane $\operatorname{Re}\lambda > d-1$.[f]

(ii) The shadow singularities. They require for their definition the Taylor expansion of the coefficients of L and $B_{\pm}$ with respect to y: Let

$$L = \sum_{|\boldsymbol{i}|+|\boldsymbol{k}|\le 2d} \ell^{\boldsymbol{i},\boldsymbol{k}}(\mathscr{x}', y)\,\partial_{\mathscr{x}'}^{\boldsymbol{i}}\,\partial_y^{\boldsymbol{k}}$$

and

$$B_{\pm,h} = \sum_{|\boldsymbol{i}|+|\boldsymbol{k}|\le\rho_{\pm,h}} b_{\pm,h}^{\boldsymbol{i},\boldsymbol{k}}(\mathscr{x}', y)\,\partial_{\mathscr{x}'}^{\boldsymbol{i}}\,\partial_y^{\boldsymbol{k}}$$

be the expressions of L and $B_{\pm}$. Then for $j \in \mathbb{N}$ we define

$$L^j[\mathscr{x}'] := \sum_{|\boldsymbol{i}|\le 2d}\ \sum_{|\boldsymbol{k}|-|\beta|=2d-j} \partial_y^{\beta}\ell^{\boldsymbol{i},\boldsymbol{k}}(\mathscr{x}', 0)\frac{y^{\beta}}{\beta!}\,\partial_{\mathscr{x}'}^{\boldsymbol{i}}\,\partial_y^{\boldsymbol{k}}$$

and

$$B_{\pm,h}^j[\mathscr{x}'] := \sum_{|\boldsymbol{i}|\le\rho_{\pm,h}}\ \sum_{|\boldsymbol{k}|-|\beta|=\rho_{\pm,h}-j} \partial_y^{\beta} b_{\pm,h}^{\boldsymbol{i},\boldsymbol{k}}(\mathscr{x}', 0)\frac{y^{\beta}}{\beta!}\,\partial_{\mathscr{x}'}^{\boldsymbol{i}}\,\partial_y^{\boldsymbol{k}}.$$

Let $\mathfrak{A}^j[\mathscr{x}']$ denote the triple $(L^j[\mathscr{x}'], B_{\pm}^j[\mathscr{x}'])$. Then the shadow singularities $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^p$ are recursively defined as

$$\boldsymbol{s}^p(\mathscr{x}', y) = -\frac{1}{2i\pi}\int_{\gamma^0+p} r^{\mu}\left[\mathfrak{A}^0[\mathscr{x}'](\mu)\right]^{-1}\mathfrak{M}\big(r^{\boldsymbol{\beta}}(\boldsymbol{f}^p, \boldsymbol{g}_{\pm}^p)\big)[\mathscr{x}'](\mu)\, d\mu, \tag{C.4}$$

$$\text{with}\quad (\boldsymbol{f}^p, \boldsymbol{g}_{\pm}^p) = \mathfrak{A}^1\boldsymbol{s}^{p-1} + \cdots + \mathfrak{A}^p\boldsymbol{s}^0.$$

Here $\boldsymbol{\beta}$ is the collection of degrees $(2d, \ldots, 2d, \rho_{\pm,1}, \ldots, \rho_{\pm,m})$ and $r^{\boldsymbol{\beta}}(\boldsymbol{f}, \boldsymbol{g}_{\pm})$ is a condensed notation for

$$(r^{2d}\boldsymbol{f}, r^{\rho_{-,1}}g_{-,1}, \ldots, r^{\rho_{-,m}}g_{-,m}, r^{\rho_{+,1}}g_{+,1}, \ldots, r^{\rho_{+,m}}g_{+,m}).$$

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[e]Defined by $\mathfrak{M}(\boldsymbol{f})[\mathscr{x}'](\lambda, \theta) = \int_0^{\infty} r^{\lambda}\boldsymbol{f}(\mathscr{x}', y)(dr/r)$ as a natural extension of the formula on $\mathbb{R}_+$.

[f]More precisely, for any $K \in \mathbb{N}$ we obtain the contribution modulo $\mathscr{O}(r^{\boldsymbol{K}})$ to the infinite asymptotic series by using a contour which surrounds the (finite set of) poles $\lambda \in \Lambda[\mathscr{x}'] \cup \mathbb{N}$ contained in the strip $d-1 < \operatorname{Re}\lambda \le K$.

(iii) The complete asymptotics. The sum $\boldsymbol{s}^0 + \boldsymbol{s}^1 + \cdots + \boldsymbol{s}^p + \cdots$ gives the asymptotics of $\boldsymbol{u}$ as $r \to 0$.

In the most general case, the structure of the $\boldsymbol{s}^p$ is quite difficult to describe because of the possible change of multiplicities in the singular exponents $\lambda[\mathscr{x}']$, see Costabel and Dauge (1993a, 1994). If hypotheses are made to avoid any change of multiplicity, see Maz'ya and Rossmann (1988), each $\boldsymbol{s}^p$ can be decomposed into elementary terms of the form $c(\mathscr{x}')\, r^{\lambda(\mathscr{x}')+p} \log^q r\, \boldsymbol{\varphi}(\mathscr{x}', \theta)$. Thus we obtain the following expansion in local cylindrical coordinates: For any $K \in \mathbb{N}$

$$\boldsymbol{u} = \sum_{\operatorname{Re}\lambda+p\leq K} \sum_{q=0}^{q(\lambda,p)} \sum_{j=1}^{j(\lambda,p,q)} c_j^{\lambda,p,q}(\mathscr{x}')\, r^{\lambda(\mathscr{x}')+p} \log^q r\, \boldsymbol{\varphi}_j^{\lambda,p,q}(\mathscr{x}', \theta) + \boldsymbol{u}_{\mathrm{rem},K}. \tag{C.5}$$

The exponents $\lambda(\mathscr{x}')$ belong to $\Lambda[\mathscr{x}'] \cup \mathbb{N}$ and their real part is $> d-1$. The coefficients $c_j^{\lambda,p,q}$ are $\mathscr{C}^\infty$ functions on $\mathscr{E}$ and depend on $\boldsymbol{f}$. The remainder $\boldsymbol{u}_{\mathrm{rem},K}$ satisfies $\partial^\beta \boldsymbol{u}_{\mathrm{rem},K} = \mathscr{o}(r^{K-|\beta|+1/2})$ as $r \to 0$ for any multi-index $\beta \in \mathbb{N}_0^{n+1}$. The $\boldsymbol{\varphi}_j^{\lambda,p,q}$ are angular N-component vector functions in $\mathscr{C}^\infty([-\omega, \omega] \times \mathscr{E})$ and depend only on the domain Ω and the operators (L, B).

The $\log r$ terms come either from nontrivial Jordan chains in $\mathfrak{A}^0[\mathscr{x}']^{-1}$, or from resonances between $\mathfrak{A}^0[\mathscr{x}']^{-1}$ and the Mellin transforms $\mathfrak{M}(r^{2d}\boldsymbol{f}, 0, 0)[\mathscr{x}']$, see Eq. (C.3), or $\mathfrak{M}\big(r^{\boldsymbol{\beta}}\mathfrak{A}^1\boldsymbol{s}^{p-1} + \cdots\big)[\mathscr{x}']$, see Eq. (C.4).

C.II. Crack Asymptotics, First Results

From now on, we concentrate on the situation of a crack, i.e., when the opening ω is π, and when the *same boundary conditions* are applied on both sides of the crack, i.e., $B_\pm = B$. Thus the boundary conditions are denoted by $B = (B_1, \ldots, B_m)$ and the order of B_h is ρ_h, $h = 1, \ldots, m$. The boundary problem takes then the form

$$\begin{cases} L\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad W_\pi \\ \gamma_\pm B\boldsymbol{u} = 0 & \text{on} \quad \partial_\pm W_\pi, \end{cases} \tag{C.6}$$

where we assume that $\boldsymbol{f} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$.

In this situation there holds

$$\forall \mathscr{x}' \in \mathscr{E}, \quad \Lambda[\mathscr{x}'] = \left\{\frac{k}{2};\, k \in \mathbb{Z}\right\}. \tag{C.7}$$

This has been known for a long time for the Laplace operator, see Grisvard (1985). It is proved for elasticity systems in Duduchava and Wendland (1995), for general second order Petrovskii-elliptic systems (such as thermoelasticity or electroelasticity for example) in Chkadua and Duduchava (2000, 2001), for general scalar elliptic Dirichlet problems of order $2m$ in Kozlov (1990), and finally in the general framework of Agmon–Douglis–Nirenberg elliptic systems in Costabel and Dauge (2002).

Therefore the assumptions on the constant multiplicity of the singular exponents are satisfied and expansion (C.5) holds with $\lambda(\mathscr{x}') = k/2$. This clear separation of the spectrum allows a decomposition of leading singularity $\boldsymbol{s}^0$ in (quasi-)homogeneous elementary parts Φ^0_λ for λ of the form $\lambda = k/2$ according to:

$$\Phi^0_\lambda[\mathscr{x}'] = \frac{1}{2i\pi}\int_{\gamma(\lambda)} r^\mu\, \mathfrak{A}^0[\mathscr{x}'](\mu)^{-1}\mathfrak{M}(r^{2d}\boldsymbol{f}, 0, 0)[\mathscr{x}'](\mu)\, d\mu, \tag{C.8}$$

where $\gamma(\lambda)$ is the circle with center λ and radius 1/4.

Definition C.1. If Φ^0 is defined by a residue formula like (C.8) on the circle $\gamma(\lambda)$, we call *sequence of shadows* associated with Φ^0, the infinite sequence Φ^p, $p \geq 1$, defined by

$$\Phi^p[\mathscr{x}'] = -\frac{1}{2i\pi}\int_{\gamma(\lambda)+p} r^\mu\, \mathfrak{A}^0[\mathscr{x}'](\mu)^{-1} \\ \times \mathfrak{M}\big(r^{\boldsymbol{\beta}}\mathbb{1}_{r\in[0,1]}(\mathfrak{A}^1\Phi^{p-1} + \cdots + \mathfrak{A}^p\Phi^0)\big)[\mathscr{x}'](\mu)\, d\mu. \tag{C.9}$$

Here $\gamma(\lambda) + p$ is the contour around $\lambda + p$ translated from $\gamma(\lambda)$ and $\mathbb{1}_{r\in[0,1]}$ is the characteristic function in r of the interval [0, 1]. ■

By linearity, we obtain that a decomposition of $\boldsymbol{s}^0$ in a sum of Φ^0_λ provides the corresponding decomposition of the shadow $\boldsymbol{s}^p$ in a sum Φ^p_λ, where $(\Phi^p)_p$ is the sequence of shadows associated with Φ^0_λ. Therefore, from now on we only consider elementary leading singularities of the form (C.8) and their sequence of shadows.

The result of Costabel and Dauge (2002, Thms. 5.2 and 5.3) gives moreover:

(i) In the *leading singularities* the noninteger exponents $k + 1/2$ have no $\log r$ terms and the corresponding basis of singular functions $\big(r^{k+1/2}\,\boldsymbol{\varphi}_j^{k+1/2}\big)_j$ has the dimension m.

(ii) Let $\rho_{\max} := \max\{\rho_1, \ldots, \rho_m\}$. For any integer $\lambda \geq \rho_{\max}$, the functions $r^\lambda\boldsymbol{\varphi}^\lambda_j(\theta)$ are polynomials in the variables (y_1, y_2). Moreover, the shadows of polynomials are polynomials.

Therefore:

(i) For an exponent $\lambda = 1/2 + k$, the elementary leading singularities have the form

$$\Phi^0_\lambda[\mathscr{x}'] = \sum_{j=1}^m c_j(\mathscr{x}')\, r^{k+1/2}\,\boldsymbol{\varphi}_j^{k+1/2}(\mathscr{x}', \theta), \quad c_j \in \mathscr{C}^\infty(\mathscr{E}).$$

(ii) For a positive integer $\lambda \geq \rho_{\max}$, Φ^0_λ is a finite sum of terms of the form $c(\mathscr{x}')\psi(y)$ with smooth c and polynomial ψ (homogeneous of degree λ). Moreover, the sequence of shadows Φ^p_λ associated with Φ^0_λ have a similar structure with homogeneous polynomials of degree $\lambda + p$.

As a consequence, we have obtained the statement of Proposition A.2 (i).

But, when $\lambda = 1/2 + k$, since $\lambda + p = 1/2 + k + p$ is a singular exponent, i.e., a pole of $\mathfrak{A}^0[\mathscr{x}']^{-1}$, we should expect resonances inside the integrand of the shadows Φ^p_λ, between $\mathfrak{M}\big(r^{\beta}\mathfrak{A}^1\Phi^{p-1}+\cdots\big)[\mathscr{x}']$ and $\mathfrak{A}^0[\mathscr{x}']^{-1}$, i.e., poles of order >1, which would yield $\log r$ factors. We are going to prove that, in fact, there are no resonances.

C.III. "Cayley" Representation Formulae

Our method is a direct continuation of Costabel and Dauge, (1993b) where "Cayley representation formulae" are introduced to describe the angular behavior (in θ) of the singular functions. It is shown there that any singularity can be expressed by combination of two fundamental types of functions which, using the complex writing ζ of the cartesian variables $y=(y_1, y_2)$

$$\zeta = y_1 + iy_2 = re^{i\theta},$$

can be written as, for any $\lambda\in\mathbb{C}$, $\zeta\in\mathbb{C}$ with $\zeta\notin\mathbb{R}^-$, and $\alpha\in\mathbb{C}$ with $|\alpha|<1$:

$$(\alpha\zeta+\overline{\zeta})^{\lambda} \quad \text{and} \quad (\zeta+\alpha\overline{\zeta})^{\lambda}.$$

The above functions have to be interpreted in the following way:

$$(\alpha\zeta+\overline{\zeta})^{\lambda} = \overline{\zeta}^{\lambda}\Big(1+\alpha\frac{\zeta}{\overline{\zeta}}\Big)^{\lambda} \quad \text{and} \quad (\zeta+\alpha\overline{\zeta})^{\lambda} = \zeta^{\lambda}\Big(1+\alpha\frac{\overline{\zeta}}{\zeta}\Big)^{\lambda}, \tag{C.10}$$

which means in polar coordinates $r>0$, $\theta\in(-\pi,\pi)$:

$$(\alpha\zeta+\overline{\zeta})^{\lambda} = r^{\lambda}e^{-i\theta\lambda}\big(1+\alpha e^{2i\theta}\big)^{\lambda} \quad \text{and} \quad (\zeta+\alpha\overline{\zeta})^{\lambda} = r^{\lambda}e^{i\theta\lambda}\big(1+\alpha e^{-2i\theta}\big)^{\lambda}. \tag{C.11}$$

The action of a partial differential operator $Q(\partial_1,\partial_2)$ on $(\alpha\zeta+\overline{\zeta})^{\lambda}$ and $(\zeta+\alpha\overline{\zeta})^{\lambda}$ exhibits its Cayley symbols $Q^+(\alpha)$ and $Q^-(\alpha)$ as follows:

$$Q^+(\alpha) := Q\big(\alpha+1, i(\alpha-1)\big) \quad \text{and} \quad Q^-(\alpha) := Q\big(1+\alpha, i(1-\alpha)\big)$$

and there holds, if Q is homogeneous of degree q

$$\begin{cases} Q(\partial_y)\,(\alpha\zeta+\overline{\zeta})^{\lambda} = P_q(\lambda)(\alpha\zeta+\overline{\zeta})^{\lambda-q}Q^+(\alpha) \\ Q(\partial_y)\,(\zeta+\alpha\overline{\zeta})^{\lambda} = P_q(\lambda)(\zeta+\alpha\overline{\zeta})^{\lambda-q}Q^-(\alpha), \end{cases}$$

where $P_q(\lambda)$ is the polynomial $\lambda(\lambda-1)\cdots(\lambda-q+1)$, of degree q.

Let us fix $\mathscr{x}'\in\mathscr{E}$. Let $L^{\pm}[\mathscr{x}'](\alpha)$ be the two Cayley symbols of $L^0[\mathscr{x}']$ and $B^{\pm}[\mathscr{x}']$ those of $B^0[\mathscr{x}']$. We have the following formulas, valid for any $\alpha\in\mathbb{C}$, which are the matrix version of the above ones: let $\boldsymbol{q}\in\mathbb{C}^N$ be a vector, there holds

$$\begin{cases} L^0[\mathscr{x}'](\partial_y)\,\big\{(\alpha\zeta+\overline{\zeta})^{\lambda}\boldsymbol{q}\big\} = P_{2d}(\lambda)(\alpha\zeta+\overline{\zeta})^{\lambda-2d}\,L^+[\mathscr{x}'](\alpha)\boldsymbol{q} \\ L^0[\mathscr{x}'](\partial_y)\,\big\{(\zeta+\alpha\overline{\zeta})^{\lambda}\boldsymbol{q}\big\} = P_{2d}(\lambda)(\zeta+\alpha\overline{\zeta})^{\lambda-2d}\,L^-[\mathscr{x}'](\alpha)\boldsymbol{q}. \end{cases} \tag{C.12}$$

These Cayley symbols allow to describe for any $\mathscr{x}'$ and λ the space $\mathfrak{Z}[\mathscr{x}'](\lambda)$ of the homogeneous functions $\boldsymbol{v}$ of degree λ, solutions of the equation without boundary conditions

$$L^0[\mathscr{x}']\boldsymbol{v} = 0.$$

Due to the ellipticity of the operator $L^0[\mathscr{x}']$, the equations

$$\det L^{\pm}[\mathscr{x}'](\alpha) = 0$$

have m roots inside the unit disc $|\alpha| < 1$, counting multiplicity, and *no roots on the unit circle* $|\alpha| = 1$. Let us denote

$$\alpha_1^-[\mathscr{x}'], \dots, \alpha_{m_-}^-[\mathscr{x}'], \qquad \alpha_1^+[\mathscr{x}'], \dots, \alpha_{m_+}^+[\mathscr{x}'] \tag{C.13}$$

the distinct roots of $\det L^-[\mathscr{x}']$ and $\det L^+[\mathscr{x}']$ inside the unit disc.

For a while let us assume that these roots are simple (i.e., $m_\pm = m$). Thus, let $\boldsymbol{q}_\ell^\pm[\mathscr{x}'] \in \mathbb{C}^N$ be nonzero elements of $\ker L^\pm(\alpha_\ell^\pm)$, and for any (noninteger) $\lambda \in \mathbb{C}$ let us define the N-component functions

$$\boldsymbol{w}_\ell^+[\mathscr{x}'](\lambda) := (\alpha_\ell^+[\mathscr{x}']\zeta + \bar{\zeta})^\lambda \, \boldsymbol{q}_\ell^+[\mathscr{x}'] \quad \text{and}$$
$$\boldsymbol{w}_\ell^-[\mathscr{x}'](\lambda) := (\zeta + \alpha_\ell^-[\mathscr{x}']\bar{\zeta})^\lambda \, \boldsymbol{q}_\ell^-[\mathscr{x}'].$$

Formulas (C.12) give immediately that these functions solve the equation $L^0[\mathscr{x}']\boldsymbol{v} = 0$, thus belong to $\mathfrak{Z}[\mathscr{x}'](\lambda)$.

As proved in Costabel and Dauge (1993b, Th. 2.1), these $2m$ functions form a basis of the space $\mathfrak{Z}[\mathscr{x}'](\lambda)$ and, moreover, we obtain "stable" expressions of $\boldsymbol{w}_\ell^\pm[\mathscr{x}'](\lambda)$ with respect to the parameter $\mathscr{x}'$ without the assumptions that the roots $\alpha_\ell^+[\mathscr{x}']$ are simple, by using contour integrals in α around the disc D_δ of radius with $\delta < 1$ such that D_δ contains all roots $\alpha_\ell^\pm[\mathscr{x}']$: There exists N-component polynomials of degree $d-1$ in α depending smoothly on $\mathscr{x}'$, denoted $\boldsymbol{q}_\ell^\pm[\mathscr{x}'](\alpha)$ for $\ell = 1, \dots, m$, which define a basis $\{\boldsymbol{w}_\ell^\pm[\mathscr{x}']\}$ of $\mathfrak{Z}[\mathscr{x}'](\lambda)$:

$$\begin{cases} \boldsymbol{w}_\ell^+[\mathscr{x}'](\lambda) = \displaystyle\int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^\lambda \, L^+[\mathscr{x}'](\alpha)^{-1} \boldsymbol{q}_\ell^+[\mathscr{x}'](\alpha)\, d\alpha \\ \boldsymbol{w}_\ell^-[\mathscr{x}'](\lambda) = \displaystyle\int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^\lambda \, L^-[\mathscr{x}'](\alpha)^{-1} \boldsymbol{q}_\ell^-[\mathscr{x}'](\alpha)\, d\alpha. \end{cases} \tag{C.14}$$

This basis allows the construction of a $2m \times 2m$ matrix $\mathscr{N}[\mathscr{x}'](\lambda)$ whose inverse has the same poles as the inverse of the Mellin symbol $\mathfrak{A}[\mathscr{x}'](\lambda)^{-1}$: For this let us introduce $\mathfrak{W}[\mathscr{x}'](\lambda)$ the $N \times 2m$ matrix the $2m$ columns of which are

$$\boldsymbol{w}_1^+[\mathscr{x}'](\lambda), \dots, \boldsymbol{w}_m^+[\mathscr{x}'](\lambda), \; \boldsymbol{w}_1^-[\mathscr{x}'](\lambda), \dots, \boldsymbol{w}_m^-[\mathscr{x}'](\lambda).$$

Let us recall that $B^0[\mathscr{x}']$ is the $m \times N$ matrix of the principal parts of the boundary operators $B(\mathscr{x}', 0; 0, \partial_y)$. Let $\mathfrak{g}_\pm$ be the trace operators (acting on *homogeneous functions*)[g]

$$\mathfrak{g}_- v = v\big|_{r=1 \text{ and } \theta=-\pi} \qquad \text{and} \qquad \mathfrak{g}_+ v = v\big|_{r=1 \text{ and } \theta=\pi}.$$

[g]The degree of homogeneity and the trace on $r = 1$ completely determine an homogeneous function: If v is homogeneous of degree μ and $V := v\big|_{r=1}$, then $v(r,\theta) = r^\mu V(\theta)$.

The *characteristic matrix* of the problem is then the $2m \times 2m$ scalar matrix given by

$$\mathscr{N}[x'](\lambda) = \begin{pmatrix} \mathfrak{g}_- B^0[x'] \\ \mathfrak{g}_+ B^0[x'] \end{pmatrix} \mathfrak{W}[x'](\lambda).$$

The formula describing $[\mathfrak{A}^0[x'](\mu)]^{-1}$ involves a right inverse to the operator L^0 on homogeneous functions of degree λ (i.e., without boundary conditions) and the inverse of the matrix $\mathscr{N}[x'](\lambda)$ allows the correction of boundary conditions.

Let $\mathfrak{H}^\lambda$ be the space of N-component vector functions homogeneous of degree λ on the plane sector Γ_π. And let $\boldsymbol{f} \mapsto \boldsymbol{v} = \mathfrak{R}[x'](\lambda)\boldsymbol{f}$ be a solution operator of the problem $L^0[x']\boldsymbol{v} = \boldsymbol{f}$, acting from $\mathfrak{H}^{\lambda-2d}$ into $\mathfrak{H}^\lambda$. According to Costabel and Dauge (1993b), it is possible to construct such an operator with $\mathscr{C}^\infty$ regularity in x' and analytic dependency in λ.

Our first representation theorem for the inverse symbol $\mathfrak{A}^0[x']^{-1}$ is the following, see Costabel and Dauge (1993b, Th. 4.4)—We write it directly for the Mellin integrand $r^\mu[\mathfrak{A}^0[x'](\mu)]^{-1}$ in view of application in formulas (C.8) and (C.9):

Theorem C.2. *Let $\mathfrak{R}[x'](\lambda)$ be a right inverse to $L^0[x']$, acting from $\mathfrak{H}^{\lambda-2d}$ into $\mathfrak{H}^\lambda$. We have for any $x' \in \mathscr{E}$, any $\mu \in \mathbb{C}$ and any $(\boldsymbol{F}, \boldsymbol{G}_\pm) \in \mathrm{L}^2(-\pi,\pi)^N \times \mathbb{C}^m \times \mathbb{C}^m$:*

$$\begin{aligned} r^\mu\big[\mathfrak{A}^0[x'](\mu)\big]^{-1}\big(\boldsymbol{F}, \boldsymbol{G}_\pm\big) &= \mathfrak{R}[x'](\mu)\big(r^{\mu-2d}\boldsymbol{F}\big) + \mathfrak{W}[x'](\mu)\,\mathscr{N}[x'](\mu)^{-1} \\ &\quad \times \Big(\boldsymbol{G}_\pm - \mathfrak{g}_\pm B^0[x']\,\mathfrak{R}[x'](\mu)\big(r^{\mu-2d}\boldsymbol{F}\big)\Big). \end{aligned} \tag{C.15}$$

Formula (C.15) will be applied recursively to special subsets of triples $(\boldsymbol{F}, \boldsymbol{G}_\pm)$ which have the property to be the traces (in $r = 1$) of homogeneous functions representable by Cayley integrals like (C.14):

Definition C.3. For any $\lambda \in \mathbb{C}$, let us denote by $\mathfrak{H}_0^\lambda$ the subspace of homogeneous N-component functions $\boldsymbol{f} \in \mathfrak{H}^\lambda$ which admit a representation as:

$$\boldsymbol{f} = \int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^\lambda\, \boldsymbol{q}^+(\alpha)\, d\alpha + \int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^\lambda\, \boldsymbol{q}^-(\alpha)\, d\alpha \tag{C.16}$$

with N-component vectors $\boldsymbol{q}^\pm$ meromorphic in α (and without pole in the annulus $\delta \le |\alpha| \le 1$). Such a representation is made *unique* if we assume that the $\boldsymbol{q}^\pm$ are holomorphic outside the unit disc and tend to 0 as $|\alpha| \to \infty$.

We can define a special solution operator $\mathfrak{R}_0[x'](\lambda)$ acting on the subspace $\mathfrak{H}_0^{\lambda-2d}$ into $\mathfrak{H}_0^\lambda$: For $\boldsymbol{f} \in \mathfrak{H}_0^{\lambda-2d}$ represented by (C.16) with the uniqueness constraint, we define $\mathfrak{R}_0[x'](\lambda)\boldsymbol{f}$ by

$$\begin{aligned} \mathfrak{R}_0[x'](\lambda)\boldsymbol{f} &= P_{2d}(\lambda)^{-1}\int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^\lambda\, L^+[x'](\alpha)^{-1}\boldsymbol{q}^+(\alpha)\, d\alpha \\ &\quad + P_{2d}(\lambda)^{-1}\int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^\lambda\, L^-[x'](\alpha)^{-1}\boldsymbol{q}^-(\alpha)\, d\alpha. \end{aligned} \tag{C.17}$$

The vector function obviously belongs to $\mathfrak{H}_0^\lambda$ and if $P_{2d}(\lambda) \neq 0$, formulae (C.12) give immediately that $L^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\lambda)\boldsymbol{f} = \boldsymbol{f}$.

C.IV. Representation of Singularities

We start from the expression (C.8) of the leading singularity Φ^0. The function

$$(\mathscr{x}', \mu) \mapsto \mathfrak{M}(r^{2d}\boldsymbol{f}, 0, 0)[\mathscr{x}'](\mu)$$

is $\mathscr{C}^\infty(\mathscr{E})$ in $\mathscr{x}'$ and analytic in μ in the disc δ_λ encircled by the contour $\gamma(\lambda)$. Using the representation (C.15) with analytic $\boldsymbol{F}$ and zero $\boldsymbol{G}_\pm$, we find that the only pole of $r^\mu\,[\mathfrak{A}^0[\mathscr{x}'](\mu)]^{-1}\mathfrak{M}(r^{2d}\boldsymbol{f}, 0, 0)[\mathscr{x}'](\mu)$ inside δ_λ is $\mu = \lambda$ and that there holds

$$\Phi^0[\mathscr{x}'] = \frac{1}{2i\pi}\int_{\gamma(\lambda)} \mathfrak{W}[\mathscr{x}'](\mu)\,\mathscr{N}[\mathscr{x}'](\mu)^{-1}\boldsymbol{\psi}^0[\mathscr{x}'](\mu)\,d\mu \tag{C.18}$$

with a $2m$-component vector function $(\mathscr{x}', \mu) \mapsto \boldsymbol{\psi}^0[\mathscr{x}'](\mu)$ which is $\mathscr{C}^\infty$ in $\mathscr{x}'$ and analytic in μ. Since the pole of $\mathscr{N}(\mu)^{-1}$ is of order 1, see Costabel and Dauge (2002), and since by construction, the columns of $\mathfrak{W}[\mathscr{x}'](\mu)$ belong to the special space $\mathfrak{H}_0^\lambda$ of homogeneous functions, we have obtained

Lemma C.4. *The leading singular function $\mathscr{x}' \mapsto \Phi^0[\mathscr{x}']$ is $\mathscr{C}^\infty(\mathscr{E})$ with values in $\mathfrak{H}_0^\lambda$, which means that there exists N-component vectors $\boldsymbol{q}_0^\pm[\mathscr{x}'](\alpha)$ meromorphic in α and $\mathscr{C}^\infty$ in $\mathscr{x}'$ such that*

$$\Phi^0[\mathscr{x}'] = \int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^\lambda\,\boldsymbol{q}_0^+[\mathscr{x}'](\alpha)\,d\alpha + \int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^\lambda\,\boldsymbol{q}_0^-[\mathscr{x}'](\alpha)\,d\alpha. \tag{C.19}$$

The first shadow singularity Φ^1 is given by

$$\Phi^1[\mathscr{x}'] = -\frac{1}{2i\pi}\int_{\gamma(\lambda)+1} r^\mu\left[\mathfrak{A}^0[\mathscr{x}'](\mu)\right]^{-1}\mathfrak{M}\big(\mathbb{1}_{r\in[0,1]}(r^{\boldsymbol{\beta}}\mathfrak{A}^1\Phi^0)\big)[\mathscr{x}'](\mu)\,d\mu. \tag{C.20}$$

The following lemmas give that the structure of $\mathfrak{A}^1\Phi^0$ is compatible with representations of the type (C.16).

Lemma C.5. *Let $\lambda \in \mathbb{C}$. For any $j \in \mathbb{N}$, the operator L^j acts from $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^\lambda)$ into $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+j-2d})$.*

Proof. The operator L^j is a linear combination with $\mathscr{C}^\infty(\mathscr{E})$ coefficients of terms of the form $y^\beta\,\partial_{\mathscr{x}'}^i\,\partial_y^\delta$ with $|\delta| - |\beta| = 2d - j$. The derivative $\partial_{\mathscr{x}'}^i$ acts only on the coefficients depending on $\mathscr{x}'$ and do not change the angular structure, so we may discard it. We are left with $y^\beta\,\partial_y^\delta$, which we can write as a linear combination of terms

$$\zeta^{\beta_1}\,\bar{\zeta}^{\beta_2}\partial^{\delta_1}\,\partial_{\bar{\zeta}}^{\delta_2} \quad \text{with} \quad \delta_1 + \delta_2 - \beta_1 - \beta_2 = 2d - j.$$

It is clear that it suffices to prove that for any δ_1, δ_2, β_1 and β_2 with $\delta_1+\delta_2-\beta_1-\beta_2=2d-j$, and for any function $q(\alpha)$ meromorphic in α, there exists $q'(\alpha)$ also meromorphic in α such that

$$\zeta^{\beta_1}\,\overline{\zeta}^{\beta_2}\,\partial^{\delta_1}\partial_{\overline{\zeta}}^{\delta_2}\int_{|\alpha|=\delta}(\alpha\zeta+\overline{\zeta})^{\lambda}\,q(\alpha)\,d\alpha=\int_{|\alpha|=\delta}(\alpha\zeta+\overline{\zeta})^{\lambda+j-2d}\,q'(\alpha)\,d\alpha.$$

We have

$$\partial^{\delta_1}\partial_{\overline{\zeta}}^{\delta_2}\int_{|\alpha|=\delta}(\alpha\zeta+\overline{\zeta})^{\lambda}\,q(\alpha)\,d\alpha=\int_{|\alpha|=\delta}\alpha^{\delta_1}(\alpha\zeta+\overline{\zeta})^{\lambda-|\delta|}\,q(\alpha)\,d\alpha.$$

With the equality $\overline{\zeta}=(\alpha\zeta+\overline{\zeta})-\alpha\zeta$, we transform $\zeta^{\beta_1}\,\overline{\zeta}^{\beta_2}$ into a linear combination of terms of the form $\zeta^{\gamma_1}(\alpha\zeta+\overline{\zeta})^{\gamma_2}$. Thus we are left with integrals of the form

$$\int_{|\alpha|=\delta}\zeta^{n}(\alpha\zeta+\overline{\zeta})^{\lambda+j-2d-n}\,q(\alpha)\,d\alpha.$$

As $\partial_\alpha^n(\alpha\zeta+\overline{\zeta})^{\lambda+j-2d}=c\,\zeta^n(\alpha\zeta+\overline{\zeta})^{\lambda+j-2d-n}$, we integrate by parts n times in the above integral and obtain the result. ■

In the same way, we obtain the corresponding result for the trace operators:

Lemma C.6. *Let $\lambda\in\mathbb{C}$. For any $j\in\mathbb{N}_0$ the operator B^j acts from $\mathscr{C}^\infty(\mathscr{E},\mathfrak{H}_0^{\lambda})$ into $\mathscr{C}^\infty(\mathscr{E},\mathfrak{H}_0^{\lambda+j-\boldsymbol{\rho}})$, where $\mathfrak{H}_0^{\lambda-\boldsymbol{\rho}}$ is the space of m-component functions homogeneous of degree $(\lambda-\rho_1,\dots,\lambda-\rho_m)$ with Cayley representation like* (C.16).

Let us return to Eq. (C.20). Let $(\boldsymbol{F}^1,\boldsymbol{G}^1_\pm)[x']$ be the traces on $r=1$ of $\mathfrak{A}^1[x']\Phi^0[x']$. We have

$$\mathfrak{M}\big(\mathbb{1}_{r\in[0,1]}(r^{\boldsymbol{\beta}}\mathfrak{A}^1\Phi^0)\big)[x'](\mu)=\frac{1}{\mu-(\lambda+1)}\,(\boldsymbol{F}^1,\boldsymbol{G}^1_\pm)[x'].$$

By Lemma C.5, $r^{\lambda+1-2d}\boldsymbol{F}^1[x'](\theta)$ belongs to $\mathscr{C}^\infty(\mathscr{E},\mathfrak{H}_0^{\lambda+1-2d})$: There exists $\boldsymbol{q}^{\pm}[x']$ such that

$$r^{\lambda+1-2d}\boldsymbol{F}^1[x']=\int_{|\alpha|=\delta}(\alpha\zeta+\overline{\zeta})^{\lambda+1-2d}\,\boldsymbol{q}^+[x'](\alpha)\,d\alpha+\int_{|\alpha|=\delta}(\zeta+\alpha\overline{\zeta})^{\lambda+1-2d}\,\boldsymbol{q}^-[x'](\alpha)\,d\alpha.$$

We define for $\mu\in\mathbb{C}$ the following element $\boldsymbol{f}_0^1[x'](\mu)\in\mathfrak{H}_0^{\mu-2d}$:

$$\boldsymbol{f}_0^1[x'](\mu):=\int_{|\alpha|=\delta}(\alpha\zeta+\overline{\zeta})^{\mu-2d}\,\boldsymbol{q}^+[x'](\alpha)\,d\alpha+\int_{|\alpha|=\delta}(\zeta+\alpha\overline{\zeta})^{\mu-2d}\,\boldsymbol{q}^-[x'](\alpha)\,d\alpha.$$

Of course, $\boldsymbol{f}_0^1[x'](\lambda+1)=r^{\lambda+1-2d}\boldsymbol{F}^1[x']$. Let us denote

$$\boldsymbol{f}^1[x'](\mu):=r^{\mu-2d}\boldsymbol{F}^1[x']-\boldsymbol{f}_0^1[x'](\mu).$$

It is clear that in the representation formula (C.15), we may take as right inverse of $r^{\mu-2d}\boldsymbol{F}^1$,

$$\mathfrak{R}_0[\mathscr{x}'](\mu)\big(\boldsymbol{f}_0^1[\mathscr{x}'](\mu)\big) + \mathfrak{R}[\mathscr{x}'](\mu)\big(\boldsymbol{f}^1[\mathscr{x}'](\mu)\big),$$

instead of $\mathfrak{R}[\mathscr{x}'](\mu)\big(r^{\mu-2d}\boldsymbol{F}^1\big)$. Therefore we have the following decomposition in four parts of the integrand of Eq. (C.20):

$$\begin{aligned} r^{\mu}\big[\mathfrak{A}^0[\mathscr{x}'](\mu)\big]^{-1}\,\mathfrak{M}\big(\mathbb{1}_{r\in[0,1]}(r^{\boldsymbol{\beta}}\mathfrak{A}^1\Phi^0)\big)(\mu) &= \frac{1}{\mu-(\lambda+1)}\Big(r^{\mu}\big[\mathfrak{A}^0[\mathscr{x}'](\mu)\big]^{-1}(\boldsymbol{F}^1, \boldsymbol{G}^1_{\pm})\Big) \\ &= \frac{1}{\mu-(\lambda+1)}\Big(U_1+U_2+U_3+U_4\Big)(\mu). \end{aligned} \tag{C.21}$$

where

$$\begin{aligned} U_1(\mu) &= \mathfrak{R}[\mathscr{x}'](\mu)\boldsymbol{f}^1[\mathscr{x}'](\mu), \\ U_2(\mu) &= \mathfrak{R}_0[\mathscr{x}'](\mu)\boldsymbol{f}_0^1[\mathscr{x}'](\mu), \\ U_3(\mu) &= \mathfrak{W}[\mathscr{x}'](\mu)\,\mathscr{N}[\mathscr{x}'](\mu)^{-1}\Big(-\mathfrak{g}_{\pm}B^0[\mathscr{x}']\,\mathfrak{R}[\mathscr{x}'](\mu)\boldsymbol{f}^1[\mathscr{x}'](\mu)\Big), \\ U_4(\mu) &= \mathfrak{W}[\mathscr{x}'](\mu)\,\mathscr{N}[\mathscr{x}'](\mu)^{-1}\Big(\boldsymbol{G}^1_{\pm} - \mathfrak{g}_{\pm}B^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\mu)\boldsymbol{f}_0^1[\mathscr{x}'](\mu)\Big). \end{aligned}$$

Coming back to Eq. (C.20), we have to compute the contour integral

$$\Phi^1[\mathscr{x}'] = -\frac{1}{2i\pi}\int_{\gamma(\lambda)+1}(U_1+U_2+U_3+U_4)(\mu)\,\frac{d\mu}{\mu-(\lambda+1)}.$$

Let us compute the residue in $\mu = \lambda + 1$ of each of the four terms.

(i) As $\boldsymbol{f}^1[\mathscr{x}'](\lambda+1) = 0$, the residue of $(\mu-(\lambda+1))^{-1}U_1(\mu)$ is 0.
(ii) The residue of $(\mu-(\lambda+1))^{-1}U_2(\mu)$ is equal to $U_2(\lambda+1)$, which coincides with $\mathfrak{R}_0[\mathscr{x}'](\lambda+1)\boldsymbol{f}_0^1[\mathscr{x}'](\lambda+1)$, so it belongs to $\mathfrak{H}_0^{\lambda+1}$.
(iii) As $\boldsymbol{f}^1[\mathscr{x}'](\lambda+1) = 0$, the pole of $(\mu-(\lambda+1))^{-1}U_3(\mu)$ is of order 1, and the residue is a linear combination of the $\boldsymbol{w}_\ell^{\pm}[\mathscr{x}'](\lambda+1)$, therefore belongs to $\mathfrak{H}_0^{\lambda+1}$.
(iv) Finally, the pole of $(\mu-(\lambda+1))^{-1}U_4(\mu)$ in $\lambda+1$ is, a priori, of order 2:

$$\begin{aligned} &\frac{1}{2i\pi}\int_{\gamma(\lambda)+1}\mathfrak{W}[\mathscr{x}'](\mu)\,\frac{\mathscr{N}[\mathscr{x}'](\mu)^{-1}}{\mu-(\lambda+1)} \\ &\quad\times\Big(\boldsymbol{G}^1_{\pm} - \mathfrak{g}_{\pm}B^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\mu)\boldsymbol{f}_0^1[\mathscr{x}'](\mu)\Big)\,d\mu. \end{aligned} \tag{C.22}$$

The term (C.22) is itself the sum of an element of $\mathfrak{H}_0^{\lambda+1}$, cf. (iii), and of

$$\frac{1}{2i\pi}\int_{\gamma(\lambda)+1} \mathfrak{W}[\mathscr{x}'](\mu)\,\frac{\mathscr{N}[\mathscr{x}'](\mu)^{-1}}{\mu-(\lambda+1)} \times \left(\boldsymbol{G}^1_\pm - \mathfrak{g}_\pm B^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\lambda+1)\left(r^{\lambda+1-2d}\boldsymbol{F}^1\right)\right) d\mu. \tag{C.23}$$

By construction $\boldsymbol{G}^1_\pm$ is the couple of traces $\mathfrak{g}_\pm B^1[\mathscr{x}']\Phi^0[\mathscr{x}']$. Therefore

$$\boldsymbol{G}^1_\pm - \mathfrak{g}_\pm B^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\lambda+1)\big(r^{\lambda+1-2d}\boldsymbol{F}\big) = \mathfrak{g}_\pm\Psi^1[\mathscr{x}'],$$

where

$$\Psi^1[\mathscr{x}] := B^1[\mathscr{x}']\Phi^0[\mathscr{x}'] - B^0[\mathscr{x}']\,\mathfrak{R}_0[\mathscr{x}'](\lambda+1)\big(L^1[\mathscr{x}']\Phi^0[\mathscr{x}']\big).$$

The m-component function $\Psi^1[\mathscr{x}']$ belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+1-\boldsymbol{\rho}})$ by virtue of Lemmas C.5 and C.6. Gathering the results for Φ^1, we have obtained

Lemma C.7. *The first shadow singularity $\Phi^1[\mathscr{x}']$ is the sum of $\Phi^1_0[\mathscr{x}']$ which belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+1})$ and of Φ^1_1:*

$$\Phi^1_1 := \frac{1}{2i\pi}\int_{\gamma(\lambda)+1} \mathfrak{W}[\mathscr{x}'](\mu)\,\frac{\mathscr{N}[\mathscr{x}'](\mu)^{-1}}{\mu-(\lambda+1)}\left(\mathfrak{g}_\pm\Psi^1[\mathscr{x}']\right) d\mu \tag{C.24}$$

where $\Psi^1[\mathscr{x}']$ belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+1-\boldsymbol{\rho}})$.

C.V. The Relation of Compatibility

Our aim is to show that, in the Laurent expansion of

$$(\mu-(\lambda+1))^{-1}\mathscr{N}[\mathscr{x}'](\mu)^{-1}\big(\mathfrak{g}_\pm\Psi^1[\mathscr{x}']\big),$$

the coefficient in front of the term $(\mu-(\lambda+1))^{-2}$ is *zero*. Since $\mathscr{N}[\mathscr{x}'](\mu)^{-1}$ has a pole of order 1 in $\lambda+1$, the necessary and sufficient condition for this coefficient to be zero is that

$$\mathfrak{g}_\pm\Psi^1[\mathscr{x}'] \in \operatorname{rg}\mathscr{N}[\mathscr{x}'](\lambda+1), \tag{C.25}$$

which is the "relation of compatibility."

Lemma C.8. *Let λ be of the form $1/2+k$ with $k\in\mathbb{Z}$. Let $\mathscr{x}'\in\mathscr{E}$. Then the range of $\mathscr{N}[\mathscr{x}'](\lambda)$ is the subspace of the*

$$(b^1_-,\dots,b^m_-,b^1_+,\dots,b^m_+)$$

which satisfy $b^h_- = -b^h_+$ for $h=1,\dots,m$.

Proof. Let us fix $\mathscr{x}'$ and let us drop it in the notations. In the case when the roots $\alpha_\ell^\pm$ are distinct, according to Costabel and Dauge (2002, §3), $\mathcal{N}(\mu)$ has the general structure, by $m \times m$ blocks:

$$\begin{pmatrix} E(\mu) & 0 \\ 0 & E(\mu) \end{pmatrix} \begin{pmatrix} e^{-i\pi\mu}\mathfrak{B}^+ & -e^{-i\pi\mu}\mathfrak{B}^- \\ -e^{-i\pi\mu}\mathfrak{B}^+ & e^{-i\pi\mu}\mathfrak{B}^- \end{pmatrix} \begin{pmatrix} F^+(\mu) & 0 \\ 0 & F^-(\mu) \end{pmatrix}$$

where $E(\mu)$ is a diagonal matrix everywhere invertible except on a finite number of integers, $F^\pm(\mu)$ are everywhere invertible and the two matrices $\mathfrak{B}^\pm$ are invertible, due to the ellipticity of the boundary value problem, see Costabel and Dauge (2002, §4). The statement of the lemma for $\mu = \lambda$ is straightforward in this case. The general case where the $\alpha_\ell^\pm$ are not supposed distinct is obtained by perturbation. ■

Lemma C.9. *Let λ be of the form $1/2 + k$ with $k \in \mathbb{Z}$. Let Ψ belong to $\mathfrak{H}_0^{\lambda-\boldsymbol{\rho}}$. Then $\mathfrak{g}_-\Psi = -\mathfrak{g}_+\Psi$.*

Proof. Let Ψ_h denote the components of Ψ, for $h = 1, \dots, m$. The component Ψ_h belongs to $\mathfrak{H}_0^{\lambda-\rho_h}$, which means that there exists functions $p_h^\pm$ meromorphic in α and such that

$$\Psi_h = \int_{|\alpha|=\delta} (\alpha\zeta + \bar\zeta)^{\lambda-\rho_h}\, p_h^+(\alpha)\, d\alpha + \int_{|\alpha|=\delta} (\zeta + \alpha\bar\zeta)^{\lambda-\rho_h}\, p_h^-(\alpha)\, d\alpha.$$

It remains to compute the traces $\mathfrak{g}_\pm$ of Ψ_h. We use the formulae

$$(\alpha\zeta + \bar\zeta)^\mu = \bar\zeta^\mu \Big(1 + \alpha\frac{\zeta}{\bar\zeta}\Big)^\mu \quad \text{and} \quad (\zeta + \alpha\bar\zeta)^\mu = \zeta^\mu\Big(1 + \alpha\frac{\bar\zeta}{\zeta}\Big)^\mu.$$

There holds (since $|\alpha| < 1$)

$$\zeta^\mu = re^{i\mu\theta}, \qquad \bar\zeta^\mu = re^{-i\mu\theta},$$

$$\Big(1 + \alpha\frac{\zeta}{\bar\zeta}\Big)^\mu = (1 + \alpha e^{2i\theta})^\mu, \quad \Big(1 + \alpha\frac{\bar\zeta}{\zeta}\Big)^\mu = (1 + \alpha e^{-2i\theta})^\mu.$$

Whence

$$\mathfrak{g}_-\Psi_h = e^{+i(\lambda-\rho_h)\pi}\int_{|\alpha|=\delta} (1+\alpha)^{\lambda-\rho_h} p_h^+(\alpha)\, d\alpha + e^{-i(\lambda-\rho_h)\pi}\int_{|\alpha|=\delta} (1+\alpha)^{\lambda-\rho_h} p_h^-(\alpha)\, d\alpha$$
$$\mathfrak{g}_+\Psi_h = e^{-i(\lambda-\rho_h)\pi}\int_{|\alpha|=\delta} (1+\alpha)^{\lambda-\rho_h} p_h^+(\alpha)\, d\alpha + e^{+i(\lambda-\rho_h)\pi}\int_{|\alpha|=\delta} (1+\alpha)^{\lambda-\rho_h} p_h^-(\alpha)\, d\alpha$$

As $\lambda = 1/2 + k$, we have obtained the lemma. ■

The consequence of Lemmas C.8 and C.9 for $\Phi^1[\mathscr{x}']$ is now clear: Eq. (C.25) holds. Therefore the function $\Psi_1^1[\mathscr{x}']$ defined in Eq. (C.24) also belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+1})$. Which means that, finally, the first shadow singularity $\Phi^1[\mathscr{x}']$ belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+1})$, i.e., satisfies at its degree of homogeneity exactly the same property as $\Phi^0[\mathscr{x}']$, see Lemma C.4.

The proof of this property can be immediately generalized to the following:

Proposition C.10. *Let $\lambda \in \mathbb{C}$ of the form $1/2 + k$ with integer k. Let $F[\mathscr{x}'](r,\theta)$ belong to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda-2d})$ and $G_\pm[\mathscr{x}'](r)$ be the traces on $\theta = \pm\pi$ of a m-component vector function $\Psi[\mathscr{x}'] \in \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda-\boldsymbol{\rho}})$. Then the N-component function $\Phi[\mathscr{x}']$ defined as*

$$\Phi[\mathscr{x}'] = \frac{1}{2i\pi}\int_{\gamma(\lambda)} r^\mu \left[\mathfrak{A}^0[\mathscr{x}'](\mu)\right]^{-1} \mathfrak{M}\big(\mathbb{1}_{r\in[0,1]}(r^{2d}F, r^{\boldsymbol{\rho}}G)\big)[\mathscr{x}'](\mu)\, d\mu$$

belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda})$.

Therefore, with the help of Lemmas C.5 and C.6, we see that the procedure for the analysis of the successive shadows $\Phi^2, \ldots, \Phi^p$ is recursive. Therefore for all $p \in \mathbb{N}_0$, Φ^p belong to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+p})$ and, thus, do not contain any logarithmic term.

C.VI. Absence of Logarithms, General Results

Examining the arguments of the proofs of Lemmas C.4 to C.6 and Proposition C.10, we can see that, in fact, the result we have proved does not use any ellipticity in the edge variable $\mathscr{x}' \in \mathscr{E}$, only the smooth dependency. In the next statement, we select the hypotheses which are sufficient to obtain our result on the absence of logarithms in shadow singularities:

Hypothesis C.11. Let $\mathscr{x}' \mapsto (L^0, B^0)[\mathscr{x}']$ be $\mathscr{C}^\infty(\mathscr{E})$ with values in the space $\mathrm{Op}_{\mathrm{Ell}}^{2\mathrm{d},\boldsymbol{\rho}}(\mathbb{R}^2)$ of $(N \times N)$ elliptic systems homogeneous of order $2d$ with constant coefficients in $\mathbb{R}^2$, with complementing boundary conditions homogeneous of degree $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ with constant coefficients. The Mellin symbol of $(L^0, \gamma_\pm B^0)[\mathscr{x}']$ is denoted by $\mathfrak{A}^0[\mathscr{x}']$ with γ_- and γ_+ the traces on $\{(y_1, y_2) \mid y_1 < 0\}$ from below and from above, respectively.

For any $j \in \mathbb{N}$, let $\mathscr{x}' \mapsto (L^j, B^j)[\mathscr{x}']$ be a matrix-function with coefficients $L^j_{k,\ell}[\mathscr{x}']$ and $B^j_{h,\ell}[\mathscr{x}']$, $\mathscr{C}^\infty(\mathscr{E})$ with values in the space of operators

$$\mathrm{Op}^{2d-j}(\mathbb{R}^2) \quad \text{for} \quad L^j_{k,\ell} \quad \text{and} \quad \mathrm{Op}^{\rho_h - j}(\mathbb{R}^2) \quad \text{for} \quad B^j_{h,\ell},$$

where for $p \in \mathbb{Z}$, $\mathrm{Op}^p(\mathbb{R}^2)$ is defined as the space of finite linear combinations with $\mathscr{C}^\infty(\mathscr{E})$ coefficients of partial differential operators of the form $y^\beta \partial^i_{\mathscr{x}'} \partial^\delta_y$ with $|\delta| - |\beta| = p$. We denote the triple $(L^j, \gamma_\pm B^j)[\mathscr{x}']$ by $\mathfrak{A}^j[\mathscr{x}']$. ∎

The proofs of Lemmas C.4 to C.6 and Proposition C.10 then yield

Theorem C.12. *Let $(L^j, B^j)_{j\geq 0}$ be a sequence of operators satisfying Hypothesis* C.11. *Let $\lambda = 1/2 + k$ with $k \in \mathbb{Z}$ and let $\gamma(\lambda)$ be the circle with center λ and radius $1/4$. With the function $(\mathscr{x}', \mu) \mapsto (\boldsymbol{F}, \boldsymbol{G}_\pm)[\mathscr{x}'](\mu)$ supposed to be $\mathscr{C}^\infty(\mathscr{E})$ in $\mathscr{x}'$ and analytic in μ, with values in $\mathrm{L}^2(-\pi, \pi) \times \mathbb{C}^m \times \mathbb{C}^m$, we define the following leading singularity, which* *is a generalization of* Eq. (C.8):

$$\Phi^0[x'] = \frac{1}{2i\pi}\int_{\gamma(\lambda)} r^{\mu}\left[\mathfrak{A}^0[x'](\mu)\right]^{-1}(\boldsymbol{F},\boldsymbol{G}_\pm)[x'](\mu)\,d\mu,$$

and its sequence of shadows $(\Phi^p[x'])_p$ *according to Definition* C.1. *Then, for any integer* $p \geq 0$, $\Phi^p[x']$ *belongs to* $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+p})$. *In particular* $\Phi^p[x'](r,\theta)$ *can be written in the form* $r^{\lambda+p}\boldsymbol{\psi}(x',\theta)$ *with* $\boldsymbol{\psi} \in \mathscr{C}^\infty(\mathscr{E}\times[-\pi,\pi])\otimes\mathbb{C}^N$.

In fact this statement extends to the wider class of Agmon–Douglis–Nirenberg systems with covering boundary conditions:

Hypothesis C.13. Let $N \in \mathbb{N}$, $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_N)$, $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_N)$,

$$m = \frac{1}{2}(\sigma_1 - \tau_1 + \cdots + \sigma_N - \tau_N)$$

and $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_m)$. Let $x' \mapsto (L^0, B^0)[x']$ be $\mathscr{C}^\infty(\mathscr{E})$ with values in the space $\mathrm{Op}^{\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\rho}}_{\mathrm{ADN}}(\mathbb{R}^2)$ of $(N\times N)$ Agmon–Douglis–Nirenberg elliptic systems homogeneous of order $\sigma_k - \tau_\ell$ with constant coefficients in $\mathbb{R}^2$, with complementing boundary conditions homogeneous of degree $\rho_h - \tau_\ell$ with constant coefficients.

For any $j \in \mathbb{N}$, let $x' \mapsto (L^j, B^j)[x'] =: \mathfrak{A}^j[x']$ be a matrix-valued function with coefficients $L^j_{k,\ell}[x']$ and $B^j_{h,\ell}[x']$, $\mathscr{C}^\infty(\mathscr{E})$ with values in the space of operators

$$\mathrm{Op}^{\sigma_k-\tau_\ell-j}(\mathbb{R}^2) \quad \text{for} \quad L^j_{k,\ell} \quad \text{and} \quad \mathrm{Op}^{\rho_h-\tau_\ell-j}(\mathbb{R}^2) \quad \text{for} \quad B^j_{h,\ell},$$

with $\mathrm{Op}^p(\mathbb{R}^2)$ as in Hypothesis C.11. ■

The Mellin transform and the Cayley representation can be used with the same success in the framework of Agmon–Douglis–Nirenberg systems, see Costabel and Dauge (1993b, 2002), which allows to obtain:

Theorem C.14. *Let* $(L^j, B^j)_{j\geq 0}$ *be a sequence of operators satisfying Hypothesis* C.13. *Let* $\lambda = 1/2 + k$, $\gamma(\lambda)$, *and* $(\boldsymbol{F},\boldsymbol{G}_\pm)[x'](\mu)$ *be as in Theorem* C.12. *We define the following leading singularity*:

$$\Phi^0[x'] = \frac{1}{2i\pi}\int_{\gamma(\lambda)} r^{\mu-\boldsymbol{\tau}}\left[\mathfrak{A}^0[x'](\mu)\right]^{-1}(\boldsymbol{F},\boldsymbol{G}_\pm)[x'](\mu)\,d\mu,$$

and its sequence of shadows $(\Phi^p[x'])_p$ *by an obvious modification of Definition* C.1, *with* $\boldsymbol{\beta} = (\sigma_1,\ldots,\sigma_N,\rho_1,\ldots,\rho_m,\rho_1,\ldots,\rho_m)$ *and* μ *replaced with* $\mu - \boldsymbol{\tau}$ *as above.*

Then $\Phi^p[x']$ *is homogeneous of multi-degree* $\lambda + p - \boldsymbol{\tau}$, *i.e., its j-th component* Φ^p_j *satisfies* $\Phi^p_j[x'](r,\theta) = r^{\lambda+p-\tau_j}\psi_j(x',\theta)$ *with* $\psi_j \in \mathscr{C}^\infty(\mathscr{E}\times[-\pi,\pi])$.

We obtain as a corollary (and a generalization of Theorem A.3) that the asymptotics along a crack edge of the solutions of Agmon–Douglis–Nirenberg systems associated with coercive bilinear forms contain no logarithmic term:

Corollary C.15. *Let* (L, B) *be an* $(N\times N)$ *Agmon–Douglis–Nirenberg elliptic system of order* $\sigma_k - \tau_\ell$ *with smooth coefficients in* $\mathbb{R}^{n+1}$, *with complementing boundary* *conditions homogeneous of degree $\rho_h - \tau_\ell$ with smooth coefficients. Let us assume that (L, B) is associated with a coercive bilinear form. Let $\rho_{\max} := \max\{\rho_1, \ldots, \rho_m\}$. Any solution $\boldsymbol{u}$ of problem* (C.6) (*with a smooth right hand side $\boldsymbol{f}$*) *which belongs to $\mathrm{H}^{s-\tau}(W_\pi)$ with $s \geq \rho_{\max}$ has the following asymptotic expansion as $r \to 0$: For any integer $K > k_0$*

$$\boldsymbol{u} = \sum_{j=1}^{m} c_j^0(\mathscr{x}')\, r^{1/2+k_0-\tau} \boldsymbol{\psi}_j^0(\mathscr{x}', \theta) + \sum_{k=k_0+1}^{K} \sum_{j=1}^{j(k)} c_j^k(\mathscr{x}')\, r^{1/2+k-\tau}\, \boldsymbol{\psi}_j^k(\mathscr{x}', \theta)$$

$$+ \boldsymbol{u}_{\mathrm{reg},K} + \boldsymbol{u}_{\mathrm{rem},K}, \tag{C.26}$$

where k_0 is the smallest integer such that $1/2 + k_0 > s - 1$. The regular part $\boldsymbol{u}_{\mathrm{reg},K}$ is in $\mathscr{C}^\infty(\mathbb{R}^{n+1})$. The remainder $\boldsymbol{u}_{\mathrm{rem},K}$ belongs to $\mathscr{C}^{K+1-\tau}(\overline{W}_\pi)$ and is flat of order $K - \tau$ near $\mathscr{E}$.

C.VII. Angular Description of Singular Functions

For simplicity, let us go back to the situation where Hypothesis C.11 is satisfied and let us consider $\Phi^0[\mathscr{x}']$ like in Theorem C.12, as well as its sequence of shadows $(\Phi^p[\mathscr{x}'])_p$. Theorem C.12 tells us that $\Phi^p[\mathscr{x}']$ belongs to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_0^{\lambda+p})$, which means that there exist meromorphic $\alpha \mapsto \boldsymbol{q}^\pm[\mathscr{x}'](\alpha)$ (with $\mathscr{C}^\infty(\mathscr{E})$ dependence on $\mathscr{x}'$) such that

$$\Phi^p[\mathscr{x}'] = \int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^{\lambda+p} \boldsymbol{q}^+[\mathscr{x}'](\alpha)\, d\alpha + \int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^{\lambda+p}\, \boldsymbol{q}^-[\mathscr{x}'](\alpha)\, d\alpha.$$

But, in fact, the vector-functions $\boldsymbol{q}^\pm[\mathscr{x}']$ are not *arbitrary* meromorphic functions in the unit disc: their poles belong to the set of the roots $\{\alpha_\ell^\pm[\mathscr{x}']\}_{\ell=1,\ldots,m_\pm}$, cf Eq. (C.13).

As a consequence, as we are going to show, it is possible to give a *modular representation* of the $\Phi^p[\mathscr{x}']$, if we assume

The multiplicities $n_\ell^\pm$ of $\alpha_\ell^\pm[\mathscr{x}']$ are constant on $\mathscr{E}$. $(\mathfrak{H}_{\mathrm{C1}})$

Let $\boldsymbol{\alpha}[\mathscr{x}']$ denote the set $\{(\alpha_\ell^\pm[\mathscr{x}'], n_\ell^\pm)\}$ of the roots with their multiplicities.

Definition C.16. Under hypothesis $(\mathfrak{H}_{\mathrm{C1}})$, for any $\mu \in \mathbb{C}$ and $p \in \mathbb{N}$, let us denote by $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_{\boldsymbol{\alpha},p}^\mu)$ the subspace of homogeneous functions $f[\mathscr{x}'] \in \mathfrak{H}^\mu$ which admit a representation as:

$$f[\mathscr{x}'] = \int_{|\alpha|=\delta} (\alpha\zeta + \bar{\zeta})^\mu q^+[\mathscr{x}'](\alpha)\, d\alpha + \int_{|\alpha|=\delta} (\zeta + \alpha\bar{\zeta})^\mu\, q^-[\mathscr{x}'](\alpha)\, d\alpha \tag{C.27}$$

where the functions $q^+[\mathscr{x}']$ and $q^-[\mathscr{x}']$ are meromorphic in α, $\mathscr{C}^\infty$ in $\mathscr{x}'$, with poles only in the roots $\alpha_\ell^+[\mathscr{x}']$ of order $\leq pn_\ell^+$ and $\alpha_\ell^-[\mathscr{x}']$ of order $\leq pn_\ell^-$, respectively. Let us denote by $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_{\boldsymbol{\alpha},p}^\mu)$ the space $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}_{\boldsymbol{\alpha},p}^\mu) \otimes \mathbb{C}^N$. ■

With these definitions, we have the following properties.

(i) By Eq. (C.14), the kernel elements $\boldsymbol{w}_\ell^\pm(\mu)$ belong to $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^\mu_{\boldsymbol{\alpha},1})$.

(ii) The definition (C.17) of $\mathfrak{R}_0$ gives us that for any $\mu \in \mathbb{C}$ and any $p \in \mathbb{N}$, $\mathfrak{R}_0$ acts:

$$\mathfrak{R}_0 : \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu-2d}_{\boldsymbol{\alpha},p}) \longrightarrow \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu}_{\boldsymbol{\alpha},p+1}).$$

(iii) The proof of Lemma C.5 yields that for any $\mu \in \mathbb{C}$ and any $p \in \mathbb{N}$, $\mathfrak{A}_j$ acts:

$$\mathfrak{A}^j : \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu}_{\boldsymbol{\alpha},p}) \longrightarrow \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu+j-2d}_{\boldsymbol{\alpha},p}) \times \mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu+j-\boldsymbol{\rho}}_{\boldsymbol{\alpha},p}).$$

Revisiting the proofs of Lemma C.4 and Proposition C.10 we obtain

Theorem C.17. *Under the assumptions of Theorem* C.12 *and under hypothesis* $(\mathfrak{H}_{\mathrm{C}1})$, *for any* $p \in \mathbb{N}_0$ *the edge singular functions* Φ^p *belong to* $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\lambda+p}_{\boldsymbol{\alpha},p+1})$.

Since there holds for any $\mu \in \mathbb{C}$, for any α and $\alpha_0 \in \mathbb{C}$

$$(\alpha\zeta + \overline{\zeta})^\mu = (\alpha_0\zeta + \overline{\zeta})^\mu + \sum_{k\geq 1} c_{\mu,k}(\alpha-\alpha_0)^k \zeta^k (\alpha_0\zeta+\overline{\zeta})^{\mu-k}$$

$$(\zeta + \alpha\overline{\zeta})^\mu = (\zeta + \alpha_0\overline{\zeta})^\mu + \sum_{k\geq 1} c_{\mu,k}(\alpha-\alpha_0)^k \overline{\zeta}^k (\zeta+\alpha_0\overline{\zeta})^{\mu-k},$$

any function Φ in $\mathscr{C}^\infty(\mathscr{E}, \mathfrak{H}^{\mu}_{\boldsymbol{\alpha},p})$ has a representation as

$$\Phi = \sum_{\ell=1}^{m_+} \sum_{k=0}^{pn_\ell^+-1} \zeta^k \big(\alpha_\ell^+[\mathscr{x}']\zeta + \overline{\zeta}\big)^{\mu-k} c^+_{k,\ell}[\mathscr{x}'] + \sum_{\ell=1}^{m_-} \sum_{k=0}^{pn_\ell^--1} \overline{\zeta}^k \big(\zeta + \alpha_\ell^-[\mathscr{x}']\overline{\zeta}\big)^{\mu-k} c^-_{k,\ell}[\mathscr{x}'],$$

with $\mathscr{C}^\infty(\mathscr{E})$ coefficients $c^\pm_{k,\ell}$. As a corollary of Theorem C.17 we obtain

Corollary C.18. *Under the assumptions of Theorem* C.12 *and under hypothesis* $(\mathfrak{H}_{\mathrm{C}1})$, *for any* $p \in \mathbb{N}_0$ *the edge singular functions* Φ^p *have representations as*

$$\begin{aligned}\Phi^p[\mathscr{x}'] = &\sum_{\ell=1}^{m_+} \sum_{k=0}^{(p+1)n_\ell^+-1} \zeta^k \big(\alpha_\ell^+[\mathscr{x}']\zeta + \overline{\zeta}\big)^{\lambda+p-k} c^{p,+}_{k,\ell}[\mathscr{x}'] \\ &+ \sum_{\ell=1}^{m_-} \sum_{k=0}^{(p+1)n_\ell^--1} \overline{\zeta}^k \big(\zeta + \alpha_\ell^-[\mathscr{x}']\overline{\zeta}\big)^{\lambda+p-k} c^{p,-}_{k,\ell}[\mathscr{x}'],\end{aligned} \tag{C.28}$$

with coefficients $c^{p,\pm}_{k,\ell} \in \mathscr{C}^\infty(\mathscr{E})$.

Let us denote by $\Psi_{\ell,\omega}$ for $\omega = \pm 1$ the fundamental functions

$$\Psi_{\ell,+}(\mathscr{x}', r, \theta) = \alpha_\ell^+[\mathscr{x}']\zeta + \overline{\zeta} \quad \text{and} \quad \Psi_{\ell,-}(\mathscr{x}', r, \theta) = \zeta + \alpha_\ell^-[\mathscr{x}']\overline{\zeta}.$$

The comparison with the fundamental angular functions introduced in Eq. (B.81) is quite simple: Since, if L is real,

$$\tau_\ell = \frac{i(\alpha_\ell^+ - 1)}{\alpha_\ell^+ + 1} \quad \text{and} \quad \overline{\tau}_\ell = \frac{i(1-\alpha_\ell^-)}{\alpha_\ell^- + 1},$$

there holds

$$\Psi_{\ell,\omega}(r,\theta) = (\alpha_\ell + 1)\, r\, \psi_{\ell,\omega}(\theta), \quad \ell = 1,\ldots,n_\ell^\omega, \quad \omega = \pm 1,$$

and conditions $(\mathfrak{H}_{B6})$ and $(\mathfrak{H}_{C1})$ are two formulations of the same assumption.

Coming back to the expansion (C.28), we note that the N-component vector functions

$$\boldsymbol{d}^p_{\ell,+}(\mathscr{x}',y) := \sum_{k=0}^{(p+1)n_\ell^+-1} \zeta^k\, \Psi_{\ell,+}^{(p+1)n_\ell^+-1-k}(\mathscr{x}',y)\, c_{k,\ell}^{p,+}[\mathscr{x}']$$
$$\boldsymbol{d}^p_{\ell,-}(\mathscr{x}',y) := \sum_{k=0}^{(p+1)n_\ell^--1} \bar{\zeta}^k\, \Psi_{\ell,-}^{(p+1)n_\ell^--1-k}(\mathscr{x}',y)\, c_{k,\ell}^{p,-}[\mathscr{x}']$$

are polynomial in y, therefore $\mathscr{C}^\infty(\mathbb{R}^{n+1})$, and there holds

$$\Phi^p[\mathscr{x}'] = \sum_{\omega=\pm 1} \sum_{\ell=1}^{m_\omega} \Psi_{\ell,\omega}^{\lambda-(p+1)(n_\ell^\omega-1)}(\mathscr{x}',y)\, \boldsymbol{d}^p_{\ell,\omega}(\mathscr{x}',y). \tag{C.29}$$

If condition $(\mathfrak{H}_{A3})$ holds (i.e., if $n_\ell^\pm = 1$, $\ell = 1,\ldots,m$) Eq. (C.29) takes the simpler form

$$\Phi^p[\mathscr{x}'] = \sum_{\omega=\pm 1} \sum_{\ell=1}^{m_\omega} \Psi_{\ell,\omega}^{\lambda}(\mathscr{x}',y)\, \boldsymbol{d}^p_{\ell,\omega}(\mathscr{x}',y), \tag{C.30}$$

which means that the singular factors $\Psi^\lambda_{\ell,\omega}$ do not depend on p.

As a final consequence of formulas (C.29) and (C.30), we obtain "modular representations" of the solutions of elliptic BVP in the domain $\Omega = \mathbb{R}^{n+1}\backslash\mathscr{M}$:

Theorem C.19. *Let the hypotheses* $(\mathfrak{H}_{A1})$, $(\mathfrak{H}_{A2})$, *and* $(\mathfrak{H}_{C1})$ *be satisfied.*

(i) *Any solution* $\boldsymbol{u}$ *of the boundary value problem* (A.1) *with smooth right hand side* $\boldsymbol{f}$ *has the following asymptotic expansion as* $r \to 0$: *For any integer* $K \geq 0$

$$\boldsymbol{u} = \sum_{\omega=\pm 1} \sum_{\ell=1}^{m_\omega} \Psi_{\ell,\omega}^{1/2-(K+1)(n_\ell^\omega-1)}(\mathscr{x}',y)\, \boldsymbol{d}^{[K]}_{\ell,\omega}(\mathscr{x}',y) + \boldsymbol{u}_{\mathrm{reg},K} + \boldsymbol{u}_{\mathrm{rem},K}, \tag{C.31}$$

where the vector-coefficients $\boldsymbol{d}^{[K]}_{\ell,\omega}$ *are* $\mathscr{C}^\infty(\mathbb{R}^{n+1})$ *and the regular parts* $\boldsymbol{u}_{\mathrm{reg},K}$ *and* $\boldsymbol{u}_{\mathrm{rem},K}$ *are as in Proposition* A.2.

(ii) *If the multiplicities* n_ℓ^ω *are all equal to* 1, *cf hypothesis* $(\mathfrak{H}_{A3})$, *then* $\boldsymbol{u}$ *admits the global decomposition*

$$\boldsymbol{u} = \sum_{\omega=\pm 1} \sum_{\ell=1}^{m} \Psi_{\ell,\omega}^{1/2}(\mathscr{x}',y)\, \boldsymbol{d}^\infty_{\ell,\omega}(\mathscr{x}',y) + \boldsymbol{u}_{\mathrm{reg},\infty}, \tag{C.32}$$

where all vector-coefficients $\boldsymbol{d}^\infty_{\ell,\omega}$ *and* $\boldsymbol{u}_{\mathrm{reg},\infty}$ *are* $\mathscr{C}^\infty(\mathbb{R}^{n+1})$.

Remark C.20. The multiplicities n_ℓ^ω are in fact the order of the poles of the inverse of the Cayley symbol $L^\omega(\alpha)^{-1}$ in α_ℓ^ω. They can be smaller than the total multiplicity of α_ℓ^ω. An example for this is the case of isotropic elasticity in $\mathbb{R}^3$ where $L^\pm(\alpha)^{-1}$ have 0 as only pole, but the multiplicity is 2 (and not 3). The fundamental functions $\Psi_{\ell,\omega}$ are simply

$$\Psi_+ = \bar{\zeta} = (y_1 - iy_2) \quad \text{and} \quad \Psi_- = \zeta = (y_1 + iy_2),$$

and expansion (C.31) takes the form, compare with Chkadua and Duduchava (2000)

$$\boldsymbol{u} = \bar{\zeta}^{1/2-(K+1)}\, \boldsymbol{d}_+^{[K]}(x', y) + \zeta^{1/2-(K+1)}\, \boldsymbol{d}_-^{[K]}(x', y) + \boldsymbol{u}_{\mathrm{reg},K} + \boldsymbol{u}_{\mathrm{rem},K}, \tag{C.33}$$

with $\mathscr{C}^\infty(\mathbb{R}^{n+1})^N$ coefficients $\boldsymbol{d}_\pm^{[K]}$. ■

REFERENCES

Agmon, S., Douglis, A., Nirenberg, L. (1964). Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.* 17:35–92.

Andersson, L., Chrusciel, P. T. (1993). Hyperboloidal Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity. *Phys. Rev. Lett.* 70:2829–2832.

Bennish, J. (1993). Asymptotics for elliptic boundary value problems for systems of pseudo-differential equations. *Journal for Mathematical Analysis and Applications* 179:417–445.

Boutet de Monvel, L. (1971). Boundary problems for pseudo-differential operators. *Acta Math.* 126:11–51.

Chazarain, J., Piriou, A. (1982). *Introduction to the Theory of Linear Partial Differential Equations*. Amsterdam: North-Holland Publishing Co.

Chkadua, O., Duduchava, R. (1998). Asymptotics of solutions to some boundary value problems of elasticity with cuspidal edges. *Memoirs on Mathematical Physics and Differential Equations* 15:29–58.

Chkadua, O., Duduchava, R. (2000). Asymptotics of functions represented by potentials. *Russian Journal of Mathematical Physics* 7(1):15–47

Chkadua, O., Duduchava, R. (2001). Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotics. *Math. Nach.* 222:79–139.

Costabel, M., Dauge, M. (1993a). General edge asymptotics of solutions of second order elliptic boundary value problems I & II. *Proc. Royal Soc. Edinburgh* 123A:109–184.

Costabel, M., Dauge, M. (1993b). Construction of corner singularities for Agmon–Douglis–Nirenberg elliptic systems. *Math. Nach.* 162:209–237.

Costabel, M., Dauge, M. (1994). Stable asymptotics for elliptic systems on plane domains with corners. *Comm. Partial Differential Equations* 9&10:1677–1726.

Costabel, M., Dauge, M. (2002). Crack singularities for general elliptic systems. *Math. Nach.* 235:29–49.

Costabel, M., Stephan, E. P. (1987). An improved boundary element Galerkin method for three-dimensional crack problems. *Integral Equations and Operator Theory* 10:467–504.

Dauge, M. (1988). Elliptic boundary value problems in corner domains—smoothness and asymptotics of solutions. In: *Lecture Notes in Mathematics*. Vol. 1341. Berlin: Springer-Verlag.

Duduchava, R. (1984). On multidimensional singular integral operators I. The half-space case. II. The case of manifolds. *Journal of Operator Theory* 11:41–76 and 199–214.

Duduchava, R., Natroshvili, D. (1998). Mixed crack type problem in anisotropic elasticity. *Mathematische Nachrichten* 191:83–107.

Duduchava, R., Natroshvili, D., Shargorodsky, E. (1995). Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts. I–II. *Georgian Mathematical Journal* 2:123–140 and 259–276.

Duduchava, R., Sändig, A. M., Wendland, W. L. (1999). Interface cracks in anisotropic composites. *Mathematical Methods in Applied Sciences* 221:413-1446.

Duduchava, R., Wendland, W. L. (1995). The Wiener–Hopf method for systems of pseudodifferential equations with an application to crack problems. *Integral Equations Operator Theory* 23(3):294–335.

Eskin, G. (1981). *Boundary Value Problems for Elliptic Pseudodifferential Equations*. Rhode Island, Providence: AMS.

Grisvard, P. (1985). *Boundary Value Problems in Non-Smooth Domains*. London: Pitman.

Grisvard, P. (1989). Singularités en élasticité. *Arch. Rational Mech. Anal.* 107(2):157–180.

Hörmander, L. (1983). *The Analysis of Linear Partial Differential Operators, v.I–IV*. Heidelberg: Springer–Verlag.

Kondrat'ev, V. (1967). Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* 16:227–313.

Kondrat'ev, V. A. (1970). Singularities of a solution of Dirichlet's problem for a second order elliptic equation in a neighborhood of an edge. *Differential Equations* 13:1411–1415.

Kozlov, V. (1990). Singularities of solutions of the Dirichlet problem for elliptic equations in a neighborhood of corner points. *Leningrad Math. J.* 4:967–982.

Kozlov, V., Maz'ya, V., Rossmann, J. (2001). *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*. Rhode Island: American Mathematical Society, (Math. Surveys and Monographs, v. 85).

Kozlov, V., Maz'ya, V. (1991). On stress singularities near the boundary of a polygonal crack. *Proc. Roy. Soc. Edinburgh Sect. A* 117(1–2):31–37.

Kupradze, V., Gegelia, T., Basheleishvili, M., Burchuladze, T. (1979). *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. Amsterdam, North-Holland.

Mazzeo, R. (1992). Edge operators in geometry. In: Symposium "Analysis on Manifolds with Singularities" (Breitenbrunn, 1990) *Teubner-Texte Math.* 131:127–137.

Maz'ya, V. G., Plamenevskii, B. A. (1984). Estimates in L^p and in Hölder classes and the Miranda–Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. *Amer. Math. Soc. Transl. (2)* 123:1–56.

Maz'ya, V., Plamenevskii, B. (1980). L^p estimates of solutions of elliptic boundary value problems in a domain with edges. *Trans. Moscow Math. Soc.* 1: 49–97.

Maz'ya, V., Rossmann, J. (1988). Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten. *Math. Nachr.* 138:27–53.

Nazarov, S., Plamenevskii, B. (1994). Elliptic problems in domains with piecewise smooth boundaries. In: *Expositions in Mathematics 13*. Berlin: Walter de Gruyter.

Nazarov, S., Plamenevskii, B. (1995a). A generalized Green's formula for elliptic problems in domains with edges. *J. Math. Sci.* 73(6):674–700.

Nazarov, S., Plamenevskii, B. (1995b). Selfadjoint elliptic problems: the scattering operator and the polarization operator on edges of the boundary. *St. Petersburg Math. J.* 6(4):839–863.

Nikishkin, A. (1979). Singularities of the solution of the Dirichlet problem for a second order equation in a neighborhood of an edge. *Moscow Univ. Math. Bull.* 34(2):53–64.

Schulze, B.-W. (1998). *Boundary Value Problems and Pseudo-Differential Operators*. Chichester: J. Wiley.

Triebel, H. (1983). *Theory of Function Spaces*. Boston: Birkhäuser Verlag.

Wendland, W. L., Stephan, E. P. (1990). A hypersingular boundary integral method for two-dimensional screen and crack problems. *Arch. Rational Mech. Anal.* 112(4):363–390.

Received October 2001
Revised January 2003